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On the maximum value of a confluent hypergeometric function

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Abstract. We study the maximum value of the confluent hypergeometric function with oscillatory conditions of parameters. As a consequence, we obtain new inequalities for the Gauss hypergeometric function.

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1. Introduction and main results

The confluent hypergeometric function ${}_1F_1(a;b;x)$, which is defined as

$$ {}_1F_1(a;b;x)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{x^n}{n!} $$

for $b\neq 0,-1,\ldots$, is a particular solution of the linear differential equation

$$ xy''+(b-x)y'-ay=0. \tag{1} $$

When a is a non-positive integer, the function ${}_1F_1(a;b;x)$ reduces to Laguerre polynomials, i.e.

$$ L_n^{(b-1)}(x)=\frac{(b)_n}{n!}\,{}_1F_1(-n;b;x),\qquad n=0,\,1,\,2,\,\ldots\,. $$

The function ${}_1F_1$ is a special case of the generalized hypergeometric function

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{x^n}{n!}, \tag{2} $$

p and q are non-negative integers, none of the numbers b_j $(j=1,\ldots,q)$ is equal to zero or to a negative integer. It is well known that the series ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x)$ converges absolutely for all x if $p\le q$ and for $|x|<1$ if $p=q+1$, and it diverges for all $x\neq 0$ if $p>q+1$. If one of the parameters a_i equals zero or a negative integer, then the series (2) reduces to a polynomial.

The confluent hypergeometric function has been studied in great detail from its mathematical point of view (see, for instance, [12, 14, 18]). In particular, the estimate of the confluent hypergeometric function ${}_1F_1(a;b;x)$ has been widely and deeply studied when $x>0$ and $b>a>0$

(see [3, 13], and references therein). For instance, Luke [13], among others, proved the following inequalities

$$e^{\frac{a}{b}x} < {}_1F_1(a;b;x) < 1 - \frac{a}{b} + \frac{a}{b}e^x, \qquad x > 0, b > a > 0,$$
$${}_1F_1(a;b;x) < \frac{(b-1)e^x}{(b-a-1)(1+x)}, \qquad x > 0, b-1 > a > 0.$$

We remark that when $x > 0$, $b > a$ and $b > \frac{1}{2}$, Love [11, Corollary 2] showed that

$$|{}_1F_1(a;b;x)| \le \frac{\Gamma(b-a)e^x}{|\Gamma(1-a)|\Gamma(b)}. \tag{3}$$

Unfortunately, the conditions given for (3) do not seem to be correct. For instance, for $a \in \mathbb{N}$ and $b > a$, (3) gives ${}_1F_1(a;b;x) = 0$ for any $x > 0$.

When the parameters verify the so called oscillatory conditions $a < 0$ and $b-a > 1$, the estimate of ${}_1F_1(a;b;x)$ is much more complicated and, to the author's knowledge, has not been studied in the literature, except for the case when a is a negative integer and $b > 0$ (see, for instance, [9, 10, 17, 21, 23], and references therein).

It is well known that, for $a < 0$ and $b > 0$, ${}_1F_1(a;b;x)$ has a finite number of real zeros (see [18, Section 13.9])

$$x^1_{a,b} < x^2_{a,b} < \cdots.$$

Based on the Kummer transformation

$${}_1F_1(a;b;x) = e^x {}_1F_1(b-a;b;-x),$$

it follows that the real zeros of ${}_1F_1(a;b;x)$ are positive when $a < 0$ and $b > 0$.

In this paper, we use the Sonin–Pólya theorem as well as the Watson–Glaeske product formula for confluent hypergeometric functions to study the maximum value of ${}_1F_1(a;b;x)$ with oscillatory conditions of parameters.

Here is our main results.

Theorem 1. *For $a < 0$ and $b > 1$*

$$\max_{x\ge 0} e^{-x}|{}_1F_1(a;b;x)| = 1. \tag{4}$$

Moreover, this maximum value is attained only when $x = 0$.

Corollary 2. *When $a < 0$ and $b > 1$, let ξ_k, $k = 1,\dots,$ be the successive maxima of $y(x) = e^{-x}{}_1F_1(a;b;x)$ arranged in increasing order, and let $j_{b,k}$ be the k-th positive zero of the Bessel function $J_b(x)$. Then,*

$$y^2(\xi_i) - y^2(\xi_j) < \frac{b-a}{b^2}\left(\frac{2b-1}{2}\Delta\xi^2_{ij} + \frac{2}{3}\Delta\xi^3_{ij}\right), \qquad i < j,$$

where $\Delta\xi^k_{ij} = \xi^k_j - \xi^k_i$ and

$$\xi_k = \frac{j^2_{b,k}}{2b-4a+2}\left(1 + \frac{2(b^2-1)+j^2_{b,k}}{3(2b-4a+2)^2}\right) + O\left(\frac{1}{a^5}\right), \qquad as \quad a \to -\infty.$$

We remark that from the asymptotics (see [14, Section 6.8.2])

$$e^{-x}{}_1F_1(a;b;x) = \frac{\Gamma(b)}{\Gamma(a)}x^{a-b}[1 + O(x^{-1})], \qquad x \to +\infty,$$

where a is not a negative integer or zero, it follows that e^{-x} in Theorem 1 cannot be replaced by any e^{-cx}, $0 \le c < 1$. In the case when a is a negative integer or zero, we have (see [23])

$$\max_{x\ge 0} e^{-\frac{x}{2}}|{}_1F_1(a;b;x)| = 1, \tag{5}$$

where $b > 1$.

On the other hand, in the oscillation region of ${}_1F_1(a;b;x)$, we have the following result analogous to (5).

Theorem 3. *Let $a<0$ and $b>1$. For $0\le x<\frac{(2b-1)(b-2a)}{b}$*

$$e^{-\frac{x}{2}}|{}_1F_1(a;b;x)|\le 1. \tag{6}$$

Furthermore, for $0\le x<\frac{(2b-1)(b-2a)}{2b}$

$$e^{-\frac{x}{2}}|{}_1F_1(a;b;x)|\le \sqrt{\mathcal{M}(x,b)}, \tag{7}$$

where

$$\mathcal{M}(x,b)=\frac{b-1}{\pi}\int_0^\pi\int_0^\pi |\cos(x\sin\theta\cos\psi)|(\sin\psi)^{2b-3}(\sin\theta)^{2b-2}\,\mathrm{d}\psi\,\mathrm{d}\theta$$

and it has the property

$$0<\mathcal{M}(x,b)<\mathcal{M}(0,x)=1,\quad x>0.$$

Consequently, by applying (4) and (5), we obtain the following inequalities for the Gauss hypergeometric function ${}_2F_1$.

Corollary 4. *Let $a<0$, $b>1$ and $\Re e(\sigma)>0$.*

(i) *If a is not a negative integer and $0\le x<\Re e(z)$*

$$\left|{}_2F_1\left(\sigma,a;b;\frac{x}{z}\right)\right|\le\sqrt{\cosh(\pi\Im m(\sigma))}\,|z^\sigma|\,[\Re e(z)-x]^{-\Re e(\sigma)}. \tag{8}$$

In particular, for $\sigma>0$ and $0\le x<1$

$$|{}_2F_1(\sigma,a;b;x)|\le(1-x)^{-\sigma}.$$

(ii) *If a is a negative integer and $0\le x<2\Re e(z)$*

$$\left|{}_2F_1\left(\sigma,a;b;\frac{x}{z}\right)\right|\le\sqrt{\cosh(\pi\Im m(\sigma))}\,|z^\sigma|\left[\Re e(z)-\frac{x}{2}\right]^{-\Re e(\sigma)}.$$

In particular, for $\sigma>0$ and $0\le x<2$

$$|{}_2F_1(-n,\sigma;b;x)|\le\left(1-\frac{x}{2}\right)^{-\sigma},\qquad n\in\mathbb{N}\cup\{0\}.$$

Under condition $a>0$, several lower and upper bound inequalities for ${}_2F_1(\sigma,a;b;x)$ have been derived in the literature using different approaches (e.g. [2–4,6,13,22] and references therein). For instance, in [13, Theorem 13], Luke gave the following two-sided bounds

$$\left(1-\frac{a}{b}x\right)^{-\sigma}<{}_2F_1(\sigma,a;b;x)<1-\frac{a}{b}+\frac{a}{b}(1-x)^{-\sigma},\quad 0<x<1, 0<\sigma, 0<a<b,$$

whereas Karp and Sitnik [6, Theorem 5] showed that

$${}_2F_1(\sigma,a;b;x)<\left(1-\frac{a}{b-1}x\right)^{-\sigma},\quad 0<x<1, 0<\sigma\le 1, 1<a+1<b.$$

On the other hand, in [22] the authors derived some inequalities for the Gauss hypergeometric function ${}_2F_1(\sigma,a;b,x)$ when $-1<a<0$, $1<b<2$, $0<\sigma<1$, and $x\in(0,1)$. We remark that when a is a negative integer or zero, the estimate of the polynomial ${}_2F_1(a,\sigma;b;x)$ has been considered in several papers from different point of views (see for instance [7,8] and references therein)

2. Proof of the main results

One of the main tool that we need for our purpose is the well-known Sonin–Pólya theorem (see [21, footnote to Theorem 7.31.1]) in the following form given by Szegö. Notice that this theorem was used by Szegö [21] in a similar context to study the successive relative maxima of classical orthogonal polynomials.

The Sonin–Pólya theorem

Suppose that a function $y = y(x)$ satisfies on an interval $I \subset \mathbb{R}$ the differential equation

$$(py')' + qy = 0, \tag{9}$$

where $p = p(x) > 0$, $q = q(x) > 0$ and both p' and q' are continuous on that interval. Define Sonin's function by

$$S(x) := y^2(x) + \frac{p(x)}{q(x)} y'^2(x), \tag{10}$$

then we observe that

$$S'(x) = -[p\,(x)q(x)]' \left[\frac{y'(x)}{q(x)}\right]^2, \tag{11}$$

by which successive relative maxima of y^2 form an increasing or decreasing sequence according as pq decreases or increases on the corresponding interval.

Now, we can prove our main results.

Proof of Theorem 1. From (1), the corresponding differential equation for $y(x) = e^{-x}{}_1F_1(a;b;x)$ is

$$xy'' + (b+x)y' + (b-a)y = 0.$$

By writing it in the self-adjoint form

$$(x^b e^x y')' + (b-a)x^{b-1}e^x y = 0,$$

we see that

$$p(x) = x^b e^x, \qquad q(x) = (b-a)x^{b-1}e^x$$

and

$$[p\,(x)q(x)]' = (b-a)(2b-1+2x)x^{2b-1}e^{2x}.$$

Thus, if $a < 0$ and $b > 1$, the successive relative maxima of $|e^{-x}{}_1F_1(a;b;x)|$ are decreasing on $[0,\infty)$ and

$$[e^{-x}{}_1F_1(a;b;x)]^2 \le S(x) \le S(0) = y^2(0) = 1, \qquad x \ge 0.$$

This proves (4). □

Proof of Corollary 2. We observe that, using the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^{-x}{}_1F_1(a;b;x)] = -\frac{b-a}{b} e^{-x}{}_1F_1(a;b+1;x), \tag{12}$$

$\xi_k = x^k_{a,b+1}$, for all $k = 1,\ldots$.

Thus, from (10) and (11) one has

$$y^2(\xi_j) - y^2(\xi_i) = -\frac{1}{b-a}\int_{\xi_i}^{\xi_j} x(2b-1+2x)[y'(x)]^2\,\mathrm{d}x, \qquad i < j.$$

Now we can apply (4) and (12) to yield

$$\begin{aligned} y^2(\xi_i) - y^2(\xi_j) &< \frac{b-a}{b^2}\int_{\xi_i}^{\xi_j} x(2b-1+2x)\,\mathrm{d}x \\ &= \frac{b-a}{b^2}\left(\frac{2b-1}{2}\Delta\xi^2_{ij} + \frac{2}{3}\Delta\xi^3_{ij}\right), \end{aligned}$$

where $\Delta\xi^k_{ij} = \xi^k_j - \xi^k_i$.

Finally, taking into account that the k-th positive zero $x^k_{a,b}$ can be approximated by (see [18, Section 13.9])

$$\frac{j^2_{b-1,k}}{2b-4a}\left(1 + \frac{2b(b-2) + j^2_{b-1,k}}{3(2b-4a)^2}\right) + O\left(\frac{1}{a^5}\right), \qquad as \quad a \to -\infty,$$

we can achieve the proof of the corollary. □

Proof of Theorem 3. For the proof of (6), we proceed as in the proof of (4). According to (1), it is straightforward to check that the function $y(x) = e^{-\frac{x}{2}}{}_1F_1(a;b;x)$ satisfies

$$xy''(x) + by'(x) + \frac{2b-4a-x}{4}y(x) = 0. \tag{13}$$

In its self-adjoint form equation (13) becomes

$$(x^b y'(x))' + \frac{2b-4a-x}{4}x^{b-1}y(x) = 0,$$

which corresponds to equation (9) with

$$p(x) = x^b, \qquad q(x) = \frac{2b-4a-x}{4}x^{b-1},$$

and

$$[p\,(x)q(x)]' = \frac{x^{2b-2}}{2}\left[(2b-1)(b-2a) - bx\right].$$

Thus, for $a < 0$ and $b > 1$, the successive relative maxima of $|e^{-\frac{x}{2}}{}_1F_1(a;b;x)|$ are decreasing if $0 < x < \frac{(2b-1)(b-2a)}{b}$ and increasing if $\frac{(2b-1)(b-2a)}{b} < x < 2b-4a$. This completes the proof of (6).

We now continue with the proof of (7). Our starting point in the proof is the Glaeske [5] product formula for Laguerre functions, which in terms of the confluent hypergeometric functions can be written as

$$\begin{aligned}{}_1F_1(a;b;x)\cdot{}_1F_1(a;b;y) = \frac{\Gamma(b)}{\sqrt{\pi}}\int_0^\pi e^{-\sqrt{xy}\cos\theta}(\sin\theta)^{2b-2}\mathscr{J}_{b-\frac{3}{2}}(\sqrt{xy}\sin\theta)\\ \times{}_1F_1(a;b;x+y+2\sqrt{xy}\cos\theta)\,\mathrm{d}\theta,\end{aligned} \tag{14}$$

where $x, y \geq 0$, $\Re e(b) > \frac{1}{2}$, and $\mathscr{J}_\nu(z) := \left(\frac{z}{2}\right)^{-\nu} J_\nu(z)$. For $a = -n$, $n \in \mathbb{N}$, equation (14) was first obtained by Watson [23] and later on by several authors using quite different methods (see [5, 7, 16, 19, 20]), whereas in [15] Markett gave another, analytic proof of Glaeske's result. In Appendix A, we give another simple proof of (14).

Using Poisson's integral (see [23, 3.3])

$$\mathscr{J}_\nu(z) = \frac{1}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\pi e^{iz\cos\psi}(\sin\psi)^{2\nu}\,\mathrm{d}\psi, \qquad \Re e(\nu) > -\frac{1}{2},$$

the product formula (14) becomes

$$\begin{aligned}{}_1F_1(a;b;x)\cdot{}_1F_1(a;b;y) &= \frac{b-1}{\pi}\int_0^\pi\int_0^\pi e^{-\sqrt{xy}\cos\theta+i\sqrt{xy}\sin\theta\cos\psi}(\sin\psi)^{2b-3}(\sin\theta)^{2b-2}\\ &\quad\times{}_1F_1(a,b,x+y+2\sqrt{xy}\cos\theta)\,\mathrm{d}\psi\,\mathrm{d}\theta\\ &= \frac{b-1}{\pi}\int_0^\pi\int_0^\pi e^{-\sqrt{xy}\cos\theta}\cos(\sqrt{xy}\sin\theta\cos\psi)(\sin\psi)^{2b-3}\\ &\quad\times(\sin\theta)^{2b-2}{}_1F_1(a,b,x+y+2\sqrt{xy}\cos\theta)\,\mathrm{d}\psi\,\mathrm{d}\theta.\end{aligned} \tag{15}$$

We put $x = y$ in (15) and multiply the obtained relation by e^{-x}. As a result, we obtain

$$\begin{aligned}\left[e^{-\frac{x}{2}}{}_1F_1(a;b;x)\right]^2 = \frac{b-1}{\pi}\int_0^\pi\int_0^\pi \cos(x\sin\theta\cos\psi)(\sin\psi)^{2b-3}(\sin\theta)^{2b-2}\\ \times e^{-x(1+\cos\theta)}{}_1F_1(a,b,2x(1+\cos\theta))\,\mathrm{d}\psi\,\mathrm{d}\theta.\end{aligned}$$

Then, taking into account (6), for $0 \leq x < \frac{(2b-1)(b-2a)}{2b}$

$$\left[e^{-\frac{x}{2}}{}_1F_1(a;b;x)\right]^2 \leq \frac{b-1}{\pi}\int_0^\pi\int_0^\pi |\cos(x\sin\theta\cos\psi)|(\sin\psi)^{2b-3}(\sin\theta)^{2b-2}\,\mathrm{d}\psi\,\mathrm{d}\theta = \mathscr{M}(x,b).$$

Finally, based on equation (20), we have

$$0 < \mathscr{M}(x,b) < \mathscr{M}(0,b), \qquad x > 0.$$

The proof of Theorem 3 is completed. □

Proof of Corollary 4. **(i).** Applying inequality (4) to the confluent hypergeometric function appearing in the Laplace transform of the Gauss hypergeometric function (see [12, p. 59])

$$ {}_2F_1\left(\sigma, a; b; \frac{x}{z}\right) = \frac{z^\sigma}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1} {}_1F_1(a;b;xt)\,\mathrm{d}t, $$

where $0 \le x < \Re e(z)$ and $\Re e(\sigma) > 0$, we have

$$ \left|{}_2F_1\left(\sigma, a; b; \frac{x}{z}\right)\right| \le \frac{|z^\sigma|}{|\Gamma(\sigma)|} \int_0^\infty e^{-t[\Re e(z)-x]}\, t^{\Re e(\sigma)-1}\,\mathrm{d}t = \frac{\Gamma(\Re e(\sigma))}{|\Gamma(\sigma)|} |z^\sigma|\, [\Re e(z)-x]^{-\Re e(\sigma)}. $$

Finally, using inequality (see [18, Section 5.6])

$$ |\Gamma(p+iq)| \ge \frac{|\Gamma(p)|}{\sqrt{\cosh(q\pi)}} $$

we get (8).

(ii). By making use of (5), the proof of case (ii) can be completed by following the proof of case (i). ☐

Appendix A. Proof of the Watson–Glaeske formula

Substituting the integral representation for ${}_1F_1$ (see [14, Section 6.5])

$$ {}_1F_1(a;b;z) = \frac{2^{1-b}\Gamma(b) e^{\frac{z}{2}}}{\Gamma(a)\Gamma(b-a)} \int_{-1}^1 e^{\frac{zt}{2}} (1-t)^{b-2} \left(\frac{1+t}{1-t}\right)^{a-1} \mathrm{d}t, \tag{16} $$

where $\Re e(b) > \Re e(a) > 0$, into Bailey's product formula for confluent hypergeometric functions (see [1])

$$ {}_1F_1(a;b;x) \cdot {}_1F_1(a;b;y) = \sum_{k=0}^\infty \frac{(-1)^k (a)_k (b-a)_k}{k!(b)_k (b)_{2k}} (xy)^k {}_1F_1(a+k;b+2k;x+y) $$

we get, after interchanging the order of summation and integration

$$ {}_1F_1(a;b;x) \cdot {}_1F_1(a;b;y) = \frac{2^{1-b}[\Gamma(b)]^2 e^{\frac{x+y}{2}}}{\Gamma(a)\Gamma(b-a)} \int_{-1}^1 e^{\frac{x+y}{2}t} (1-t)^{b-2} \left(\frac{1+t}{1-t}\right)^{a-1} \times \left[\sum_{k=0}^\infty \frac{(-1)^k \left(\frac{\sqrt{xy(1-t^2)}}{2}\right)^{2k}}{k!\Gamma(b+k)}\right] \mathrm{d}t. $$

Using the series expansion

$$ \mathscr{J}_\nu(z) = \sum_{k=0}^\infty \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k!\Gamma(\nu+k+1)}, $$

we obtain

$$ {}_1F_1(a;b;x) \cdot {}_1F_1(a;b;y) = \frac{2^{1-b}[\Gamma(b)]^2 e^{\frac{x+y}{2}}}{\Gamma(a)\Gamma(b-a)} \int_{-1}^1 e^{\frac{x+y}{2}t} (1-t)^{b-2} \left(\frac{1+t}{1-t}\right)^{a-1} \times \mathscr{J}_{b-1}(\sqrt{xy(1-t^2)})\,\mathrm{d}t. \tag{17} $$

On the other hand, from Gegenbauer's double integral representation for J_ν (see [23, Section 3.33])

$$ \mathscr{J}_\nu(\omega)) = \frac{1}{\pi\Gamma(\nu)} \int_0^\pi \int_0^\pi e^{iZ\cos\theta - iz(\cos\Phi\cos\theta + \sin\Phi\sin\theta\cos\psi)} (\sin\psi)^{2\nu-1} (\sin\theta)^{2\nu}\,\mathrm{d}\psi\,\mathrm{d}\theta, $$

where $\omega^2 = Z^2 + z^2 - zZ\cos\Phi$ and $\nu > 0$, for $Z = -it\sqrt{xy}$, $z = -\sqrt{xy}$, $\Phi = \frac{\pi}{2}$ and $\nu = b-1$ we have

$$\mathscr{J}_{b-1}\left(\sqrt{xy(1-t^2)}\right) = \frac{1}{\pi\Gamma(b-1)}\int_0^\pi\int_0^\pi e^{t\sqrt{xy}\cos\theta + i\sqrt{xy}\sin\theta\cos\psi} \times (\sin\psi)^{2b-3}(\sin\theta)^{2b-2}\,\mathrm{d}\psi\,\mathrm{d}\theta. \quad (18)$$

Now substituting (18) into (17) and taking into account (16) yields

$$\begin{aligned} {}_1F_1(a;b;x)\cdot{}_1F_1(a;b;y) &= \frac{2^{1-b}(b-1)\Gamma(b)e^{\frac{x+y}{2}}}{\pi\Gamma(a)\Gamma(b-a)}\int_0^\pi\int_0^\pi e^{i\sqrt{xy}\sin\theta\cos\psi}(\sin\psi)^{2b-3}(\sin\theta)^{2b-2} \\ &\quad\times\left[\int_{-1}^1 e^{\left(\frac{x+y}{2}+\sqrt{xy}\cos\theta\right)t}(1-t)^{b-2}\left(\frac{1+t}{1-t}\right)^{a-1}\mathrm{d}t\right]\mathrm{d}\psi\,\mathrm{d}\theta \\ &= \frac{b-1}{\pi}\int_0^\pi\int_0^\pi e^{-\sqrt{xy}\cos\theta+i\sqrt{xy}\sin\theta\cos\psi}(\sin\psi)^{2b-3}(\sin\theta)^{2b-2} \\ &\quad\times{}_1F_1(a,b,x+y+2\sqrt{xy}\cos\theta)\,\mathrm{d}\psi\,\mathrm{d}\theta. \end{aligned} \quad (19)$$

By using analytic continuation, equation (19) can be extended to $a \in \mathbb{C}$ and $\Re e(b) > 1$. This proves equation (15) and completes the proof of Glaeske's result.

In particular, putting $x = y = 0$ in (19) yields

$$1 = \frac{b-1}{\pi}\int_0^\pi\int_0^\pi(\sin\psi)^{2b-3}(\sin\theta)^{2b-2}\,\mathrm{d}\psi\,\mathrm{d}\theta. \quad (20)$$

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