INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Bing-Ling Wu and Xiao-Hui Yan

On the denominators of harmonic numbers. IV

Volume 360 (2022), p. 53-57

Published online: 26 January 2022

https://doi.org/10.5802/crmath.282

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2022, 360, p. 53-57
https://doi.org/10.5802/crmath.282

Number theory / *Théorie des nombres*

On the denominators of harmonic numbers. IV

Bing-Ling Wu [a] **and Xiao-Hui Yan** [*, b]

[a] School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, P. R. China

[b] School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, P. R. China

E-mails: wubingling911219@163.com, yanxiaohui_1992@163.com

Abstract. Let $\mathscr{L}$ be the set of all positive integers n such that the denominator of $1+1/2+\cdots+1/n$ is less than the least common multiple of $1,2,\ldots,n$. In this paper, under a certain assumption on linear independence, we prove that the set $\mathscr{L}$ has the upper asymptotic density 1. The assumption follows from Schanuel's conjecture.

Keywords. harmonic numbers, least common multiples, upper asymptotic density.

Mathematical subject classification (2010). 11B05, 11B75.

Funding. This work is supported by the National Natural Science Foundation of China (Grant Nos. 12101332 and 12101009), the Natural Science Foundation in Jiangsu Province (Grant No. BK20200748), the Natural Science Foundation for Colleges and Universities in Jiangsu Province (Grant No. 20KJB110003), the Anhui Provincial Natural Science Foundation (Grant No. 2108085QA02) and the Talent Foundation of Anhui Normal University (Grant No.752038).

Manuscript received 11th August 2021, revised 25th August 2021 and 18th September 2021, accepted 12th October 2021.

1. Introduction

For any positive integer n, let

$$H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\frac{u_n}{v_n},\quad (u_n,v_n)=1,\ v_n>0.$$

The number H_n is called *n-th harmonic number*. Shiu [6] proved that $v_n = v_{n+1}$ for infinitely many positive integers n. Recently, Wu and Chen [9] showed that the set of positive integers n with $v_n = v_{n+1}$ has asymptotic density one. For related research, one may refer to [1, 2, 5, 7, 8]. Especially, Eswarathasan and Levine [2] conjectured that the set J_p of positive integers n such that $p \mid u_n$ is finite for any prime number p. Sanna [5] proved that $J_p(x) \le 129p^{2/3}x^{0.765}$, where

[*] Corresponding author.

$J_p(x)$ denotes the number of integers which are in J_p and do not exceed x. Later, Wu and Chen [7] improved this result to $J_p(x) \le 3x^{2/3+1/(25\log p)}$.

Let $\mathscr{L}$ be the set of all positive integers n such that v_n is less than the least common multiple of $1,2,\dots,n$. Note that v_n divides the least common multiple of $1,2,\dots,n$. Let

$$\bar{d}(\mathscr{L}) = \limsup_{x\to\infty} \frac{\mathscr{L}(x)}{x}.$$

In this paper, under a certain assumption on linear independence, we prove that the set $\mathscr{L}$ has the upper asymptotic density 1. Firstly, we introduce a special case of Schanuel's conjecture [4, p. 30–31].

Weak Schanuel's Conjecture. *If $\beta_1,\dots,\beta_m$ are non-zero, multiplicatively independent algebraic numbers, then $\log\beta_1,\dots,\log\beta_m$ are algebraically independent.*

It is clear that the set of prime numbers is multiplicatively independent. For any distinct primes $q_1, q_2,\dots,q_l$, it follows from weak Schanuel's conjecture that $\log q_1,\dots,\log q_l$ are algebraically independent and so are $1/\log q_1,\dots,1/\log q_l$. So we mention the following conjecture.

Conjecture 1. *For any distinct primes $q_1, q_2,\dots,q_l$, the l real numbers $1/\log q_1,\dots,1/\log q_l$ are linear independent over $\mathbb{Q}$.*

In this paper, we prove the following result.

Theorem 2. *Assuming Conjecture 1, we have $\bar{d}(\mathscr{L}) = 1$.*

2. Preliminaries

Lemma 3 ([3, Theorem 429], Mertens' theorem).

$$\prod_{\substack{p\le x\\ p \text{ is a prime}}} \left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x} \text{ as } x\to+\infty.$$

Lemma 4 ([3, Theorem 442], Kronecker's theorem). *If $\vartheta_1,\vartheta_2,\dots,\vartheta_k,1$ are real numbers and linearly independent over $\mathbb{Q}$, $\alpha_1,\alpha_2,\dots,\alpha_k$ are arbitrary real numbers, and N and ϵ are positive real numbers, then there are integers $n>N$, $s_1,s_2,\dots,s_k$ such that $|n\vartheta_m - s_m - \alpha_m| < \epsilon$ $(m=1,2,\dots,k)$.*

Lemma 5. *Let a,b,c,d be positive real numbers. Then*

$$|(a,b)\setminus(c,d)| \le |c-a|+|b-d|+2,$$

where (x,y) denotes the set of all integers n with $x<n<y$.

Proof. If $b\le c$ or $d\le a$, then the claim is obvious, since $(a,b)\cap(c,d)=\emptyset$. So assuming $b>c$ and $d>a$ one gets

$$\begin{aligned}|(a,b)\setminus(c,d)| &= \sum_{\substack{a<n<b\\ n\le c \text{ or } n\ge d}} 1 \le \sum_{a<n\le\min\{b,c\}} 1 + \sum_{\max\{a,d\}\le n<b} 1 = \sum_{a<n\le c} 1 + \sum_{d\le n<b} 1\\ &\le |c-a|+|b-d|+2.\end{aligned}$$

This completes the proof of Lemma 5. ☐

3. Proof of Theorem 2

Let L_n be the least common multiple of $1,2,\dots,n$. Let p_i be i-th prime, $a_1 = 1$ and

$$a_i = \prod_{j=2}^{i}\left(1-\frac{1}{p_j}\right), \quad i = 2,3,\dots.$$

By Lemma 3, $a_i \to 0$ as $i \to +\infty$. For any $0 < \varepsilon < 1$, we can choose k (fixed) such that $a_k < \frac{1}{2}\varepsilon$. In view of the assumption,

$$\frac{\log p_2}{\log p_i}, \quad i = 2,3,\dots,k$$

are linear independent over $\mathbb{Q}$. By Lemma 4, for any

$$0 < \delta < \frac{\varepsilon}{16k\log p_k}, \tag{1}$$

there are infinitely many k tuples $q > 0$, $s_2,\dots,s_k$ of integers such that

$$\left|q\frac{\log p_2}{\log p_i} - s_i + \frac{\log a_{i-1}}{\log p_i}\right| \le \delta, \quad i = 2,3,\dots,k.$$

That is,

$$p_i^{s_i-\delta} \le a_{i-1}p_2^q \le p_i^{s_i+\delta}, \quad i = 2,3,\dots,k. \tag{2}$$

It is clear that $s_i \to +\infty$ as $q \to +\infty$ $(i = 2,3,\dots,k)$. Now we prove that

$$((p_i-1)p_i^{s_i-1}, p_i^{s_i}) \subseteq \mathscr{L}. \tag{3}$$

Let $n \in ((p_i-1)p_i^{s_i-1}, p_i^{s_i})$. Then $p_i^{s_i-1} | L_n$. Since

$$1+\frac{1}{2}+\cdots+\frac{1}{p_i-1} = \sum_{1\le j\le (p_i-1)/2}\left(\frac{1}{j}+\frac{1}{p_i-j}\right) = \sum_{1\le j\le (p_i-1)/2}\frac{p_i}{j(p_i-j)},$$

it follows that

$$\begin{aligned} H_n &= \sum_{j=1,\, p_i^{s_i-1}\nmid j}^{n}\frac{1}{j} + \frac{1}{p_i^{s_i-1}}\left(1+\frac{1}{2}+\cdots+\frac{1}{p_i-1}\right)\\ &= \sum_{j=1,\, p_i^{s_i-1}\nmid j}^{n}\frac{1}{j} + \frac{1}{p_i^{s_i-2}}\sum_{1\le j\le (p_i-1)/2}\frac{1}{j(p_i-j)}\\ &= \frac{a}{b}+\frac{1}{p_i^{s_i-2}}\frac{c}{d}, \end{aligned}$$

where $p_i^{s_i-1} \nmid b$ and $p_i \nmid d$. Then $\nu_{p_i}(H_n) \ge -(s_i-2)$. Hence, v_n is not divisible by $p_i^{s_i-1}$, that is, $p_i^{s_i-1} \nmid v_n$. Noting that $v_n | L_n$ and $p_i^{s_i-1} | L_n$, we have $v_n < L_n$. Hence $n \in \mathscr{L}$. So (3) holds. It follows from (3) that

$$\begin{aligned}
\mathscr{L}(p_2^q) &\ge \sum_{i=2}^{k} |\mathscr{L} \cap (a_i p_2^q, a_{i-1} p_2^q)| \\
&\ge \sum_{i=2}^{k} |((p_i-1)p_i^{s_i-1}, p_i^{s_i}) \cap (a_i p_2^q, a_{i-1} p_2^q)| \\
&= \sum_{i=2}^{k} \left(|(a_i p_2^q, a_{i-1} p_2^q)| - |(a_i p_2^q, a_{i-1} p_2^q) \setminus ((p_i-1)p_i^{s_i-1}, p_i^{s_i})| \right) \\
&\ge \sum_{i=2}^{k} \left(a_{i-1} p_2^q - a_i p_2^q - 1 - |(a_i p_2^q, a_{i-1} p_2^q) \setminus ((p_i-1)p_i^{s_i-1}, p_i^{s_i})| \right) \\
&= a_1 p_2^q - a_k p_2^q - k + 1 - \sum_{i=2}^{k} |(a_i p_2^q, a_{i-1} p_2^q) \setminus ((p_i-1)p_i^{s_i-1}, p_i^{s_i})| \\
&\ge p_2^q - \frac{1}{2}\varepsilon p_2^q - k - \sum_{i=2}^{k} |(a_i p_2^q, a_{i-1} p_2^q) \setminus ((p_i-1)p_i^{s_i-1}, p_i^{s_i})|.
\end{aligned}$$

It follows from Lemma 5 that

$$\begin{aligned}
|(a_i p_2^q, a_{i-1} p_2^q) \setminus ((p_i-1)p_i^{s_i-1}, p_i^{s_i})| &\le |a_i p_2^q - (p_i-1)p_i^{s_i-1}| + |a_{i-1} p_2^q - p_i^{s_i}| + 2 \\
&= \left(1 - \frac{1}{p_i}\right) |a_{i-1} p_2^q - p_i^{s_i}| + |a_{i-1} p_2^q - p_i^{s_i}| + 2 \\
&\le 2|a_{i-1} p_2^q - p_i^{s_i}| + 2.
\end{aligned}$$

If $a_{i-1} p_2^q \ge p_i^{s_i}$, then by (2) we have

$$0 \le a_{i-1} p_2^q - p_i^{s_i} \le p_i^{s_i+\delta} - p_i^{s_i} = \left(p_i^{\delta} - 1\right) p_i^{s_i}.$$

If $a_{i-1} p_2^q < p_i^{s_i}$, then by (2) we have

$$0 \le p_i^{s_i} - a_{i-1} p_2^q \le p_i^{s_i} - p_i^{s_i-\delta} = p_i^{-\delta}\left(p_i^{\delta} - 1\right) p_i^{s_i} \le \left(p_i^{\delta} - 1\right) p_i^{s_i}.$$

In all cases, we have

$$|a_{i-1} p_2^q - p_i^{s_i}| \le \left(p_i^{\delta} - 1\right) p_i^{s_i}.$$

By (2),

$$p_i^{s_i} \le a_{i-1} p_2^q p_i^{\delta} \le p_2^q p_i^{\delta}.$$

Hence,

$$|a_{i-1} p_2^q - p_i^{s_i}| \le \left(p_i^{\delta} - 1\right) p_i^{\delta} p_2^q.$$

In view of (1),

$$0 < \delta \log p_i \le \delta \log p_k < \frac{\varepsilon}{16k} < \frac{1}{16}.$$

It follows from $e^x - 1 \le 2x$ $(0 \le x \le \frac{1}{2})$ that

$$\left(p_i^{\delta} - 1\right) p_i^{\delta} \le 2\delta(\log p_i) e^{\delta \log p_i} < \frac{\varepsilon}{8k} e^{1/16} < \frac{\varepsilon}{4k}.$$

Thus,

$$|a_{i-1} p_2^q - p_i^{s_i}| < \frac{\varepsilon}{4k} p_2^q.$$

It follows that

$$\sum_{i=2}^{k} |(a_i p_2^q, a_{i-1} p_2^q) \setminus ((p_i-1)p_i^{s_i-1}, p_i^{s_i})| < k\left(\frac{\varepsilon}{2k} p_2^q + 2\right) = \frac{1}{2}\varepsilon p_2^q + 2k.$$

Hence,

$$\mathscr{L}(p_2^q) \ge p_2^q - \frac{1}{2}\varepsilon p_2^q - k - \frac{1}{2}\varepsilon p_2^q - 2k = p_2^q - \varepsilon p_2^q - 3k.$$

Thus,

$$\bar{d}(\mathscr{L}) \ge \lim_{q\to\infty} \frac{\mathscr{L}(p_2^q)}{p_2^q} \ge 1-\varepsilon,$$

where $q \to \infty$ and q satisfying (2). Therefore, $\bar{d}(\mathscr{L}) = 1$. This completes the proof.

Acknowledgments

We would like to thank the referee for his/her helpful comments.

References

[1] D. W. Boyd, "A p-adic study of the partial sums of the harmonic series", *Exp. Math.* **3** (1994), no. 4, p. 287-302.
[2] A. Eswarathasan, E. Levine, "p-integral harmonic sums", *Discrete Math.* **91** (1991), no. 3, p. 249-257.
[3] G. H. Hardy, E. M. Wright, *An introduction to the theory of numbers*, 5th ed., Oxford University Press, 1979.
[4] S. Lang, *Introduction to transcendental numbers*, Addison-Wesley Series in Mathematics, Addison-Wesley Publishing Group, 1966.
[5] C. Sanna, "On the p-adic valuation of harmonic numbers", *J. Number Theory* **166** (2016), p. 41-46.
[6] P. Shiu, "The denominators of harmonic numbers", https://arxiv.org/abs/1607.02863v1, 2016.
[7] B.-L. Wu, Y.-G. Chen, "On certain properties of harmonic numbers", *J. Number Theory* **175** (2017), p. 66-86.
[8] ———, "On the denominators of harmonic numbers", *C. R. Math. Acad. Sci. Paris* **356** (2018), no. 2, p. 129-132.
[9] ———, "On the denominators of harmonic numbers. II", *J. Number Theory* **200** (2019), p. 397-406.