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Algebraic geometry / *Géométrie algébrique*

# Note on absolute sets of rigid local systems

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**Abstract.** In this note we give a description up to a quasi-finite morphism of the absolute sets of simple cohomologically rigid local systems on a smooth complex quasi-projective algebraic variety. In dimension one or rank two, this proves a conjecture of Budur–Wang on the structure of these sets.

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## 1. Introduction

The aim of this note is to state the results of [2], give brief indications on their proofs, and provide the motivation behind them. Let  $X$  be a smooth complex algebraic variety and  $r > 0$  an integer. The Betti moduli space  $M_B(X, r)$  is an affine scheme defined over  $\mathbb{Q}$  whose complex points parametrize semisimple local systems on  $X^{\text{an}}$  of rank  $r$ , cf. Section 2.1. Certain points in  $M_B(X, r)(\mathbb{C})$  are of particular interest: they correspond to the local systems of *geometric origin*. For instance, local systems that are direct summands of higher pushforwards  $R^i f_* \mathbb{C}$ , where  $i \geq 0$  and  $f: Y \rightarrow X$  is a smooth projective morphism, are of geometric origin (the precise definition of geometric origin is given in [1, 6.2.4]).

It is a subtle problem to describe the distribution of these “special” points in  $M_B(X, r)$ . A natural question, analogous to the André–Oort problem in the case of Shimura varieties, is to describe the Zariski closure  $\overline{\Sigma}^{\text{Zar}}$  where  $\Sigma \subset M_B(X, r)(\mathbb{C})$  is a subset of points of geometric origin. We recall here the corresponding picture for the Shimura variety  $\mathcal{A}_g$  parametrizing principally polarized complex abelian varieties. The role of the points of geometric origin in  $M_B(X, r)$

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is played by the CM points in  $\mathcal{A}_g$ , i.e. points in  $\mathcal{A}_g(\mathbb{C})$  parametrizing abelian varieties with complex multiplication. If  $\Sigma \subset \mathcal{A}_g(\mathbb{C})$  is a subset of CM points then the André–Oort conjecture (now a theorem) states that the irreducible components of  $\overline{\Sigma}^{\text{Zar}}$  are connected Shimura subvarieties of  $\mathcal{A}_g$ . Roughly speaking, Shimura subvarieties of  $\mathcal{A}_g$  parametrize abelian varieties whose associated weight one Hodge structure has more tensor Hodge classes than a very general abelian variety. Shimura subvarieties of  $\mathcal{A}_g$  also contain a dense set of CM points for the Euclidean topology. The counterparts in  $M_B(X, r)$  of the Shimura subvarieties of  $\mathcal{A}_g$  are the absolute  $\overline{\mathbb{Q}}$ -constructible subsets, that we define in Section 2. They were first defined by Simpson in [12] when  $X$  is a projective variety. We follow a less constrained definition of [3] available also in the non-projective case. The essential ingredient needed for their definition is the Riemann–Hilbert correspondence between complex local systems on  $X^{\text{an}}$  and regular algebraic flat connections on  $X$ . The absolute  $\overline{\mathbb{Q}}$ -constructible subsets of dimension 0 give yet another collection of interesting points: as conjectured by Simpson, they should correspond exactly to the local systems of geometric origin (see point (A3) below). One can formulate a set of conjectures in analogy with the Shimura case. This is done in [3]:

**Conjecture A.** *Let  $X$  be a smooth complex algebraic variety, and  $r > 0$  an integer. The following hold in  $M_B(X, r)$ :*

- (A1) *The collection of absolute  $\overline{\mathbb{Q}}$ -constructible subsets is generated from the absolute  $\overline{\mathbb{Q}}$ -closed subsets via finite sequences of taking unions, intersections, and complements; the Euclidean (or equivalently, the Zariski) closure of an absolute  $\overline{\mathbb{Q}}$ -constructible set is absolute  $\overline{\mathbb{Q}}$ -closed; and an irreducible component of an absolute  $\overline{\mathbb{Q}}$ -closed subset is absolute  $\overline{\mathbb{Q}}$ -closed.*
- (A2) *Any non-empty absolute  $\overline{\mathbb{Q}}$ -constructible subset contains a Zariski dense subset of absolute  $\overline{\mathbb{Q}}$ -points.*
- (A3) *A point is an absolute  $\overline{\mathbb{Q}}$ -point if and only if it is a local system of geometric origin.*
- (A4) *The Zariski closure of a set of absolute  $\overline{\mathbb{Q}}$ -points is  $\overline{\mathbb{Q}}$ -pseudo-isomorphic to an absolute  $\overline{\mathbb{Q}}$ -constructible subset of some  $M_B(X', r')$ , for possibly a different smooth complex algebraic variety  $X'$  and rank  $r'$ . Moreover, the  $\overline{\mathbb{Q}}$ -pseudo-isomorphism restricts to a bijection between the sets of absolute  $\overline{\mathbb{Q}}$ -points. (See Definition 3 for the notion of  $\overline{\mathbb{Q}}$ -pseudo-isomorphisms.)*

In this note we address a modification of Conjecture A for certain subvarieties of  $M_B(X, r)$ , which are themselves absolute  $\overline{\mathbb{Q}}$ -constructible subsets  $M_B(X, r)$ . They correspond to the submoduli of simple local systems (Section 3) and cohomologically rigid local systems (Section 4). We give a description up to a quasi-finite morphism of the absolute  $\overline{\mathbb{Q}}$ -constructible sets of simple cohomologically rigid local systems. Conjecture A for this kind of absolute sets is then: reduced to (A3) (Theorem 7), and shown to hold if  $\dim X = 1$  or if  $r = 2$  (Theorem 8). If  $r = 1$ , or if  $X$  is a complex affine torus or an abelian variety, Conjecture A is known by [3]. One can find in [5] an arithmetic analog of our results.

## 2. The Betti moduli space and absolute sets

### 2.1. Betti moduli

Let  $X$  be a smooth complex algebraic variety with a base point  $x_0 \in X(\mathbb{C})$ . Let  $r > 0$  be an integer. The space of representations of rank  $r$  of the (topological) fundamental group  $\pi_1(X, x_0)$  is

$$R_B(X, x_0, r) := \text{Hom}(\pi_1(X, x_0), \text{GL}_r),$$

the  $\mathbb{Q}$ -scheme representing the functor associating to a  $\mathbb{Q}$ -scheme  $T$  the set  $\text{Hom}(\pi_1(X, x_0), \text{GL}_r(\Gamma(T, \mathcal{O}_T)))$ . There is an action of  $\text{GL}_r$  on  $R_B(X, x_0, r)$  by conjugation: on  $T$ -valued points  $R_B(X, x_0, r)(T)$ , the group  $\text{GL}_r(\Gamma(T, \mathcal{O}_T))$  acts by conjugation on the target.

The *moduli space of local systems of rank  $r$*  is defined as the affine GIT quotient of  $R_B(X, x_0, r)$  with respect to this action,

$$M_B(X, r) := R_B(X, x_0, r) // \mathrm{GL}_r.$$

It is an affine scheme defined over  $\mathbb{Q}$ . The complex points of  $M_B(X, r)$  are in one-to-one correspondence with the isomorphism classes of complex semisimple local systems of rank  $r$  on  $X^{\mathrm{an}}$ . For a representation  $\rho: \pi_1(X, x_0) \rightarrow \mathrm{GL}_r(\mathbb{C})$  we denote the associated local system on  $X^{\mathrm{an}}$  by  $L_\rho$ . We denote by

$$q_B: R_B(X, x_0, r) \rightarrow M_B(X, r)$$

the quotient morphism. The fiber over a complex point  $L \in M_B(X, r)(\mathbb{C})$  is the set of representations  $\rho$  such that  $L_{\rho^{ss}} \cong L$ , where  $\rho^{ss}$  is the semisimplification of  $\rho$ . By  $R_B(X, x_0)$  we will mean the disjoint union over all  $r$  of  $R_B(X, x_0, r)$  and similarly for  $M_B(X)$ .

### 2.2. Unispaces

Unispaces, defined in [3], enlarge the class of spaces for which constructibility can be defined in a useful way. Let  $\mathrm{Alg}_{ft,reg}(\mathbb{C})$  be the category of finite type regular  $\mathbb{C}$ -algebras and  $\mathrm{Set}$  the category of sets. A *unispace*  $\mathcal{C}$  is a functor

$$\mathcal{C}: \mathrm{Alg}_{ft,reg}(\mathbb{C}) \rightarrow \mathrm{Set}.$$

Given  $R \in \mathrm{Alg}_{ft,reg}(\mathbb{C})$  elements of  $\mathcal{C}(R)$  will be denoted by  $\mathcal{F}_R$ . If  $R \rightarrow R'$  is a morphism in  $\mathrm{Alg}_{ft,reg}(\mathbb{C})$  and  $\mathcal{F}_R \in \mathcal{C}(R)$ , the image of  $\mathcal{F}_R$  in  $\mathcal{C}(R')$  under the morphism  $\mathcal{C}(R) \rightarrow \mathcal{C}(R')$  will be denoted  $\mathcal{F}_R \star_R R'$ . We think of an element of  $\mathcal{C}(R)$  as a family of objects in  $\mathcal{C}(\mathbb{C})$  parametrized by  $\mathrm{Spec}(R)$ . A subset  $S \subset \mathcal{C}(\mathbb{C})$  is called  $\mathbb{C}$ -*constructible* (respectively,  $\mathbb{C}$ -*closed*) with respect to the unispace  $\mathcal{C}$  if for every  $R \in \mathrm{Alg}_{ft,reg}(\mathbb{C})$  and every  $\mathcal{F}_R \in \mathcal{C}(R)$ , the subset of maximal ideals

$$\{m \in \mathrm{Spec}(R)(\mathbb{C}) \mid \mathcal{F}_R \star_R R/m \in S\}$$

is the set of complex points of a constructible subscheme of  $\mathrm{Spec}(R)$ .

#### Examples

Let  $X$  be a complex algebraic variety.

- (1) We denote by  $\mathrm{LocSys}_{free}(X, \_)$  the unispace associating to  $R \in \mathrm{Alg}_{ft,reg}(\mathbb{C})$  the set of isomorphism classes of local systems of free  $R$ -modules of finite rank on  $X^{\mathrm{an}}$ .
- (2) We denote by  $D_c^b(X, \_)$  the unispace associating to  $R \in \mathrm{Alg}_{ft,reg}(\mathbb{C})$  the set of isomorphism classes of objects of  $D_c^b(X, R)$ , the bounded derived category of constructible sheaves of finite type  $R$ -modules on  $X^{\mathrm{an}}$ .

In the notation of the previous example, there is a natural transformation of functors

$$\mathrm{LocSys}_{free}(X, \_) \rightarrow D_c^b(X, \_). \tag{1}$$

For  $R \in \mathrm{Alg}_{ft,reg}(\mathbb{C})$ , it sends a local system  $\mathcal{L}_R \in \mathrm{LocSys}_{free}(X, R)$  to the complex in  $D_c^b(X, R)$  whose only nonzero term is  $\mathcal{L}_R$  placed in degree zero.

### 2.3. $\overline{\mathbb{Q}}$ -constructibility

Consider the unispace  $\mathrm{LocSys}_{free}(X, \_)$  (the same discussion applies verbatim to  $D_c^b(X, \_)$ ). We can modify its definition in the obvious way to define a functor  $\mathrm{Alg}_{ft,reg}(K) \rightarrow \mathrm{Set}$ , where  $K$  is a field of characteristic zero and  $\mathrm{Alg}_{ft,reg}(K)$  is the category of finite type regular  $K$ -algebras. Fix an embedding  $\iota: \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ . We write  $\overline{\mathbb{Q}} \subset L \subset \mathbb{C}$  to denote a tower of field extensions whose composite is  $\iota$ . A subset  $S \subset \mathrm{LocSys}_{free}(X, \mathbb{C})$  is  $\overline{\mathbb{Q}}$ -*constructible* (resp.  $\overline{\mathbb{Q}}$ -*closed*) if it is  $\mathbb{C}$ -constructible (resp.

$\mathbb{C}$ -closed) and for all field extensions  $\overline{\mathbb{Q}} \subset L \subset \mathbb{C}$ , for all  $R \in \text{Alg}_{ft,reg}(L)$  and  $\mathcal{L}_R \in \text{LocSys}_{free}(X, R)$  the set

$$\{m \in \text{Hom}_L(R, \mathbb{C}) : \mathcal{L}_R \otimes_{R,m} \mathbb{C} \in S\}$$

is the set  $\text{Hom}_L(\text{Spec}(\mathbb{C}), Y_R)$  of complex points of a constructible (resp. closed)  $L$ -subscheme  $Y_R$  of  $\text{Spec}(R)$ . One checks that the choice of a different embedding  $l' : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$  gives rise to the same notion of  $\overline{\mathbb{Q}}$ -constructibility for subsets of  $\text{LocSys}_{free}(X, \mathbb{C})$ . We record here that there is a general theory of  $K$ -structures for unispaces, where  $K$  is a subfield of  $\mathbb{C}$ , developed in [3, Section 3]. The previous definition coincides with [3, Definition 3.4.1], when phrased in terms of the indicator function of  $S$ .

### 2.4. Absolute sets

Assume now that  $X$  is a *smooth* complex algebraic variety. We denote by  $\mathcal{D}_X$  the sheaf (of non-commutative  $\mathbb{C}$ -algebras) of differential operators on  $X$ . The Riemann–Hilbert correspondence is an equivalence of categories

$$RH : D_{rh}^b(\mathcal{D}_X) \rightarrow D_c^b(X, \mathbb{C})$$

where the left hand side denotes the category of regular holonomic  $\mathcal{D}_X$ -modules, cf. [6].

For each  $\sigma \in \text{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$ , let  $X^\sigma \rightarrow X$  be the base change of  $X$  via  $\sigma$ . Consider the diagram

$$\begin{array}{ccc} D_{rh}^b(\mathcal{D}_X) & \xrightarrow{p_\sigma} & D_{rh}^b(\mathcal{D}_{X^\sigma}) \\ RH \downarrow & & RH \downarrow \\ D_c^b(X, \mathbb{C}) & & D_c^b(X^\sigma, \mathbb{C}), \end{array} \tag{2}$$

where  $p_\sigma$  is the pullback of  $\mathcal{D}_X$ -modules under the base change over  $\sigma$  (it is an equivalence of categories as well). For a subset  $T$  of the isomorphism classes of objects in  $D_c^b(X, \mathbb{C})$ , define

$$T^\sigma := RH \circ p_\sigma \circ RH^{-1}(T), \tag{3}$$

a subset of the isomorphism classes of objects in  $D_c^b(X^\sigma, \mathbb{C})$ . In the remainder of the section we let  $K = \mathbb{C}$  or  $\overline{\mathbb{Q}}$ . The key definition is the following:

**Definition 1.**

- (1) A set  $T$  of isomorphism classes of objects in  $D_c^b(X, \mathbb{C})$  is *absolute  $K$ -constructible* (resp. *absolute  $K$ -closed*) with respect to the unispace  $D_c^b(X, \_)$  if the set  $T^\sigma$  of isomorphism classes of objects in  $D_c^b(X^\sigma, \mathbb{C})$  is  *$K$ -constructible* (resp.  *$K$ -closed*) with respect to  $D_c^b(X^\sigma, \_)$  for all  $\sigma \in \text{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$ .
- (2) A set  $T$  of isomorphism classes of objects in  $\text{LocSys}_{free}(X, \mathbb{C})$  is *absolute  $K$ -constructible* (resp. *absolute  $K$ -closed*) with respect to the unispace  $\text{LocSys}_{free}(X, \_)$  if its image in  $D_c^b(X, \mathbb{C})$  under the natural transformation (1) is *absolute  $K$ -constructible* (resp. *absolute  $K$ -closed*) with respect to the unispace  $D_c^b(X, \_)$ .

Any subset of isomorphism classes in  $D_c^b(X, \mathbb{C})$  or  $\text{LocSys}_{free}(X, \mathbb{C})$  defined via a composition of the usual functors on derived categories of bounded  $K$ -constructible complexes on smooth complex algebraic varieties is *absolute  $K$ -constructible* in the above sense, by [3, Section 6].

The previous definition deals with abstract isomorphism classes of objects. We now define the notion of absoluteness for subsets of  $M_B(X)(\mathbb{C})$ . Let

$$l : R_B(X, x_0)(\mathbb{C}) \rightarrow \text{LocSys}(X, \mathbb{C})$$

be the map sending a representation  $\rho$  to the isomorphism class of  $L_\rho$ .

**Definition 2.** A subset  $S \subset M_B(X)(\mathbb{C})$  is absolute  $K$ -constructible (resp. absolute  $K$ -closed) if  $S = q_B(l^{-1}(T))$  where  $T \subset \text{LocSys}(X, \mathbb{C})$  is absolute  $K$ -constructible (resp. absolute  $K$ -closed) with respect to the unispace  $\text{LocSys}_{free}(X, \_)$ . If  $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$  the set  $S^\sigma \subset M_B(X^\sigma)(\mathbb{C})$  is defined as in (3) with the points of  $S$  corresponding to isomorphism classes of semisimple local systems.

Notice that an absolute  $K$ -constructible subset of  $M_B(X, r)(\mathbb{C})$  is indeed the set of complex points of a constructible  $K$ -subscheme of  $M_B(X, r)$ . For equivalent definitions of absoluteness and some basic properties, see [2, Section 4.3]. In Conjecture A we use the notion of “ $\overline{\mathbb{Q}}$ -pseudo-isomorphism”. The definition is as follows and it is related with the theory of motivic measures and the Grothendieck group of varieties, see [8, 9]:

**Definition 3.** Let  $K$  be a field,  $Y$  (resp.  $Y'$ ) a  $K$ -scheme, and  $S$  (resp.  $S'$ ) a constructible subscheme of  $Y$  (resp.  $Y'$ ). The sets  $S(\mathbb{C})$  and  $S'(\mathbb{C})$  are  $K$ -pseudo-isomorphic if there exist  $n \in \mathbb{N}$ , constructible  $K$ -subschemes  $S_i$  of  $S$  (resp.  $S'_i$  of  $S'$ ), for  $i = 1, \dots, n$ , such that  $S(\mathbb{C}) = \coprod_{i=1}^n S_i(\mathbb{C})$  (resp.  $S'(\mathbb{C}) = \coprod_{i=1}^n S'_i(\mathbb{C})$ ) and such that there exist  $K$ -isomorphisms  $S_i \rightarrow S'_i$ ,  $i = 1, \dots, n$ . The ambient schemes  $Y$  and  $Y'$  will be clear from the context.

### 2.5. Restricting Conjecture A to submoduli

Let  $M \subset M_B(X)(\mathbb{C})$  be an absolute  $K$ -constructible subset. Let  $\widetilde{M} = l(q_B^{-1}(M)) \subset \text{LocSys}(X, \mathbb{C})$ . Define the unispace

$$\mathcal{M} : R \mapsto \{ \mathcal{L}_R \in \text{LocSys}_{free}(X, R) \mid \mathcal{L}_R \otimes_R R/m \in \widetilde{M}, \forall m \in \text{Spec}(R)(\mathbb{C}) \}$$

together with the usual base change. Clearly  $\mathcal{M}(\mathbb{C}) = \widetilde{M}$ . There is a natural transformation

$$\mathcal{M} \rightarrow \text{LocSys}_{free}(X, \_) \tag{4}$$

of unispaces defined over  $K$  given by inclusion. For  $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ , we let  $\mathcal{M}_{X^\sigma}$  be the unispace associated to  $M^\sigma$  (the latter set is defined in Definition 2). Note that  $(\_)^\sim$  and  $(\_)^\sigma$  commute, and  $\mathcal{M} = \mathcal{M}_X$ . We adapt Definition 1 to the collection of unispaces  $\mathcal{M}_{X^\sigma}$ . We say that a subset  $S \subset \widetilde{M}$  is absolute  $K$ -constructible (resp. absolute  $K$ -closed) with respect to  $\mathcal{M}$  if  $S^\sigma$  is  $K$ -constructible (resp.  $K$ -closed) with respect to  $\mathcal{M}_{X^\sigma}$  for all  $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ . At the level of the Betti moduli space, the corresponding definition is:

**Definition 4.** A subset  $S$  of  $M$  is called absolute  $K$ -constructible (resp. absolute  $K$ -closed) in  $M$  if  $S = q_B(l^{-1}(T))$  for some subset  $T$  of  $\widetilde{M}$  that is absolute  $K$ -constructible (resp. absolute  $K$ -closed) with respect to the unispace  $\mathcal{M}$ .

One can then restrict the formulation of Conjecture A to  $M$ : we replace  $M_B(X)$  by  $M$  and replace all instances of the phrase “absolute  $\overline{\mathbb{Q}}$ -constructible subset” (resp. “absolute  $\overline{\mathbb{Q}}$ -closed subset”) by “absolute  $\overline{\mathbb{Q}}$ -constructible subset of  $M$ ” (resp. “absolute  $\overline{\mathbb{Q}}$ -closed subset of  $M$ ”). Conjecture (A3) will deal only with the equivalence for absolute  $\overline{\mathbb{Q}}$ -points in  $M$ . We shall refer to the restricted version of the conjecture as “Conjecture A for  $M$ ”.

### 3. Simple local systems

Let  $M_B^s(X, r)$  be the Zariski open subscheme of  $M_B(X, r)$  whose complex points parametrize simple local systems. It is easy to see that  $M_B^s(X, r)(\mathbb{C})$  is absolute  $\overline{\mathbb{Q}}$ -constructible. Our main result concerning the locus of simple local systems is:

**Theorem 5.** Conjecture (A1) for  $M_B^s(X, r)(\mathbb{C})$  holds.

Let us say some words on the proof. The notion of absolute  $\overline{\mathbb{Q}}$ -constructibility given in Definition 2 has a more geometric version, which however is less flexible. Embed  $X$  in a smooth projective variety  $\overline{X}$  with  $D := \overline{X} \setminus X$  a simple normal crossing divisor. It is proved by Nitsure [10, Theorem 3.5] and Simpson [13, Theorem 4.10] that there exists a coarse moduli space  $M_{DR}(\overline{X}/D, r)$  for Jordan equivalence classes of rank  $r$  semistable logarithmic connections on  $\overline{X}$  with poles on  $D$ . It is a countable disjoint union of finite type  $\mathbb{C}$ -schemes. The underlying  $\mathcal{O}_{\overline{X}}$ -module of a logarithmic connection is, by definition, only a torsion-free  $\mathcal{O}_{\overline{X}}$ -module, locally free over  $X$ . The Riemann-Hilbert correspondence induces a complex analytic map

$$RH^{\text{an}} : M_{DR}(\overline{X}/D, r)^{\text{an}} \rightarrow M_B(X, r)^{\text{an}}. \tag{5}$$

One can then give a definition of absoluteness as in 2 but using  $M_B(X, r)$  (resp.  $M_{DR}(\overline{X}/D, r)$ ) in place of  $D_c^b(X, \mathbb{C})$  (resp.  $D_{rh}^b(\mathcal{D}_X)$ ), see [2, Section 5.3]. The advantage of this definition is the possibility of working with geometric objects rather than abstract sets of isomorphism classes. On the other hand, there are some drawbacks. First, it is much easier to define functors at the categorical level rather than at the level of coarse moduli spaces. Second, the condition of semistability on connections, needed to ensure the existence of a coarse moduli space, leads to some complications: for example, it is not known whether  $RH^{\text{an}}$  is surjective (notice that without semistability the surjectivity is clear, e.g. using Deligne extensions). However, many difficulties disappear, and the two definitions of absoluteness agree, if we restrict  $RH^{\text{an}}$  to the analytic open subspace  $M_{DR}^{\text{good}}(\overline{X}/D, r) \subset M_{DR}(\overline{X}/D, r)^{\text{an}}$  which parametrizes simple logarithmic connections such that no two distinct eigenvalues of the residue along a component of  $D$  differ by an integer (it is the complement of a locally finite, countable collection of Zariski closed subschemes). Then, the map  $RH_0 := RH^{\text{an}} : M_{DR}^{\text{good}}(\overline{X}/D, r) \rightarrow M_B^s(X, r)^{\text{an}}$  is surjective. The key ingredient for the proof of Theorem 5 is a result of Nitsure-Sabbah [11, Section 8] stating that  $RH_0$  is a local analytic isomorphism.

#### 4. Cohomologically rigid local systems

We continue to consider a smooth complex variety  $X$  with a good compactification  $j : X \rightarrow \overline{X}$ .

**Definition 6.** *Let  $L$  be a simple complex local system on  $X$ . We say that  $L$  is cohomologically rigid if  $\mathbb{H}^1(\overline{X}, j_{!*} \text{End}^0(L)) = 0$  where  $\text{End}^0(L)$  denotes the local system of traceless endomorphisms of  $L$  and  $j_{!*}$  is the intermediate extension functor.*

In general the vector space  $\mathbb{H}^1(\overline{X}, j_{!*} \text{End}^0(L))$  is the Zariski tangent space at  $L$  of the Betti moduli space of complex local systems with prescribed determinant and prescribed local monodromies at infinity. Thus a cohomologically rigid complex local system is rigid, in the sense that  $L$  is an isolated point of this moduli space, and in addition it is a smooth point. Without fixing the determinant nor the monodromy at infinity, the set of simple cohomologically rigid local systems in  $M_B(X, r)(\mathbb{C})$  is the set of complex points of a possibly high-dimensional constructible subscheme  $M_B^{\text{cohrig}}(X, r) \subset M_B(X, r)$ . From the results of [3] and the cohomological characterization  $\mathbb{H}^1(\overline{X}, j_{!*} \text{End}^0(L)) = 0$ , one deduces that  $M_B^{\text{cohrig}}(X, r)(\mathbb{C})$  is an absolute  $\overline{\mathbb{Q}}$ -constructible subset, see [2, Proposition 6.2 (2)]. Our second main result is:

**Theorem 7.** *Conjectures (A1), (A2), (A4) for  $M_B^{\text{cohrig}}(X, r)$  hold. Conjecture (A3) is equivalent to Simpson’s conjecture that cohomologically rigid simple local systems with quasi-unipotent determinant and conjugacy classes at infinity are of geometric origin.*

Simpson’s conjecture is proved when  $X$  is a curve and the rank is arbitrary in [7], and when  $X$  is arbitrary and the rank is 2 in [4]. Therefore we obtain:

**Theorem 8.** *Conjecture A for  $M_B^{\text{cohrig}}(X, r)$  holds if  $X$  is a curve or if the rank  $r = 2$ .*

We provide some indications on the proof. Let  $\bar{X}$  be a smooth compactification of  $X$  with complement  $D$ , a simple normal crossing divisor. Let  $D = \cup_{i=1}^s D_i$  be the irreducible decomposition. We consider the open subscheme  $M_{DR}(\bar{X}/D, r)^{lf} \subset M_{DR}(\bar{X}/D, r)$  of logarithmic connections of rank  $r$  whose underlying  $\mathcal{O}_{\bar{X}}$ -module is locally free. Consider the diagram:

$$\begin{array}{ccccc}
 M_{DR}(\bar{X}/D, r)^{lf}(\mathbb{C}) & \xrightarrow{\det \times res} & M_{DR}(\bar{X}/D, 1)(\mathbb{C}) \times \mathbb{C}^{rs} & \xleftarrow{id \times root} & M_{DR}(\bar{X}/D, 1)(\mathbb{C}) \times \mathbb{C}^{rs} \\
 \downarrow RH^{an} & & \downarrow RH^{an} \times f & & \downarrow id \times Exp \\
 M_B(X, r)(\mathbb{C}) & \xrightarrow{\det \times mon} & M_B(X, 1)(\mathbb{C}) \times ((\mathbb{C}^*)^{(r)})^s & \xleftarrow{id \times root} & M_B(X, 1)(\mathbb{C}) \times (\mathbb{C}^*)^{rs}.
 \end{array} \tag{6}$$

defined as follows. Here  $res = (res_1, \dots, res_s)$  and  $res_i : M_{DR}(\bar{X}/D, r)^{lf} \rightarrow \mathbb{C}^r$  is the algebraic morphism which associates to a locally free connection  $(E, \nabla)$  the  $r$ -tuple of coefficients, excluding the top degree, of the characteristic polynomial of the residue homomorphism  $\Gamma_i \in \text{Hom}(E|_{D_i}, E|_{D_i})$ . This is well-defined, since over a point of  $D_i$  the residue homomorphism determines a conjugacy class in  $GL_r(\mathbb{C})$  which remains constant as the point varies. Similarly,  $mon = (mon_1, \dots, mon_s)$  and  $mon_i : M_B(X, r) \rightarrow \mathbb{C}^{r-1} \times \mathbb{C}^*$  is the algebraic morphism that associates to a representation  $\rho : \pi_1(X, x_0) \rightarrow GL_r(\mathbb{C})$  the coefficients of the characteristic polynomial of the matrix associated to a small loop around  $D_i$ . The constant term coefficient is nonzero. By identifying a monic polynomial with the unordered set of its roots, one identifies  $\mathbb{C}^{r-1} \times \mathbb{C}^*$  with the  $r^{\text{th}}$  symmetric product  $(\mathbb{C}^*)^r/S_r$  denoted  $(\mathbb{C}^*)^{(r)}$ . The two maps  $det$  correspond to taking the determinant of a logarithmic connection and of a local system, respectively.

Each of the two maps  $root$  is the product  $s$  times of the quotient maps by the action of the symmetric group  $S_r$ . The quotient map is a finite map of algebraic varieties, given by the symmetric polynomials, such that the fiber over  $(a_0, \dots, a_{r-1})$  is the set of roots of the polynomial  $t^r + a_{r-1}t^{r-1} + \dots + a_0$  and their distinct permutations. The map  $Exp$  takes  $z$  to  $e^{2\pi iz}$  component-wise. The map  $f$  is the unique map that makes the diagram commutative. We can restrict the left-most column of (6) to  $RH^{an} : M_{DR}(\mathbb{C}) \rightarrow M_B^s(X, r)(\mathbb{C})$ , where  $M_{DR}(\mathbb{C}) \subset M_{DR}(\bar{X}/D, r)^{lf}(\mathbb{C})$  is the inverse image of the set  $M_B^s(X, r)(\mathbb{C})$  of simple local systems. We keep using the same names for the restricted maps, which will be the ones used below. Viewing  $(\mathbb{C}^*)^{rs}$  as the complex points of the Betti moduli of rank one local systems on the affine torus  $\mathbb{G}_m^{rs}$ , the first step in the proof of Theorem 7 is:

**Proposition 9.**

- (1) If  $S \subset M_B^s(X, r)(\mathbb{C})$  is an absolute  $\bar{\mathbb{Q}}$ -constructible subset, then  $(id \times root)^{-1}((det \times mon)(S))$  an absolute  $\bar{\mathbb{Q}}$ -constructible subset of  $M_B(X \times \mathbb{G}_m^{rs}, 1)(\mathbb{C})$ .
- (2) If  $S' \subset M_B(X \times \mathbb{G}_m^{rs}, 1)(\mathbb{C})$  is an absolute  $\bar{\mathbb{Q}}$ -constructible subset, then  $(det \times mon)^{-1}((id \times root)(S'))$  is an absolute  $\bar{\mathbb{Q}}$ -closed subset of  $M_B^s(X, r)(\mathbb{C})$ .

This allows one to pass from an absolute  $\bar{\mathbb{Q}}$ -constructible subset of Betti moduli of higher rank to one in Betti moduli of rank one. In rank one, one has a precise description of absolute  $\bar{\mathbb{Q}}$ -constructible subsets: these are the subsets generated from torsion-translated algebraic subtori via a finite sequence of taking unions, intersections, and complements, [3, Theorem 9.1.2]. The last step is to observe that the restriction of  $det \times mon$  to  $M_B^{coh\,rig}(X, r)(\mathbb{C})$  has finite fibers. These ingredients prove Theorem 7.

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