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On the Thom–Sebastiani Property of Quasi-Homogeneous Isolated Hypersurface Singularities

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Abstract. Let $(V,0) \subset (\mathbb{C}^n,0)$ be a quasi-homogeneous isolated hypersurface singularity. In this paper we prove under certain weight conditions a relation between the property of $(V,0)$ being of Thom–Sebastiani type and the dimension of toral Lie subalgebras contained in the Yau algebra $L(V)$.

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1. Introduction

Let $n \in \mathbb{N}_{>0}$. We denote by $\mathbb{C}\{\mathbf{x}\} = \mathbb{C}\{x_1,\dots,x_n\}$ the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$ and by $\mathfrak{m}$ its unique maximal ideal, which is generated by germs of holomorphic functions which vanish at the origin. For a power series $f \in \mathbb{C}\{\mathbf{x}\}$ we denote by J_f the Jacobian ideal $J_f = \left\langle f, \frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right\rangle$.

By definition, the isomorphism class of a hypersurface germ $(V,0) \subset (\mathbb{C}^n,0)$, where $V = \{f = 0\}$, is given by the isomorphism class of the algebra $X_f = \mathbb{C}\{\mathbf{x}\}/I$, where $I = \langle f \rangle$ is the principal ideal generated by f. If f defines an isolated hypersurface singularity Mather and Yau have shown, in their celebrated paper [12] from 1982, that the isomorphism class of $(V,0) \subset (\mathbb{C}^n,0)$ is determined by the isomorphism class of the moduli algebra $A(V) = \mathbb{C}\{\mathbf{x}\}/J_f$. This result is known as the Mather–Yau Theorem. Hauser and Müller studied in [5] the Lie algebra of derivations of the moduli algebra $L(V) = \mathrm{Der}(A(V), A(V))$ and showed that the isomorphism class for isolated hypersurface singularities of dimension $n \geq 3$ is determined by the isomorphism class of $L(V)$. Although we are not using the result by Hauser and Müller in this paper, it motivates the study of $L(V)$. In order to distinguish $L(V)$ from Lie algebras of other types of singularities, $L(V)$ is called Yau algebra.

In the last years Hussain, Yau, Zuo and collaborators have constructed many new natural connections between the set of isolated hypersurface singularities and the set of finite dimensional solvable (respectively nilpotent) Lie algebras (see for example[1, 7–9]).

In this paper we are going to investigate the Yau algebra itself in the context of quasi-homogeneous isolated hypersurface singularities. In [16] Xu and Yau have stated an equivalence between the quasi-homogeneity of a defining equation and the existence of a positive grading of the moduli algebra. In the same article this equivalence has been extended to properties of the Yau algebra.

We are going to establish a relation between the dimension of a toral Lie subalgebra of $L(V)$ and the property of $(V,0)$ being a Thom–Sebastiani singularity under certain constraints on the weights of a quasi-homogeneous defining equation f. Let us make these notions more precise:

Definition 1. *Let $(V,0) \subset (\mathbb{C}^n, 0)$ be an isolated hypersurface singularity.*

(1) *We say $(V,0)$ is* quasi-homogeneous*, if there exists coordinates $x_1,\dots,x_n$, weights $w_1,\dots,w_n \in \mathbb{N}_{>0}$ and a polynomial $f \in \mathbb{C}\{x_1,\dots,x_n\}$, which is weighted-homogeneous with respect to $\mathbf{w} = (w_1,\dots,w_n)$, such that*

$$(V,0) \cong (V(f),0).$$

If the weight vector is known, we also say that $(V,0)$ is **w**-homogeneous.

(2) *We say $(V,0)$ is of* Thom–Sebastiani type*, if there exists coordinates $x_1,\dots,x_k$ and $y_{k+1},\dots,y_n$ and power series $h \in \mathbb{C}\{x_1,\dots,x_k\}, g \in \mathbb{C}\{y_{k+1},\dots,y_n\}$, such that*

$$(V,0) \cong (V(h+g),0).$$

We say $(V,0)$ has quasi-homogeneous summands*, if h and g can be chosen to be quasi-homogeneous.*

Then our conjecture is the following:

Conjecture 2. *Let $(V,0) \subset (\mathbb{C}^n,0)$ be a quasi-homogeneous isolated hypersurface singularity. Then $(V,0)$ is of Thom–Sebastiani type with quasi-homogeneous Thom–Sebastiani summands if, and only if, $L(V)$ contains a toral subalgebra $\mathfrak{t}$ with $\dim_{\mathbb{C}} \mathfrak{t} \geq 2$.*

The goal of this paper is to prove the following theorem, which proves a special version of Conjecture 2:

Theorem 3. *Let $\mathbf{w} = (w_,\dots,w_n) \in \mathbb{N}^n_{>0}$ and $(V,0)$ be a $\mathbf{w}$-homogeneous isolated hypersurface singularity. Assume that one of the following properties holds:*

(a) *$A(V)$ is $(\mathbb{Z}^n,+)$ graded.*

(b) *$\mathbf{w}$ satisfies, after possibly permuting the variables,*

$$w_1 > \ldots > w_n > \frac{w_1}{2}.$$

Then $(V,0)$ is of Thom–Sebastiani type with quasi-homogeneous Thom–Sebastiani summands if, and only if, $L(V)$ contains a toral Lie subalgebra $\mathfrak{t}$ with $\dim_{\mathbb{C}} \mathfrak{t} \geq 2$.

Remark 4. In [18] Yau and Zuo showed that under the hypothesis from Theorem 3 (b) the Halperin conjecture holds true for $L(V)$.

This paper is based on work contained in the authors Ph.D. thesis [3].

2. Analytic Gradings and Derivations

In order to understand the relation between the toral Lie subalgebra of the Yau algebra $L(V)$ and the Thom–Sebastiani property, we first need to introduce the theory of analytic gradings by Scheja and Wiebe. This theory relates analytic gradings to a certain type of derivations which then give a connection to a toral Lie subalgebra of the Yau algebra. The theory presented here works for singularities in general, so we do not restrict ourselves to isolated hypersurface singularities.

Definition 5. *Let A be an analytic $\mathbb{C}$-algebra and $(G,+)$ an abelian group. A is a* graded algebra *if we have a system of group homomorphisms $\pi_g^A : A \to A$ for $g \in G$, which induce group homomorphisms $\overline{\pi_g^A} : A/\mathfrak{m}_A^n \to A/\mathfrak{m}_A^n$ that define a finite grading on $A/\mathfrak{m}_A^n$ for all $n \in \mathbb{N}$. A is a* multigraded algebra *if $G = \mathbb{C}^k$ for a $k \in \mathbb{N}$.*

Remark 6. In case $A = \mathbb{C}\{\mathbf{x}\}/I$ and $G = \mathbb{Z}$ or $G = \mathbb{Z}^k$ the notion of analytic gradings by Scheja and Wiebe can be understood as in the case of polynomial rings: By choosing a suitable system of coordinates, the ideal I can be generated by weighted (multi)homogeneous power series.

In order to establish a connection between the Yau algebra and the notion of analytic gradings, we need the following definitions involving the Lie algebra of derivations $\operatorname{Der}(A) := \operatorname{Der}(A,A)$.

Definition 7. *Let A be an analytic $\mathbb{C}$-algebra and $I \subset A$ an ideal. Then*

$$\operatorname{Der}_I(A) = \{\delta \in \operatorname{Der}(A) \mid \delta(I) \subset I\}.$$

In case $I = \mathfrak{m}_A$ we write $\operatorname{Der}'(A)$ instead of $\operatorname{Der}_{\mathfrak{m}_A}(A)$.

Definition 8. *Let $A = \mathbb{C}\{\mathbf{x}\}$. We call a derivation $\delta \in \operatorname{Der}(A)$* diagonalizable *if there exists a coordinate system $y_1,\dots,y_n$, such that*

$$\delta = \sum_{i=1}^{n} a_i y_i \frac{\partial}{\partial y_i},$$

where $a_i \in \mathbb{C}$ for $i = 1,\dots,n$. We say that pairwise different diagonalizable derivations $\delta_1,\dots,\delta_k \in \operatorname{Der}(A)$ are simultaneously diagonalizable *if there exists a coordinate system $y_1,\dots,y_n$, such that*

$$\delta_j = \sum_{i=1}^{n} a_{ij} y_i \frac{\partial}{\partial y_i},$$

where $a_{ij} \in \mathbb{C}$ for $i = 1,\dots,n$ and for $j = 1,\dots,k$.

We obtain the following correspondence between a $(\mathbb{C},+)$ grading of A and the existence of diagonalizable derivation.

Theorem 9 ([15, (2.2) and (2.3)]). *Let $A = \mathbb{C}\{\mathbf{x}\}/I$ be an analytic $\mathbb{C}$-algebra. Then A is $(\mathbb{C},+)$ graded if, and only if, there exists a diagonalizable derivation $\delta \in \operatorname{Der}_I(\mathbb{C}\{\mathbf{x}\})$.*

Using simultaneously diagonalizable derivations we obtain the following generalization to multigraded algebras.

Theorem 10 ([3, Theorem 2.26 and Theorem 2.27]). *Let $A = \mathbb{C}\{\mathbf{x}\}/I$ be an analytic $\mathbb{C}$-algebra. Then A is $(\mathbb{C}^k,+)$ graded if, and only if, there exist k simultaneously diagonalizable derivations $\delta_1,\dots,\delta_k \in \operatorname{Der}_I(\mathbb{C}\{\mathbf{x}\})$ for $i = 1,\dots,k$.*

To obtain a relation between A being multigraded and derivations in $L(V)$ we need the following result.

Theorem 11 ([19, Theorem 2.2]). *Let $A = \mathbb{C}\{\mathbf{x}\}$ and $I \subset A$ an ideal. Then there exists a natural isomorphism of Lie algebras*

$$\operatorname{Der}_I(A)/I\operatorname{Der}(A) \cong \operatorname{Der}(A/I).$$

The previous theorems imply the following proposition:

Proposition 12. *Let A be an analytic $\mathbb{C}$-algebra. Then A is (multi)graded if, and only if, $\operatorname{Der}(A)$ contains a toral Lie subalgebra $\mathfrak{t}$ with $\dim_{\mathbb{C}} \mathfrak{t} \geq 1$.*

Proof. Let $A \cong \mathbb{C}\{\mathbf{x}\}/I$. First we assume that A is (multi)graded. Theorem 11 implies the existence of a canonical map from $\mathrm{Der}_I(\mathbb{C}\{\mathbf{x}\})$ into $\mathrm{Der}(A)$. The image of a diagonalizable derivation under this map is still diagonalizable and reduces to the notion of diagonalizability of endomorphisms of vector spaces. Thus a (multi)grading of A implies the existence of toral Lie subalgebra $\mathfrak{t} \subset \mathrm{Der}(A)$ with $\dim_{\mathbb{C}} \mathfrak{t} \geq 1$. Let us now assume that there exists a toral Lie subalgebra $\mathfrak{t} \subset \mathrm{Der}(A)$ with $\dim_{\mathbb{C}} \mathfrak{t} \geq 1$. Due to [15, (2.1)] we can lift $\mathfrak{t}$ to $\mathrm{Der}_I(\mathbb{C}\{\mathbf{x}\})$. By Theorem 10 we obtain that A is (multi)graded. □

By Proposition 12 the property of $A(V)$ being (multi)graded is reduced to the investigation of the existence of toral Lie subalgebras of $L(V)$. We obtain the following special version of Proposition 12:

Corollary 13. *Let $(V,0) \subset (\mathbb{C}^n,0)$ be an isolated hypersurface singularity. Then $A(V)$ is (multi)graded if, and only if, $L(V)$ contains a toral Lie subalgebra $\mathfrak{t}$ with $\dim_{\mathbb{C}} \mathfrak{t} \geq 1$.*

Remark 14. So far the gradings we considered are complex valued. Due to [15, 3.2] we can restrict ourselves to rational gradings, which we do from now on.

Next we want to consider computational aspects in order to provide the reader with examples. Let A be an analytic $\mathbb{C}$-algebra. Then every derivation $\delta \in \mathrm{Der}'(A)$ induces a linear map $\bar{\delta}$ on the $\mathbb{C}$-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$. Denote by $\mathfrak{gl}(\mathfrak{m}_A/\mathfrak{m}_A^2)$ the Lie algebra of endomorphisms of $\mathfrak{m}_A/\mathfrak{m}_A^2$. We define the following:

Definition 15. *Let A be an analytic $\mathbb{C}$-algebra. Denote by ρ the morphism of Lie algebras $\rho : \mathrm{Der}'(A) \to \mathfrak{gl}(\mathfrak{m}_A/\mathfrak{m}_A^2), \delta \mapsto \bar{\delta}$. We call $\mathrm{Der}(A)_0 = \rho(\mathrm{Der}'(A))$ the* linearization of *$\mathrm{Der}'(A)$. In case A is the moduli algebra of an isolated hypersurface singularity $(V,0) \subset (\mathbb{C}^n,0)$, we denote the linearization of $L(V)$ by $L(V)_0$.*

The next lemma will make it possible for us to compute possible gradings of a zero-dimensional analytic $\mathbb{C}$-algebra A using $\mathrm{Der}(A)_0$, without knowing the coordinate system in which the algebra is (multi)homogeneous. It also shows that the dimension of any maximal toral Lie subalgebra is an invariant associated to A, which can be obtain from $\mathrm{Der}(A)_0$. This result follows from [6, Corollary 10.7], [6, Corollary 21.3C] and [13, Lemma 1.39].

Lemma 16. *Let A be a zero-dimensional analytic $\mathbb{C}$-algebra. Furthermore, let $\mathfrak{t} \subset \mathrm{Der}(A)$ and $\mathfrak{t}' \subset \mathrm{Der}(A)_0$ be arbitrary maximal toral Lie subalgebras. Then $\dim_{\mathbb{C}} \mathfrak{t} = \dim_{\mathbb{C}} \mathfrak{t}'$. In particular, the dimension of maximal toral Lie subalgebras is an invariant of $\mathrm{Der}(A)$ respectively $\mathrm{Der}(A)_0$.*

Now we are able to consider an example, where we compute possible weight-vectors of an analytic algebra as well as the dimension of the maximal toral Lie subalgebras.

Example 17. Let $f_1 = 7x^6y \in \mathbb{C}[x,y]$ and $f_2 = x^7 - 11y^{10} \in \mathbb{C}[x,y]$. Define $I = \langle f_1, f_2 \rangle$ and $A = \mathbb{C}[x,y]/I$. Furthermore, let

$$M = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \in \mathbb{C}[x,y]^{2\times 2}.$$

Denote by $\overline{M}$ the matrix resulting from M by considering the entries as elements of A. Then $\mathrm{Der}(A) = \mathrm{syz}_A(\overline{M})$. Using the computer algebra system Singular (see [2]) we obtain that $\mathrm{Der}(A)$ is generated as an A-module by the derivations

$$\delta_1 = 10x\frac{\partial}{\partial x} + 7y\frac{\partial}{\partial y}, \quad \delta_2 = y^2\frac{\partial}{\partial y}, \quad \delta_3 = -11y^9\frac{\partial}{\partial x} + 6x^6\frac{\partial}{\partial y}.$$

We can see that δ_1 is a diagonalizable derivation, hence A is homogeneous with respect to the weight-vector $\mathbf{w} = (10,7)$. A further computation shows that $\dim_{\mathbb{C}} \mathrm{Der}(A)_0 = 1$. Then Lemma 16 implies that $\mathfrak{t}$ is a maximal toral Lie subalgebra $\mathrm{Der}(A)$.

3. Proof of Theorem 3

So far, we have seen that the notion of grading of the moduli algebra $A(V)$ is related to the existence of a toral Lie subalgebra of the Yau algebra $L(V)$. In [16] it was shown by Yau and Xu that $(V,0)$ is quasi-homogeneous if, and only if, $A(V)$ admits a positive grading. Due to this we are going to work in the setup of quasi-homogeneous isolated hypersurface singularities to establish a relationship between the Thom–Sebastiani property of $(V,0)$ and the existence of an at least two-dimensional toral Lie subalgebra $\mathfrak{t} \subset L(V)$. In the following we use the classical notion of (multi)graded polynomial rings, see for example [11] for more details. This notion coincides with the notion introduced in Section 2.

This section is dedicated to the proof of Theorem 3. We split the section into two parts, each proving one of the statements of the theorem.

3.1. *Proof of Theorem 3 (a)*

Before proving the first part of the theorem, we show that we can assume that the order of the defining equation f of V is at least 3.

Proposition 18. *Let* $(V,0) \subset (\mathbb{C}^n,0)$ *be a quasi-homogeneous isolated hypersurface singularity and* $f \in \mathbb{C}\{\mathbf{x}\}$ *with* $(V(f),0) \cong (V,0)$*. Assume that* $\operatorname{ord}(f) = 2$*. Then* $(V,0)$ *is of Thom–Sebastiani type with quasi-homogeneous Thom–Sebastiani summands.*

Proof. By the Splitting Lemma (see [10, Lemma 9.2.10]) we know that f is right-equivalent to $x_1^2 + \ldots + x_k^2 + g \in \mathbb{C}\{\mathbf{x}\}$, where $g \in \mathbb{C}\{x_{k+1},\ldots,x_n\}$ defines an isolated hypersurface singularity and $\operatorname{ord}(g) \geq 3$. We obtain the isomorphism of moduli algebras $A(V(f)) \cong A(V(g))$. In particular, $A(V(g))$ is positively graded, since f is quasi-homogeneous. It follows from [16, Theorem 1.2] and the Mather–Yau Theorem that g is right-equivalent to a quasi-homogeneous polynomial $h \in \mathbb{C}\{x_{k+1},\ldots,x_n\}$. Thus $(V(f),0)$ is of Thom–Sebastiani type with quasi-homogeneous Thom–Sebastiani summands. □

Remark 19. Proposition 18 justifies to consider Theorem 3 only for the case $\operatorname{ord}(f) \geq 3$.

For the next result, we need the following definition.

Definition 20. *Let* $f \in \mathbb{C}\{\mathbf{x}\}$*. We say* f *defines a* Brieskorn–Pham singularity *if there exist integers* $a_1,\ldots,a_n \in \mathbb{N}_{>0}^n$*, such that* f *is right-equivalent to* $x_1^{a_1} + \ldots + x_n^{a_n}$*.*

Theorem 3 (a) follows from the following proposition, which gives a characterization of Brieskorn–Pham singularities.

Proposition 21. *Let* $f \in \mathbb{C}\{\mathbf{x}\}$ *define an isolated hypersurface singularity. Then the following are equivalent:*

1. f *defines a Brieskorn–Pham singularity.*
2. $A(V(f))$ *is* $(\mathbb{Z}^n,+)$ *graded.*
3. *There exists an automorphism* $\varphi \in \operatorname{Aut}(\mathbb{C}\{\mathbf{x}\})$*, such that* $\varphi(J_f)$ *is a monomial ideal.*

Proof. To keep notation short we set $A := A(V(f))$.

We first show that (1) implies (2). Let f define a Brieskorn–Pham singularity, i.e. there exist integers $a_i \in \mathbb{N}_{>0}, i = 1,\ldots,n$ with $f = \sum_{i=1}^n x_i^{a_i}$. Then $J_f = \langle x_1^{a_1-1},\ldots,x_n^{a_n-1}\rangle$. Denote by $e_i \in \mathbb{Z}^n, i = 1,\ldots,n$ the canonical basis of $\mathbb{Z}^n$. We define for arbitrary $k \in \mathbb{Z}$ the group homomorphism $\pi^A_{k\cdot e_i} : A \to A$ via

$$\sum_{\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{N}^n} c_\beta x_1^{\beta_1}\cdot\ldots\cdot x_n^{\beta_n} \mapsto \sum_{\beta\in\mathbb{N}^n \text{ s.th. } \beta_i=k} c_\beta x_1^{\beta_1}\cdot\ldots\cdot x_i^k\cdot\ldots\cdot x_n^{\beta_n}.$$

For general $v = \sum_{i=1}^n k_i e_i$ we define $\pi_v^A = \pi_{k_1 e_1}^A \circ \ldots \circ \pi_{k_n e_n}^A$. It follows immediately that π_v^A satisfies the requirements Definition 5, hence Statement (2) follows.

Next we show (2) implies (3). Since A is $(\mathbb{Z}^n, +)$ graded, there exists an isomorphism $\varphi \in \mathrm{Aut}(\mathbb{C}\{\mathbf{x}\})$, such that $\varphi(J_f)$ is generated by polynomials which are multihomogeneous with respect to the weights $\mathbf{w}_i = e_i$ for $i = 1, \ldots, n$. Thus $\varphi(J_f)$ is a monomial ideal.

Finally, we show (3) implies (1). By assumption there exists an isomorphism $\varphi \in \mathrm{Aut}(\mathbb{C}\{\mathbf{x}\})$, such that $\varphi(J_f) = J_{\varphi(f)}$ is a monomial ideal. Since $J_{\varphi(f)}$ defines a zero-dimensional complete intersection ideal, we obtain that it is generated by $x_1^{a_1}, \ldots, x_n^{a_n}$ for certain $a_1, \ldots, a_n \in \mathbb{N}_{>0}^n$. Then $g = x_1^{a_1+1} + \ldots + x_n^{a_n+1}$ satisfies $J_g = J_{\varphi(f)}$. By the Mather–Yau Theorem, f is right-equivalent to g, hence f defines a Brieskorn–Pham singularity. □

Remark 22. In [4] we characterize, together with M. Schulze, up to analytic isomorphism all hypersurface singularities with monomial Jacobian ideal J_f.

3.2. *Proof of Theorem 3 (b)*

Before we prove our result, we state a characterization of a zero-dimensional algebra to be the moduli algebra of a quasi-homogeneous isolated hypersurface singularity.

Remark 23. In order to keep our notation short, we will write ∂_{x_i} instead of $\frac{\partial}{\partial x_i}$.

To prove our result we need the following version of the Poincaré Lemma:

Lemma 24. *Let $F_1, \ldots, F_n \in \mathbb{C}\{\mathbf{x}\}$ with*

$$\partial_{x_j} F_i = \partial_{x_i} F_j$$

for all $1 \le i, j \le n$. Then there exists an $f \in \mathbb{C}\{\mathbf{x}\}$, such that $F_i = \partial_{x_i} f$.

Furthermore, we need the following auxiliary lemma, which is part of the proof of [17, Theorem 2]:

Lemma 25. *Let $f \in \mathfrak{m} \subset \mathbb{C}\{\mathbf{x}\}$. Assume that there exists a weight-vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Q}_{>0}^n$, such that the partial derivatives of f are $\mathbf{w}$-homogeneous. Then $f = \sum_{i=1}^n \frac{w_i}{\deg_{\mathbf{w}}(\partial_{x_i} f) + w_i} x_i \partial_{x_i} f$. In particular, f is quasi-homogeneous.*

Proof. The result follows immediately from the following computation:

$$f = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(t^{w_1} x_1, \ldots, t^{w_n} x_n) \mathrm{d}t = \sum_{i=1}^n \frac{w_i}{\deg_{\mathbf{w}}(\partial_{x_i} f) + w_i} x_i \partial_{x_i} f. \qquad \square$$

Using the Poincaré-Lemma and Lemma 25, Yau proved the following theorem:

Theorem 26 ([17, Theorem 3]). *Let $I \subset \mathbb{C}\{\mathbf{x}\}$ be an ideal and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}_{>0}^n$. Then $A := \mathbb{C}\{\mathbf{x}\}/I$ is the moduli algebra of a $\mathbf{w}$-homogeneous hypersurface singularity $f \in \mathbb{C}\{\mathbf{x}\}$ if and only if I is generated by $F_1, \ldots, F_n \in \mathbb{C}\{\mathbf{x}\}$ with the following properties:*

1. *the F_i are weighted-homogeneous with respect to* $\mathbf{w}$,
2. *$\partial_{x_j} F_i = \partial_{x_i} F_j$ for all $1 \le i, j \le n$, and*
3. *$\partial_{x_i} f = F_i$ for all $1 \le i \le n$.*

In the following we are going to work with polynomials, which are homogeneous with respect to two weight-vectors. Therefore we state the following definition

Definition 27. *Let $F \in \mathbb{C}[\mathbf{x}]$ be a polynomial and $\mathbf{w}, \mathbf{v} \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$ $\mathbb{Q}$-linearly independent. We say F is* multihomogeneous with respect to $\mathbf{w}$ and $\mathbf{v}$ *if F is $\mathbf{w}$-homogeneous and $\mathbf{v}$-homogeneous.*

If F is multihomogeneous with respect to $\mathbf{w}$ and $\mathbf{v}$ then F is also $\lambda\mathbf{w}+\mu\mathbf{v}$ - homogeneous for arbitrary $\lambda,\mu\in\mathbb{Q}$. In the next lemma we are going to show that we can assume certain properties of at least one of the weight-vectors.

Lemma 28. *Let* $\mathbf{w}=(w_1,\dots,w_n)\in\mathbb{N}^n_{>0}$ *and* $\mathbf{v}'=(v'_1,\dots,v'_n)\in\mathbb{Q}^n\setminus\{\mathbf{0}\}$, *where* $\mathbf{w}$ *and* $\mathbf{v}'$ *are* $\mathbb{Q}$*-linearly independent. Then there exists a* $\mathbf{v}=(v_1,\dots,v_n)\in\mathbb{Q}^n\setminus\{\mathbf{0}\}$, *such that*

(1) $v_i\geq w_i$ *for all* $1\leq i\leq n$, *and*
(2) *there exists a* $1\leq k\leq n$ *with* $v_k=w_k$.

Proof. Since $w_i>0$ for all $i=1,\dots,n$, we can find a $\lambda\in\mathbb{N}$, such that $\mathbf{v}=\lambda\cdot\mathbf{w}+\mathbf{v}'$ satisfies $v_i\geq w_i$ for all $1\leq i\leq n$. Let $\mathbf{v}=(v_1,\dots,v_n)$. By choosing k with $v_k-w_k\leq v_i-w_i$ for all $i\neq k$ and scaling $\mathbf{v}$ by $\frac{w_k}{v_k}$, we can additionally assume that $v_k=w_k$. □

Remark 29. Let $\mathbf{w}=(w_1,\dots,w_n)\in\mathbb{N}^n_{>0}$ and $\mathbf{v}=(v_1,\dots,v_n)\in\mathbb{Q}^n\setminus\{\mathbf{0}\}$, where $\mathbf{w}$ and $\mathbf{v}$ are linearly independent. Furthermore, let $F\in\mathbb{C}[\mathbf{x}]$ be multihomogeneous with respect to $\mathbf{w}$ and $\mathbf{v}$. By Lemma 28 we can assume without loss of generality that $\mathbf{v}$ satisfies $v_i\geq w_i$ for all $1\leq i\leq n$ and that there exists a $k\in\mathbb{N}$ such that $v_k=w_k$. This yields that $\deg_{\mathbf{v}}F\geq\deg_{\mathbf{w}}F$.

Now we are able to prove a weak version of our result.

Proposition 30. *Let* $I\subset\mathbb{C}\{\mathbf{x}\}$ *be an ideal. Then* $A:=\mathbb{C}\{\mathbf{x}\}/I$ *is the moduli algebra of a quasi-homogeneous isolated hypersurface singularity* $(V,0)$ *of Thom–Sebastiani type with quasi-homogeneous Thom–Sebastiani summands if, and only if,* I *is generated by* $F_1,\dots,F_n\in\mathfrak{m}^2$ *with the following properties:*

(1) I *is zero-dimensional,*
(2) *There exist weight-vectors* $\mathbf{w}\in\mathbb{N}^n_{>0}$ *and* $\mathbf{v}\in\mathbb{Q}^n\setminus\{\mathbf{0}\}$, *where* $\mathbf{w}$ *and* $\mathbf{v}$ *are* $\mathbb{Q}$*-linearly independent, such that the* F_i *are multihomogeneous with respect to* $\mathbf{w}$ *and* $\mathbf{v}$, *and*
(3) $\partial_{x_j}F_i=\partial_{x_i}F_j$ *for all* $1\leq i,j\leq n$.

Proof. First we show the "only if" direction. Since $(V,0)$ is quasi-homogeneous and of Thom–Sebastiani type with quasi-homogeneous Thom–Sebastiani summands, we can assume that there exist $2\leq r<n, f_1\in\mathbb{C}\{x_1,\dots,x_r\}$ and $f_2\in\mathbb{C}\{x_{r+1},\dots,x_n\}$ such that $f=f_1+f_2$ satisfies $(V,0)\cong(V(f),0)$. Denote by $\mathbf{w}=(w_1,\dots,w_n)\in\mathbb{N}^n_{>0}$ a weight-vector of f and define $\mathbf{v}=(0,\dots,0,w_{r+1},\dots,w_n)$ and $F_i=\partial_{x_i}f$. Then F_i is $\mathbf{w}$-homogeneous and $\mathbf{v}$-homogeneous for $i=1,\dots,n$. Then Condition (1) and (2) are satisfied, since $I=J_f$. Condition (3) follows by symmetry of second derivatives.

Now we show the "if" direction. Let $\mathbf{w}=(w_1,\dots,w_n)$ and $\mathbf{v}=(v_1,\dots,v_n)$. Due to Theorem 26 there exists a $\mathbf{w}$-homogeneous $f\in\mathbb{C}\{\mathbf{x}\}$ satisfying $\partial_{x_i}f=F_i$ for $1\leq i\leq n$. In particular, it holds that $\operatorname{ord}(f)\geq 3$. By assumption f defines an isolated hypersurface singularity, since I is zero-dimensional. Due to Lemma 28 we can assume that $v_i\geq w_i$ and that there exists an index k, such that $v_k=w_k$. By assumption, the partial derivatives of f are $\mathbf{w}$-homogeneous, as well as $\mathbf{v}$-homogeneous. Due to Lemma 25 and due to the uniqueness of the weights of f, see [14, Satz 1.3], we obtain

$$\frac{w_i}{\deg_{\mathbf{w}}(\partial_{x_i}f)+w_i}=\frac{v_i}{\deg_{\mathbf{v}}(\partial_{x_i}f)+v_i}\tag{1}$$

for all $1\leq i\leq n$. For any index k with $w_k=v_k$ Equation (1) implies

$$\deg_{\mathbf{w}}(\partial_{x_k}f)=\deg_{\mathbf{v}}(\partial_{x_k}f).\tag{2}$$

Assume $\partial^2_{x_i,x_k}f\neq 0$. For any $1\leq i\leq n$ with $v_i>w_i$ Equation (2) yields

$$\deg_{\mathbf{w}}(\partial^2_{x_i,x_k}f)=\deg_{\mathbf{w}}(\partial_{x_k}f)-w_i>\deg_{\mathbf{v}}(\partial_{x_k}f)-v_i=\deg_{\mathbf{v}}(\partial^2_{x_i,x_k}f).\tag{3}$$

Inequality (3) and Remark 29 yield a contradiction, hence

$$\partial^2_{x_i,x_k}f=0.$$

This means that the partial derivatives with respect to variables x_k which satisfy $w_k = v_k$ do not depend on variables x_i with $v_i > w_i$ and vice versa. We reorder the $\mathbf{x}$ variables together with the corresponding weights, such that $v_i = w_i$ for $1 \le i \le r$ and $v_i > w_i$ for $r+1 \le i \le n$. Under our assumptions, Lemma 24 and Lemma 25 imply the existence of $\mathbf{w}$-homogeneous $f_1(x_1,\dots,x_r)$ and $f_2(x_{r+1},\dots,x_n)$, such that $\partial_{x_i} f_1 = \partial_{x_i} f$ for $1 \le i \le r$ and $\partial_{x_i} f_2 = \partial_{x_i} f$ for $r+1 \le i \le n$. This proves the claim. □

Before we prove Theorem 3 we state an example to see how Proposition 30 can be applied.

Example 31. Let $F_1 = 7x^6 y$, $F_2 = x^7 - 11y^{10}$ and $F_3 = 13z^{12}$ be polynomials in $\mathbb{C}[x,y,z]$. The F_i are multihomogeneous with respect to the weight-vectors $\mathbf{w} = (10,7,1)$ and $\mathbf{v} = (0,0,1)$. Consider the ideal $I = \langle F_1, F_2, F_3\rangle \subset \mathbb{C}[x,y,z]$. Using SINGULAR we obtain that I is a zero-dimensional ideal. The condition that $\partial_{x_j} F_i = \partial_{x_i} F_j$ for all $1 \le i, j \le 3$ can also be easily verified. Then Proposition 30 tells us, that $\mathbb{C}[x,y,z]/I$ is the moduli algebra of a quasi-homogeneous polynomial.

Indeed, one can verify that $f = yx^7 - y^{11} + z^{13}$ satisfies $I = J_f$ and f is quasi-homogeneous with respect to the weight-vector $\mathbf{w}' = (130,91,77)$.

Using Proposition 30, which is highly coordinate dependent, we are able to prove Theorem 3 (b).

Proof of Theorem 3 (b). Denote by f a $\mathbf{w}$-homogeneous defining equation of $(V,0)$. If $(V,0)$ is of Thom–Sebastiani type with quasi-homogeneous summands it follows immediately that $A(V)$ is multihomogeneous and hence $\dim_{\mathbb{C}} \mathfrak{t} \ge 2$. Let $\dim_{\mathbb{C}} \mathfrak{t} \ge 2$ or, equivalently, $A(V)$ be multihomogeneous. Due to [14, Satz 4.1] we know that $f \in J_f$, hence J_f is minimaly generated by $\partial_{x_1} f, \dots, \partial_{x_n} f$. Since f is $\mathbf{w}$-homogeneous, all of the partial derivatives of f are $\mathbf{w}$-homogeneous. Due to this we can assume that $A(V)$ is $\mathbf{w}$-homogeneous. Denote by $F_1,\dots,F_n$ a multihomogeneous generating system of J_f and by $d = \deg_{\mathbf{w}}(f)$ the $\mathbf{w}$-degree of f. Since $\mathbf{w}$ is a positive weight-vector, the weighted degrees of a minimal (multi)homogeneous generating system are uniquely determined (see [11, Proposition 4.7.8]). Thus we can assume that, after possibly renaming the F_i, the following holds:

$$\deg_{\mathbf{w}}(F_i) = \deg_{\mathbf{w}}(\partial_{x_i} f) = d - w_i$$

for $1 \le i \le n$. Write $\partial_{x_i} f = \sum_{j=1}^n m_{ij} F_j$. Due to the $\mathbf{w}$-homogeneity, we obtain

$$\deg_{\mathbf{w}}(m_{ij}) = w_j - w_i. \tag{4}$$

The fact that $w_1 > \ldots > w_n$ implies together with Equation (4) that the matrix $M = (m_{ij}) \in \mathbb{C}\{\mathbf{x}\}^{n\times n}$ is an invertible, upper triangular matrix. Our assumption implies

$$w_j - w_i < w_n - \frac{w_n}{2} = \frac{w_n}{2} < w_1,$$

hence the matrix M has only constant entries and is diagonal. Due to this, the partial derivatives of f are already multihomogeneous and Proposition 30 implies that $(V,0)$ is of Thom–Sebastiani type with quasi-homogeneous summands. □

Let us consider two examples. We start by proving that a given quasi-homogeneous isolated hypersurface singularity is not of Thom–Sebastiani type.

Example 32. Consider the isolated hypersurface singularity defined by $f = yx^7 - y^{11} \in \mathbb{C}[x,y]$. The polynomial f is quasi-homogeneous with respect to the weight-vector $\mathbf{w} = (10,7)$. We obtain $J_f = \langle 7x^6 y, x^7 - 11y^{10}\rangle$, which is the same ideal as in Example 17. This implies that for every maximal toral Lie subalgebra $\mathfrak{t} \subset L(V)$ it holds that $\dim_{\mathbb{C}} \mathfrak{t} = 1$. Since $10 > 7 > \frac{10}{2} = 5$, Theorem 3 implies that f cannot define a hypersurface singularity of Thom–Sebastiani type.

In our next example we want use Theorem 3 to show that a given defining equation is of Thom–Sebastiani type.

Example 33. Consider $f = y\cdot(x-z)^7 - y^{11} + z^{13} \in \mathbb{C}[x,y,z]$ and $(V,0)\subset(\mathbb{C}^3,0)$, where $V=\{f=0\}$. One can use Singular to verify that f defines an isolated hypersurface singularity. Next we need to see that $(V,0)$ is quasi-homogeneous and that $(V,0)$ satisfies the assumptions of Theorem 3. We proceed as in Example 17. It holds that

$$\delta = (130x-53z)\frac{\partial}{\partial x} + 91y\frac{\partial}{\partial y} + 77z\frac{\partial}{\partial z} \in \mathrm{Der}(X_f).$$

The derivation δ satisfies $\delta(x-z) = 130(x-z)$, $\delta(y) = 91y$ and $\delta(z) = 77z$. This implies that $(V,0)$ is quasi-homogeneous with respect to the weight-vector $\mathbf{w} = (130,91,77)$, hence $(V,0)$ satisfies the assumptions of Theorem 3. Next we need to show that $(V,0)$ is of Thom–Sebastiani type. Using Singular we obtain that $L(V)$ contains the derivations

$$\delta_1 = (10x-10z)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{and} \quad \delta_2 = z\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}.$$

Define $\mathfrak{t} = \langle\delta_1,\delta_2\rangle$. The derivations δ_1 and δ_2 are simultaneously diagonalizable with eigenvectors $x-z$, y and z and satisfy $[\delta_1,\delta_2] = 0$. This implies $\mathfrak{t}$ is a toral Lie subalgebra of $L(V)$ with $\dim_C \mathfrak{t} = 2$. Theorem 3 implies that $(V,0)$ is of Thom–Sebastiani type.

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