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Weak-type endpoint bounds for Bochner–Riesz means for the Hermite operator

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Abstract. We obtain weak-type (p, p) endpoint bounds for Bochner–Riesz means for the Hermite operator $H = -\Delta + |x|^2$ in $\mathbb{R}^n$, $n \geq 2$ and for other related operators, for $1 \leq p \leq 2n/(n+2)$, extending earlier results of Thangavelu and of Karadzhov.

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1. Introduction

Convergence of the Bochner–Riesz means on Lebesgue L^p spaces is one of the classical problems in harmonic analysis. Let us begin with recalling the Bochner–Riesz means operator S_R^δ on $\mathbb{R}^n$ which is defined by, for $\delta \geq 0$ and $R > 0$,

$$\widehat{S_R^\delta f}(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\delta \widehat{f}(\xi), \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{1}$$

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Here $\widehat{f}$ denotes the Fourier transform of f and $(x)_+ := \max\{0,x\}$ for $x \in \mathbb{R}$. A natural problem is to characterize the optimal range of δ for which S_R^δ is bounded on $L^p(\mathbb{R}^n)$. The Bochner–Riesz conjecture is that, for $n \geq 2$ and $1 \leq p \leq 2n/(n+1)$, S_R^δ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\delta > \delta(p) = n\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}. \tag{2}$$

It was shown by Herz that for a given p the above condition on δ is necessary; see [14]. Carleson and Sjölin [2] proved the conjecture when $n = 2$. Afterward substantial progress has been made [1, 11, 18, 30], but the conjecture still remains open for $n \geq 3$ and p close to $2n/(n+1)$. We refer the reader to Stein's monograph [25, Chapter IX] and Tao [29] for historical background and more on the Bochner–Riesz conjecture.

Concerning the endpoint estimates (for $\delta = \delta(p)$) of the Bochner–Riesz means, it is natural to conjecture that $S_R^{\delta(p)}$ is of weak-type (p,p) for $1 \leq p < 2n/(n+1)$. In the special case $n = 2$ the weak-type endpoint conjecture was proved by Seeger [21] for the full range of $p \in [1,4/3)$. In higher dimensions the weak-type endpoint estimate was proved by Christ [6, 7] for the range $1 \leq p < 2(n+1)/(n+3)$, making use of the well-known $(p,2)$ restriction theorem of Stein–Tomas [25, p. 386]. The weak-type endpoint estimate for $p = 2(n+1)/(n+3)$ was proved by Tao [27]. As shown by Tao [28], the weak-type endpoint Bochner–Riesz conjecture is equivalent to the standard Bochner–Riesz conjecture.

Inspired by the works of Christ and Tao [6, 7, 27], Ouhabaz, Sikora and the first and fourth authors of this paper extended the above results to the Bochner–Riesz means associated to second-order elliptic differential operators L on $\mathbb{R}^n$ which are self-adjoint and formally non-negative; see [5]. Such an operator L admits a spectral resolution

$$Lf = \int_0^\infty \lambda dE_L(\lambda)f, \quad f \in L^2\left(\mathbb{R}^n\right),$$

where $E_L(\lambda)$ is the projection-valued measure supported on the spectrum of L. Notice that the spectrum of L may be continuous, discrete, or a combination of both. By the spectral theorem, the Bochner–Riesz means for L of order $\delta \geq 0$ with $R > 0$ are defined by

$$S_R^\delta(L)f = \int_0^{R^2} \left(1 - \frac{\lambda}{R^2}\right)_+^\delta dE_L(\lambda)f \tag{3}$$

for $f \in L^2(\mathbb{R}^n)$. In the special case when $-L$ is the standard Laplace operator $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ on $\mathbb{R}^n$, $S_R^\delta(-\Delta)$ coincides with the usual S_R^δ. It was proved in [5, Theorem I.24] that if L satisfies the finite speed of propagation property (FS) (see Section 2 below), then for some p with $1 \leq p < 2$, the spectral measure estimate

$$\left\| dE_{\sqrt{L}}(\lambda)\right\|_{L^p(\mathbb{R}^n)\to L^{p'}(\mathbb{R}^n)} \leq C\,\lambda^{n\left(\frac{1}{p}-\frac{1}{p'}\right)-1},\ \lambda > 0 \tag{$\mathrm{R_p}$}$$

implies weak-type (p,p) estimates for $S_R^{\delta(p)}(L)$, uniformly in R. This recovers the known results in [6,7,27]. To understand the condition $(\mathrm{R_p})$, we recall that for $\lambda > 0$, the restriction operator R_λ is given by $R_\lambda(f)(\omega) = \widehat{f}(\lambda\omega)$, where $\omega \in \mathbf{S}^{n-1}$ (the unit sphere). Then $dE_{\sqrt{-\Delta}}(\lambda) = (2\pi)^{-n}\lambda^{n-1}R_\lambda^* R_\lambda$, and for p with $1 \leq p \leq 2(n+1)/(n+3)$ the Stein–Tomas $(p,2)$ restriction theorem [25, p. 386] is equivalent to the estimate $(\mathrm{R_p})$. The condition $(\mathrm{R_p})$ is valid for a broad class of second-order elliptic operators such as scattering operators on $\mathbb{R}^3$ and Schrödinger operators $-\Delta + V$ on $\mathbb{R}^n$, where V is smooth and decays sufficiently fast at infinity. See [5, Propositions III.3 and III.6].

The condition $(\mathrm{R_p})$ implies that the point spectrum of L is empty. In particular, $(\mathrm{R_p})$ does not hold for elliptic operators on compact manifolds, nor for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. In the case of the Laplace–Beltrami operator Δ_g on a compact smooth Riemannian manifold (M,g) of dimension $n \geq 2$, Sogge [24] used a Fourier transform side argument to prove that under an additional curvature assumption, one has a (discrete) $(p,2)$ restriction theorem for all p with $1 \le p \le 2(n+1)/(n+3)$, namely

$$\left\| E_{\sqrt{\Delta_g}}[\lambda,\lambda+1) \right\|_{L^p(M)\to L^2(M)} \le C(1+\lambda)^{\delta(p)}, \quad \lambda \ge 0. \tag{S$_\mathrm{p}$}$$

In Christ and Sogge [8], it was shown using the condition (S$_\mathrm{p}$) that $S_R^{\delta(1)}(\Delta_g)$ is weak $(1,1)$ uniformly in R. Later, weak-type (p,p) estimates for $S_R^{\delta(p)}(\Delta_g)$ were proved by Seeger [20] when $1 < p < 2(n+1)/(n+3)$ and by Tao [27] when $p = 2(n+1)/(n+3)$. See also [5, Proposition III.2].

The purpose of this paper can be viewed as a continuation of the above body of work on the weak-type L^p mapping properties of the Bochner–Riesz summation for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$, $n \ge 2$, and for other related operators. For the Hermite operator, it is known that the spectral decomposition of H is given by the Hermite expansion; see [33]. Let $h_\alpha(x), \alpha \in \mathbb{N}^n$, be the normalized Hermite functions which are eigenfunctions for H with eigenvalues $(2|\alpha|+n)$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Thus every $f \in L^2(\mathbb{R}^n)$ has the Hermite expansion

$$f = \sum_\alpha \left(f, h_\alpha\right) h_\alpha, \tag{4}$$

where the sum is extended over all multi-indices $\alpha \in \mathbb{N}^n$. Then the Bochner–Riesz means for H of order $\delta \ge 0$ with $R > 0$ as defined in equation (3) with $L = H$ coincide with

$$S_R^\delta(H) f = \sum_{k=0}^\infty \left(1 - \frac{2k+n}{R^2}\right)_+^\delta P_k f, \tag{5}$$

where P_k are the projections

$$P_k f = \sum_{|\alpha|=k} \left(f, h_\alpha\right) h_\alpha. \tag{6}$$

The Hermite expansion (4) and the corresponding Bochner–Riesz means (5) were studied in [33]. When $n \ge 2$, the conjecture is that the operators $S_R^\delta(H)$ are bounded on $L^p(\mathbb{R}^n)$, uniformly in $R > 0$, if and only if $\delta > \delta(p)$, where $\delta(p)$ is the same critical index as defined in (2) for the Bochner–Riesz means in the case of the standard Laplacian on $\mathbb{R}^n$ (see [34, p. 259]). In [33], Thangavelu proved that the conjecture is true when $p = 1$ and that for a given p the above condition on δ is necessary. In 1994, Karadzhov [15] proved the conjecture in the range $1 \le p \le 2n/(n+2)$. The main ingredient in the proof of these results is to establish the following restriction type theorem

$$\left\|P_k f\right\|_{L^2(\mathbb{R}^n)} \le C k^{(\delta(p)-1/2)/2} \left\|f\right\|_{L^p(\mathbb{R}^n)}, \quad \text{for all } k \in \mathbb{N} \tag{7}$$

for the spectral projection operators P_k for $1 \le p \le 2n/(n+2)$, which is an adaptation of the arguments from [12, 23] that the restriction theorem implies Bochner–Riesz summation theorems for $L^p(\mathbb{R}^n)$. For more on the Bochner–Riesz summation for the Hermite operator, see also [31, 32, 34].

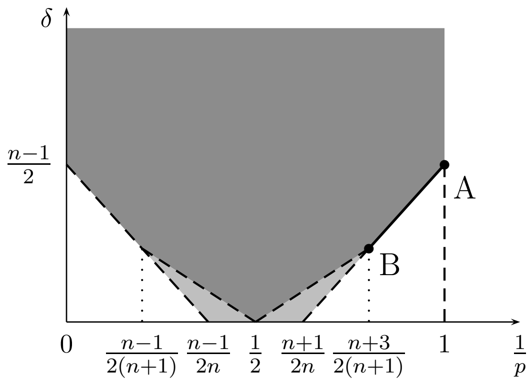
The main goal of this paper is to extend the results of [15, 33] to weak-type endpoint results for the range $1 \le p \le 2n/(n+2)$. We first recall that for $1 \le p < \infty$, a function f is said to be in weak $L^p(\mathbb{R}^n)$, written $f \in L^{p,\infty}(\mathbb{R}^n)$, if

$$\left\|f\right\|_{L^{p,\infty}(\mathbb{R}^n)} := \sup_{\alpha>0} \alpha \left|\left\{x : \left|f(x)\right| > \alpha\right\}\right|^{1/p} < \infty.$$

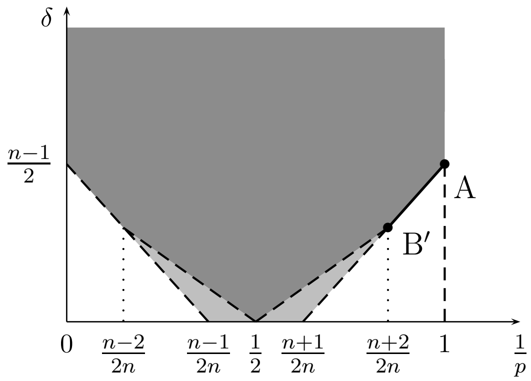
We can now state our main result, which we put in context in Figure 1b.

Theorem 1. *For $n \ge 2$ and $1 \le p \le 2n/(n+2)$, the Bochner–Riesz means $S_R^{\delta(p)}(H)$ are of weak-type (p,p) uniformly in R. That is, there exists a constant $C > 0$ independent of R such that*

$$\left\| S_R^{\delta(p)}(H) f \right\|_{L^{p,\infty}(\mathbb{R}^n)} \le C \left\|f\right\|_{L^p(\mathbb{R}^n)}, \qquad \textit{for all } f \in L^p\left(\mathbb{R}^n\right) \textit{ and all } R > 0.$$



(a) Results for Laplacian operator on $\mathbb{R}^n$ or on compact manifolds.



(b) Results for Hermite operator, including our result.

Figure 1. Schematic diagrams, in dimension $n \geq 3$, summarizing known results on boundedness of Bochner–Riesz means S_R^δ, in Figure 1a for the Laplacian operator and in Figure 1b for the Hermite operator. For each point $(1/p, \delta)$ in the dark gray regions, S_R^δ is bounded on L^p; for $(1/p, \delta)$ in the light gray triangles, the boundedness of S_R^δ is unknown or partial results are known; and for $(1/p, \delta)$ in the white triangles, S_R^δ is not bounded on L^p. For $(1/p, \delta)$ on the line segment AB, where $\delta = \delta(p)$, in Figure 1a, the (Laplacian) Bochner–Riesz means $S_R^\delta = S_R^{\delta(p)}$ satisfy the weak-type endpoint estimate. Our result in Theorem 1 is that for $(1/p, \delta)$ on the line segment AB' in Figure 1b, the (Hermite) Bochner–Riesz means $S_R^\delta(H) = S_R^{\delta(p)}(H)$ satisfy the weak-type endpoint estimate. Note that A represents the same point in both figures, but $B \neq B'$.

As a consequence of this theorem, we have that when $f \in L^p(\mathbb{R}^n)$, the operator $S_R^{\delta(p)}(H)f$ converges in measure to f. By this we mean that for each $\alpha > 0$,

$$\left|\left\{x : \left|S_R^{\delta(p)}(H)f(x) - f(x)\right| > \alpha\right\}\right| \to 0 \quad \text{as } R \to \infty.$$

In fact $S_R^{\delta(p)}(H)f \to f$ in the $L^{p,\infty}$ quasi-norm, that is,

$$\left\| S_R^{\delta(p)}(H)f - f \right\|_{L^{p,\infty}(\mathbb{R}^n)} \to 0.$$

This result is of course considerably weaker than almost-everywhere convergence, and, in fact, at the critical index $\delta(p)$ one does not generally have almost-everywhere convergence of the Riesz means to a given L^1 function; see Stein and Weiss [26].

We would like to mention the following:

(1) Our Theorem 1 shows that the Bochner–Riesz means $S_R^{\delta(p)}(H)$ are of weak type (p,p). This result is optimal. In fact, the operator $S_R^{\delta(p)}(H)$ can not be bounded on $L^p(\mathbb{R}^n)$. Otherwise, along with a transplantation theorem of Mitjagin [19] (see also Kenig–Stanton–Tomas [16, pp. 29-31]), the L^p-boundedness of $S_R^{\delta(p)}(H)$ would imply that the classical Bochner–Riesz means $S_R^{\delta(p)}(-\Delta)$ are bounded on $L^p(\mathbb{R}^n)$, which is already proved to be false (see, for example, [25]).

(2) Our restriction-type condition (7) is weaker than the classical restriction-type condition ($\mathrm{R_p}$). To compensate for this difference, when proving the weak-type L^p estimates for $S_R^{\delta(p)}(H)$ in our Theorem 1, we need an a priori estimate

$$\left\|(1+H)^{-(\delta(p)+1/2)/2}\right\|_{L^2(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)} \le C \tag{8}$$

for $1 \le p \le 2n/(n+2)$, and this is a crucial observation in our paper. Then Theorem 1 is proved by using the a priori estimate (8), along with the L^p eigenfunction bounds (7) for the Hermite operator, and the approach in the work of Christ [6, 7] and Tao [27]. Their approach is based on L^2 Calderón–Zygmund techniques (as used in Fefferman [11]), a spatial decomposition of the Bochner–Riesz summation, and the fact that if the inverse Fourier transform F is supported on a set of width R, then by the finite speed of propagation property the operator $F(\sqrt{L})$ is supported in a CR-neighbourhood of the diagonal.

We outline our proof of Theorem 1 here, highlighting the point where it differs from the approach of Christ [6, 7] and Tao [27]. We first use L^2 Calderón–Zygmund techniques to decompose the function f into $f = g + \sum_j b_j$. Next we make a decomposition (Lemmas 5 and 6) of the Bochner–Riesz multiplier function, corresponding to this Calderón–Zygmund decomposition, in such a way that the main contribution acting on b_j is a multiplier operator $n_j(\sqrt{H})$, where the support of $n_j : \mathbb{R} \to \mathbb{R}$ is mostly concentrated on a set whose radius goes like the reciprocal of the radius of the support of b_j. For the "good" part g and for those b_j which have small support, the argument is similar to that in [27]. However for those b_j with large support, following the argument in [27], we get an extra factor in the upper bound for the L^2 estimate of $n_j(\sqrt{H})b_j$ (see estimate (34) below), compared to the situation treated in Christ [6,7] and Tao [27], where the operator L satisfies the restriction type estimate ($\mathrm{R_p}$) or the manifold on which f is defined is compact. We overcome the obstacle posed by this extra factor by applying our a priori estimate (8) and a modification of the argument in [27]. See Section 3 for details, specifically where we use the a priori estimate (8) to deduce estimate (33) from estimate (34).

The paper is organized as follows. In Section 2 we provide some preliminary results, which we need later, mainly to prove (8) and a few technical lemmas. The proof of Theorem 1 is given in Section 3. In Section 4 we discuss some extensions of Theorem 1 for other operators related to the Hermite operator H.

2. Preliminaries

For brevity, in the rest of the whole paper, for $1 \le p \le +\infty$, we write L^p for $L^p(\mathbb{R}^n)$, $L^{p,\infty}$ for $L^{p,\infty}(\mathbb{R}^n)$, and so on. We denote the norm of a function $f \in L^p$ by $\|f\|_p$ and if T is a bounded linear operator from L^p to L^q, $1 \le p,\ q \le +\infty$, we write $\|T\|_{p\to q}$ for the operator norm of T.

In this section, we mainly consider Schrödinger operators H_V similar to the Hermite operator H, that is, $H_V = -\Delta + V$ on $\mathbb{R}^n$ for $n \geq 2$, with a positive potential V which satisfies the conditions

$$V \sim |x|^2, \quad |\nabla V| \sim |x|, \quad \left|\partial_x^2 V\right| \leq 1. \tag{9}$$

We first state some basic properties of H_V in Lemmas 2 and 3. Then we prove estimate (8) for H_V in Lemma 4. Finally, we state two technical lemmas, Lemmas 5 and 6, for decompositions of Bochner–Riesz multiplier functions.

The operator H_V is a self-adjoint operator on L^2. Since the potential V is nonnegative, the semigroup kernels $\mathscr{K}_t(x,y)$ of the operators e^{-tH_V} satisfy

$$0 \leq \mathscr{K}_t(x,y) \leq h_t(x-y) \tag{10}$$

for all $x, y \in \mathbb{R}^n$ and $t > 0$, where

$$h_t(x-y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \tag{11}$$

is the kernel of the classical heat semigroup $\{T_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$ on $\mathbb{R}^n$.

To formulate the finite speed of propagation property for the wave equation corresponding to an operator H_V, we set

$$\mathscr{D}_r := \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \leq r\right\}.$$

Given an operator T from L^p to L^q, we write

$$\operatorname{supp} K_T \subseteq \mathscr{D}_r \tag{12}$$

if $\langle T f_1, f_2\rangle = 0$ whenever $f_1 \in L^p(B(x_1, r_1))$ and $f_2 \in L^{q'}(B(x_2, r_2))$ with $r_1 + r_2 + r < |x_1 - x_2|$, where as usual $1/q + 1/q' = 1$. Note that if T is an integral operator with kernel K_T, then (12) coincides with the standard meaning of $\operatorname{supp} K_T \subseteq \mathscr{D}_r$, namely, $K_T(x, y) = 0$ for all $(x, y) \notin \mathscr{D}_r$.

Following [3], given a nonnegative self-adjoint operator L on L^2 we say that L satisfies the *finite speed of propagation property* if

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subseteq \mathscr{D}_t, \quad \text{for all } t > . \tag{FS}$$

From (10) and (11), it follows (see for example [9]) that the operator H_V satisfies the finite speed of propagation property (FS). Then we have the following result.

Lemma 2. *Assume that F is an even bounded Borel function with Fourier transform $\widehat{F} \in L^1(\mathbb{R})$ and that* $\operatorname{supp} \widehat{F} \subseteq [-r, r]$. *Then the kernel $K_{F(\sqrt{H_V})}$ of the operator $F(\sqrt{H_V})$ satisfies*

$$\operatorname{supp} K_{F(\sqrt{H_V})} \subseteq \mathscr{D}_r.$$

Proof. If F is an even function, then by the Fourier inversion formula,

$$F\left(\sqrt{H_V}\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{F}(t) \cos\left(t\sqrt{H_V}\right) dt.$$

But $\operatorname{supp} \widehat{F} \subseteq [-r, r]$, and the lemma then follows from (FS). □

We also have the following result.

Lemma 3. *Let $\{Q_k\}_{k\in\mathbb{N}}$ be a family of continuous real-valued functions such that $\sum_k |Q_k(\lambda)|^2 \leq A$ for some constant A independent of $\lambda \in \mathbb{R}$. Then for every sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ on $\mathbb{R}^n$,*

$$\left\| \sum_{k\in\mathbb{N}} Q_k\left(\sqrt{H_V}\right) f_k \right\|_2^2 \leq A \sum_{k\in\mathbb{N}} \left\|f_k\right\|_2^2. \tag{13}$$

Proof. The proof of Lemma 3 is given in [5, Lemma I.28]. □

Let $H_V = -\Delta + V$ with a positive potential V satisfying (9). It is shown in [4, Corollary 6.3] that for each $\nu > 0$,

$$\left\| (1+H_V)^{-\gamma/2} \right\|_{2\to p} \le C, \quad \text{where } \gamma = n(1/p - 1/2) + \nu \tag{14}$$

for $1 \le p \le 2n/(n+2)$ (see also [10, Lemma 7.9] for $p = 1$). To prove Theorem 1, we need the following endpoint version of (14).

Lemma 4. *Let $H_V = -\Delta + V$ with a positive potential V satisfying* (9)*. Then*

$$\left\| (1+H_V)^{-\gamma/2} \right\|_{L^2\to L^{p,\infty}} \le C, \quad \textit{where } \gamma = n(1/p - 1/2), \tag{15}$$

for $1 \le p \le 2n/(n+2)$.

Proof. To prove (15), we put $M_g(f) := fg$ and $M := M_{\sqrt{1+V}}$. We observe that

$$\left\| (1+H_V)^{1/2} f \right\|_2^2 = \left\langle (1+H_V) f, f \right\rangle \ge \left\langle M^2 f, f \right\rangle = \| M f \|_2^2.$$

Now by the Löwner–Heinz inequality for arbitrary quadratic forms B_1 and B_2, if $B_1 \ge B_2 \ge 0$, then $B_1^\alpha \ge B_2^\alpha$ for $0 \le \alpha \le 1$. Hence

$$\left\langle (1+H_V)^\alpha f, f \right\rangle \ge \left\langle M^{2\alpha} f, f \right\rangle.$$

Thus, for $\alpha \in [0,1]$,

$$\left\| M^\alpha (1+H_V)^{-\alpha/2} \right\|_{2\to 2} \le C. \tag{16}$$

For $\alpha = 1$ the operator $M^\alpha (1+H_V)^{-\alpha/2}$ is of first-order Riesz transform type, and a standard argument yields,

$$\left\| M (1+H_V)^{-1/2} \right\|_{L^1\to L^{1,\infty}} \le C;$$

see [22, Theorem 11]. Then by an interpolation theorem for Lorentz spaces [13, Theorem 1.4.19], which as noted there can be seen as the off-diagonal extension of Marcinkiewicz's interpolation theorem, we have for each $q \in (1,2)$,

$$\left\| M (1+H_V)^{-1/2} \right\|_{L^{q,\infty}\to L^{q,\infty}} \le C. \tag{17}$$

By Hölder's inequality for weak spaces (see for example [13, Exercise 1.1.15]), for all $q_1 \ge q_2 \ge 1$ with $s = (1/q_2 - 1/q_1)^{-1}$,

$$\| M^{-\alpha} \|_{L^{q_1,\infty}\to L^{q_2,\infty}} \le C \left\| \left(\sqrt{1+V}\right)^{-\alpha} \right\|_{L^{s,\infty}}. \tag{18}$$

Recall that $\gamma = n(1/p - 1/2)$. Write

$$(1+H_V)^{-\gamma/2} = \left(M^{-1} M (1+H_V)^{-1/2}\right)^{[\gamma]} M^{[\gamma]-\gamma} M^{\gamma-[\gamma]} (1+H_V)^{([\gamma]-\gamma)/2}. \tag{19}$$

Since $V(x) \sim |x|^2$, choosing $s = n/\alpha$ in (18) gives

$$\left\| \left(\sqrt{1+V}\right)^{-\alpha} \right\|_{L^{s,\infty}}^s \le C \sup_{\lambda>0} \lambda^s \left\{ x \in \mathbb{R}^n : \left(\sqrt{1+V}\right)^{-\alpha} > \lambda \right\} \le C \lambda^s \lambda^{-n/\alpha} \le C.$$

Define p_0 by $1/p_0 = (\gamma - [\gamma])/n + 1/2$, and for each $1 \le i \le [\gamma] - 1$ define p_i by setting $1/p_{i+1} - 1/p_i = 1/n$, so $p_{[\gamma]} = p$. Now multiple composition of the operators from (16), (17) and (18), in combination with (19), yields

$$\begin{aligned} &\left\| (1+H_V)^{-\gamma/2} \right\|_{L^2\to L^{p,\infty}} \\ &\qquad \le \left\| M^{\gamma-[\gamma]} (1+H_V)^{([\gamma]-\gamma)/2} \right\|_{L^2\to L^{2,\infty}} \left\| M^{[\gamma]-\gamma} \right\|_{L^{2,\infty}\to L^{p_0,\infty}} \\ &\qquad\qquad \prod_{i=0}^{[\gamma]-1} \left\| M^{-1} M (1+H_V)^{-1/2} \right\|_{L^{p_i,\infty}\to L^{p_{i+1},\infty}} \le C. \end{aligned}$$

This completes the proof of (15). □

The proof of Theorem 1 also requires the following two technical lemmas for decompositions of Bochner–Riesz multiplier functions.

Lemma 5. *For each integer $k \le 0$ there exists a decomposition of the Bochner–Riesz multiplier function $S_R^{\delta(p)}(\lambda^2)$ as follows:*

$$S_R^{\delta(p)}\left(\lambda^2\right) := \left(1-\frac{\lambda^2}{R^2}\right)_+^{\delta(p)} = \eta_k(\lambda) n_k(\lambda) + S_R^{\delta(p)}\left(\lambda^2\right) n_k(\lambda), \quad \lambda \in \mathbb{R}, \tag{20}$$

such that

(a) *The functions n_k are even and their Fourier transforms are supported in $[-2^k/R, 2^k/R]$, that is,* $\operatorname{supp} \widehat{n_k} \subset [-2^k/R, 2^k/R]$*;*
(b) *The functions η_k are continuous and even, and $\sum_{k=-\infty}^{0} |\eta_k(\lambda)|^2 \le C$ with C independent of λ and R;*
(c) *For arbitrary large $N \in \mathbb{N}$ there exists a constant C such that*

$$\left|n_k(\lambda)\right| \le C\left(1+\frac{2^k|\lambda|}{R}\right)^{-N}.$$

Proof. For the proof, we refer the reader to [5, Lemma I.26]. See also [27, Lemma 2.1]. □

Lemma 6. *For each integer $k > 0$, there exists a decomposition of the Bochner–Riesz multiplier function $S_R^{\delta(p)}(\lambda^2)$ as follows:*

$$S_R^{\delta(p)}\left(\lambda^2\right) = \left(1-\frac{\lambda^2}{R^2}\right)_+^{\delta(p)} = m_k(\lambda) + \eta_k(\lambda) n_k(\lambda), \quad \lambda \in \mathbb{R}, \tag{21}$$

such that:

(a) *The functions $\widehat{m_k}$ and $\widehat{n_k}$ are even and supported on $[-2^k/R, 2^k/R]$;*
(b) *The functions η_k are continuous and $\sum_{k=1}^{\infty} |\eta_k(\lambda)|^2 \le C$ uniformly in $\lambda > 0$ and in $R > 0$. In addition, we have that for $R > 1$ and $\lambda > 0$,*

$$\sum_{k=1}^{\infty} \left|\eta_k(\lambda)\right|^2 \left(1+\lambda^2\right)^{\gamma} \le C R^{2\gamma}, \quad \textit{for all } \gamma > 0 \tag{22}$$

with C independent of λ and R;
(c) *For arbitrary large $N \in \mathbb{N}$ there exists a constant $C = C(N)$ such that*

$$\left|n_k(\lambda)\right| \le C 2^{-\delta(p)k}\left(1+2^k\left|1-\frac{|\lambda|}{R}\right|\right)^{-N}.$$

Proof. We follow [27, Lemma 2.1] to obtain a decomposition $S_R^{\delta(p)}(\lambda^2) = m_k(\lambda) + \eta_k(\lambda) n_k(\lambda)$ such that properties (a), (b) and (c) of Lemma 6 hold, except that inequality (22) in (b) remains to be verified. Indeed, from the construction of η_k in [27, Lemma 2.1], it follows that for $|1-|\lambda|/R| > 2^{-k}$,

$$\eta_k(\lambda) \le C_N \left(2^k\left|1-\frac{|\lambda|}{R}\right|\right)^{-N}$$

for each $N \in \mathbb{N}$, and for $|1-|\lambda|/R| \le 2^{-k}$,

$$\eta_k(\lambda) \le C\left(2^{-k}+2^k\left|1-\frac{|\lambda|}{R}\right|\right)^{\varepsilon}$$

for some $\varepsilon > 0$. Then we write

$$\sum_{k=1}^{\infty} \left|\eta_k(\lambda)\right|^2 \left(1+\lambda^2\right)^{\gamma} \le C \sum_{k:2^{-k}<\left|1-\frac{|\lambda|}{R}\right|} \left(2^k\left|1-\frac{|\lambda|}{R}\right|\right)^{-2N} \left(1+\lambda^2\right)^{\gamma}$$

$$
\begin{aligned}
&+C \sum_{k:2^{-k}\geq\left|1-\frac{|\lambda|}{R}\right|} \left(2^{-k}+2^{k}\left|1-\frac{|\lambda|}{R}\right|\right)^{2\varepsilon} \left(1+\lambda^{2}\right)^{\gamma} \\
&=: (\mathrm{I})+(\mathrm{II}).
\end{aligned}
$$

Since $R>1$, we have

$$
\begin{aligned}
(\mathrm{I}) &\leq \begin{cases} C\sum_{k:2^{-k}<\left|1-\frac{|\lambda|}{R}\right|}\left(2^{k}\left|1-\frac{|\lambda|}{R}\right|\right)^{-2N} R^{2\gamma}, & \text{if } \frac{|\lambda|}{R}<2; \\ C\sum_{k:2^{-k}<\left|1-\frac{|\lambda|}{R}\right|}\left(2^{k}\left|1-\frac{|\lambda|}{R}\right|\right)^{-2(N-\gamma)}\left(\frac{|\lambda|}{R}\right)^{-2\gamma}2^{-2\gamma k}\lambda^{2\gamma}, & \text{if } \frac{|\lambda|}{R}\geq 2; \end{cases} \\
&\leq CR^{2\gamma},
\end{aligned}
$$

as long as $N\in\mathbb{N}$ is chosen so that $N>\gamma$. For the term (II), we note that $|1-|\lambda|/R|\leq 2^{-k}\leq 1$, and so $|\lambda|\leq 2R$. Then we have

$$
(\mathrm{II}) \leq C \sum_{|1-|\lambda|/R|\leq 2^{-k}} \left(2^{-k}+2^{k}\left|1-\frac{|\lambda|}{R}\right|\right)^{2\varepsilon} R^{2\gamma} \leq CR^{2\gamma}.
$$

This proves (22), and completes the proof of Lemma 6. □

3. Proof of Theorem 1

For clarity, we prove Theorem 1 for the Hermite operator $H=-\Delta+|x|^{2}$. Actually the conclusion of Theorem 1 also holds for $H_V=-\Delta+V$ where V satisfies (9). See Section 4 for details. For the proof, as stated in the introduction, we first use Calderón–Zygmund techniques to decompose the function f as $f=g+\sum_j b_j$. For the "good" part g and for those b_j that have small support, the argument is similar to that in [27] or that in [5]. The main difference happens when the support of b_j is large; here we apply Lemma 4. See estimate (33) and its proof below for details.

Proof of Theorem 1. First we consider the case that $R\leq 4$. Fix $n\geq 2$ and p with $1\leq p\leq 2n/(n+2)$. We show that $S_R^{\delta(p)}(H)$ is of weak type (p,p) uniformly in $R\leq 4$. In this case, we apply Lemma 4 to obtain that for $\gamma=n(1/p-1/2)$,

$$
\begin{aligned}
\left\| S_R^{\delta(p)}(H)f \right\|_{L^{p,\infty}} &\leq \left\|(I+H)^{-\gamma/2}\right\|_{L^2\to L^{p,\infty}} \left\| S_R^{\delta(p)}(H)(1+H)^{\gamma/2}f \right\|_2 \\
&\leq C\left\| S_R^{\delta(p)}(H)(1+H)^{\gamma/2}f \right\|_2.
\end{aligned}
$$

Since $R\leq 4$, we have supp $S_R^{\delta(p)}(\lambda^2)\subset[-16,16]$. So it follows from the Hermite expansion (4) and equality (5) that

$$
S_R^{\delta(p)}(H)(1+H)^{\gamma/2}f=\sum_{k=0}^{7} S_R^{\delta(p)}(2k+n)(1+2k+n)^{\gamma/2}P_k f.
$$

We apply the above equality and the restriction estimate (7) to obtain

$$
\left\| S_R^{\delta(p)}(H)f \right\|_{L^{p,\infty}} \leq C\sum_{k=0}^{7} S_R^{\delta(p)}(2k+n)(1+2k+n)^{\gamma/2}k^{\delta(p)/2-1/4}\left\|f\right\|_p \leq C\left\|f\right\|_p,
$$

as required.

Next we consider the remaining case $R>4$. Fix $f\in L^p$ and $\alpha>0$, and apply the Calderón–Zygmund decomposition at height α to $|f|^p$. There exist constants C and K so that

(i) $f=g+b=g+\sum_j b_j$;

(ii) $\|g\|_p\leq C\|f\|_p$, $\|g\|_\infty\leq C\alpha$;

(iii) b_j is supported in B_j and $\#\{j : x\in 4B_j\}\leq K$ for all $x\in\mathbb{R}^n$;

(iv) $\int_{\mathbb{R}^n} |b_j|^p dx \le C\alpha^p |B_j|$, and $\sum_j |B_j| \le C\alpha^{-p}\|f\|_p^p$.

Note that by (ii), we have $\alpha^{p-2}\|g\|_2^2 \le C\|f\|_p^p$.

Let r_{B_j} be the radius of B_j and let

$$J_k := \left\{ j : 2^k/R \le r_{B_j} < 2^{k+1}/R \right\}, \quad \text{for } k \in \mathbb{Z}.$$

Write

$$\begin{aligned} f = g + \sum_j b_j &= g + \sum_{k\le 0}\sum_{j\in J_k} b_j + \sum_{k>0}\sum_{j\in J_k} b_j \\ &=: g + h_1 + h_2. \end{aligned}$$

Then it is enough to show that there exists a constant $C>0$ independent of R and α such that

$$\left|\left\{x : S_R^{\delta(p)}(H)(g)(x) > \alpha\right\}\right| \le C\alpha^{-p}\|f\|_p^p \tag{23}$$

and such that for $i = 1,2$,

$$\left|\left\{x : S_R^{\delta(p)}(H)\,(h_i)\,(x) > \alpha\right\}\right| \le C\alpha^{-p}\|f\|_p^p. \tag{24}$$

Note that

$$\sup_{\lambda, R>0} \left(1-\lambda^2/R^2\right)_+^{\delta(p)} = 1 \ \text{ and that } \ \alpha^{p-2}\|g\|_2^2 \le C\|f\|_p^p.$$

Hence by the spectral theorem

$$\left|\left\{x : S_R^{\delta(p)}(H)(g)(x) > \alpha\right\}\right| \le \alpha^{-2}\left\|S_R^{\delta(p)}(H)(g)\right\|_2^2 \le \alpha^{-2}\|g\|_2^2 \le C\alpha^{-p}\|f\|_p^p, \tag{25}$$

which proves (23).

Next we prove (24) for $i=1$. By the decomposition (20),

$$\begin{aligned} \sum_{k\le 0}\sum_{j\in J_k} S_R^{\delta_q(p)}(H) b_j = \sum_{k\le 0} \eta_k\left(\sqrt{H}\right)&\left(\sum_{j\in J_k} n_k\left(\sqrt{H}\right) b_j\right) \\ &+ S_R^{\delta(p)}(H)\left(\sum_{k\le 0}\sum_{j\in J_k} n_k\left(\sqrt{H}\right) b_j\right). \end{aligned}$$

Applying the spectral theorem and Lemma 3 with $Q_k(\lambda) = \eta_k(\lambda)$ yields

$$\left\|\sum_{k\le 0}\sum_{j\in J_k} S_R^{\delta(p)}(H) b_j\right\|_2^2 \le C\sum_{k\le 0}\left\|\sum_{j\in J_k} n_k\left(\sqrt{H}\right) b_j\right\|_2^2 + C\left\|\sum_{k\le 0}\sum_{j\in J_k} n_k\left(\sqrt{H}\right) b_j\right\|_2^2. \tag{26}$$

Next, since $\operatorname{supp}\widehat{n_k} \subseteq [-2^k/R, 2^k/R]$, by Lemma 2, we have

$$\operatorname{supp} K_{n_k(\sqrt{H})} \subseteq \left\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : |x-y| \le 2^k/R\right\}.$$

Hence if $j\in J_k$, then $\operatorname{supp} n_k(\sqrt{H}) b_j \subseteq 4B_j$. Thus by (iii) there exists a constant $C>0$ such that

$$\sum_{k\le 0}\left\|\sum_{j\in J_k} n_k\left(\sqrt{H}\right) b_j\right\|_2^2 + \left\|\sum_{k\le 0}\sum_{j\in J_k} n_k\left(\sqrt{H}\right) b_j\right\|_2^2 \le C\sum_{k\le 0}\sum_{j\in J_k}\left\|n_k\left(\sqrt{H}\right) b_j\right\|_2^2. \tag{27}$$

Next, noting that $R>4$ and $k\le 0$, by Lemma 5 (c) and the restriction type estimate (7)

$$\begin{aligned} \left\|n_k\left(\sqrt{H}\right) b_j\right\|_2^2 = \left\langle n_k^2\left(\sqrt{H}\right) b_j, b_j\right\rangle &= \left\langle \sum_{\ell\in\mathbb{N}} n_k^2\left(\sqrt{2\ell+n}\right) P_\ell b_j, b_j\right\rangle \\ &\le C\sum_{\ell\in\mathbb{N}}\left(1+\frac{2^k\sqrt{2\ell+n}}{R}\right)^{-2N} \left\|P_\ell b_j\right\|_2^2 \\ &\le C\sum_{\ell\in\mathbb{N}}\left(1+\frac{2^k\sqrt{2\ell+n}}{R}\right)^{-2N} \ell^{\delta(p)-1/2}\left\|b_j\right\|_p^2. \end{aligned} \tag{28}$$

Since

$$\begin{aligned}\sum_{\ell\in\mathbb{N}}\left(1+\frac{2^k\sqrt{2\ell+n}}{R}\right)^{-2N}\ell^{\delta(p)-1/2} &\le \sum_{\ell\in\mathbb{N}}\left(1+\frac{2^k\sqrt{\ell}}{R}\right)^{-2N}\ell^{\delta(p)-1/2}\\ &\le \sum_{\ell\ge(R/2^k)^2}\left(\frac{2^k\sqrt{\ell}}{R}\right)^{-2N}\ell^{\delta(p)-1/2}+\sum_{0<\ell<(R/2^k)^2}\ell^{\delta(p)-1/2}\\ &\le C\left(\frac{2^k}{R}\right)^{2n(1/2-1/p)},\end{aligned}$$

we have

$$\left\|n_k\left(\sqrt{H}\right)b_j\right\|_2\le C\left(\frac{2^k}{R}\right)^{n(1/2-1/p)}\|b_j\|_p\le C\left|B_j\right|^{1/2-1/p}\|b_j\|_p\le C\alpha\left|B_j\right|^{1/2}.$$

Hence by (26), (27) and (iv),

$$\left|\left\{x:\left|S_R^{\delta(p)}(H)\left(\sum_{k\le0}\sum_{j\in J_k}b_j\right)\right|>\alpha\right\}\right|\le C\alpha^{-2}\left\|S_R^{\delta(p)}(H)\left(\sum_{k\le0}\sum_{j\in J_k}b_j\right)\right\|_2^2\le C\alpha^{-p}\|f\|_p^p,\tag{29}$$

which proves (24) for $i=1$.

Now, we prove (24) for $i=2$. Let $\Omega^*:=\bigcup_{j\in\mathbb{N}}4B_j$. From (iii) and (iv), it follows that

$$\left|\Omega^*\right|\le C\sum_j\left|B_j\right|\le C\alpha^{-p}\|f\|_p^p.$$

Hence it is enough to show that

$$\left|\left\{x\in\mathbb{R}^n\backslash\Omega^*:\left|S_R^{\delta(p)}(H)\left(\sum_{k>0}\sum_{j\in J_k}b_j\right)\right|>\alpha\right\}\right|\le C\alpha^{-p}\|f\|_p^p.\tag{30}$$

Using the decomposition from Lemma 6 we write

$$S_R^{\delta(p)}(H)\left(\sum_{k>0}\sum_{j\in J_k}b_j\right)=\sum_{k>0}\sum_{j\in J_k}m_k\left(\sqrt{H}\right)b_j+\sum_{k>0}\eta_k\left(\sqrt{H}\right)n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right).\tag{31}$$

Recall that $\widehat{m_k}$ is even and supported in $[-2^k/R,2^k/R]$. By Lemma 2,

$$\operatorname{supp}K_{m_k(\sqrt{H})}\subset\left\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:|x-y|\le2^k/R\right\}.$$

This implies that if $x\in\mathbb{R}^n\backslash\Omega^*$, then $m_k(\sqrt{H})b_j(x)=0$ for each $j\in J_k$ and $k>0$. So the first term on the right hand of equality (31) makes no contribution to estimate (30). So the proof of (30) reduces to showing that

$$\left|\left\{x:\left|\sum_{k>0,2^k\le R^2}\eta_k\left(\sqrt{H}\right)n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right)\right|>\alpha\right\}\right|\le C\alpha^{-p}\|f\|_p^p\tag{32}$$

and

$$\left|\left\{x:\left|\sum_{k>0,2^k>R^2}\eta_k\left(\sqrt{H}\right)n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right)\right|>\alpha\right\}\right|\le C\alpha^{-p}\|f\|_p^p.\tag{33}$$

We claim that

$$\left\|n_k\left(\sqrt{H}\right)b_j\right\|_2\le C\alpha\left|B_j\right|^{1/2}\max\left\{2^{k/2}R^{-1},1\right\},\quad\text{for } j\in J_k,k>0.\tag{34}$$

We note that for operators satisfying $(\mathrm{S_p})$ on compact manifolds, or for operators satisfying $(\mathrm{R_p})$, there is no need for the extra factor $\max\{2^{k/2}R^{-1},1\}$ in the above estimate. For the Hermite operator, this extra factor may be unavoidable.

Before we prove estimate (34), let us see how it implies (32) and (33). We handle (32) first.

3.1. *Estimate for* (32)

This estimate follows from a similar argument to that in [27] or that in [5]. By Lemma 3 and Lemma 6 (b),

$$\begin{aligned}\left|\left\{x:\left|\sum_{k>0,2^k\le R^2}\eta_k\left(\sqrt{H}\right)n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right)\right|>\alpha\right\}\right| \\ \le \alpha^{-2}\left\|\sum_{k>0,2^k\le R^2}\eta_k\left(\sqrt{H}\right)n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right)\right\|_2^2 \\ \le C\alpha^{-2}\sum_{k>0,2^k\le R^2}\left\|n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right)\right\|_2^2.\end{aligned}\tag{35}$$

Next, since $\widehat{n_k}$ is even and supported on $[-2^k/R,2^k/R]$, by Lemma 2

$$\operatorname{supp}K_{n_k(\sqrt{H})}\subseteq\left\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:|x-y|\le 2^k/R\right\}.$$

Hence $\operatorname{supp}n_k(\sqrt{H})b_j\subseteq 4B_j$ for each $j\in J_k$. By (iii) and (34) there exists a constant $C>0$ such that

$$\begin{aligned}\sum_{k>0,2^k\le R^2}\left\|n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right)\right\|_2^2 &\le C\sum_{k>0,2^k\le R^2}\sum_{j\in J_k}\left\|n_k\left(\sqrt{H}\right)b_j\right\|_2^2\\ &\le C\sum_{k>0,2^k\le R^2}\sum_{j\in J_k}\alpha^2\left|B_j\right|\max\left\{2^kR^{-2},1\right\}\\ &= C\sum_{k>0,2^k\le R^2}\sum_{j\in J_k}\alpha^2\left|B_j\right|\\ &\le C\alpha^2\sum_j\left|B_j\right|\le C\alpha^{2-p}\left\|f\right\|_p^p,\end{aligned}$$

where in the last inequality we have used (iv). This, in combination with (35), implies (32).

3.2. *Estimate for* (33)

As explained in the introduction, because of the extra factor $\max\{2^{k/2}R^{-1},1\}$ in estimate (34), the proof of this estimate relies on the a priori estimate (8). Recall that $\gamma=n(1/p-1/2)$ where $1\le p\le 2n/(n+2)$. We apply Lemma 4 (which states estimate (8)), (3) and inequality (22) to obtain

$$\begin{aligned}&\left|\left\{x:\left|\sum_{k>0,2^k>R^2}\eta_k\left(\sqrt{H}\right)n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right)\right|>\alpha\right\}\right|\\ &\le C\alpha^{-p}\left\|(1+H)^{-\gamma/2}\right\|_{L^2\to L^{p,\infty}}\left\|\sum_{k>0,2^k>R^2}\eta_k\left(\sqrt{H}\right)(1+H)^{\gamma/2}n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right)\right\|_2^p\\ &\le C\alpha^{-p}\left(\sum_{k>0,2^k>R^2}R^{2\gamma}\left\|n_k\left(\sqrt{H}\right)\left(\sum_{j\in J_k}b_j\right)\right\|_2^2\right)^{p/2}.\end{aligned}\tag{36}$$

As noted above, $\operatorname{supp}n_k(\sqrt{L})b_j\subseteq 4B_j$ for each $j\in J_k$. By (iii) of the Calderón–Zygmund decomposition of $|f|$ and (34) there exists a constant $C>0$ such that

$$\begin{aligned}\sum_{k>0,2^k>R^2}\left\|n_k(\sqrt{H})\left(\sum_{j\in J_k}b_j\right)\right\|_2^2 &\le C\sum_{k>0,2^k>R^2}\sum_{j\in J_k}\left\|n_k\left(\sqrt{H}\right)b_j\right\|_2^2\\ &\le C\sum_{k>0,2^k>R^2}\sum_{j\in J_k}\alpha^2\left|B_j\right|\max\left\{2^kR^{-2},1\right\}\end{aligned}$$

$$= C\sum_{k>0,2^k>R^2}\sum_{j\in J_k}\alpha^2\left|B_j\right|2^kR^{-2}.$$

This estimate, in combination with the fact that $p/2\le 1$, shows that

$$\begin{aligned}\text{RHS of (36)} &\le C\alpha^{-p}\left(\sum_{k>0,2^k>R^2}\sum_{j\in J_k}R^{2\gamma}\alpha^2\left|B_j\right|2^kR^{-2}\right)^{p/2}\\ &\le C\sum_{k>0,2^k>R^2}\sum_{j\in J_k}R^{\gamma p}\left(\frac{2^k}{R}\right)^{np/2}2^{kp/2}R^{-p}.\end{aligned}$$

On the other hand, since $2^k>R^2$, $R>4$, $\gamma=n(1/p-1/2)$ and $n(1/p-1/2)-1/2\ge 0$, we have

$$R^{\gamma p}\left(\frac{2^k}{R}\right)^{np/2}2^{kp/2}R^{-p}=R^{p\left(\gamma-\frac12\right)}\left(\frac{2^k}{R}\right)^{\left(\frac n2+\frac12\right)p}\le\left(\frac{2^k}{R}\right)^{p\left(n\left(\frac1p-\frac12\right)-\frac12\right)+\left(\frac n2+\frac12\right)p}\le\left(\frac{2^k}{R}\right)^{n}\le C\left|B_j\right|,$$

which implies

$$\text{LHS of (36)}\le C\sum_j\left|B_j\right|\le C\alpha^{-p}\|f\|_p^p$$

as desired in (33).

It remains only to prove (34). Note that in this case $k>0$ and $R>4$.

By the Hermite expansion and the functional calculus of H, we can rewrite

$$n_k\left(\sqrt{H}\right)b_j=\sum_{\ell\in\mathbb{N}}n_k\left(\sqrt{2\ell+n}\right)P_\ell b_j,$$

where P_ℓ are the projections defined by (6). Then by Lemma 6 (c) and the restriction type estimate (7), we see that

$$\begin{aligned}\left\|n_k\left(\sqrt{H}\right)b_j\right\|_2^2&=\left\langle\sum_{\ell\in\mathbb{N}}n_k^2\left(\sqrt{2\ell+n}\right)P_\ell b_j,b_j\right\rangle\\ &\le C\sum_{\ell\in\mathbb{N}}2^{-2\delta(p)k}\left(1+2^k\left|\frac{\sqrt{2\ell+n}}{R}-1\right|\right)^{-2N}\left\|P_\ell b_j\right\|_2^2\\ &\le C\sum_{\ell\in\mathbb{N}}2^{-2\delta(p)k}\left(1+2^k\left|\frac{\sqrt{2\ell+n}}{R}-1\right|\right)^{-2N}\ell^{\delta(p)-1/2}\left\|b_j\right\|_p^2\\ &\le C\sum_{\ell\in\mathbb{N}}2^{-2\delta(p)k}\left(1+2^k\left|\frac{\sqrt{\ell}}{R}-1\right|\right)^{-2N}\ell^{\delta(p)-1/2}\left\|b_j\right\|_p^2.\end{aligned}\tag{37}$$

We split this sum into three parts:

$$\begin{aligned}\sum_{\ell\in\mathbb{N}}&=\sum_{R^2(1-2^{-k})^2-1<\ell<R^2(1+2^{-k})^2+1}+\sum_{\ell\ge R^2(1+2^{-k})^2+1}+\sum_{0<\ell\le R^2(1-2^{-k})^2-1}\\ &=:\text{(I)}+\text{(II)}+\text{(III)}.\end{aligned}\tag{38}$$

For (I), we can control each term in the summation by the same bound, namely $C2^{-2\delta(p)k}R^{2\delta(p)-1}$, because for ℓ in this range, the expression with exponent $-2N$ is almost 1. So the key point is to count how many terms there are in the summation. If $2^k\le R^2$, then there are at most R^22^{-k} terms (up to multiplication by an absolute constant) in the summation, and if $2^k>R^2$, then there are at most six terms in the summation. Thus we see that

$$\text{(I)}\ \le\begin{cases}C2^{-2\delta(p)k}R^{2\delta(p)-1}, & \text{if } 2^k>R^2;\\ CR^22^{-k}2^{-2\delta(p)k}R^{2\delta(p)-1}, & \text{if } 2^k\le R^2;\end{cases}$$

$$= \begin{cases} C\left(\frac{2^k}{R}\right)^{2n(1/2-1/p)} \frac{2^k}{R^2}, & \text{if } 2^k > R^2; \\ C\left(\frac{2^k}{R}\right)^{2n(1/2-1/p)}, & \text{if } 2^k \le R^2; \end{cases}$$

$$= C\left(\frac{2^k}{R}\right)^{2n(1/2-1/p)} \max\left\{1, \frac{2^k}{R^2}\right\}.$$

We briefly highlight why the extra factor $\max\{1, 2^k/R^2\}$ is present here. When the manifold on which f is defined is compact, since $R > 4$, $2^k/R^2$ is less than $2^k/R$ which is the radius of the support of b_j and so $2^k/R^2$ is less than the diameter of the manifold. So in this situation the factor $\max\{1, 2^k/R^2\}$ is controlled by an absolute constant. When the operator L satisfies the restriction estimate ($\mathrm{R_p}$), we can use integration over the continuous spectrum, instead of summation over the eigenvalues, in the expression for $n_k(\sqrt{L})b_j$. Then when $2^k/R^2$ is large, the interval of integration is correspondingly small. So in this situation, the extra factor $\max\{1, 2^k/R^2\}$ is canceled out, by a factor involving the length of the interval of integration. However, for our Hermite operator H, no matter how small the interval of integration or summation, there still may be an eigenvalue in it, so we do have the extra factor $\max\{1, 2^k/R^2\}$ in our estimate.

To estimate the term (II), we note that the function

$$x^{\delta(p)-1/2}\left(2^k\left(\frac{\sqrt{x}}{R}-1\right)\right)^{-2N}$$

is decreasing for $x > R^2$ and N sufficiently large, and thus

$$\begin{aligned}
\text{(II)} &\le \int_{R^2(1+2^{-k})^2}^{\infty} 2^{-2\delta(p)k}\left(2^k\left(\frac{\sqrt{x}}{R}-1\right)\right)^{-2N} x^{\delta(p)-1/2}\,dx \\
&\le C2^{-2\delta(p)k}2^{-2Nk}R^{2\delta(p)+1}\int_{1+2^{-k}}^{\infty}(t-1)^{-2N}\,t^{2\delta(p)}\,dt \\
&\le C\left(\frac{2^k}{R}\right)^{2n(1/2-1/p)}.
\end{aligned}$$

By symmetry, a similar argument to that in (II) shows that (III) $\le C(2^k/R)^{2n(1/2-1/p)}$.

Collecting the estimates of the terms (I), (II) and (III), together with (37), (iv) of Calderón–Zygmund decomposition of function f and the fact $j \in J_k$, we arrive at the conclusion that

$$\left\|n_k\left(\sqrt{H}\right)b_j\right\|_2 \le C\left(\frac{2^k}{R}\right)^{n(1/2-1/p)} \max\left\{1, 2^{k/2}R^{-1}\right\}\|b_j\|_p \le C\alpha\left|B_j\right|^{1/2}\max\left\{1, 2^{k/2}R^{-1}\right\}.$$

This proves (34), and completes the proof of Theorem 1. □

4. Extensions

In the previous section, we proved Theorem 1, where the potential is $V = |x|^2$. However, the precise form of this potential does not play a fundamental role in the estimates. Here we consider instead the operators $H_V = -\Delta + V$ with a positive potential V which satisfies the following conditions:

$$V \sim |x|^2, \quad |\nabla V| \sim |x|, \quad \left|\partial_x^2 V\right| \le 1. \tag{39}$$

Under these assumptions the operator H_V is a nonnegative self-adjoint operator acting on the space L^2. Such an operator admits a spectral resolution

$$H_V = \int_0^{\infty} \lambda\, dE_{H_V}(\lambda).$$

Now, the Bochner–Riesz means of order $\delta \geq 0$ can be defined by

$$S_R^\delta(H_V) f := \int_0^{R^2} \left(1 - \frac{\lambda}{R^2}\right)^\delta dE_{H_V}(\lambda) f, \quad f \in L^2. \tag{40}$$

Then, just as for the Hermite operator H, the Bochner–Riesz means $S_R^\delta(H_V)$ are of weak-type (p,p) uniformly in $R > 0$, as we now show.

Theorem 7. *Suppose the potential V satisfies* (39)*. For $n \geq 2$ and $1 \leq p \leq 2n/(n+2)$, there is a constant C independent of R for which*

$$\left\| S_R^{\delta(p)}(H_V) f \right\|_{L^{p,\infty}} \leq C \left\| f \right\|_p.$$

Proof. It follows from [17, Theorem 4] that for all $\lambda \geq 0$ and all $1 \leq p \leq 2n/(n+2)$

$$\left\| E_{H_V}[\lambda^2, \lambda^2+1) \right\|_{p \to 2} \leq C(1+\lambda)^{n\left(\frac{1}{p} - \frac{1}{2}\right) - 1}. \tag{41}$$

With Lemma 4 and the spectral projection estimate (41), the argument in the proof of Theorem 1 also establishes Theorem 7. □

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