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Some examples of algebraic surfaces with canonical map of degree 20

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Abstract. In this note, we construct two minimal surfaces of general type with geometric genus $p_g = 3$, irregularity $q = 0$, self-intersection of the canonical divisor $K_X^2 = 20, 24$ such that their canonical map is of degree 20. In one of these surfaces, the canonical linear system has a non-trivial fixed part. These surfaces, to our knowledge, are the first examples of minimal surfaces of general type with canonical map of degree 20.

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1. Introduction

If $X$ is a minimal smooth complex projective surface, we denote by $\varphi_{[K_X]} : X \dasharrow \mathbb{P}^{p_g(X)-1}$ the canonical map of $X$, where $K_X$ is the canonical divisor of $X$ and $p_g(X) = \dim H^0(X, K_X)$ is the geometric genus. It is interesting to know which positive integers $d$ occur as the degree of such canonical maps for surfaces of general type. This problem is motivated by the work of A. Beauville [1]. One knows that, for surfaces of general type, the degree $d$ of the canonical map is at most 36 [9, Proposition 5.7]. While surfaces with $d = 1, 2, 3, \ldots, 8$ are easy to construct, only few surfaces with $d > 8$ have been known so far. The first example was found by U. Persson [9] in 1977; in this example, the canonical map has degree 16. Then, a surface with $d = 9$ was constructed by S. L. Tan [14] in 1992. In the last decade, some surfaces with $d = 12, 16, 24, 27, 32, 36$ were constructed by C. Rito [10–13], C. Gleissner, R. Pignatelli and C. Rito [4], Ching-Jui Lai and Sai-Kee Yeung [5], and the author [2]. In this paper, we present a way to construct surfaces with $d = 20$ as $\mathbb{Z}_2^4$-covers of the Del Pezzo surface $Y_4$ of degree 5.

Throughout this paper all surfaces are projective algebraic over the complex numbers. The linear equivalence of divisors is denoted by $\equiv$. We call a surface $X$ no non-trivial 2-torsion if the
only 2-torsion in \( \text{Pic}(X) \) is \( \mathcal{O}_X \). A character \( \chi \) of the group \( \mathbb{Z}^4_2 \) is a homomorphism from \( \mathbb{Z}^4_2 \) to \( \mathbb{C}^* \), the multiplicative group of the non-zero complex numbers. We also use the following notations for Del Pezzo surfaces of degree 5:

**Notation 1.** We denote by \( Y_4 \) the blow-up of \( \mathbb{P}^2 \) at four points in general position \( P_1, P_2, P_3, P_4 \). Let us denote by \( l \) the pull-back of a general line in \( \mathbb{P}^2 \), by \( e_1, e_2, e_3, e_4 \) the exceptional divisors corresponding to \( P_1, P_2, P_3, P_4 \), respectively, by \( f_1, f_2, f_3, f_4 \) the strict transforms of a general line through \( P_1, P_2, P_3, P_4 \), respectively and by \( h_{ij} \) the strict transforms of the line \( P_i P_j \), for all \( i \neq j \) in \{1, 2, 3, 4\}, respectively. The anti-canonical class

\[-K_{Y_4} \equiv f_1 + f_2 + f_3 - e_4 \equiv f_1 + f_2 + f_4 - e_3 \equiv f_1 + f_3 + f_4 - e_2 \equiv f_2 + f_3 + f_4 - e_1\]

is very ample and the linear system \( |-K_{Y_4}| \) embeds \( Y_4 \) as a smooth Del Pezzo surface of degree 5 in \( \mathbb{P}^5 \).

The construction of abelian covers was studied by R. Pardini in [7]. For details about the building data of abelian covers and their notations, we refer the reader to Section 1 and Section 2 of R. Pardini’s work ([7]). For the sake of completeness, we recall some facts on \( \mathbb{Z}^4_2 \)-covers, in a form which is convenient for our later constructions. We will denote by \( \chi_{j_1 j_2 j_3 j_4} \) the character of \( \mathbb{Z}^4_2 \) defined by

\[\chi_{j_1 j_2 j_3 j_4}(a_1, a_2, a_3, a_4) := e^{(\pi a_1 j_1)\sqrt{-1}} e^{(\pi a_2 j_2)\sqrt{-1}} e^{(\pi a_3 j_3)\sqrt{-1}} e^{(\pi a_4 j_4)\sqrt{-1}}\]

for all \( j_1, j_2, j_3, j_4, a_1, a_2, a_3, a_4 \in \mathbb{Z}^4_2 \). A \( \mathbb{Z}^4_2 \)-cover \( X \to Y \) can be determined by a collection of non-trivial divisors \( L \) labelled by characters of \( \mathbb{Z}^4_2 \) and effective divisors \( D_{\sigma} \) labelled by elements of \( \mathbb{Z}^4_2 \) of the surface \( Y \). More precisely, from [7, Theorem 2.1] we can define \( \mathbb{Z}^4_2 \)-covers as follows:

**Proposition 2.** Given \( Y \) a smooth projective surface with no non-trivial 2-torsion, let \( L_\chi \) be divisors of \( Y \) such that \( L_\chi \neq \mathcal{O}_Y \) for all non-trivial characters \( \chi \) of \( \mathbb{Z}^4_2 \) and let \( D_\sigma \) be effective divisors of \( Y \) for all \( \sigma \in \mathbb{Z}^4_2 \setminus \{(0,0,0,0)\} \) such that the total branch divisor \( B := \sum_{\sigma \neq 0} D_\sigma \) is reduced. Then \( \{L_\chi, D_\sigma\}_{\chi, \sigma} \) is the building data of a \( \mathbb{Z}^4_2 \)-cover \( f : X \to Y \) if and only if

\[2L_\chi \equiv \sum_{\chi(\sigma) = -1} D_\sigma \]  \hspace{1cm} (1)

for all non-trivial characters \( \chi \) of \( \mathbb{Z}^4_2 \).

The following theorem is a result of this note:

**Theorem 3.** Let \( f : X \to Y_4 \) be a \( \mathbb{Z}^4_2 \)-cover with the building data \( \{L_\chi, D_\sigma\}_{\chi, \sigma} \) such that the following hold:

(a) Each branch component \( D_\sigma \) is smooth, the total branch locus \( B \) is a simple normal crossings divisor and no more than two of these divisors \( D_\sigma \) go through the same point;
(b) \( D_{0010} + D_{0101} + D_{0110} + D_{1011}, D_{1000} + D_{1010} + D_{1101}, D_{1100} + D_{1110} + D_{1111} \in \mathcal{O}(-K_{Y_4}) \);
(c) \( h^0(K_Y + L_X) = 0 \) for all \( \chi \in \{\chi_{1000}, \chi_{0100}, \chi_{1100}\} \);
(d) The divisor \( D_{0001} + D_{0010} + D_{0011} - K_{Y_4} \) is nef and big.

Then \( X \) is a minimal surface of general type with canonical map of degree 20 satisfying the following:

\[p_g(X) = 3, \quad K_X^2 = 4(D_{0001} + D_{0010} + D_{0011} - K_{Y_4})^2.\]

Moreover, the reduced divisor supported on \( f^* (D_{0001} + D_{0010} + D_{0011}) \) is the fixed part of the canonical system \( |K_X| \).
Let us summarize the proof of Theorem 3. Assumptions (a), (b) and (d) show that the surface \( X \) is a minimal surface of general type. Assumption (c) implies that the following diagram commutes (see Remark 6 for the proof):

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_{1, X}} & Z^2 \\
\downarrow & & \downarrow \phi \\
Z & \xrightarrow{\varphi_{2, X}} & Y_4 \\
\end{array}
\]

In the above diagram, the intermediate surface \( Z := X/\Gamma \) is the quotient surface of \( X \), where \( \Gamma := \langle (0,0,0,1), (0,0,1,0) \rangle \) is the subgroup of \( Z_4^2 \). The surface \( Z \) is the bidouble cover of \( Y_4 \) ramified on \((D_{0100} + D_{0101} + D_{0110} + D_{0111}) + (D_{1000} + D_{1001} + D_{1010} + D_{1011}) + (D_{1100} + D_{1101} + D_{1110} + D_{1111})\).

Assumption (b) shows that the canonical map of \( Z \) is of degree 5 (see Remark 6 for the proof). Therefore, the canonical map of \( X \) is of degree 20. As application of Theorem 3, we construct two surfaces with \( d = 20 \) described as follows:

**Theorem 4.** There exist minimal surfaces of general type \( X \) satisfying the following

| \( d \) | \( K_X^2 \) | \( p_g(X) \) | \( q(X) \) | \( |K_X| \) |
|---|---|---|---|---|
| 20 | 20 | 3 | 0 | base point free |
| 20 | 24 | 3 | 0 | has a non-trivial fixed part |

2. \( Z_4^2 \)-coverings

For the convenience of the reader, we leave here the relations (1) of the building data of \( Z_4^2 \)-covers:

By [7, Theorem 3.1] if each branch component \( D_{\sigma} \) is smooth and the total branch locus \( B \) is a simple normal crossings divisor, the surface \( X \) is smooth.

Also from [7, Lemma 4.2, Proposition 4.2] we have:
Proposition 5. If $Y$ is a smooth surface and $f : X \to Y$ is a smooth $\mathbb{Z}_2^4$-cover with the building data $\{L_X, D_\sigma\}_{\chi, \sigma}$, the surface $X$ satisfies the following:

\[ 2K_X \equiv f^* \left( 2K_Y + \sum_{\sigma \neq 0} D_\sigma \right); \tag{2} \]
\[ f_* \mathcal{O}_X = \mathcal{O}_Y \oplus \bigoplus_{x \neq 0} L_x^{-1}; \tag{3} \]
\[ H^0 (X, K_X) = H^0 (Y, K_Y) \oplus \bigoplus_{x \neq \emptyset} H^0 (Y, K_Y + L_x); \tag{4} \]
\[ K_X^2 = 4 \left( 2K_Y + \sum_{\sigma \neq 0} D_\sigma \right)^2; \tag{5} \]
\[ p_g (X) = p_g (Y) + \sum_{x \neq \emptyset} h^0 (L_x + K_Y); \tag{6} \]
\[ \chi (\mathcal{O}_X) = 16 \chi (\mathcal{O}_Y) + \sum_{x \neq \emptyset} \frac{1}{2} \sum_{x \neq \emptyset} h^0 (L_x + K_Y). \tag{7} \]

Moreover, the canonical linear system $|K_X|$ is generated by

\[ f^* (|K_Y + L_x|) + \sum_{x \neq \emptyset} R_\sigma, \quad \forall x \in J \tag{8} \]

where $J := \{ x : |K_Y + L_x| \neq \emptyset \}$ and $R_\sigma$ is the reduced divisor supported on $f^* (D_\sigma)$.

For the proof of the last statement of Proposition 5, we refer the reader to [4, p. 3].

3. Surfaces with $d = 20$ as $\mathbb{Z}_2^4$-covers

3.1. Proof of Theorem 3

The surface $X$ is smooth because each branch component $D_\sigma$ is smooth, the total branch locus $B$ is a normal crossings divisor and no more than two of these divisors $D_\sigma$ go through the same point. Moreover, by Proposition 5, the surface $X$ satisfies the following:

\[ 2K_X \equiv f^* \left( 2K_Y + \sum_{\sigma} D_\sigma \right) \]
\[ \equiv f^* (D_{0001} + D_{0010} + D_{0011} - K_Y). \]

We notice that a surface is of general type and minimal if the canonical divisor is big and nef (see e.g. [6, Section 2]). We remark that the divisor $D_{0001} + D_{0010} + D_{0011} - K_Y$ is nef and big by Assumption (d). Since the divisor $2K_X$ is the pull-back of a nef and big divisor, the canonical divisor $K_X$ is nef and big. Thus, the surface $X$ is of general type and minimal. Furthermore, from Proposition 5, the surface $X$ possesses the following invariants:

\[ p_g (X) = 3, \quad K_X^2 = 4 \left( D_{0001} + D_{0010} + D_{0011} - K_Y \right)^2. \]

We show that the canonical map $\varphi_{|K_X|}$ has degree 20. By Assumptions (b) and (c), we have

\[ L_{1000} + K_{Y_1} \equiv L_{0100} + K_{Y_1} \equiv L_{1100} + K_{Y_1} \equiv \mathcal{O}_{Y_1}, \]
\[ h^0 (L_x + K_{Y_1}) = 0, \quad \forall x \notin \{ x_{1000}, x_{0100}, x_{1100} \}. \]

By (8), the linear system $|K_X|$ is generated by the three following divisors:

\[ D_{0001} + D_{0010} + D_{0011} + D_{0100} + D_{0110} + D_{0111} + D_{1000} + D_{1001} + D_{1010} + D_{1011}, \]
\[ D_{0001} + D_{0010} + D_{0011} + D_{0100} + D_{1000} + D_{1001} + D_{1010} + D_{1011}, \]
\[ D_{0001} + D_{0010} + D_{0011} + D_{1100} + D_{1101} + D_{1110} + D_{1111}. \]
where $D_\sigma$ are the reduced divisors supported $f^* (D_\sigma)$, for all $\sigma$. Because the divisors $D_{0001}, D_{0010}, D_{0011}$ are common components of the three above divisors, these divisors $D_{0001}, D_{0010}, D_{0011}$ are fixed components of $|K_X|$.

On the other hand, by Assumption (a) the three divisors $D_{0100} + D_{0101} + D_{0110} + D_{0111}, D_{1000} + D_{1010} + D_{1011}, D_{1100} + D_{1101} + D_{1110} + D_{1111}$ have no common intersection. So the linear system $|M|$ is base point free, where $M := D_{0100} + D_{0101} + D_{0110} + D_{0111}$. This together with $M^2 = 4(3l - e_1 - e_2 - e_3 - e_4)^2 = 20 > 0$ implies that the linear system $|K_X|$ is not composed with a pencil. Thus, the canonical image is $\mathbb{P}^2$, the canonical map is of degree 20, and the divisor $D_{0001} + D_{0010} + D_{0011}$ is the fixed part of $|K_X|$.

**Remark 6.** The canonical map $\varphi|_{K_X}$ of $X$ is the composition of the quotient map $X \to Z := X/\Gamma$ with the canonical map $\varphi|_{K_Z}$ of $Z$. Moreover, the canonical map of $Z$ is of degree 5.

In fact, by (4), we have the following decomposition:

$$H^0 (X, K_X) = H^0 (Y_4, K_{Y_4}) \oplus \bigoplus_{\chi \neq 1000} \left( H^0 (Y_4, K_{Y_4} + L_Y) \right).$$

The group $\Gamma := \langle (0, 0, 0, 1) \rangle \subset \mathbb{Z}_2^4$ is the subgroup of $\mathbb{Z}_2^4$. Let $\Gamma^\perp$ denote the kernel of the restriction map $(\mathbb{Z}_2^4)^* \to \Gamma^*$, where $\Gamma^*$ is the character group of $\Gamma$. We have $\Gamma^\perp = \langle \chi_{0100}, \chi_{0010}, \chi_{1100} \rangle$. The subgroup $\Gamma$ acts trivially on $H^0 (X, K_X)$ since $h^0 (L_Y + K_{Y_4}) = 0$ for all $\chi \notin \Gamma^\perp$ by Assumption (c). So the canonical map $\varphi|_{K_X}$ is the composition of the quotient map $X \to Z := X/\Gamma$ with the canonical map $\varphi|_{K_Z}$ of $Z$ (see e.g. [8, Example 2.1]).

The intermediate surface $Z$ is the bidouble cover of $Y_4$ with the building data $\{D_1, D_2, D_3, L_1, L_2, L_3\}$ determined as follows:

$$D_1 := D_{0100} + D_{0101} + D_{0110} + D_{0111} \equiv -K_{Y_4},$$

$$D_2 := D_{1000} + D_{1001} + D_{1010} + D_{1011} \equiv -K_{Y_4},$$

$$D_3 := D_{1100} + D_{1101} + D_{1110} + D_{1111} \equiv -K_{Y_4},$$

$$L_1 := L_{1000} \equiv -K_{Y_4},$$

$$L_2 := L_{0100} \equiv -K_{Y_4},$$

$$L_3 := L_{1100} \equiv -K_{Y_4}.$$

Assumption (a) shows that the singularities of $Z$ are nodes and the canonical map of $Z$ is of degree $(3l - e_1 - e_2 - e_3 - e_4)^2 = 5$.

### 3.2. Constructions of the surfaces in Theorem 4

#### 3.2.1. A surface with $d = 20$, $p_g = 3$, $q = 0$, $K^2 = 20$

In this section, we construct the surface described in the first row of Theorem 4. Let $Y_4$ be a Del Pezzo surface of degree 5 (see Notation 1). We consider the following smooth divisors of $Y_4$:

$$D_{1001} := h_{14}, \quad D_{0110} := f_{31} + e_1, \quad D_{0111} := h_{12},$$

$$D_{1101} := f_{11} + e_2, \quad D_{1010} := h_{23}, \quad D_{1011} := h_{24},$$

$$D_{1110} := h_{34}, \quad D_{1111} := f_{21} + e_3.$$
and $D_\sigma = 0$ for the other $\sigma$, where $f_1 \in |f_1|$, $f_{21} \in |f_{21}|$ and $f_{31} \in |f_{31}|$ such that no more than two of these divisors $D_\sigma$ go through the same point. We consider the following non-trivial divisors of $Y_4$:

$$L_{0001} := 2f_1 + f_2 - e_4$$
$$L_{0010} := 2f_2 + f_3 - e_4$$
$$L_{0100} := f_1 + f_2 + f_3 - e_4$$
$$L_{1000} := f_1 + f_2 + f_3 - e_4$$
$$L_{0011} := f_1 + 2f_2 - e_4$$
$$L_{0101} := f_3 + f_4$$
$$L_{1010} := h_{12} + h_{34}$$
$$L_{0111} := f_1 + f_3$$
$$L_{1101} := f_2 + f_3$$
$$L_{1110} := f_1 + f_4$$
$$L_{1111} := h_{12} + h_{34}.$$  

These divisors $D_\sigma, L_X$ satisfy the following relations:

Thus by Proposition 2, the divisors $D_\sigma, L_X$ define a $\mathbb{Z}_2^4$-cover $g : X \to Y_4$. Moreover, this $\mathbb{Z}_2^4$-cover fulfills the hypotheses of Theorem 3. In fact, we have that

$$D_{0100} + D_{0010} + D_{0110} + D_{1010} + D_{1011} + D_{1101} = h_{14} + f_{31} + e_4 + h_{12} \equiv 3l - e_1 - e_2 - e_3 - e_4$$
$$D_{1000} + D_{1001} + D_{1010} + D_{1011} + D_{1100} + D_{1110} = f_1 + e_2 + h_{23} + h_{24} \equiv 3l - e_1 - e_2 - e_3 - e_4$$
$$D_{1100} + D_{1101} + D_{1110} + D_{1111} = h_{13} + h_{34} + f_2 + e_3 \equiv 3l - e_1 - e_2 - e_3 - e_4.$$  

$h^0(K_Y + L_X) = 0$ for all $\chi \in \{\chi_{1000}, \chi_{0100}, \chi_{1100}\}$, and the divisor $D_{0001} + D_{0010} + D_{0011} - K_{Y_4} \equiv 3l - e_1 - e_2 - e_3 - e_4$ is nef and big. Thus by Theorem 3 and Proposition 5, the surface $X$ is a minimal surface of general type and possesses the following invariants:

$$K_X^2 = 20, p_g(X) = 3, \chi(\mathcal{O}_X) = 4, q(X) = 0.$$  

Moreover, the canonical map $q_{|K_X|}$ is of degree 20 and the linear system $|K_X|$ is base point free.

**Remark 7.** The surface $X$ has four pencils of genus 9 corresponding to the fibres $f_1, f_2, f_3, f_4$.

In the above construction, for each choice of $f_{11} \in |f_1|, f_{21} \in |f_{21}|$ and $f_{31} \in |f_{31}|$, we obtain a natural deformation of the surface $X$ (we refer [7, Definition 5.1] for the definition of natural deformations of an abelian cover). It is worth pointing out that a natural deformation of an abelian cover $X \to Y$ is a deformation of the map $X \to Y$ by [7, Proposition 5.1].
Remark 8. The surface $X$ admits natural deformations. Moreover, all the natural deformations of $X$ are Galois.

In fact, by [7, Definition 5.1] the natural deformations of the $\mathbb{Z}_2^4$-cover $g : X \to Y_4$ are parametrized by the direct sum of the vector spaces

$$\bigoplus_{\sigma \neq 0} H^0(Y_4, D_\sigma) \bigoplus_{\chi \neq 0} H^0(Y_4, D_\sigma - L_X).$$

Moreover, all the natural deformations of $X$ are Galois if the second summand $\bigoplus_{\chi \neq 0} H^0(Y_4, D_\sigma - L_X)$ is zero (see [3, Definition 3.2]). We have that

$$H^0(Y_4, D_{0110}) = H^0(Y_4, f_{31}) \equiv \mathbb{C}^2$$
$$H^0(Y_4, D_{1001}) = H^0(Y_4, f_{11}) \equiv \mathbb{C}^2$$
$$H^0(Y_4, D_{1111}) = H^0(Y_4, f_{21}) \equiv \mathbb{C}^2$$

and $H^0(Y_4, D_\sigma) \equiv \mathbb{C}$ for the other non-trivial $D_\sigma$. So the family of natural deformations of $g : X \to Y_4$ is parametrized by the base space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, all natural deformations of $X$ are Galois since $\bigoplus_{\chi \neq 0} H^0(Y_4, D_\sigma - L_X) = 0$.

3.2.2. A surface with $d = 20, p_g = 3, q = 0, K^2 = 24$

In this section, we construct the surface described in the second row of Theorem 4. We consider the following smooth divisors of a del Pezzo surface $Y_4$ of degree 5:

\begin{align*}
D_{0011} &:= e_4 \\
D_{0101} &:= h_{14} \\
D_{1000} &:= e_2 \\
D_{1100} &:= h_{34}
\end{align*}

\begin{align*}
D_{0110} &:= f_{21} \\
D_{1010} &:= h_{23} \\
D_{1110} &:= e_1 \\
D_{1111} &:= e_3
\end{align*}

\begin{align*}
D_{0011} &:= f_{31} \\
D_{1010} &:= h_{24} \\
D_{1111} &:= f_{11}
\end{align*}

and the other $D_\sigma = 0$, where $f_{11} \in |f_1|$, $f_{21} \in |f_2|$ and $f_{31} \in |f_3|$ such that no more than two of these divisors $D_\sigma$ go through the same point. We consider the following non-trivial divisors of $Y_4$:

\begin{align*}
L_{0001} &:= 2f_1 + f_2 - e_3 \\
L_{0010} &:= f_2 + l \\
L_{0100} &:= f_1 + f_2 + f_3 - e_4 \\
L_{1000} &:= f_1 + f_2 + f_3 - e_4 \\
L_{0011} &:= f_1 + 2f_2 - e_3 - e_4 \\
L_{0110} &:= f_2 + f_3 - e_3 - e_4 \\
L_{1010} &:= f_1 + f_2 + f_3 - e_3 \\
L_{1011} &:= f_1 + f_2 + f_3 - e_4 \\
L_{1100} &:= f_3 + f_4 \\
L_{1101} &:= f_1 + f_2 + f_3 - e_4 \\
L_{1110} &:= f_1 + f_2 + f_3 - e_4 \\
L_{1111} &:= f_1 + f_3.
\end{align*}
These divisors $D_\sigma, L_X$ satisfy the following relations:

\[
\begin{align*}
2D_{0011} & \equiv D_{0011} + D_{0011} + D_{0011} + D_{0011} + D_{0111} + D_{1111} + D_{1111} = 4f_1 + 2f_2 - 2e_3 \\
2D_{0101} & \equiv D_{0101} + D_{0011} + D_{0110} + D_{1010} + D_{1110} + D_{1111} + D_{1111} = 2f_2 + 2f_3 - 2e_4 \\
2D_{1101} & \equiv D_{1101} + D_{0101} + D_{1010} + D_{1101} + D_{1110} + D_{1111} + D_{1111} = 2f_1 + 2f_2 + 2f_3 - 2e_4 \\
2L_{1101} & \equiv D_{1101} + D_{0101} + D_{1010} + D_{1101} + D_{1101} + D_{1110} + D_{1110} = 2f_1 + 2f_2 + 2f_3 - 2e_4 \\
2L_{1110} & \equiv D_{1110} + D_{0110} + D_{1011} + D_{1110} + D_{1110} + D_{1111} + D_{1111} = 2f_1 + 2f_2 + 2f_3 - 2e_4 \\
2L_{1111} & \equiv D_{1111} + D_{0111} + D_{1011} + D_{1111} + D_{1111} + D_{1111} + D_{1111} = 2f_1 + 2f_2 + 2f_3 \\
\end{align*}
\]

Thus by Proposition 2, the divisors $D_\sigma, L_X$ define a $\mathbb{Z}_2^4$-cover $g : X \to Y$. Moreover, this $\mathbb{Z}_2^4$-cover fulfills the hypotheses of Theorem 3. In fact, we have

\[
\begin{align*}
D_{0100} + D_{0101} + D_{0110} + D_{1111} & = h_{14} + f_2 + f_3 + f_4 = 3l - e_1 - e_2 - e_3 - e_4 \\
D_{1000} + D_{1001} + D_{1010} + D_{1111} & = e_2 + h_{13} + h_{24} + f_{11} = 3l - e_1 - e_2 - e_3 - e_4 \\
D_{1100} + D_{1101} + D_{1110} + D_{1111} & = h_{34} + h_{12} + h_{13} + e_1 + e_3 = 3l - e_1 - e_2 - e_3 - e_4,
\end{align*}
\]

$h^0(K_Y + L_X) = 0$ for all $\chi \notin \{\chi_{1000}, \chi_{1010}, \chi_{1100}\}$, and the divisor $D_{0001} + D_{0010} + D_{0101} - K_Y \equiv 3l - e_1 - e_2 - e_3$ is nef and big. Thus by Theorem 3 and Proposition 5, the surface $X$ is a minimal surface of general type and possesses the following invariants:

\[K_X^2 = 24, p_g(S) = 3, \chi(\mathcal{O}_S) = 4, q(S) = 0.\]

Moreover, the canonical map $\varphi_{K_X}$ is of degree 20 and the two $(-2)$-curves coming from $\bar{e}_4$ are the fixed part of $|K_X|$. Therefore, we obtain the surface in the second row of Theorem 4.

**Remark 9.** The surface $X$ has three pencils of genus 9 corresponding the fibres $f_1, f_2, f_3$ and a pencil of genus 13 corresponding to the fibre $f_4$.

**Remark 10.** The surface $X$ admits natural deformations. Moreover, all the natural deformations of $X$ are Galois.

Similarly to Remark 8, we have that $H^0(Y_4, D_{0010}) \equiv H^0(Y_4, D_{0111}) \equiv H^0(Y_4, D_{1011}) \equiv C^2$ and $H^0(Y_4, D_0) \equiv C$ for the other non-trivial $D_\sigma$. This implies that the family of natural deformations of $g : X \to Y_4$ is parametrized by the base space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, all natural deformations of $X$ are Galois since \[H^0(Y_4, D_\sigma - L_X) = 0.\]

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**References**


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