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Complex analysis and geometry / *Analyse et géométrie complexes*

Canonical metrics on generalized Hartogs triangles

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Abstract. This paper is concerned with the canonical metrics on generalized Hartogs triangles. As main contributions, we first show the existence of a Kähler–Einstein metric on generalized Hartogs triangles. On the other hand, we calculate the explicit expression for Rawnsley's ε-function, and then we give the sufficient and necessary condition for the canonical metric to be balanced. As an application, we also find that there exist canonical metrics on generalized Hartogs triangles being both Kähler–Einstein and balanced.

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1. Introduction

Let $\Omega_{k,m}$ be a Hartogs type domain in $\mathbb{C}^{k+m}$ which is defined by

$$\Omega_{k,m} := \left\{(z,w) = (z_1,\cdots,z_k,w_1,\cdots,w_m) \in \mathbb{C}^{k+m} : \|z\|^2 < |w_1|^2 < \cdots < |w_m|^2 < 1\right\}, \tag{1}$$

where $\|z\|^2 = |z_1|^2 + \cdots + |z_k|^2$. In this paper we call $\Omega_{k,m}$ the generalized Hartogs triangle. It is not hard to see that $\Omega_{k,m}$ is a bounded nonhomogeneous pseudoconvex domain without a smooth boundary.

Recently, the Hartogs triangles have attracted a lot of attentions and many deep results have been obtained on it from different points of view. For example, Edholm [9] studied the L^p boundedness of the Bergman projection on fat Hartogs triangles. In 2017, Zapałowski [17] characterized the existence of proper holomorphic mappings between generalized Hartogs triangles and gave their explicit forms. Later on, Bi–Su [2] explored the geometric properties of a special Hartogs triangle (that is $\Omega_{1,m}$). There are also some other beautiful works on Hartogs triangles, the reader is referred to [4, 8, 16] and references therein.

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In the following, we equip $\Omega_{k,m}$ with a Kähler metric $g(\mu)$ associated with the Kähler potential $\Phi_{k+m}(z,w)$ defined by

$$\Phi_{k+m}(z,w) := -\mu_0 \ln\left(|w_1|^2 - \|z\|^2\right) - \sum_{j=1}^{m-1} \mu_j \ln\left(\left|w_{j+1}\right|^2 - \left|w_j\right|^2\right) - \mu_m \ln\left(1 - |w_m|^2\right), \quad (2)$$

in which $\mu = (\mu_0, \cdots, \mu_m)$ with $\mu_j > 0 (0 \le j \le m)$. Therefore, we know that the associated Kähler form $\omega_{g(\mu)}$ on $\Omega_{k,m}$ can be expressed as

$$\omega_{g(\mu)} := \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\Phi_{k+m}.$$

A Kähler metric on a complex manifold (M,g) is said to be Kähler–Einstein if the Ricci tensor form Ric_g is proportional to the associated Kähler form ω, namely

$$\mathrm{Ric}_g = \lambda\omega.$$

In 1980, Cheng-Yau [5] proved that there exist complete Kähler–Einstein metrics with negative Ricci curvature on strictly pseudoconvex domains (M,g) in $\mathbb{C}^n$. And their solution was exactly the solution of the Monge–Ampère equation:

$$\begin{cases} \det\left(\frac{\partial^2\Phi}{\partial z_i \bar{\partial} z_j}\right) = e^{(n+1)\Phi}, & z \in M, \\ \Phi(z) = \infty, & z \in \partial M, \end{cases} \quad (3)$$

where the metric g is given by $\partial^2\Phi/\partial z_i \bar{\partial} z_j dz^i \otimes d\overline{z^j}$. So the existence of Kähler–Einstein metric on M is equivalent to the existence of solution for the above Monge–Ampère equation on M up to a constant. However the solution of (3) is not easy to obtain.

It is well known that the Bergman metric of homogeneous domain was a Kähler–Einstein metric, and the explicit solution of (3) was also obtained on homogeneous domain. For the nonhomogeneous case, people were mainly focused on the Hartogs domains. Bland [3] described the Kähler–Einstein metric on $\{|z|^2 + |w|^{2p} < 1\}$ without giving the explicit solution of (3). Wang et al. [15] firstly obtained the explicit solution of the Monge–Ampère equation on nonhomogeneous domain called Cartan–Hartogs domain. Zedda [18] also gave an equivalent description of Kähler–Einstein on Cartan–Hartogs domain by using the scalar curvature. For other studies on Hartogs domains, please see [1, 11] and [14]. So it is natural to ask whether the Monge–Ampère equation can be explicitly solved on some other Hartogs domains.

Hence one purpose of our paper is to show that there exists a Kähler–Einstein metric on generalized Hartogs triangles $\Omega_{k,m}$ or equivalently, a solution of the Monge–Ampère equation on generalized Hartogs triangles $\Omega_{k,m}$ can be explicitly found. One of our results is as follows:

Theorem 1. *The generalized Hartogs triangle $(\Omega_{k,m}, g(\mu))$ is a Kähler–Einstein manifold if and only if $\mu_0/\mu_j = (k+1)/2$ for $j = 1, \cdots, m$.*

Therefore it is easy to obtain a solution of Monge–Ampère equation on generalized Hartogs triangles $\Omega_{k,m}$.

Theorem 2. *For $\nu > 0, \mu_j > 0 (j = 0, \cdots, m)$, the following $\Phi'(z,w)$*

$$\Phi'(z,w) = \frac{1}{n+m+1} \ln \nu \prod_{j=1}^{m} \left|w_j\right|^2 - \mu_0 \ln\left(|w_1|^2 - \|z\|^2\right) - \sum_{j=1}^{m-1} \mu_j \ln\left(\left|w_{j+1}\right|^2 - \left|w_j\right|^2\right) - \mu_m \ln\left(1 - |w_m|^2\right)$$

is the explicit solution of Monge–Ampère equation on generalized Hartogs triangles $\Omega_{k,m}$ if and only if

$$\nu = \mu_0^k \mu_1 \cdots \mu_m,\ \mu_0 = \frac{k+1}{n+m+1},\ \mu_j = \frac{2}{n+m+1},\ j = 1, \cdots, m.$$

In the rest of this paper, we will discuss the balanced metrics. Firstly, we define a complex Hilbert space $H_\alpha(\Omega_{k,m})$ as follows:

$$H_\alpha\left(\Omega_{k,m}\right) := \left\{ f \in \mathscr{O}\left(\Omega_{k,m}\right) : \int_{\Omega_{k,m}} |f|^2 \exp\{-\alpha\Phi_{k+m}\} \frac{\omega_{g(\mu)}^{k+m}}{(k+m)!} < +\infty \right\},$$

where $\mathscr{O}(\Omega_{k,m})$ denotes the space of holomorphic functions on $\Omega_{k,m}$. If $H_\alpha(\Omega_{k,m}) \neq \{0\}$, let $K_\alpha(Z,\overline{Z})$ be its weighted reproducing kernel. Then the Rawnsley's ε-function can be expressed as

$$\varepsilon_{(\alpha,g)}(Z) := \exp\left\{-\alpha\Phi_{k+m}(Z)\right\} K_\alpha\left(Z,\overline{Z}\right),\ Z = (z_1,\cdots,z_k,w_1,\cdots,w_m) \in \Omega_{k,m}.$$

Definition 3. *The metric $g(\mu)$ on $\Omega_{k,m}$ is balanced if the Rawnsley's ε-function $\varepsilon_{(1,g)}(Z)(Z \in \Omega_{k,m})$ is a positive constant on $\Omega_{k,m}$.*

The definition of balanced metrics was originally given by Donaldson [7] in the case of a compact polarized Kähler manifold (M,g) in 2001. After that, there has been tremendous interest in developing the existence of balanced metrics (refer to [1,11] and [12]). Moreover, many studies also indicate that balanced metric plays a great role in the geometric quantization, the asymptotic expansion of Bergman kernel and the stability of projective algebraic varieties, please see [10] and [13] for reference.

Recently, Bi–Su [2] proved that there exist a balanced metric on $\Omega_{1,m}$. Inspired by this, we want to solve the existence of balanced metric on generalized Hartogs domain $\Omega_{k,m}$ for general k. In fact, we also obtain the following result.

Theorem 4. *The Kähler metric $g(\mu)$ on $\Omega_{k,m}$ is balanced if and only if $\mu_0 \geq k+1$ is an integer, $\mu_j \geq 2$ are integers for all $j = 1,\cdots,m-1$, and $\mu_m > 1$.*

By Theorems 1 and 4, we easily have

Corollary 5. *The metrics $g(\mu)$ on $\Omega_{k,m}$ are both Kähler–Einstein and balanced if and only if $\mu_0 = (k+1)p$ and $\mu_j = 2p$ $(j = 1,\cdots,m)$ where $p \in \mathbb{N}^+$. In particular, the Bergman metric on the generalized Hartogs triangles $\Omega_{k,m}$ is both Kähler–Einstein and balanced.*

The paper is organized as follows. In Section 2, we will compute the determinant of $g(\mu)$, and then give the proof of Theorem 1 and Theorem 2. In Section 3, we will calculate the explicit expression of the Rawnsley's ε-function. By using this explicit form, we will complete the proof of Theorem 4.

2. The existence of Kähler–Einstein metric

In order to prove the existence of Kähler–Einstein metric on $\Omega_{k,m}$, we first give a key lemma.

Lemma 6. *Let $(\Omega_{k,m}, g(\mu))$ be the generalized Hartogs triangles. Then we have*

$$\det(g(\mu)) = \frac{\mu_0^k \prod_{j=1}^{m} \mu_j \left|w_j\right|^2}{\left(|w_1|^2 - \|z\|^2\right)^{k+1} \cdot \prod_{j=1}^{m-1} \left(\left|w_{j+1}\right|^2 - \left|w_j\right|^2\right)^2 \cdot \left(1 - |w_m|^2\right)^2}. \tag{4}$$

Proof. By definition, a straightforward computations imply that

$$\frac{\partial^2 \Phi_{k+m}(z,w)}{\partial z_j \partial \overline{z_l}} = \begin{cases} \mu_0 \frac{|w_1|^2 - \|z\|^2 + \left|z_j\right|^2}{\left(|w_1|^2 - \|z\|^2\right)^2} & \left(j = l, l = 1,\cdots,k\right), \\ \mu_0 \frac{\overline{z_j} \cdot z_l}{\left(|w_1|^2 - \|z\|^2\right)^2} & \left(j \neq l; j,l = 1,\cdots,k\right). \end{cases}$$

Taking the derivative of Φ_{k+m} with respect to z_j and $\overline{w_1}$, we have

$$\frac{\partial^2\Phi_{k+m}(z,w)}{\partial z_j\partial\overline{w_1}} = \mu_0\frac{-\overline{z_j}\cdot w_1}{\left(|w_1|^2-\|z\|^2\right)^2},\ \left(j=1,\cdots,k\right).$$

Similarly, we can also see that

$$\frac{\partial^2\Phi_{k+m}(z,w)}{\partial w_j\partial\overline{w_l}} = \begin{cases} \mu_{j-1}\frac{|w_{j-1}|^2}{\left(|w_j|^2-|w_{j-1}|^2\right)^2}+\mu_j\frac{|w_{j+1}|^2}{\left(|w_{j+1}|^2-|w_j|^2\right)^2} & \left(j=l,l=2,\cdots,m-1\right),\\ \mu_l\frac{-\overline{w_j}w_l}{\left(|w_j|^2-|w_l|^2\right)^2} & \left(l=1,\cdots,m-1;j=l+1\right),\\ \mu_l\frac{-w_j\overline{w_l}}{\left(|w_j|^2-|w_l|^2\right)^2} & \left(j=1,\cdots,m-1;l=j+1\right),\\ \mu_{m-1}\frac{|w_{m-1}|^2}{(|w_m|^2-|w_{m-1}|^2)^2}+\mu_m\frac{1}{(1-|w_m|^2)^2} & \left(j=l=m\right), \end{cases}$$

and

$$\frac{\partial^2\Phi_{k+m}(z,w)}{\partial w_1\partial\overline{w_1}} = \mu_0\frac{\|z\|^2}{\left(|w_1|^2-\|z\|^2\right)^2}+\mu_1\frac{|w_2|^2}{\left(|w_2|^2-|w_1|^2\right)^2}.$$

Therefore we get

$$g(\mu) = \begin{pmatrix} \mu_0\frac{|w_1|^2-\|z\|^2+|z_1|^2}{(|w_1|^2-\|z\|^2)^2} & \cdots & 0\\ \mu_0\frac{\overline{z}_2 z_1}{(|w_1|^2-\|z\|^2)^2} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mu_{m-1}\frac{-\overline{w_{m-1}}w_m}{(|w_m|^2-|w_{m-1}|^2)^2}\\ 0 & \cdots & \mu_{m-1}\frac{|w_{m-1}|^2}{(|w_m|^2-|w_{m-1}|^2)^2}+\mu_m\frac{1}{(1-|w_m|^2)^2} \end{pmatrix}.$$

It is not hard to see that

$$\det(g(\mu)) = \frac{\mu_0^k\prod_{j=1}^{m}\mu_j\left|w_j\right|^2}{\left(|w_1|^2-\|z\|^2\right)^{k+1}\cdot\prod_{j=1}^{m-1}\left(\left|w_{j+1}\right|^2-\left|w_j\right|^2\right)^2\cdot\left(1-|w_m|^2\right)^2}.$$

The proof of Lemma 6 is finished. □

Proof of Theorem 1. It is well known that the Ricci curvature Ric_g is given in local coordinates by

$$\mathrm{Ric}_{j\bar{l}} = -\frac{\partial^2\ln\left(\det\left(g(\mu)\right)\right)}{\partial Z_j\partial\overline{Z_l}},\left(j,l=1,\cdots,k+m\right).$$

The formula (4) implies that

$$\ln\det g(\mu) = \ln\mu_0^k\prod_{j=1}^{m}\mu_j\left|w_j\right|^2-\ln\left(|w_1|^2-\|z\|^2\right)^{k+1}-\ln\prod_{j=1}^{m-1}\left(\left|w_{j+1}\right|^2-\left|w_j\right|^2\right)^2\cdot\left(1-|w_m|^2\right)^2.$$

Combining with (2), we can see that the metric $g(\mu)$ is Kähler–Einstein if and only if $\mu_0=\frac{k+1}{\lambda},\mu_j=\frac{2}{\lambda},\lambda$ is a constant, that is $\frac{\mu_0}{\mu_j}=\frac{k+1}{2}$ for $j=1,\cdots,m$. We finish the proof. □

Proof of Theorem 2. By definition and after a direct calculation, we can see that

$$e^{(n+m+1)\Phi'} = \frac{\nu\prod_{j=1}^{m}\left|w_j\right|^2}{\left(|w_1|^2-\|z\|^2\right)^{(n+m+1)\mu_0}\cdot\prod_{j=1}^{m-1}\left(\left|w_{j+1}\right|^2-\left|w_j\right|^2\right)^{\mu_j(n+m+1)}\cdot\left(1-|w_m|^2\right)^{\mu_m(n+m+1)}}.$$

By (2), we have

$$\Phi'(z,w)-\Phi_{k+m}(z,w)=\frac{1}{n+m+1}\prod_{j=1}^{m}\ln\nu\left|w_j\right|^2.$$

Since $\ln\nu|w_j|^2$ are pluriharmonic terms. Therefore we get $\partial\overline{\partial}\Phi'=\partial\overline{\partial}\Phi_{k+m}$. It follows that

$$\det\left(\frac{\partial^2\Phi'}{\partial Z_i\overline{\partial}Z_j}\right)=\det g(\mu).$$

Then the conclusion follows by (4). The proof is completed. □

3. The existence of balanced metric

Now we are in a position to discuss the balanced metrics on $\Omega_{k,m}$. We at first give some useful lemmas.

Lemma 7 (see D'Angelo [6, Lemma 1]). *Suppose $\alpha\in(\mathbb{R}_+)^n$, then we have*

$$\int_{B_+^n} r^{2\alpha-1}dV(r)=\frac{\beta(\alpha)}{2^n|\alpha|},$$

$$\int_{S_+^{n-1}}\omega^{2\alpha-1}d\sigma(\omega)=\frac{\beta(\alpha)}{2^{n-1}}.$$

For $\alpha\in(\mathbb{R}_+)^n$, $\beta(\alpha)$ is defined by

$$\beta(\alpha)=\frac{\prod_{i=1}^{n}\Gamma(\alpha_i)}{\Gamma(|\alpha|)},$$

where Γ is the usual Euler gamma function. Here $dV(r)$ is the Euclidean n-dimensional volume form, d is the Euclidean $(n-1)$ dimensional volume form, and the subscript + denotes that all the variables are positive.

Lemma 8 (see D'Angelo [6] Lemma 2). *Let $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, $\|x\|^2<1$ and $s\in\mathbb{R}$ with $s>0$. Then*

$$\sum_{q\in N^n}\frac{\Gamma(|q|+s)}{\Gamma(s)\prod_{i=1}^{n}\Gamma\left(q_i+1\right)}x^{2q}=\frac{1}{\left(1-\|x\|^2\right)^s}.$$

Theorem 9. *Let $Z=(z_1,\cdots,z_k,w_1,\cdots,w_m)$, $p=(p_{01},\cdots,p_{0k},p_1,\cdots,p_m)\in\mathbb{Z}^{k+m}$, then we have*

$$\begin{aligned}\left\|Z^p\right\|^2=&\pi^{k+m}\mu_0^k\prod_{j=1}^{m}\mu_j\frac{\prod_{j=1}^{k}\Gamma\left(p_{0j}+1\right)}{\Gamma\left(\left|p_0\right|+k\right)}B\left(\left|p_0\right|+k,\mu_0-k\right)\\&\times\prod_{v=1}^{m}B\left(\left|p_0\right|+\mu_0+\sum_{j=1}^{v}\left(p_j+\mu_j\right)-\mu_v+1,\mu_v-1\right),\end{aligned}$$

where $B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$ is the beta function.

Proof. According to the definition, we get

$$\begin{aligned}\|Z^p\|^2 &= \int_{\Omega_{k+m}} |Z|^{2p} \exp\{-\Phi_{k+m}(z,w)\} \frac{\omega^{k+m}_{g(\mu)}}{(k+m)!} \\ &= \int_{\Omega_{k+m}} |Z|^{2p} \left(|w_1|^2 - \|z\|^2\right)^{\mu_0} \prod_{j=1}^{m-1} \left(\left|w_{j+1}\right|^2 - \left|w_j\right|^2\right)^{\mu_j} \left(1-|w_m|^2\right)^{\mu_m} \\ &\quad \times \frac{\mu_0^k \prod\limits_{j=1}^{m} \mu_j \left|w_j\right|^2}{\left(|w_1|^2 - \|z\|^2\right)^{k+1} \cdot \prod\limits_{j=1}^{m-1} \left(\left|w_{j+1}\right|^2 - \left|w_j\right|^2\right)^2 \cdot \left(1-|w_m|^2\right)^2} dm(Z),\end{aligned} \tag{5}$$

where $\mathrm{dm}(Z)$ is the standard Euclidean measure. Let

$$z_q = t_{0q} e^{i\theta_q} \left(1 \le q \le k\right), t_0^2 = \sum_{q=1}^{k} t_{0q}^2,$$

and $w_j = t_j e^{i\theta_j}\,(1 \le j \le m)$. Combining with the Fubini theorem, we get

$$\begin{aligned}(5) = (2\pi)^{k+m} \mu_0^k \prod_{j=1}^{m} \mu_j \underbrace{\int_{0 \le \frac{t_0}{t_1} < 1} \prod_{q=1}^{k} \left(\frac{t_{0q}}{t_1}\right)^{2p_{0q}+1} \left(1 - \left(\frac{t_0}{t_1}\right)^2\right)^{\mu_0 - k - 1} d\frac{t_{01}}{t_1} \cdots d\frac{t_{0k}}{t_1}}_{I} \\ \times \underbrace{\int_{0<t_1<\cdots<t_m<1} t_1^{2|p_0|+2\mu_0-2} \prod_{j=1}^{m} t_j^{2p_j+3} \prod_{l=1}^{m-1} \left(t_{l+1}^2 - t_l^2\right)^{\mu_l - 2} \left(1-t_m^2\right)^{\mu_m-2} dt_1 \cdots dt_m}_{J}.\end{aligned}$$

Let $u = \frac{t_0}{t_1} = (u_1, \cdots, u_k)$ and using the spherical coordinates in the variable $u = r\xi$, we get

$$\begin{aligned}I &= \int_{0 \le u < 1} \prod_{q=1}^{k} u_q^{2p_{0q}+1} \left(1-u^2\right)^{\mu_0-k-1} du \\ &= \int_0^1 r^{2|p_0|+k} r^{k-1} \left(1-r^2\right)^{\mu_0-k-1} dr \int_{\partial B^k} \prod_{q=1}^{k} \xi_q^{2p_{0q}+1} d\xi.\end{aligned}$$

Next we set $s_0 = r^2$, and by Lemma 7, we have

$$\begin{aligned}I &= \frac{1}{2}\int_0^1 s^{|p_0|+k-1} (1-s)^{\mu_0-k-1} ds \frac{\beta\left(p_0+1\right)}{2^{k-1}} \\ &= \frac{1}{2^k} B\left(\left|p_0\right| + k, \mu_0 - k\right) \frac{\prod\limits_{j=1}^{k} \Gamma\left(p_{0j}+1\right)}{\Gamma\left(\left|p_0\right|+k\right)}.\end{aligned}$$

Similarly, we can also obtain that

$$J = \frac{1}{2^m} \int_{0<s_1<\cdots<s_m<1} s_1^{|p_0|+\mu_0+p_1} \prod_{j=2}^{m} s_j^{p_j+1} \prod_{l=1}^{m-1} \left(s_{l+1}-s_l\right)^{\mu_l-2} (1-s_m)^{\mu_m-2} ds,$$

where $s = (s_1, \cdots, s_m)$ and $s_j = t_j^2$ $(1 \le j \le m)$. A direct computation implies that

$$J = \frac{1}{2^m} \prod_{v=1}^{m} B\left(\left|p_0\right| + \mu_0 + \sum_{j=1}^{v} \left(p_j + \mu_j\right) - \mu_v + 1, \mu_v - 1\right).$$

Therefore

$$\|Z^p\|^2 = \pi^{k+m}\mu_0^k \prod_{j=1}^{m}\mu_j \frac{\prod_{j=1}^{k}\Gamma\left(p_{0j}+1\right)}{\Gamma\left(|p_0|+k\right)} B\left(|p_0|+k,\mu_0-k\right) \times \prod_{v=1}^{m} B\left(|p_0|+\mu_0+\sum_{j=1}^{v}\left(p_j+\mu_j\right)-\mu_v+1,\mu_v-1\right).$$

The proof of Theorem 9 is completed. □

Theorem 10. *Suppose that* $(\Omega_{k+m}, g(\mu))$ *is a Hartogs triangle endowed with the Kähler metric* $g(\mu)$*. Let* $\mu_0 \geq k+1$ *be an integer,* $\mu_j \geq 2$ *be integers for all* $j = 1,\cdots,m-1$*, and let* $\mu_m > 1$*. Then* $H_1(\Omega_{k+m}) \neq \{0\}$*, and the Bergman kernel is given by*

$$K_1\left(Z,\bar{Z}\right) = \frac{1}{\pi^{k+m}} \frac{\prod_{v=1}^{k}\left(\mu_0-v\right)}{\mu_0^k\left(|w_1|^2-\|z\|^2\right)^{\mu_0}} \prod_{j=1}^{m-1} \frac{\mu_j-1}{\mu_j\left(\left|w_{j+1}\right|^2-\left|w_j\right|^2\right)^{\mu_j}} \frac{\mu_m-1}{\mu_m\left(1-|w_m|^2\right)^{\mu_m}}.$$

Proof. Since Ω_{k+m} is a Reinhardt domain, thus $\{Z^p/\|Z^p\|^2\}$ forms a complete orthonormal basis of $H_1(\Omega_{k+m})$. Together with Theorem 9, we can conclude that the corresponding multi-index $p = \left(p_0,\cdots,p_m\right)$ ranges all integers that satisfy the following inequalities for all $v = 1,\cdots,m$,

$$|p_0|+\mu_0+\sum_{j=1}^{v}\left(p_j+\mu_j\right)-\mu_v \geq 0.$$

Let $\mathbf{N}$ denote the set of all the multi-index $p = (p_0,\cdots,p_m)$ satisfying such inequalities. Hence we get

$$\begin{aligned} &K_1\left(Z,\bar{Z}\right) \\ &= \sum_{p\in\mathbf{N}} \frac{|Z^p|^2}{\|Z^p\|^2_{L^2_{\Phi_{k+m}}}} \\ &= \frac{1}{\pi^{k+m}\mu_0^k\prod_{j=1}^{m}\mu_j} \sum_{p_0\in\mathbb{N}^k}^{+\infty} \frac{\Gamma\left(|p_0|+k\right)}{\prod_{j=1}^{k}\Gamma\left(p_{0j}+1\right)} \frac{|z|^{2p_0}}{B\left(|p_0|+k,\mu_0-k\right)} \sum_{p_1=-|p_0|-\mu_0}^{+\infty} \frac{|w_1|^{2p_1}}{B\left(|p_0|+p_1+\mu_0+1,\mu_1-1\right)} \\ &\times\cdots\times \sum_{p_m=-|p_0|-\mu_0-\sum_{j=1}^{m-1}(p_j+\mu_j)}^{+\infty} \frac{|w_m|^{2p_m}}{B\left(|p_0|+\mu_0+\sum_{j=1}^{m}\left(p_j+\mu_j\right)-\mu_m+1,\mu_m-1\right)}. \end{aligned}$$

Then one can see that

$$\begin{aligned} &\sum_{p_m=-|p_0|-\mu_0-\sum_{j=1}^{m-1}(p_j+\mu_j)}^{+\infty} \frac{|w_m|^{2p_m}}{B\left(|p_0|+\mu_0+\sum_{j=1}^{m}\left(p_j+\mu_j\right)-\mu_m+1,\mu_m-1\right)} \\ &= |w_m|^{-2\left(|p_0|+\mu_0+\sum_{j=1}^{m-1}(p_j+\mu_j)\right)} \sum_{A=0}^{+\infty} \frac{|w_m|^{2A}}{B\left(A+1,\mu_m-1\right)} \\ &= |w_m|^{-2\left(|p_0|+\mu_0+\sum_{j=1}^{m-1}(p_j+\mu_j)\right)} \frac{\mu_m-1}{\left(1-|w_m|^2\right)^{\mu_m}}. \end{aligned}$$

Similarly, we have

$$\sum_{p_{m-1}=-|p_0|-\mu_0-\sum_{j=1}^{m-2}(p_j+\mu_j)}^{+\infty} \frac{|w_{m-1}|^{2p_{m-1}}|w_m|^{-2\left(|p_0|+\mu_0+\sum_{j=1}^{m-1}(p_j+\mu_j)\right)}}{B\left(|p_0|+\mu_0+\sum_{j=1}^{m-1}(p_j+\mu_j)-\mu_{m-1}+1,\mu_{m-1}-1\right)}$$
$$=|w_{m-1}|^{-2\left(|p_0|+\mu_0+\sum_{j=1}^{m-2}(p_j+\mu_j)\right)}\frac{\mu_{m-1}-1}{\left(|w_m|^2-|w_{m-1}|^2\right)^{\mu_{m-1}}}.$$

By induction, we finally get

$$K_1\left(Z,\bar{Z}\right)=\frac{1}{\pi^{k+m}}\frac{\prod_{v=1}^{k}(\mu_0-v)}{\mu_0^k\left(|w_1|^2-\|z\|^2\right)^{\mu_0}}\prod_{j=1}^{m-1}\frac{\mu_j-1}{\mu_j\left(\left|w_{j+1}\right|^2-\left|w_j\right|^2\right)^{\mu_j}}\frac{\mu_m-1}{\mu_m\left(1-|w_m|^2\right)^{\mu_m}}.$$

The proof of Theorem 10 is completed. □

Now we are able to prove Theorem 4.

Proof of Theorem 4. By the definition of balanced metric and Theorem 10, we can get

$$\varepsilon_{(1,g)}(Z)=\exp\left\{-\Phi_{k+m}\right\}K_1\left(Z,\bar{Z}\right)$$
$$=\frac{\prod_{v=1}^{k}(\mu_0-v)\prod_{j=1}^{m}(\mu_j-1)}{\pi^{k+m}\mu_0^k\prod_{j=1}^{m}\mu_j}.$$

Thus, the metric $g(\mu)$ is balanced.

On the other hand, assume that $g(\mu)$ is balanced. So there exists a constant $C>0$ such that

$$K_1\left(Z,\bar{Z}\right)=C\exp\left\{\Phi_{k+m}\right\}$$
$$=C\left(|w_1|^2-\|z\|^2\right)^{-\mu_0}\prod_{j=1}^{m-1}\left(\left|w_{j+1}\right|^2-\left|w_j\right|^2\right)^{-\mu_j}\left(1-|w_m|^2\right)^{-\mu_m}.$$

According to Lemma 8, we get

$$\left(\left|w_{j+1}\right|^2-\left|w_j\right|^2\right)^{-\mu_j}=\sum_{p_j=0}^{+\infty}\frac{\Gamma\left(p_j+\mu_j\right)}{\Gamma\left(\mu_j\right)\Gamma\left(p_j+1\right)}\left|w_j\right|^{2p_j}\left|w_{j+1}\right|^{-2\left(p_j+\mu_j\right)}.$$

Thus for any $p_0\in\mathbb{N}^k$, consider the coefficient of $|z|^{2p_0}$ in the expansion of $K_1(Z,\bar{Z})$, we can see that there exists a constant $\widetilde{C}$ such that the corresponding coefficient equals

$$\widetilde{C}\sum_{p_0\in\mathbb{N}^k}^{+\infty}\frac{\Gamma\left(|p_0|+\mu_0\right)}{\Gamma\left(\mu_0\right)\prod_{q=1}^{k}\Gamma\left(p_{0q}+1\right)}|w_1|^{-2(|p_0|+\mu_0)}\cdot\prod_{j=1}^{m-1}\sum_{p_j=0}^{+\infty}\frac{\Gamma\left(p_j+\mu_j\right)}{\Gamma\left(\mu_j\right)\Gamma\left(p_j+1\right)}\left|w_j\right|^{2p_j}\left|w_{j+1}\right|^{-2\left(p_j+\mu_j\right)}$$
$$\times\sum_{p_m=0}^{+\infty}\frac{\Gamma\left(p_m+\mu_m\right)}{\Gamma\left(\mu_m\right)\Gamma\left(p_m+1\right)}|w_m|^{2p_m}.$$

It follows that there exist p_1 such that $\mu_0=p_1-|p_0|$. Therefore μ_0 is forced to be an integer.

Similarly, we can see that $\mu_1=p_2-p_1$. Then μ_1 must be an integer. Then by induction, we can easily conclude that for any $j=1,\cdots,m-1$, there exist p_j and p_{j+1} such that

$$\mu_0=p_1-|p_0|,\mu_j=p_{j+1}-p_j,$$

where $|p_0|$ and p_j $(j=1,\cdots,m)$ are integers, therefore $\mu_0,\cdots,\mu_{m-1}$ are forced to be integers.

The proof of the Theorem 4 is finished. □

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