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Number theory / *Théorie des nombres*

Fractional parts of powers of real algebraic numbers

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Abstract. Let α be a real algebraic number greater than 1. We establish an effective lower bound for the distance between an integral power of α and its nearest integer.

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1. introduction

For a real number x, let

$$\|x\| = \min\{|x-m| : m \in \mathbb{Z}\}$$

denote the distance to its nearest integer. Not much is known on the distribution of the sequence $(\|\alpha^n\|)_{n\ge1}$ for a given real number α greater than 1. For example, we do not know whether the sequence $(\|(3/2)^n\|)_{n\ge1}$ is dense in $[0,1/2]$, nor whether $\|\mathrm{e}^n\|$ tends to 0 as n tends to infinity. In 1957 Mahler [15] applied Ridout's p-adic extension of Roth's theorem to prove the following result.

Theorem 1. *Let r/s be a rational number greater than* 1 *and which is not an integer. Let ε be a positive real number. Then, there exists an integer n_0 such that*

$$\|(r/s)^n\| > s^{-\varepsilon n},$$

for every integer n exceeding n_0.

In a breakthrough paper, Corvaja and Zannier [11] applied ingeniously the p-adic Schmidt Subspace Theorem to extend Theorem 1 to real algebraic numbers. Recall that a Pisot number is a real algebraic integer greater than 1 with the property that all of its Galois conjugates (except itself) lie in the open unit disc.

Theorem 2. *Let α be a real algebraic number greater than* 1 *and ε a positive real number. If there are no positive integers h such that the real number α^h is a Pisot number, then there exists an integer n_0 such that*

$$\|\alpha^n\| > \alpha^{-\varepsilon n},$$

for every integer n exceeding n_0.

Let $\alpha > 1$ be a real algebraic number and h a positive integer such that α^h is a Pisot number of degree d. Then there exists a positive real number η such that the modulus of any Galois conjugate (except itself) of α^h is no greater than $\alpha^{-\eta}$. Let n be a positive integer. Since the trace of α^{hn} is a rational integer, we get $\|\alpha^{hn}\| \le d\alpha^{-\eta n}$. This shows that the restriction in Theorem 2 is necessary.

Theorems 1 and 2 are ineffective in the sense that their proof does not yield an explicit value for the integer n_0. To get an effective improvement on the trivial estimate $\|(r/s)^n\| \ge s^{-n}$, Baker and Coates [1] (see also [7] and [8, Section 6.2]) used the theory of linear forms in p-adic logarithms, for a prime number p which divides s.

Theorem 3. *Let r/s be a rational number greater than* 1 *and which is not an integer. Then, there exist an effectively computable positive real number τ and an effectively computable integer n_0 such that*

$$\|(r/s)^n\| > s^{-(1-\tau)n},$$

for every integer n exceeding n_0.

The purpose of this note is to extend Theorem 3 to real algebraic numbers exceeding 1. At first, we have to see which estimate follows from a Liouville-type inequality, which boils down to say that any nonzero rational integer has absolute value at least 1. To simplify the discussion, take α a real algebraic integer greater than 1 and of degree $d \ge 2$ such that each of its Galois conjugates has absolute value at most equal to α. For a positive integer n, let A_n be the integer such that

$$\|\alpha^n\| = |\alpha^n - A_n|.$$

Observe that every Galois conjugate of $\alpha^n - A_n$ has modulus less than $3\alpha^n$. Noticing that the absolute value of the norm of the nonzero algebraic integer $\alpha^n - A_n$ is at least equal to 1, we derive that

$$\|\alpha^n\| \ge 3^{-(d-1)}\,\alpha^{-n(d-1)}. \tag{1}$$

This is much weaker than what follows from Theorem 2, but this is effective. For an arbitrary real algebraic number greater than 1, a similar argument gives the following statement. In the sequel, an empty product is understood to be equal to 1.

Lemma 4. *Let α be a real algebraic number greater than* 1 *and of degree $d \ge 1$. Let a_d denote the leading coefficient of its minimal defining polynomial over $\mathbb{Z}$ and $\alpha_1, \ldots, \alpha_d$ its Galois conjugates, ordered in such a way that $|\alpha_1| \le \cdots \le |\alpha_d|$. Let j be such that $\alpha = \alpha_j$. Set*

$$C(\alpha) = a_d\,\alpha^{d-1} \prod_{i>j} \frac{|\alpha_i|}{\alpha}.$$

If α is not the d-th root of an integer, then we have

$$\|\alpha^n\| \ge 3^{-(d-1)}\,C(\alpha)^{-n}, \quad \textit{for } n \ge 1. \tag{2}$$

Otherwise, (2) *holds only for the positive integers n such that α^n is not an integer.*

We will see how the theory of linear forms in logarithms allows us to slightly improve (2), unless there exists a positive integer h such that α^h is an integer or a quadratic Pisot unit. In the latter case, α^h is a root of an integer polynomial of the shape $X^2 - aX + b$, with $a \ge 1$, $b \in \{-1, 1\}$, and $(a, b) \notin \{(1,1),(2,1)\}$, thus $\alpha = (a + \sqrt{a^2 - 4b})/2$ and $\|\alpha^{hn}\| = \alpha^{-hn}$. Except in these cases, we establish the following effective strengthening of Lemma 4.

Theorem 5. *Let α be a real algebraic number greater than 1. Let $C(\alpha)$ be as in the statement of Proposition 4. Let h be the smallest positive integer such that α^h is an integer or a quadratic Pisot unit and put $\mathcal{N}_\alpha = \{hn : n \in \mathbb{Z}_{\geq 1}\}$. If no such integer exists, then $\mathcal{N}_\alpha$ is the empty set. There exist a positive, effectively computable real number $\tau = \tau(\alpha)$ and an effectively computable integer $n_0 = n_0(\alpha)$, both depending only on α, such that*

$$\|\alpha^n\| \geq C(\alpha)^{-(1-\tau)n}, \quad \textit{for } n > n_0 \textit{ not in } \mathcal{N}_\alpha.$$

Theorem 5 should be compared with the effective improvement of Liouville's upper bound for the irrationality exponent of an irrational, algebraic real number. Recall that the irrationality exponent $\mu(\xi)$ of an irrational real number ξ is given by

$$\mu(\xi) = 1 + \limsup_{q \to +\infty} \frac{-\log \|q\xi\|}{\log q}.$$

Let $\mu_{\text{eff}}(\xi)$ denote the infimum of the real numbers μ for which there exists an effectively computable positive integer q_0 such that $1 + (-\log \|q\xi\|)/(\log q) \leq \mu$ holds for $q \geq q_0$. Let α be an algebraic real number of degree $d \geq 2$. Roth's theorem asserts that $\mu(\alpha) = 2$, while Liouville's theorem says that $\mu_{\text{eff}}(\alpha) \leq d$. By means of the theory of linear forms in logarithms, Feldman [14] proved the existence of an effectively computable positive real number $\tau' = \tau'(\alpha)$, depending on α, such that $\mu_{\text{eff}}(\alpha) \leq (1 - \tau')d$.

Here, the situation is similar. For a real number ξ not an integer, nor a root of an integer, define

$$\nu(\xi) = \limsup_{n \to +\infty} \frac{-\log \|\xi^n\|}{n}$$

and let $\nu_{\text{eff}}(\xi)$ denote the infimum of the real numbers ν for which there exists an effectively computable integer n_0 such that $(-\log \|\xi^n\|)/n \leq \nu$ for $n \geq n_0$.

Let $\alpha > 1$ be an algebraic real number. Theorem 2 asserts that $\nu(\alpha) = 0$, unless α is an integer root of a Pisot number. Lemma 4 says that $\nu_{\text{eff}}(\alpha) \leq \log C(\alpha)$, while Theorem 5 slightly improves the latter inequality. Furthermore, the positive real number $\tau(\alpha)$ occurring in Theorem 5 is very small and of comparable size as the real number $\tau'(\alpha)$, when α is an algebraic integer (otherwise, it also depends on the prime factors of the leading coefficient of the minimal defining polynomial of α over $\mathbb{Z}$).

Among the many open questions on the function ν, let us mention that we do not know whether $\nu(\mathrm{e})$ is finite or not (see [8, Problem 13.20] and [10] for further results and questions). Mahler and Szekeres [16] established that, with respect to the Lebesgue measure, almost all real numbers ξ satisfy $\nu(\xi) = 0$. Furthermore, the set of real numbers ξ such that $\nu(\xi)$ is infinite has Hausdorff dimension zero [10, Theorem 3].

Sometimes, the hypergeometric method yields effective improvements of (2). This is the case for the algebraic numbers $\sqrt{2}$ and $3/2$, see Beuker's seminal papers [4, 5] and the subsequent works [2, 19] where it is shown that

$$\nu_{\text{eff}}(\sqrt{2}) \leq 0.595, \quad \nu_{\text{eff}}(3/2) < 0.5443,$$

respectively.

As pointed out by the referee, we have $C(\alpha^n) \leq C(\alpha)^n$ for $n \geq 1$, with equality if α^n has the same degree as α. However, in some cases, the inequality can be strict, for example, for $\alpha = (3/2)^{1/3}$, we have $C(\alpha^3) = 2$ and $C(\alpha)^3 = 18$. Following Dubickas [12], if the set of positive integers

$$U(\alpha) := \{n : \deg(\alpha^n) < \deg(\alpha)\}$$

is nonempty, then there is a finite set $F(\alpha) = \{m_1, \ldots, m_k\}$ of positive integers such that

$$U(\alpha) = \{\ell m : \ell \geq 1, m \in F(\alpha)\}.$$

Applying Theorem 5 to α^m for m in $F(\alpha)$, we get that there exist a positive, effectively computable real number τ_m and an effectively computable integer ℓ_m, both depending only on α, such that

$$\|\alpha^{m\ell}\| \geq C(\alpha^m)^{-(1-\tau_m)\ell}, \quad \text{for } \ell > \ell_m \text{ such that } m\ell \text{ is not in } \mathcal{N}_{\alpha^m}.$$

Continuing like this with the sets $U(\alpha^m)$ for m in $F(\alpha)$, and the sets $U\big((\alpha^m)^{m'}\big)$ for m in $F(\alpha)$ and m' in $F(\alpha^m)$, etc., we conclude that in Theorem 5 and also in Lemma 4 the quantity $C(\alpha)^n$ can be replaced by $C(\alpha^n)$. Thereby, we obtain slightly stronger statements when $U(\alpha)$ is non-empty. However, we have decided not to highlight these minor improvements, mainly because the definition of ν_{eff} is very natural in view of the current statement of Theorem 5.

2. Proofs

Proof of Lemma 4. We keep the notation of the lemma and follow the proof of [16, Assertion (a)] with a slight improvement.

Let n be a positive integer. Observe that the polynomial

$$f_n(X) = a_d^n(X-\alpha_1^n)\cdots(X-\alpha_d^n)$$

has integer coefficients and denote by A_n the integer such that

$$\|\alpha^n\| = |\alpha^n - A_n|.$$

If α^n is not an integer, then $f_n(A_n)$ is a nonzero integer and we get

$$|f_n(A_n)| \geq 1, \tag{3}$$

thus,

$$|\alpha^n - A_n| \geq a_d^{-n} \prod_{1\leq i\leq d, i\neq j} |\alpha_i^n - A_n|^{-1}.$$

For $i = 1,\ldots,d$, note that

$$|\alpha_i^n - A_n| \leq |\alpha_i|^n + \alpha^n + 1 \leq 3(\max\{|\alpha_i|,\alpha\})^n.$$

Consequently, we obtain the lower bound

$$\|\alpha^n\| \geq 3^{-(d-1)}\, a_d^{-n}\, \alpha^{-(d-1)n} \prod_{i>j} \frac{\alpha^n}{|\alpha_i|^n},$$

as claimed. This inequality reduces to (1) if $a_d = 1$ and $j = d$. □

The proof of Theorem 5 makes use of the following result of Boyd [6].

Lemma 6. *Let $f(X)$ be an irreducible polynomial of degree d with integer coefficients. Let m denote the number of roots of $f(X)$ of maximal modulus. Assume that one of these roots is real and positive. Then m divides d and there is an irreducible polynomial $g(X)$ with integer coefficients such that $f(X) = g(X^m)$.*

Proof of Theorem 5. We proceed in a similar way as when dealing with Thue equations. In view of Theorem 3 we assume that α is irrational. Let K denote the number field $\mathbb{Q}(\alpha)$. Let h denote the logarithmic absolute Weil height. For convenience, we define the function $h^*(\cdot) = \max\{h(\cdot), 1\}$. The constants $c_1, c_2, \ldots$ below are positive, effectively computable, and depend only on α.

Let a_d denote the leading coefficient of the minimal defining polynomial of α over $\mathbb{Z}$ and S the set of places of K composed of all the infinite places and all the places corresponding to a prime ideal dividing a_d. Let N_S denote the S-norm. We direct the reader to [13, Chapter 1] for definitions and basic results. Let us only mention that if the absolute value of the norm of a nonzero element β in K is written as $|\mathrm{Norm}_{K/\mathbb{Q}}(\beta)| = a_S b$, where a_S and b are positive integers, every prime divisor of a_S divides a_d, and no prime divisor of b divides a_d, then $N_S(\beta) = b$. In particular, if $a_d = 1$, then N_S is the absolute value of the norm $\mathrm{Norm}_{K/\mathbb{Q}}$.

Let n be a positive integer and A_n denote the integer such that

$$\|\alpha^n\| = |\alpha^n - A_n|.$$

Put $\delta_n = \alpha^n - A_n$. We will establish a lower bound of the form κ^n with $\kappa > 1$ for the S-norm of the nonzero S-integer δ_n. Since the S-norm is multiplicative, $N_S(\delta_n)$ divides $N_S(f_n(A_n))$, thus it divides $|f_n(A_n)|^d$. Consequently, by replacing in the proof of Lemma 4 the right hand side of (3) by $\kappa^{n/d}$, we then obtain the expected improvement.

Let $\eta_1, \ldots, \eta_s$ be a fundamental system of S-units in K. By [13, Proposition 4.3.12], there exist integers $b_1, \ldots, b_s$ such that

$$h\big(\delta_n \eta_1^{-b_1} \cdots \eta_s^{-b_s}\big) \le \frac{\log N_S(\delta_n)}{d} + c_1. \tag{4}$$

Since

$$h(\delta_n) \le n h(\alpha) + \log A_n + \log 2 \le n h(\alpha) + n \log \alpha + 2 \log 2,$$

it follows from [13, Proposition 4.3.9 (iii)] and (4) that

$$B := \max\{|b_1|, \ldots, |b_s|\} \le c_2 h^*(\delta_n) \le c_3 n.$$

Set $\gamma_n = \delta_n \eta_1^{-b_1} \cdots \eta_r^{-b_r}$.

Assume first that there exists a Galois conjugate β of α such that $|\beta| > \alpha$ and consider the quantity

$$\Lambda_n = \frac{\beta^n - A_n}{\beta^n}.$$

Observe that

$$0 < |\Lambda_n - 1| \le 2^{-c_4 n}.$$

Let σ denote the embedding sending α to β and observe that

$$\Lambda_n - 1 = \sigma(\gamma_n) \beta^{-n} \sigma(\eta_1)^{b_1} \cdots \sigma(\eta_r)^{b_r}.$$

We apply the theory of linear forms in logarithms: it follows from [17, Theorem 9.1] (or see [8, Theorem 2.1]) that

$$\log|\Lambda_n - 1| \ge -c_5 h^*(\gamma_n) \log\left(\frac{B+n}{h^*(\gamma_n)}\right), \tag{5}$$

giving

$$n \le c_6 h^*(\gamma_n) \log\left(\frac{n}{h^*(\gamma_n)}\right).$$

We derive that

$$n \le c_7 h^*(\gamma_n) \le c_8 \log N_S(\delta_n) + c_9,$$

by (4), thus

$$N_S(\delta_n) \ge 2^{c_{10} n}, \quad \text{for } n \ge c_{11}.$$

This improves the trivial lower bound $N_S(\delta_n) \ge 1$ used in the proof of Lemma 4.

Like for Feldman's result [14], the key point for our improvement of the trivial bound is the quantity B' occurring in the estimates for linear forms in logarithms (see [8, Theorem 2.1]), which allows us to get in (5) the factor $\log \frac{B+n}{h^*(\gamma_n)}$ instead of $\log(B+n)$, and similarly in (9). For other consequences of the quantity B', see [8, 9] and the references quoted therein.

Secondly, we assume that the modulus of every Galois conjugate of α is less than or equal to α. By Lemma 6, there exist a divisor m of d and an irreducible integer polynomial $g(X)$ of degree d/m such that $f(X)$ has exactly m roots of modulus α and the minimal defining polynomial $f(X)$ over $\mathbb{Z}$ satisfies $f(X) = g(X^m)$.

Assume that $d/m \geq 2$. If $f(X)$ has a root β of modulus at least equal to 1 and different from α, then $A_n - \alpha^n$ cannot be equal to β^n, thus the quantity

$$\Lambda'_n = \frac{A_n - \beta^n}{\alpha^n} \tag{6}$$

satisfies

$$0 < |\Lambda'_n - 1| \leq 2^{-c_{12} n}. \tag{7}$$

We get a lower bound for $|\Lambda'_n - 1|$ by proceeding exactly as above, and it takes the same shape as our lower bound for $|\Lambda_n - 1|$. We then deduce the lower bound

$$|N_S(\delta_n)| \geq 2^{c_{13} n}, \quad \text{for } n \geq c_{14}.$$

Now, we assume that all the roots of $f(X)$, except α, lie in the open unit disc.

If α has two real Galois conjugates in the open unit disc, then one of them, denoted by β, is such that the quantity Λ'_n defined as in (6) is not equal to 1 and (7) holds. We argue as above to get a similar lower bound for $N_S(\delta_n)$.

If $d/m \geq 3$ and α^m has a complex nonreal Galois conjugate β^m in the open unit disc, then β^j is complex nonreal for every positive integer j and we proceed as above, since the quantity Λ'_n defined as in (6) is not equal to 1.

Consequently, we can assume that $d/m = 2$ and $g(X)$ is the minimal defining polynomial over $\mathbb{Z}$ of the quadratic number α^m.

If n is not a multiple of m, then there exists a Galois conjugate β of α such that β^n is complex nonreal, thus the quantity Λ'_n defined above is not equal to 1, and we can proceed exactly as above to get a similar lower bound for $N_S(\delta_n)$.

Assume now that n is a multiple of m. Write $g(X) = a_2 X^2 - uX - v$. Denote by $\sigma(\alpha)$ a Galois conjugate of α such that α^m and $\sigma(\alpha^m)$ are the distinct roots of $g(X)$. If α^m is not an algebraic integer, then there exists a prime number p such that $v_p(\alpha^m) < 0$. Since $v_p(\alpha^m) \leq -1/2$, it follows from [8, Theorem B.11] that the p-adic valuation of $\alpha^n + \sigma(\alpha)^n$ satisfies

$$v_p\big(\alpha^n + \sigma(\alpha)^n\big) = v_p\big((\alpha^m)^{n/m} + \sigma(\alpha^m)^{n/m}\big) \leq \frac{n}{m} v_p(\alpha^m) + c_{15} \log n \leq -\frac{n}{3m},$$

for $n \geq c_{16}$. In particular, for n greater than c_{16}, the algebraic number $\alpha^n + \sigma(\alpha^n)$ cannot be a rational integer. Then, the quantity Λ'_n defined above is not equal to 1, and we can proceed exactly as above to get a similar lower bound for $N_S(\delta_n)$.

If α^m is an algebraic integer, then $a_2 = 1$ and $\alpha^n + \sigma(\alpha^n)$ is equal to the nearest integer A_n to α^n. Thus, we have

$$\|\alpha^n\| = |\sigma(\alpha^n)| = \frac{|v|^{n/m}}{\alpha^n},$$

while Lemma 4 asserts that

$$\|\alpha^n\| \geq 3^{-1} \alpha^{-n}.$$

Consequently, we obtain the desired improvement on (2) if $|v| \geq 2$. As already noticed, (2) is essentially best possible if $|v| = 1$.

It only remains for us to consider the case $d = m$. Then, there exist coprime nonzero integers u, v with $u > v > 0$ such that the minimal defining polynomial of α over $\mathbb{Z}$ is $vX^d - u$. If $v = 1$, then α is the d-th root of the integer u. If $d = 2$, then

$$\|\sqrt{u}\, u^m\| \geq u^{-(1-c_{17})m}, \quad \text{for } m \geq 1,$$

by [3, Theorem 1.2] (see also [8, Theorem 6.3]). If $d \geq 3$ and $j = 1, \dots, d-1$, then it follows from an effective improvement of Liouville's bound d for the irrationality exponent of $u^{j/d}$ (see [8, Section 6.3]) that

$$\|u^{j/d} u^m\| \geq u^{-(1-c_{18})(d-1)m}, \quad \text{for } m \geq 1.$$

In both cases, noticing that $C(\sqrt[d]{u}) = u^{(d-1)/d}$, we get

$$\|(\sqrt[d]{u})^n\| \geq C(\sqrt[d]{u})^{-(1-c_{19})n}, \quad \text{for } n \geq 1 \text{ not a multiple of } d,$$

as expected. Now, assume that $v \geq 2$. We argue in a similar way as in the proof of Theorem 3. Let p be a prime divisor of v. Write

$$\frac{\delta_n}{v^{n/d}} = \left(\sqrt[d]{\frac{u}{v}}\right)^n - A_n$$

and note that the p-adic valuation of

$$\Omega_n = v^{n/d} A_n = (u^{1/d})^n - \delta_n$$

satisfies $v_p(\Omega_n) \geq c_{20} n$. Let L denote the number field generated by $u^{1/d}$ and $v^{1/d}$. Let S be the set of places of L composed of all the infinite places and all the places corresponding to a prime ideal dividing v. Let $\eta_1, \ldots, \eta_s$ be a fundamental system of units in L. By [13, Proposition 4.3.12], there exist integers $b_1, \ldots, b_r$ such that

$$h\big(\delta_n \eta_1^{-b_1} \cdots \eta_r^{-b_r}\big) \leq \frac{\log N_S(\delta_n)}{d} + c_{21}. \tag{8}$$

Since $h(\delta_n) \leq c_{22} n$, it follows from [13, Proposition 4.3.9 (iii)] and (8) that

$$B := \max\{|b_1|, \ldots, |b_r|\} \leq c_{23} h^*(\delta_n) \leq c_{24} n.$$

Set $\gamma_n = \delta_n \eta_1^{-b_1} \cdots \eta_r^{-b_r}$ and note that

$$\Omega_n = (u^{1/d})^n - \gamma_n \eta_1^{b_1} \cdots \eta_r^{b_r}.$$

It follows from the theory of linear forms in p-adic logarithms, more precisely, from an estimate of [18] (or see [8, Theorem 2.11]), that

$$v_p(\Omega_n) \leq c_{25} h^*(\gamma_n) \log\left(\frac{B+n}{h^*(\gamma_n)}\right). \tag{9}$$

This gives

$$n \leq c_{26} h^*(\gamma_n) \log\left(\frac{n}{h^*(\gamma_n)}\right),$$

and we derive that

$$n \leq c_{27} h^*(\gamma_n) \leq c_{28} \log N_S(\delta_n) + c_{29},$$

thus

$$N_S(\delta_n) \geq 2^{c_{30} n}, \quad \text{for } n \geq c_{31}.$$

This improves the trivial lower bound $N_S(\delta_n) \geq 1$ used in the proof of Lemma 4. This concludes the proof of the theorem. □

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