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Mathématique

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Volume 360 (2022), p. 431-438

<https://doi.org/10.5802/crmath.308>

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Numerical analysis, Partial differential equations / *Analyse numérique, Équations aux dérivées partielles*

Finite element error estimates for a mixed degenerate parabolic model

Des estimations d'erreur par éléments finis pour un modèle parabolique dégénéré mixte

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Abstract. The aim of this note is to deduce error estimates for a fully-discrete finite element method approximation of a kind of degenerate mixed parabolic equations. The obtained results consider regularity assumptions about the main variable according to the degenerate character of the problem, given by the term involving the time-derivative, which is represented with a non-invertible linear operator R. We show two different approaches to obtain the error estimates. The first one needs to introduce an extension operator of R and the second one requires to add a new ellipticity property for this operator. These error estimates can be applied to analyze the fully-discrete finite element method approximation of an *eddy current model*.

Résumé. Le but de cette note est de déduire des estimations d'erreur pour une approximation par la méthode des éléments finis entièrement discrets d'un type d'équations paraboliques mixtes dégénérées. Les résultats obtenus considèrent des hypothèses de régularité sur la variable principale selon le caractère dégénéré du problème, donné par le terme impliquant la dérivée temporelle, qui est représentée par un opérateur linéaire non inversible R. Nous présentons deux approches différentes pour obtenir les estimations d'erreur. La première nécessite d'introduire un opérateur d'extension de R et la seconde nécessite d'ajouter une nouvelle propriété d'ellipticité pour cet opérateur. Ces estimations d'erreur peuvent être appliquées pour analyser l'approximation par la méthode des éléments finis entièrement discrets d'un modèle de *courants de Foucault*.

2020 Mathematics Subject Classification. 65N30, 35K65, 78M10.

Funding. This work was partially supported by University of Cauca through project VRI ID 5679.

Electronic supplementary material. Supplementary material for this article is supplied as a separate archive available from the journal's website under article's URL or from the author.

Manuscript received 17 September 2021, revised 8 November 2021, accepted 30 November 2021.

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ISSN (electronic) : 1778-3569 https://comptes-rendus.academie-sciences.fr/mathematique/

1. Introduction

We start introducing the mixed degenerate parabolic problem: given $u_0 \in Y$, $f \in L^2(0,T;X')$ and $g \in L^2(0,T;M')$, find $u \in L^2(0,T;X)$ and $\lambda \in L^2(0,T;M)$ satisfying the following equations:

$$\frac{d}{dt}\left[\left\langle Ru(t),v\right\rangle_Y + b(v,\lambda(t))\right] + \left\langle Au(t),v\right\rangle_X = \left\langle f(t),v\right\rangle_X \qquad \forall\, v \in X \quad \text{in } \mathscr{D}'(0,T), \tag{1}$$

$$b\big(u(t),\eta\big) = \left\langle g(t),\eta\right\rangle_M \qquad \forall\, \eta \in M, \tag{2}$$

$$\left\langle Ru(0),v\right\rangle_Y = \left\langle Ru_0,v\right\rangle_Y \qquad \forall\, v \in Y. \tag{3}$$

where X and Y are two real Hilbert spaces such that X is contained in Y with a continuous and dense imbedding; M is a real reflexive Banach space; $\langle\cdot,\cdot\rangle_Z$ denotes the duality pairing in Z ($Z \in \{X,Y,M\}$); $R : Y \to Y'$, $A : X \to X'$ are continuous *linear* operator with R *not necessarily* invertible; and $b : X \times M \to \mathbb{R}$ is a continuous bilinear form. A motivation for the study of this kind of equations comes from the *eddy current models* considered in [3–5]. These models allow to consider the *time-primitive* of the electric field as the main variable of the problem and to compute it by using the finite element method (FEM). Furthermore, to obtain the convergence of FEM approximation, in [3,4] have been assumed that $\partial_{tt}u$ is square integrable in $\Omega \times (0,T)$ where Ω is the computational domain, while in [5] *it* has been supposed that this property holds in a parabolic subdomain $\Omega_C \times (0,T)$ where Ω_C represents the conductor domain which is contained in Ω.

Recently, a FEM approximation for the general problem (1)–(3) has been analyzed in [1] and to obtain the convergence of the method has been assumed that $u \in H^2(0,T;Y)$. This condition for the concrete case of problems in [3–5] is equivalent to suppose that $\partial_{tt}u$ is square integrable in $\Omega \times (0,T)$. The goal of this paper is to obtain error estimates for the approximation of the general problem (1)–(3) by considering a time regularity that in the practice problems is equivalent to the regularity in the subdomain $\Omega_C \times (0,T)$ (see Remark 4 below).

With this aim, we present two alternatives. The first one requires to build a vector space that allows the linear operator R to be adequately extended in order to improve the aforementioned regularity. The second one consists in assuming a kind of elliptic condition for the operator R. In both cases, it is necessary to redefine the error term for the approximation of the time-derivatives, which is based on a projection operator of Y onto the orthogonal space of the kernel of the operator R.

The outline of this note is as follows: in Section 2 we recall results obtained in [2] about the well-posedness of Problem (1)–(3) and we present a fully-discrete approximation for that problem. Next, some preliminary notation and relevant results for the error estimate analysis are presented in Section 3. Finally, two different approaches to deduce the error estimates of the fully-discrete approximation with a (new) time regularity for the problem are shown in Sections 4 and 5.

2. The mixed degenerate parabolic problem

Let V be the kernel of the bilinear form b, i.e., $V := \left\{v \in X : b(v,\eta) = 0 \quad \forall\, \eta \in M\right\}$ and denote by W its *closure* with respect to the Y-norm. The well-posedness of problem (1)-(3) is obtained by assuming the following assumptions [2]:

(H1) R is self-adjoint and monotone on V, i.e., $\langle Rv,w\rangle_Y = \langle Rw,v\rangle_Y$ and $\langle Rv,v\rangle_Y \geq 0$ for any $v,w \in V$.

(H2) The bilinear form b satisfies a continuous *inf-sup* condition, i.e., there exists $\beta > 0$ such that

$$\sup_{v\in X}\frac{b(v,\eta)}{\|v\|_X} \geq \beta\,\|\eta\|_M \quad \forall\, \eta \in M.$$

(H3) The operator A is self-adjoint on V, i.e., $\langle Av, w\rangle_X = \langle Aw, v\rangle_X$ for any $v, w \in V$.
(H4) There exist $\xi > 0$ and $\alpha > 0$ such that $\langle Av, v\rangle_X + \xi\langle Rv, v\rangle_Y \geq \alpha\|v\|_X^2$ for any $v \in V$.
(H5) The initial data u_0 belongs to W.
(H6) The data function g belongs to $\mathrm{H}^1(0, T; M')$.

Theorem 1. *Let us assume that assumptions (H1)–(H6) hold true. Then the Problem* (1)–(3) *has a unique solution* $(u, \lambda) \in \mathrm{L}^2(0, T; X) \times \mathscr{C}^0([0, T]; M)$ *and there exists a constant* $C > 0$ *such that*

$$\|u\|_{\mathrm{L}^2(0,T;X)} + \|\lambda\|_{\mathrm{L}^2(0,T;M)} \leq C\left\{\|f\|_{\mathrm{L}^2(0,T;X')} + \|g\|_{\mathrm{H}^1(0,T;M')} + \|u_0\|_Y\right\}.$$

Moreover, $\lambda(0) = 0$.

Proof. See [2, Theorem 2.1]. □

Now, we introduce the following fully-discrete approximation for the mixed degenerate parabolic problem (1)-(3). Let $\{X_h\}_{h>0}$ and $\{M_h\}_{h>0}$ be sequences of finite-dimensional subspaces of X and M, respectively (*the choice* for the subspaces X_h and M_h correspond to element finite spaces) and let $\{t_n := n\Delta t : n = 0, \ldots, N\}$ be a uniform partition of $[0, T]$ with a step size $\Delta t := T/N$. For any finite sequence $\{\theta^n : n = 0, \ldots, N\}$ we denote

$$\bar{\partial}\theta^n := \frac{\theta^n - \theta^{n-1}}{\Delta t}, \qquad n = 1, \ldots, N.$$

Let $u_{0,h} \in X_h$ and assume that $f \in \mathscr{C}([0, T]; X')$. Then, the fully-discrete approximation of the Problem (1)-(3) reads as follows:

Find $u_h^n \in X_h$, $\lambda_h^n \in M_h$, $n = 1, \ldots, N$, such that:

$$\left\langle R\bar{\partial}u_h^n, v\right\rangle_Y + b\left(v, \bar{\partial}\lambda_h^n\right) + \left\langle Au_h^n, v\right\rangle_X = \left\langle f(t_n), v\right\rangle_X \qquad \forall\, v \in X_h, \tag{4}$$

$$b\left(u_h^n, \eta\right) = \left\langle g(t_n), \eta\right\rangle_M \qquad \forall\, \eta \in M_h, \tag{5}$$

$$u_h^0 = u_{0,h}, \tag{6}$$

$$\lambda_h^0 = 0. \tag{7}$$

We can notice that this scheme is obtained by using a backward Euler discrete approximation for the time-derivatives. Furthermore, the third equation of Problem (4)-(7) will require a suitable approximation $u_{0,h}$ of the initial data u_0 to obtain the convergence of the scheme (see Theorems 3 and 7). The existence and uniqueness of the problem (4)-(7) is obtained from the classical Babuška–Brezzi Theory by assuming the following conditions:

(H7) There exist $\xi_h > 0$ and $\alpha_h > 0$ such that

$$\langle Av, v\rangle_X + \xi_h\langle Rv, v\rangle_Y \geq \alpha_h\|v\|_X^2 \qquad \forall\, v \in V_h, \tag{8}$$

where V_h denotes the discrete kernel of b in X_h, i.e., $V_h := \{v \in X_h : b(v, \eta) = 0\ \forall\, \eta \in M_h\}$.

(H8) The bilinear form $b : X_h \times M_h \to \mathbb{R}$ is bounded and it satisfies the discrete *inf-sup* condition, i.e., there exists $\beta_h > 0$ such that

$$\sup_{v \in X_h} \frac{b(v, \eta)}{\|v\|_X} \geq \beta_h \|\eta\|_M \quad \forall\, \eta \in M_h. \tag{9}$$

3. Preliminary concepts for the analysis of error estimates

We start by considering the projection operator $\Pi_h : X \to X_h$ characterized by

$$\Pi_h w \in X_h : \left(\Pi_h w, z\right)_X = (w, z)_X \qquad \forall\, z \in X_h,$$

then there is $C > 0$ independent on h satisfying

$$\left\|w - \Pi_h w\right\|_X \leq C \inf_{z \in X_h} \|w - z\|_X.$$

Now, *we introduce the discrete orthogonal of* $V_h^{\perp}$*, i.e,*

$$V_h^{\perp} := \left\{ v \in X_h : (v, w)_X = 0 \quad \forall\, w \in V_h \right\} \subset X_h.$$

By proceeding as in [1, Subsection 4.1], we can define $\mathscr{P}_h : X \to X_h$ given by

$$\mathscr{P}_h w := \widetilde{\mathscr{P}}_h w + \Pi_h w \quad \forall\, w \in X,$$

where $\widetilde{\mathscr{P}}_h : X \to V_h^{\perp}$ is defined by

$$b\left(\widetilde{\mathscr{P}}_h w, \mu\right) = b\left(w - \Pi_h w, \mu\right) \quad \forall\, \mu \in M_h.$$

The operator $\widetilde{\mathscr{P}}_h$ is well defined thanks to the discrete *inf-sup* condition (9) and there holds

$$\left\| \widetilde{\mathscr{P}}_h w \right\|_X \le \frac{\|b\|}{\beta_h} \left\| w - \Pi_h w \right\|_X,$$

thus, from the triangle inequality, it follows that

$$\left\| w - \mathscr{P}_h w \right\|_X \le C \left(1 + \frac{\|b\|}{\beta_h} \right) \inf_{z \in X_h} \|w - z\|_X \qquad \forall\, w \in X.$$

Therefore, we can use the operator $\mathscr{P}_h$ to consider the split of the error given by

$$e_h^n := u(t_n) - u_h^n = \rho_h^n + \sigma_h^n, \qquad n = 1, \ldots, N, \tag{10}$$

where

$$\rho_h(t) := u(t) - \mathscr{P}_h u(t), \quad \rho_h^n := \rho_h(t_n), \quad \sigma_h^n := \mathscr{P}_h u(t_n) - u_h^n.$$

In order to define the error term for time derivatives and obtain the error estimates with the desired regularity, we will need to introduce some notations and deduce some relevant results. In fact, let $\Lambda_Y : Y \to Y'$ be the Riesz isomorphism and define $\widehat{R} := \Lambda_Y^{-1} R$. Thus, by following the ideas from [6, Section 4], we can observe $\widehat{R} : Y \to Y$ is a monotone, linear and bounded self-adjoint operator. Therefore, the operator $\widehat{R}$ admits a unique square root $\widehat{R}^{1/2}$ which is also a monotone, linear and bounded self-adjoint operator (see [7]). Let $Y_0 := \ker \widehat{R}$, Y_+ be the orthogonal space of Y_0 and $Y_+^{1/2}$ be the completion of Y_+ with respect the topology induced by the norm

$$\|v\|_+ := \left\| \widehat{R}^{1/2} v \right\|_Y.$$

Therefore, we can notice that

$$\langle R v, v \rangle_Y = \left\| \widehat{R}^{1/2} v \right\|_Y^2 \quad \forall\, v \in Y.$$

Denote $P_+ : Y \to Y_+$ the orthogonal projection *satisfies* $\widehat{R} v = \widehat{R}(P_+ v)$ for any $v \in Y$ and observe that

$$\left\| \widehat{R}^{1/2} P_+ v \right\|_Y = \|P_+ v\|_+ \quad \forall\, v \in Y.$$

In summary, we have

$$\langle R v, v \rangle_Y = \left\| \widehat{R}^{1/2} v \right\|_Y^2 = \left\| \widehat{R}^{1/2} P_+ v \right\|_Y^2 = \|P_+ v\|_+^2 \quad \forall\, v \in Y.$$

Now, since the operator R is monotone and self-adjoint, it is easy to check the following Cauchy–Schwarz type inequality

$$|\langle R v, w \rangle_Y| \le \langle R v, v \rangle_Y^{1/2} \langle R w, w \rangle_Y^{1/2} \quad \forall\, v, w \in Y, \tag{11}$$

which implies

$$|\langle R v, w \rangle_Y| \le \|R\|_{\mathscr{L}(Y, Y')}^{\frac{1}{2}} \|w\|_Y \|v\|_+ \quad \forall\, v \in Y_+ \ \forall\, w \in Y$$

and consequently

$$\|R v\|_{Y'} \le \|R\|_{\mathscr{L}(Y, Y')}^{\frac{1}{2}} \|v\|_+ \quad \forall\, v \in Y_+.$$

4. Error estimates by using an extension operator of R

Now we present the first approach to obtain the error estimates with the corresponding regularity of the solution. We start by introducing the operator

$$\widetilde{R} : Y_+^{1/2} \to Y',$$

which is the linear and continuous extension operator of the operator $R|_{Y_+} : Y_+ \to Y'$ (the restriction of the operator R to Y_+). Then, from inequality (11) and by means of an standard argument of continuity and density, we can deduce

$$\left|\left\langle \widetilde{R}v, w\right\rangle_Y\right| \leq \|v\|_+ \langle Rw, w\rangle_Y^{1/2} \quad \forall\, v \in Y_+^{1/2} \quad \forall\, w \in Y. \tag{12}$$

Furthermore, we can easily notice that

$$Rv = R\,(P_+ v) = \widetilde{R}\,(P_+ v) \quad \forall\, v \in Y. \tag{13}$$

On the other hand, we can prove that if $u \in \mathrm{H}^1(0,T;X)(\subseteq \mathrm{H}^1(0,T;Y))$ then $P_+ u \in \mathrm{H}^1(0,T;Y_+)(\subseteq \mathrm{H}^1(0,T;Y))$ and therefore

$$\partial_t\,(P_+ u) = P_+\,(\partial_t u) \quad \text{in} \quad \mathrm{L}^2\,(0,T;Y_+)\,.$$

Thus, by using (13) and the continuity of R, it follows $RP_+ u \in \mathrm{H}^1(0,T;Y')$ and consequently

$$\partial_t(Ru) = \partial_t\,(RP_+ u) = R\partial_t\,(P_+ u) = RP_+ \partial_t u \quad \text{in} \quad \mathrm{L}^2\left(0,T;Y'\right). \tag{14}$$

Now, since the embedding $Y_+ \subseteq Y_+^{1/2}$ is continuous and $P_+ u \in \mathrm{H}^1(0,T;Y_+)$, we have $P_+ u \in \mathrm{H}^1(0,T;Y_+^{1/2})$. Hence, by recalling the fact that R and $\widetilde{R}$ coincide in Y_+ and $\widetilde{R} \in \mathscr{L}(Y_+^{1/2}, Y')$, we deduce $R(P_+ u) = \widetilde{R}(P_+ u) \in \mathrm{H}^1(0,T;Y')$. Moreover, if $P_+ u \in \mathscr{C}^1([0,T];Y_+^{1/2})$ then

$$\partial_t\,(RP_+ u) = \widetilde{R}\partial_t\,(P_+ u) \quad \text{in}\, \mathscr{C}^0\left([0,T];Y'\right)\left(\subseteq \mathscr{C}^0\left([0,T];X'\right)\right).$$

Consequently, if $u \in \mathrm{H}^1(0,T;X)$ and $P_+ u \in \mathscr{C}^1([0,T];Y_+^{1/2})$, from (14) and the previous identity, we can conclude

$$\partial_t(Ru) = \partial_t\,(RP_+ u) = \widetilde{R}\partial_t\,(P_+ u) \quad \text{in}\, \mathscr{C}^0\left([0,T];Y'\right). \tag{15}$$

Now, we are able to define the corresponding error term to estimate the time-derivatives approximation. In fact, by assuming $P_+ u(t) \in \mathscr{C}^1([0,T];Y_+^{1/2})$ we denote

$$\tau^n := \frac{P_+ u\,(t_n) - P_+ u\,(t_{n-1})}{\Delta t} - \partial_t P_+ u\,(t_n)\,, \qquad n = 1,\ldots,N.$$

Note that $\tau^n \in Y_+^{1/2}$, but in general $\tau^n \notin Y$. Moreover, we can notice the following relationship between τ^n, ρ_h^n and σ_h^n (see (10)) for $n = 1,\ldots,N$:

$$\partial_t P_+ u\,(t_n) - \overline{\partial}\left(P_+ u_h^n\right) = \overline{\partial}\left(P_+ e_h^n\right) - \tau^n = \overline{\partial}\left(P_+ \rho_h^n\right) + \overline{\partial}\left(P_+ \sigma_h^n\right) - \tau^n. \tag{16}$$

In the following, we denote C as a generic positive constant that is not necessarily the same at each occurrence.

Lemma 2. *If $u \in \mathrm{H}^1(0,T;X)$ with $P_+ u \in \mathscr{C}^1([0,T];Y_+^{1/2})$, $\lambda \in \mathscr{C}^1(0,T;M)$ and $\{\xi_h\}_{h>0}$, $\{\alpha_h\}_{h>0}$ (see H7) are bounded uniformly in h, then provided Δt is small enough, there exists a constant $C > 0$ independent of h and Δt, such that*

$$\begin{aligned}\left\langle R\sigma_h^n, \sigma_h^n\right\rangle_Y &+ \Delta t \sum_{k=1}^{n} \left\|\sigma_h^k\right\|_X^2 \\ &\leq C\left(\left\langle R\sigma_h^0, \sigma_h^0\right\rangle_Y + \Delta t \sum_{k=1}^{N}\left[\left\|\tau^k\right\|_+^2 + \left\|\overline{\partial}\rho_h^k\right\|_Y^2 + \left\|\rho_h^k\right\|_X^2 + \left(\sup_{v \in V_h} \frac{b\left(v, \partial_t \lambda\,(t_k)\right)}{\|v\|_X}\right)^2\right]\right).\end{aligned} \tag{17}$$

Proof. Let $k \in \{1,\dots,N\}$. By combining (1) and (15), we have

$$\left\langle \widetilde{R}\big(\partial_t (P_+ u)(t_k)\big), v \right\rangle_Y + b\big(v, \partial_t \lambda(t_k)\big) + \langle Au(t_k), v\rangle_X = \langle f(t_k), v\rangle_X \quad \forall\, v \in V_h.$$

Moreover, by using (4) and (13), we obtain

$$\left\langle \widetilde{R}\left(\bar{\partial}\left(P_+ u_h^k\right)\right), v \right\rangle_Y + \left\langle A u_h^k, v\right\rangle_X = \langle f(t_k), v\rangle_X \quad \forall\, v \in V_h.$$

Then, by subtracting these two identities and using (16), we deduce

$$\begin{aligned}&\left\langle \widetilde{R}\bar{\partial}\left(P_+ \sigma_h^k\right), v \right\rangle_Y + \left\langle A\sigma_h^k, v\right\rangle_X \\ &\qquad = \left\langle \widetilde{R}\tau^k, v\right\rangle_Y - \left\langle \widetilde{R}\left(P_+ \bar{\partial}\rho_h^k\right), v\right\rangle_Y - \left\langle A\rho_h^k, v\right\rangle_X - b\big(v, \partial_t \lambda(t_k)\big) \qquad \forall\, v \in V_h,\end{aligned}$$

hence, from (13) it follows

$$\begin{aligned}&\left\langle R\bar{\partial}\sigma_h^k, v \right\rangle_Y + \left\langle A\sigma_h^k, v\right\rangle_X \\ &\qquad = \left\langle \widetilde{R}\tau^k, v\right\rangle_Y - \left\langle R\bar{\partial}\rho_h^k, v\right\rangle_Y - \left\langle A\rho_h^k, v\right\rangle_X - b\big(v, \partial_t \lambda(t_k)\big) \qquad \forall\, v \in V_h.\end{aligned}$$

Thus, by taking $v := \sigma_h^k \in V_h$ in this last identity, we have

$$\left\langle R\bar{\partial}\sigma_h^k, \sigma_h^k \right\rangle_Y + \left\langle A\sigma_h^k, \sigma_h^k\right\rangle_X = \left\langle \widetilde{R}\tau^k, \sigma_h^k\right\rangle_Y - \left\langle R\bar{\partial}\rho_h^k, \sigma_h^k\right\rangle_Y - \left\langle A\rho_h^k, \sigma_h^k\right\rangle_X - b\left(\sigma_h^k, \partial_t \lambda(t_k)\right). \quad (18)$$

Now, we can estimate the first term in the right hand term of this equality by recalling that $\tau^k \in Y_+^{1/2}$ and using (12) and Young inequality, to obtain

$$\left|\left\langle \widetilde{R}\tau^k, \sigma_h^k\right\rangle_Y\right| \le \frac{1}{4}\left\langle R\sigma_h^k, \sigma_h^k\right\rangle_Y + \left\|\tau^k\right\|_+^2.$$

Similarly, it is easy to check that

$$\left|\left\langle R\bar{\partial}\rho_h^k, \sigma_h^k\right\rangle_Y\right| \le \frac{1}{4}\left\langle R\sigma_h^k, \sigma_h^k\right\rangle_Y + C\left\|\bar{\partial}\rho_h^k\right\|_Y^2, \qquad \left\langle A\rho_h^k, \sigma_h^k\right\rangle_X \le \frac{\alpha}{4}\left\|\sigma_h^k\right\|_X^2 + C\left\|\rho_h^k\right\|_X^2.$$

Therefore, having in mind the estimate

$$b\left(\sigma_h^k, \partial_t \lambda(t_k)\right) \le \left\|\sigma_h^k\right\|_X \sup_{v \in V_h} \frac{b\big(v, \partial_t \lambda(t_k)\big)}{\|v\|_X} \le \frac{\alpha}{4}\left\|\sigma_h^k\right\|_X^2 + C\left(\sup_{v \in V_h} \frac{b\big(v, \partial_t \lambda(t_k)\big)}{\|v\|_X}\right)^2,$$

the Lemma 2 is obtained from (18), by using (8) and following the standard arguments for the error estimates of mixed degenerate parabolic problems (see, for instance, [1, Lemma 4.1]). □

Theorem 3. *Under the assumptions of Lemma 2, if* $\{\beta_h\}_{h>0}$ *is bounded uniformly in h and* $u \in \mathrm{H}^1(0,T;X)$ *with* $P_+ u \in \mathrm{H}^2(0,T;Y_+^{1/2})$ *then there exists a constant* $C > 0$ *independent of h and* Δt*, such that for* Δt *small enough there holds*

$$\begin{aligned}&\max_{1\le n\le N}\left\langle R\left(u(t_n) - u_h^n\right), u(t_n) - u_h^n\right\rangle_Y + \Delta t\sum_{n=1}^{N}\|u(t_n) - u_h^n\|_X^2 \\ &\le C\left\{\langle R\left(u_0 - u_{0,h}\right), u_0 - u_{0,h}\rangle_Y + \max_{0\le n\le N}\left(\inf_{z\in X_h}\|u(t_n) - z\|_X\right)^2\right. \\ &\left. + \int_0^T \left(\inf_{z\in X_h}\left\|\partial_t u(t) - z\right\|_X\right)^2 dt + (\Delta t)^2\left\|\partial_{tt}P_+ u\right\|^2_{\mathrm{L}^2(0,T;Y_+^{1/2})} + \Delta t\sum_{n=1}^{N}\left(\inf_{\mu\in M_h}\left\|\partial_t\lambda(t_n) - \mu\right\|_M\right)^2\right\}.\end{aligned}$$

Proof. A Taylor's expansion shows that

$$\sum_{k=1}^{N}\left\|\tau^k\right\|_+^2 = \sum_{k=1}^{N}\left\|\frac{1}{\Delta t}\int_{t_{k-1}}^{t_k}\left(t_{k-1} - t\right)\partial_{tt}P_+ u(t)\,dt\right\|_+^2 \le \Delta t\int_0^T \left\|\partial_{tt}P_+ u(t)\right\|_+^2 dt.$$

Now, by using $\sigma_h^0 = e_h^0 - \rho_h^0$ and recalling that R is self-adjoint and monotone, it follows that

$$\left\langle R\sigma_h^0, \sigma_h^0\right\rangle_Y \le 2\left\langle R\left(u_0 - u_{0,h}\right), u_0 - u_{0,h}\right\rangle_Y + 2\left\langle R\rho_h^0, \rho_h^0\right\rangle_Y.$$

Thus, by using the Lemma 2, we can proceed as in [1, Theorem 4.1] to conclude the result. □

Remark 4. For the case of formulations of the *eddy current model* in terms of a time primitive of the electric field (see [3–5]), we have $X := \mathbf{H}(\mathbf{curl};\Omega)$, $Y := \mathrm{L}^2(\Omega)^3$ and

$$\langle R\mathbf{v},\mathbf{w}\rangle_Y := \int_{\Omega_{\mathrm{C}}} \sigma\mathbf{v}\cdot\mathbf{w} \quad \forall\, \mathbf{v},\mathbf{w}\in Y,$$

where $\Omega\subset\mathbb{R}^3$ is the computational domain, Ω_{C} is the conductor domain satisfying $\Omega_{\mathrm{C}}\subsetneq\Omega$ and σ is the physical parameter called the *electric conductivity* that is assumed to be a positive and bounded function in the conductor (i.e., $0<\sigma_0\le\sigma(\mathbf{x})\le\sigma_1$ a.e. $\mathbf{x}\in\Omega_{\mathrm{C}}$, with σ_0 and σ_1 positive constants) and zero in the insulator $\Omega_{\mathrm{D}} := \Omega\setminus\overline{\Omega}_{\mathrm{C}}$. Therefore, the operators defined above are given by

$$\widehat{R}\mathbf{v} := \sigma\chi_{\Omega_{\mathrm{C}}}\mathbf{v}, \quad \widehat{R}^{1/2}\mathbf{v} := \sigma^{1/2}\chi_{\Omega_{\mathrm{C}}}\mathbf{v} \qquad \forall\,\mathbf{v}\in Y := \mathrm{L}^2(\Omega)^3,$$

where $\chi_{\Omega_{\mathrm{C}}}$ is the characteristic function of Ω_{C}. Furthermore,

$$Y_+^{1/2} = Y_+ := \left\{\mathbf{v}\in\mathrm{L}^2(\Omega)^3 :\ \mathbf{v}|_{\Omega_{\mathrm{D}}} = 0\right\} \cong \mathrm{L}^2\left(\Omega_{\mathrm{C}}\right)^3$$

and $P_+\mathbf{v} := \chi_{\Omega_{\mathrm{C}}}\mathbf{v}$ for all $\mathbf{v}\in Y$. Consequently, the assumption $P_+u\in\mathrm{H}^2(0,T;Y_+^{1/2})$ in Theorem 3 is equivalent to *the fact that* the solution $\mathbf{u}$ of the eddy current problem satisfies

$$\mathbf{u}|_{\Omega_{\mathrm{C}}} \in H^2\left(0,T;\mathrm{L}^2\left(\Omega_{\mathrm{C}}\right)^3\right).$$

5. Error estimates by assuming an elliptic condition for R

Another alternative to deduce error estimates for solution is obtained by assuming the following property about the operator R: there exists a constant $\gamma>0$ such that

$$\langle Rv,v\rangle_Y \ge \gamma\|v\|_Y^2 \quad \forall\, v\in Y_+, \tag{19}$$

where Y_+ is the ortogonal space to $Y_0 := \ker\widehat{R}$ (see Section 3). We can easily check that (19) implies the norms $\|\cdot\|_Y$ and $\|\cdot\|_+$ are equivalent on Y_+ and consequently $Y_+^{1/2} = Y_+$.

Remark 5. Let us notice that for the case of the eddy current model application (see Remark 4), the operator R satisfies (19). More precisely,

$$\langle R\mathbf{v},\mathbf{v}\rangle_Y = \int_{\Omega_{\mathrm{C}}}\sigma|\mathbf{v}|^2 \ge \sigma_0\|\mathbf{v}\|^2_{\mathrm{L}^2\left(\Omega_{\mathrm{C}}\right)^3} \qquad \forall\,\mathbf{v}\in Y_+\cong\mathrm{L}^2\left(\Omega_{\mathrm{C}}\right)^3.$$

For the rest of this section we assume $u\in\mathrm{H}^1(0,T;X)$ and $P_+u\in\mathscr{C}^1([0,T];Y_+)$. Then $R(P_+u)\in\mathscr{C}^1([0,T];Y')$ and

$$\partial_t\left(RP_+u\right) = R\partial_t\left(P_+u\right)\in\mathscr{C}^0\left([0,T];Y'\right)\subset\mathscr{C}^0\left([0,T];X'\right).$$

Furthermore, if $\lambda\in\mathscr{C}^1(0,T;M)$ since $f(\cdot)-Au(\cdot)\in\mathscr{C}^0([0,T];X')$, the identity

$$\left\langle R\big(\partial_t\left(P_+u\right)(t)\big),v\right\rangle_Y + b\big(v,\partial_t\lambda(t)\big) + \left\langle Au(t),v\right\rangle_X = \left\langle f(t),v\right\rangle_X \quad \forall\, v\in V_h \tag{20}$$

holds for all $t\in[0,T]$. Besides, by using (4) and (13), we obtain

$$\left\langle R\left(\overline{\partial}\left(P_+u_h^n\right)\right),v\right\rangle_Y + \left\langle Au_h^n,v\right\rangle_X = \left\langle f(t_n),v\right\rangle_X \quad \forall\, v\in V_h \tag{21}$$

for $n=1,\ldots,N$.

Now, we define

$$\tau^n := \frac{u(t_n)-u(t_{n-1})}{\Delta t} - \partial_t P_+u(t_n), \qquad n=1,\ldots,N.$$

We can notice $\tau^n\in Y$. Moreover, by recalling (10) we obtain

$$\partial_t P_+u(t_n) - \overline{\partial}\left(P_+u_h^n\right) = \overline{\partial}\left(P_+\rho_h^n\right) + \overline{\partial}\left(P_+\sigma_h^n\right) - P_+\tau^n, \tag{22}$$

for $n=1,\ldots,N$.

Next, by using (20), (21) and (22), we can proceed as in the proof of Lemma 2 to obtain the following similar result.

Lemma 6. *If $u \in \mathrm{H}^1(0,T;X)$ with $P_+u \in \mathscr{C}^1([0,T];Y_+)$, $\lambda \in \mathscr{C}^1(0,T;M)$ and $\{\xi_h\}_{h>0}$, $\{\alpha_h\}_{h>0}$ (see H7) are bounded uniformly in h, then provided Δt is small enough, there exists a constant $C>0$ independent of h and Δt, such that*

$$\left\langle R\sigma_h^n, \sigma_h^n \right\rangle_Y + \Delta t \sum_{k=1}^n \left\| \sigma_h^k \right\|_X^2 \leq C\left(\left\langle R\sigma_h^0, \sigma_h^0 \right\rangle_Y + \Delta t \sum_{k=1}^N \left[\left\| P_+\tau^k \right\|_+^2 + \left\| \bar{\partial}\rho_h^k \right\|_Y^2 + \left\| \rho_h^k \right\|_X^2 + \left(\sup_{v\in V_h} \frac{b\left(v, \partial_t \lambda(t_k)\right)}{\|v\|_X} \right)^2 \right] \right).$$

Finally, in virtue of the previous lemma and by proceeding as in the proof of Theorem 3, we deduce the following error estimate.

Theorem 7. *Under the assumptions of Lemma 6, if $\{\beta_h\}_{h>0}$ is bounded uniformly in h and $u \in \mathrm{H}^1(0,T;X)$ with $P_+u \in \mathrm{H}^2(0,T;Y_+)$ then there exists a constant $C>0$ independent of h and Δt, such that*

$$\max_{1\leq n\leq N} \left\langle R\left(u(t_n) - u_h^n\right), u(t_n) - u_h^n \right\rangle_Y + \Delta t \sum_{n=1}^N \left\| u(t_n) - u_h^n \right\|_X^2 \leq C\Bigg\{ \left\langle R\left(u_0 - u_{0,h}\right), u_0 - u_{0,h} \right\rangle_Y + \max_{0\leq n\leq N} \left(\inf_{z\in X_h} \left\| u(t_n) - z \right\|_X \right)^2 + \int_0^T \left(\inf_{z\in X_h} \left\| \partial_t u(t) - z \right\|_X \right)^2 dt + (\Delta t)^2 \left\| \partial_{tt} P_+ u \right\|_{\mathrm{L}^2(0,T;Y_+)}^2 + \Delta t \sum_{n=1}^N \left(\inf_{\mu\in M_h} \left\| \partial_t \lambda(t_n) - \mu \right\|_M \right)^2 \Bigg\}.$$

Remark 8. Theorems 3 and 7 are similar to [1, Theorem 4.1], but this last result requires the solution u belongs to $\mathrm{H}^2(0,T;Y)$, instead of the hypothesis $P_+u \in \mathrm{H}^2(0,T;Y_+)$, which is a more reasonable assumption for a solution of a degenerate problem. In particular, for the case of eddy current applications (see Remark 4), the assumption $P_+u \in \mathrm{H}^2(0,T;Y_+)$ means $\mathbf{u}|_{\Omega_{\mathrm{C}}} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{C}})^3)$, that is in accordance with the functional space to which the solution $\mathbf{u}$ belongs (see [3–5]):

$$\mathscr{W} := \left\{ \mathbf{v} \in \mathrm{L}^2\left(0,T;\mathbf{H}\left(\mathbf{curl};\Omega_{\mathrm{C}}\right)\right) : \partial_t \mathbf{v}|_{\Omega_{\mathrm{C}}} \in \mathrm{L}^2\left(0,T;\mathbf{H}\left(\mathbf{curl};\Omega_{\mathrm{C}}\right)'\right) \right\}.$$

Consequently, by proceeding as in [1, Remark 5.7], we can apply Theorem 3 or Theorem 7 to obtain the asymptotic convergence of the fully-discrete approximation for the eddy current problems studied in [3–5].

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