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Free boundary problems in the spirit of Sakai's theorem

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Abstract. A Schwarz function on an open domain Ω is a holomorphic function satisfying $S(\zeta) = \bar{\zeta}$ on Γ, which is part of the boundary of Ω. Sakai in 1991 gave a complete characterization of the boundary of a domain admitting a Schwarz function. In fact, if Ω is simply connected and $\Gamma = \partial\Omega \cap D(\zeta_0, r)$, then Γ has to be regular real analytic (with possible cusps). Sakai's result has natural applications to 1) quadrature domains, 2) free boundary problem for $\Delta u = 1$ equation. In our scenarios Γ can be, respectively, from real-analytic to just C^∞, regular except for a harmonic-measure-zero set, or regular except finitely many points.

Résumé. Dans le présent article, nous considérons la pléthore de résultats dans l'esprit de théorème de Sakai concernant les fonctions de Schwarz, c'est-à-dire les fonctions holomorphes dans un domaine ouvert Ω satisfaisant $S(\zeta) = \bar{\zeta}$ sur Γ, qui fait partie de la frontière de Ω. Sakai en 1991 a donné une caractérisation complète de la frontière d'un domaine admettant une fonction de Schwarz. Les résultats ci-dessous concernent trois scénarios de généralisation du résultat de Sakai, motivés plutôt par l'application au problème de dynamique complexe étudié dans [13]. À la fin de cette note, nous mentionnons quelques problèmes encore ouverts.

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Le résultat principal de cet article consiste en des généralisations du résultat de Sakai ayant été motivées par une application à la dynamique complexe. Ici, nous nous intéressons à trois scénarios différents sur un domaine simplement connexe Ω :

(1) lorsque $f_1(\zeta) = \bar{\zeta} f_2(\zeta)$ sur Γ, avec f_1, f_2 holomorphes et continues jusqu'au bord,
(2) lorsque $\mathscr{U}/\mathscr{V}$ est égal à certaine fonction analytique réelle sur Γ avec $\mathscr{U}, \mathscr{V}$ positives et harmoniques sur Ω et s'annulant sur Γ, et
(3) lorsque $S(\zeta) = \Phi(\zeta, \bar{\zeta})$ sur Γ, avec Φ une fonction holomorphe de deux variables.

Il s'avère que la partie Γ de la frontière peut être, respectivement et indifféremment, une partie analytique réelle ou seulement C^∞, régulière à l'exception d'un nombre fini de points, ou même régulière à l'exception d'un ensemble de mesure harmonique zéro.

Introduction

In 1991, Sakai proved a beautiful theorem [12, Theorem 5.2] which fully characterizes the boundaries of sets admitting a Schwarz function. Before we dwell on his result and its applications, we introduce some necessary notation: The disk of radius $r > 0$ and centre ζ_0 will be denoted by $D(\zeta_0, r)$, while the unit disk and the unit circle by $\mathbb{D}$ and $\mathbb{T}$, respectively. In general, D will be a (small) open domain. Ω will stand for a bounded open set on the complex plane and $\zeta_0 \in \partial\Omega$ for a non-isolated point of its boundary $\Gamma = \partial\Omega \cap D$ (for some D). For convenience, we will sometimes work with $\zeta_0 = 0$.

Now, suppose there is a function S holomorphic on Ω, continuous on $\Omega \cup \Gamma$, and which satisfies that $S(\zeta) = \bar{\zeta}$ for all $\zeta \in \Gamma$, where $\zeta_0 \in \Gamma = \partial\Omega \cap D(\zeta_0, r)$. The function S is called a *Schwarz function* on $\Omega \cup \Gamma$ at ζ_0. Sakai proved the following theorem:

Theorem 1. *Let Ω be an arbitrary bounded open set and S a Schwarz function on $\Omega \cup \Gamma$ at 0. Then, there exists a small neighbourhood of 0, D, so that exactly one of the following must be true:*

- (1) *$\Omega \cap D$ is simply connected and $\Gamma \cap D$ is a regular real analytic simple arc through 0.*
- (2a) *$\Gamma \cap D$ determines uniquely a regular real analytic arc through 0. The set $\Gamma \cap D$ is either an infinite proper subset of this arc with 0 as an accumulation point or equal to it. Also, $\Omega \cap D = D \setminus \Gamma$.*
- (2b) *$\Omega \cap D = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are simply connected and $\partial\Omega_1 \cap D$ and $\partial\Omega_2 \cap D$ are regular real analytic simple arcs through 0 and tangent at 0.*
- (2c) *$\Omega \cap D$ is simply connected and $\Gamma \cap D$ is a regular real analytic simple arc except for a cusp at 0. The cusp points into Ω.*

The converse is also true in the sense that if any of these four cases hold, then there exists a Schwarz function S on $\Omega \cup \Gamma$ at 0.

His proof is heavily dependent on the Phragmén–Lindelöf Principle and on certain auxiliary functions. In particular, depending on which one of $\Phi_1(z) = zS(z)$ and $\Phi_2(z) = \sqrt{zS(z)}$ is univalent around 0, we get the cases (1), or (2a), (2b) and (2c), respectively.

Driven by Sakai's results and by motivation from complex dynamics, we ask whether we can relax the condition $S(\zeta) = \bar{\zeta}$ and get the same regularity for the boundary Γ, and if not, how regular can Γ be. We are interested in three different scenarios, but we always restrict our attention to a simply connected domain Ω since otherwise too many complications arise.

Nevanlinna Domains and Model Spaces

Initially, we investigate a scenario where we have two holomorphic functions f_1 and f_2 (instead of only one) on the simply connected Ω continuous on the boundary $\Gamma = \partial\Omega \cap D(\zeta_0, r)$, where $\zeta_0 \in \partial\Omega$, which satisfy

$$f_1(\zeta) = \bar{\zeta} f_2(\zeta) \qquad \text{whenever } \zeta \in \Gamma. \tag{1}$$

We may assume this happens around any point ζ_0 simply by substituting

$$f_1^T(z) = f_1(z+\zeta_0) - \overline{\zeta_0} f_2(z+\zeta_0) \quad \text{and} \quad f_2^T(z) = f_2(z+\zeta_0),$$

which translates our problem at 0.

If it so happens that f_2 is a polynomial or it somehow extends analytically over the boundary Γ, then the ratio $f = f_1/f_2$ turns out to be a Schwarz function on Ω thanks to the Phragmén–Lindelöf principle. However, if this is not the case, we need stronger tools in order to proceed.

Suppose Ω is a simply connected open domain and that there exist bounded holomorphic functions f and g on Ω satisfying that $\bar{\xi} = f(\xi)/g(\xi)$ for $\xi \in \partial\Omega$ in the sense of conformal mappings, i.e. $\overline{\varphi(\zeta)} = f(\varphi(\zeta))/g(\varphi(\zeta))$ for almost every $\zeta \in \mathbb{T}$ where φ is a conformal mapping from

$\mathbb{D}$ onto Ω. In this case, Ω is called a *Nevanlinna domain* (cf. [5, Definition 2.1]). This definition is independent of the choice of φ and does not guarantee additional regularity for f or g; not even continuity [5, Example 5.8], which is in our interest in this note.

Now, consider such a conformal $\varphi : \mathbb{D} \to \Omega$ onto Ω and construct the functions $F_1 = f_1 \circ \varphi$ and $F_2 = f_2 \circ \varphi$ which, because of (1), satisfy

$$\frac{F_1(\zeta)}{F_2(\zeta)} = \overline{\varphi(\zeta)} \qquad \text{for almost every } \zeta \in \mathbb{T}.$$

This turns Ω into a Nevanlinna domain.

The question arises as to whether such functions f_1, f_2 and φ actually exist and if they do, how "good" or "bad" can Γ be? It turns out Sakai's result does not in general hold in this setup as (we will see below) we can find functions f_1, f_2 holomorphic on Ω and continuous on $\Omega \cup \Gamma$ satisfying (1), but such that Γ is only C^∞-smooth and *not* real analytic.

Decomposing F_1 and F_2 into inner and outer parts, we see that φ satisfies

$$\varphi = \frac{\overline{F}}{\overline{\theta}} = \frac{\widetilde{F}}{\widetilde{\theta}} \qquad \text{almost everywhere on } \mathbb{T},$$

where θ is an inner function and $F \in H^\infty$, and where $\widetilde{h}(z) = \overline{h(\overline{z})}$. This means that φ admits a (Nevanlinna-type) pseudo-continuation and as such it belongs to some model space K_θ. See [8] for the proof of this and [6] for more details about pseudo-continuations.

A lot of results are already known for conformal maps in model spaces when the inner function θ has a Blaschke part. In fact, the boundaries of such domains may be C^1 but not $C^{1,\alpha}$ for any $\alpha \in (0,1)$ [2, Theorem 2], may be unrectifiable [11, Example 1] or nowhere analytic [10], or may be rectifiable but admitting no such functions continuable up to the boundary [5, Example 5.8]. Furthermore, the Hausdorff dimension of Nevanlinna domains' boundaries can be any number from 1 up to 2 [3, Theorem 1]. Additionally, it is always possible to find conformal maps smooth up to $\mathbb{T}$, for example $\varphi(z) = 1/(1 - \overline{z_0} z) \in K_\theta \cap C^\infty(\mathbb{T})$ where $\theta(z_0) = 0$.

In the case of a purely singular inner function θ, it was recently found through a series of papers that K_θ contains bounded univalent functions if, and only if, the singular measure, μ_θ, associated with θ is positive on some Carleson set $E \subset \mathbb{T}$. When the latter happens we say θ has the *Carleson property*. (See [1] or [4] for a nice exposition on this.)

More explicitly, take singular θ with the Carleson property and a non-trivial $g \in K_\theta \cap C^\infty(\mathbb{T})$ (such functions always exist thanks to [7, Theorem 2.1]) and let $\mathscr{G}$ be its analytic extension on $\widehat{\mathbb{C}} \setminus \operatorname{supp}(\mu_\theta)$. Then, as shown in [4], for appropriate $|\alpha| > 1$ the function

$$\varphi(z) = \frac{\mathscr{G}(z) - \mathscr{G}(\alpha)}{z - \alpha}$$

lives, in fact, inside $K_\theta \cap C^\infty(\mathbb{T})$ and it is additionally univalent there yet *not* real-analytic on $\mathbb{T}$.

This $\varphi \in K_\theta$ sends $\mathbb{D}$ onto some simply connected domain Ω and we will see that Sakai's type real analyticity result fails for this Ω. Indeed, it is true that $F = \overline{\varphi}\theta$ for some $F \in H^\infty$. Multiplying this equality with a suitable outer function $\mathscr{H}$, we find that the functions $f_1 = (\mathscr{H} F) \circ \varphi^{-1}$ and $f_2 = (\mathscr{H}\theta) \circ \varphi^{-1}$ are holomorphic on Ω, continuous on $\Omega \cup \Gamma$ and satisfy (1). However, by construction Γ cannot be real analytic.

Holomorphic function on the complex 2-plane

Next, we want to describe the boundary $\Gamma = \partial\Omega \cap D(\zeta_0, r)$ of Ω under a different setup; we replace the $\overline{\zeta} f_2(\zeta)$ from the equality $f_1(\zeta) = \overline{\zeta} f_2(\zeta)$ with $\Phi(\zeta, \overline{\zeta})$ where Φ a function of two complex variables this time holomorphic *in a neighbourhood of* $(\zeta_0, \overline{\zeta_0})$. In particular, we prove the following:

Theorem 2. *Let $\Omega \subset D(\zeta_0, r)$ be a bounded simply connected domain such that the set $\Gamma = \partial\Omega \cup D(\zeta_0, r)$ is a (union of) Jordan arc(s) for some (small) $r > 0$. Also, let Φ be a (non-trivial) function of two variables holomorphic on a $\mathbb{C}^2$ neighbourhood of $(\zeta_0, \overline{\zeta_0})$, and suppose there exists a function R*

(i) *holomorphic on Ω,*
(ii) *continuous on $\overline{\Omega}$, and such that*
(iii) *$R(\zeta) = \Phi(\zeta, \bar{\zeta})$ for all $\zeta \in \Gamma$.*

Then, there exists a closed set, $E \subset \Gamma$, of zero harmonic measure so that $\Gamma \setminus E$ is a countable union of regular real-analytic simple arcs except possibly for some cusps. The cusps (if they exist) point into Ω and may only accumulate on E.

The first thing to note is that the problem can be "translated" from any ζ_0 to 0. Also, $\Phi(z, w)$ must have a non-zero partial derivative of some order in terms of w.

The proof is essentially a "chase" of the roots of the function $\Phi(z, \bar{z}) - R(z)$ and particularly those lying on the boundary Γ. To begin with, we would want to "isolate" the zeroes of the function $\Phi(z, w) - R(z)$ using the Weierstraß Approximation Theorem around 0, but R is not holomorphic on Γ. So, instead we first deploy Mergelyan's Theorem to find a polynomial sequence, p_n, approaching R and then use Weierstraß' theorem on $\Phi(z, w) - p_n(z)$.

Taking limits, this leads to a polynomial of two variables of the form

$$P(z, \bar{z}) = \bar{z}^k + a_{k-1}(z)\bar{z}^{k-1} + \cdots + a_0(z),$$

where a_n are holomorphic functions on Ω and continuous on $\Omega \cup \Gamma$. Additionally, $P(\zeta, \bar{\zeta}) = 0$ for $\zeta \in \Gamma$.

Eventually, the roots of $P(z, \cdot)$ for any fixed z give us a multivalued holomorphic function. Therefore, we need to consider several simply connected domains around Γ in order to split the different branches. Each such domain and branch gives us a Schwarz function which, using Sakai's theorem, gives us the desired regularity for Γ. The set E which was excluded is simply the the zeroes of the discriminant, $\mathscr{D}(z)$ of $P(z, \cdot)$ that fall on Γ.

Harmonic functions and the one-phase problem

The final part of this note deals with a one-phase problem of harmonic functions that appears naturally in a problem of dimension of harmonic measure on limits sets of a wide class of holomorphic dynamical systems; see [13]. The theorem we show is the following.

Theorem 3. *Let $\Omega \subset D(\zeta_0, r)$ be a simply-connected domain of $\mathbb{C}$ and let Γ be an open Jordan arc of its boundary with $\zeta_0 \in \Gamma$. Suppose there are two positive non-proportional harmonic functions $\mathscr{U}$ and $\mathscr{V}$ on Ω, continuous on $\Omega \cup \Gamma$ which satisfy*

$$\mathscr{U}(\zeta) = \mathscr{V}(\zeta) = 0 \quad \textit{and} \quad \frac{\mathscr{U}(\zeta)}{\mathscr{V}(\zeta)} = |A(\zeta)|^2$$

for all $\zeta \in \Gamma$, where A is a non-trivial analytic function on a neighborhood of the boundary piece Γ.

Then, for all but possibly finitely many points $\zeta_0 \in \Gamma$ there exist some small neighborhood D of ζ_0 such that the following holds:

$$\Gamma \cap D \textit{ is a regular real-analytic simple arc through } \zeta_0 \textit{ except possibly a cusp at } \zeta_0. \tag{2}$$

The finitely many points around which (2) might fail are the points $\zeta \in \Gamma$ where $A'(\zeta) = 0$, i.e. where A might not to be invertible.

At first, using Harnack's principle we show that the ratio $\mathscr{U}/\mathscr{V}$ is indeed well defined and (positive) real-analytic on Γ. What is more, this real-analytic function gives rise to a holomorphic function, R, on Ω which is continuous on $\Omega \cup \Gamma$ and satisfies $R = |A|^2$ on Γ.

We immediately conclude that Γ is a union of regular real-analytic simple arcs except possibly a closed set $E \subset \Gamma$ of zero harmonic measure. This comes directly from Theorem 2 for the function $\Phi(z,w) = A(z)\overline{A(\overline{w})}$.

However, it is possible to say even more. Indeed, around any point where A is invertible the function

$$S(z) = \frac{1}{z} R \circ A^{-1}(z)$$

turns out to be a Schwarz function on $\Omega \cup \Gamma$ at ζ_0. Again by Sakai, we conclude that Γ is either a regular real-analytic simple arc through ζ_0 itself or two regular real-analytic simple arcs forming a cusp at ζ_0.

Moreover, it possible to differentiate between the two cases. Following through Sakai's work we find that the following is true:

Proposition 4. *With assumptions as above, around any $\zeta_0 \in \Gamma$ with $A'(\zeta_0) \neq 0$ either*

(1) *there exist a function Ψ_1 holomorphic and univalent on $\Omega \cap D$ such that Ψ_1 is continuous on $(\Omega \cup \Gamma) \cap D$, and $\Psi_1(\zeta) = |A(\zeta) - A(\zeta_0)|^2$ for $\zeta \in \Gamma \cap D$, or*

(2) *there exist a function Ψ_2 holomorphic and univalent on $\Omega \cap D$ such that Ψ_2^2 is continuous on $(\Omega \cup \Gamma) \cap D$, and $\Psi_2^2(\zeta) = |A(\zeta) - A(\zeta_0)|^2$ for $\zeta \in \Gamma \cap D$.*

The functions Ψ_1 and Ψ_2 are given by

$$\Psi_1(z) = (A(z) - A(\zeta_0)) \left(\frac{R(z)}{A(z)} - \overline{A(\zeta_0)} \right) \text{ and}$$
$$\Psi_2(z) = \sqrt{\Psi_1(z)}.$$

It turns out Γ has a cusp at ζ_0 if, and only if, (2) of Proposition 4 holds.

Open "free boundary" problems in the spirit of Sakai

All problems treated above are examples of the so-called (non-variational) free boundary problems.

We would like to call the attention of the reader to the open question: what if the domain Ω for positive harmonic functions is not simply connected? Finitely connected situation presents no difficulties, but what if, for example, Γ is a Cantor set and $\Omega = \mathbb{D} \setminus \Gamma$? Suppose we know that the ratio of two positive harmonic (non-proportional) functions $\mathscr{U}, \mathscr{V}$ in Ω vanishing on the Cantor set Γ has a well defined ratio on Γ (this happens for a wide class of Cantor sets, for instance for all regular Cantor sets of positive Hausdorff dimension).

Suppose this ratio is non-constant and equal to $|A(\zeta)|^2$ for $\zeta \in \Gamma$, where A is a holomorphic function on $\mathbb{D}$. What we can say about the Cantor set Γ? The "desired" answer is that this is impossible to happen on any Cantor set.

This type of problems (we may call them *one-phase free boundary problems*) appear naturally in certain problems of complex dynamics, see e.g. [13]. If we would know the aforementioned answer (we conjecture that no Cantor set would allow such a triple $(\mathscr{U}, \mathscr{V}, A)$), then a long standing problem of dimension of harmonic measure on Cantor repellers would be solved.

Another similar one-phase boundary problem concerns functions in $\mathbb{R}^n$ for $n > 2$. Let Ω be a connected bounded domain in $\mathbb{R}^n$, $n > 2$, and $\Gamma = \partial\Omega \cap D(x,r)$, where $x \in \partial\Omega$. Again, let $\mathscr{U}, \mathscr{V}$ be two positive (non-proportional) harmonic functions in Ω vanishing continuously on Γ. If Ω is assumed to be a Lipschitz domain, then [9] claims that $\mathscr{U}/\mathscr{V}$ makes sense on Γ and is additionally a Hölder function on Γ (boundary Harnack principle).

Here is a question: Let R be a real analytic function on $D(x,r)$, $x \in \Gamma$, and let $\mathscr{U}/\mathscr{V} = R$ on $\Gamma \cap D(x,r)$. Is it true that $\Gamma \cap D(x,r)$ is real analytic, maybe with the exception of some lower dimensional singular set?

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