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Complex analysis and geometry / *Analyse et géométrie complexes*

On the GAGA principle for algebraic affine hypersurfaces

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Abstract. For any *complete* $\mathbb{C}$-algebraic variety Y and its underlying *compact* $\mathbb{C}$-analytic space $\mathscr{Y}$, it follows from the well known GAGA principle that the *algebraic* Picard group $Pic(Y)$ and the *analytic* Picard group $\mathbb{P}ic(\mathscr{Y})$ are isomorphic. Our main purpose here is to provide a simple proof of an analogous situation for non complete $\mathbb{C}$-algebraic varieties, namely $\mathbb{C}$-algebraic affine hypersurfaces with at most isolated singularities.

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1. Introduction

Unless the contrary is explicitly stated, all $\mathbb{C}$-analytic spaces $\mathscr{X}$ are assumed to be equipped with an analytic structural sheaf $\mathscr{O}_{\mathscr{X}}$. For any $\mathbb{C}$-algebraic variety X, let us denote by $Pic(X)$ (resp. by $\mathbb{P}ic(\mathscr{X}) = H^1(\mathscr{X}, \mathscr{O}^*_{\mathscr{X}})$), the *algebraic* (resp. *analytic*) Picard group of X (resp. of $\mathscr{X}$), where $\mathscr{X}$ is the $\mathbb{C}$-analytic space associated to X. Any 1-dimensional $\mathbb{C}$-analytic spaces will be referred to as *curves*. Assume that a given *compact* $\mathbb{C}$-analytic space $\mathscr{Y}$ is biholomorphic to an underlying topological space of some complete $\mathbb{C}$-algebraic variety Y; since there is a 1-1 correspondence between linear equivalent classes of Cartier divisors and locally free sheaves of rank 1, it follows from Serre *GAGA* principle, (see e.g. [7, Chapitre XII, Théorème 4.4] that the *analytic* Picard group $\mathbb{P}ic(\mathscr{Y})$ and the *algebraic* Picard group $\mathrm{Pic}(Y)$ are isomorphic.

On the other hand, let $\mathscr{X}$ be a $\mathbb{C}$-analytic space which is an underlying topological space of some *affine* algebraic variety X defined over $\mathbb{C}$. Then it is well known that

(1) $\mathscr{X}$ is Stein.
(2) we have the following exact sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_{\mathscr{X}} \to \mathscr{O}^*_{\mathscr{X}} \to 0$$

(3) $\mathbb{P}ic(\mathscr{X}) = H^1(\mathscr{X}, \mathscr{O}^*_{\mathscr{X}}) \simeq H^2(X, \mathbb{Z})$.

In that direction, we have the following well known result:

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Theorem 1. *Let $\mathbb{A}^n$ (resp. $\mathbb{C}^n$) be the* affine *n-space (resp. the* complex *n-space). Then*

$$Pic\left(\mathbb{A}^n\right) \quad \text{and} \quad \mathbb{P}ic\left(\mathbb{C}^n\right) \quad \text{are trivial.}$$

As far as *reduced* non-singular affine curves are concerned, a glimpse of GAGA principle does enter into this picture, namely

Proposition 2 ([16, Corollary 2.2]). *Biholomorphically equivalent non-singular affine algebraic curves are algebraically isomorphic.*

Unfortunately, all the similarities cease from there; indeed, one has

Example 3. Let C be a fixed non-singular projective curve of genus $g \geq 0$ together with a finite set of points $p_j \in C$ and let $A := C \setminus \cup_{j\geq 1} p_j$ be the affine curve. By abuse of notations, let us denote also by A its associated (non compact) Riemann surface.

Therefore, since $H^2(A, \mathbb{Z}) = 0$, we infer from (3) that

$$\mathbb{P}ic(A) = 0$$

On the other hand, one has

Proposition 4 ([8, Corollary 1.3]). $Pic(A) = 0$ *iff* $g = 0$.

Example 5. Let $X := \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\}$ and $\mathscr{X} \simeq \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\}$. Then it is easy to see that

$$Pic(X) = 0 \quad \text{and} \quad \mathbb{P}ic(\mathscr{X}) \simeq \mathbb{Z}.$$

Example 6. Let B be a fixed non-singular affine curve of genus $g > 0$. For $i = 1, 2$, let $L_i \in Pic(B)$ be 2 *non-equivalent* algebraic line bundles. Let X_i be the total space of L_i and let $\mathscr{X}_i$ be its associated Stein surfaces. Then, from [16], one obtains the following biholomorphisms

$$\mathscr{X}_1 \simeq \mathbb{C} \times B \simeq \mathscr{X}_2 \tag{1}$$

Therefore, from (1) one has

$$\mathbb{P}ic\,(\mathscr{X}_1) \simeq \mathbb{P}ic\,(\mathscr{X}_2)$$

However, in contrast with Proposition 2, it is known that X_i are *not algebraically isomorphic* [16, Proposition 3.1]. In spite of this fact and against all expectations, we have the following interesting result which was communicated to us by the referee which we gratefully acknowledge.

Theorem 7. $Pic(X_1) \simeq Pic(B) \simeq Pic(X_2)$

Proof. Let V be an algebraic variety over an algebraically closed field $\mathbf{k}$. Let $\mathbf{k}_V^*$ be the constant sheaf on V associated to $\mathbf{k}^*$, let $\mathbb{G}_{m,V}$ be the units sheaf on V, and let $U_{\mathbf{k},V} :=$ the presheaf cokernel of $(\mathbf{k}_V^* \to \mathbb{G}_{m,V})$. Then it is known [14, Lemma 2] that

(1) $U_{\mathbf{k},V}$ is a sheaf on V,
(2) $\mathrm{Pic}(V) = H^1(V, U_{\mathbf{k},V}) = H^1(V, \mathbb{G}_{m.V})$, and
(3) for a smooth curve B and a Zariski fibration [14, Definition 3] $f\colon E \longrightarrow B$ with fibre F, one has [14, Theorem 5] the following exact sequence

$$\begin{aligned} 0 \longrightarrow U_{\mathbf{k},B}(B) \longrightarrow U_{\mathbf{k},E}(E) \longrightarrow U_{\mathbf{k},F}(F) \longrightarrow Pic(B) \longrightarrow \\ \longrightarrow Pic(E) \longrightarrow Pic(F) \longrightarrow \end{aligned} \tag{2}$$

provided, for all sufficiently small open sets W of B the natural map

$$Pic(F) \times Pic(W) \longrightarrow Pic(F \times W) \quad \text{is an isomorphism.} \tag{3}$$

Now, from a more general result in [4, Corollary 6, p. 11] we have, for any smooth algebraic variety V,

$$Pic\left(\mathbb{A}^n \times V\right) \simeq Pic(V) \tag{4}$$

From (4) it follows that, the assumption (3) is fulfilled for any line bundle $E \longrightarrow B$; in particular for X_i with $i = 1$ or 2. Therefore the exact sequence (2) can be applied. Furthermore in our case $F = \mathbb{A}^1$, the groups $U_{\mathbf{k},F}(F)$ and Pic(F) are trivial. Hence one obtains

$$Pic(X_1) \simeq Pic(B) \simeq Pic(X_2)$$ □

Remark 8. Confronted with this state of affairs, we are looking at a class of affine algebraic hypersurfaces X with $\dim.X \geq 3$.

2. The non-singular hypersurfaces

Despite such an adverse situation, one has the following important result:

Theorem 9 ([11, Corollary 2.3]). *Let $Y \subset \mathbb{P}_{n+1}$ with $n \geq 3$, be a non-singular hypersurface, let $\Gamma \subset Y$ be a transverse hyperplane section and let $X := Y \setminus \Gamma$. Then one has*

$$H^i(X, \mathbb{Z}) = 0 \quad for \quad i \neq 0,\ n.$$

Since the underlying $\mathbb{C}$-analytic variety of $X =: \mathscr{X}$ is a Stein manifold, we infer from Theorem 9, the following:

Corollary 10. *Let X be as in Theorem 9. Then*

$$\mathbb{P}ic(\mathscr{X}) := H^1\left(\mathscr{X}, \mathscr{O}^*_{\mathscr{X}}\right) \simeq H^2(X, \mathbb{Z}) \quad \textit{is trivial.}$$

Dimensionwise, this result is optimal. In fact, let us look at the following

Example 11. Let $Y_0 \subset \mathbb{P}_3$ be a non-singular hypersurface and let $\Gamma \subset Y_0$ be a transverse hyperplane section. Then it is known [2, Lemma 1.2] that $X_0 := Y_0 \setminus \Gamma$ is homeomorphic to the Milnor fiber of the singularity $(\mathscr{C}, 0)$ where $\mathscr{C}$ is the affine cone over Γ with 0 as its vertex. Consequently

$$\mathbb{P}ic\left(\mathscr{X}_0\right) = H^2(\mathscr{X}_0, \mathbb{Z}) = \mathbb{Z}^{\mu}$$

where μ is the Milnor number of $(\mathscr{C}, 0)$.

As far as an algebraic analogue of Corollary 10 is concerned, we have the following result:

Proposition 12. *Let $Y \subset \mathbb{P}_{n+1}$ be a non-singular hypersurface, let $\Gamma \subset Y$ be a transverse hyperplane section and let $X := Y \setminus \Gamma$. Then*

$$Pic(X) = 0$$

provided $n = \dim.Y \geq 3$.

Proof. By [9, Chapter IV, Corollary 3.2] one has $Pic(Y) \simeq \mathbb{Z}[\Gamma]$. Then from the following exact sequence [10, Chapter II, Proposition 6.5 (c)]

$$\begin{array}{l} \mathbb{Z} \longrightarrow Pic(Y) \simeq \mathbb{Z} \xrightarrow{\delta} Pic(X) \longrightarrow 0 \\ 1 \longmapsto \quad 1.\Gamma \end{array}$$

and the surjectivity of δ, we infer the exact sequence

$$0 \longrightarrow Pic(X) \longrightarrow 0$$

Hence our desired conclusion will follow. □

Remark 13. The proof of Theorem 9 relies heavily on Poincare duality for Γ and Alexander duality for the pair $(Y, Y \setminus \Gamma)$ [5] which depend entirely on the fact that both Γ and Y are non singular and the transversal intersection of Γ. In this situation, it is natural to wonder how such results could be generalized to the context of an ambient space $Y \subset \mathbb{P}_{n+1}$ with only *mild* singularities; that is the purpose of the next section.

3. Hypersurfaces with isolated singularities

With those examples as guidelines, various endeavors were devoted to generalize Theorem 9 within the framework of hypersurfaces $Y \subset \mathbb{P}_N$ with only isolated singularities and with $N \geq 4$. In the same spirit as Theorem 9, we are now in a position to provide an elementary and complete proof of the following result:

Theorem 14. *Let $Y \subset \mathbb{P}_{n+1}$ be an irreducible hypersurface, with only isolated singularities, say $\{p_j\}_{1\leq j\leq k}$ and with $n \geq 3$. Let $\Gamma \subset Y$ be a transverse hyperplane section, in particular $p_j \notin \Gamma$ for $1 \leq j \leq k$ and let $X := Y \setminus \Gamma$. Then one has*

(1) $H_i(X, \mathbb{Z}) = 0$ *for* $1 \leq i \leq n-2$.

(2)
$$H_{n-1}(X, \mathbb{Z}) = \begin{cases} \mathbf{0} & \text{if } n \text{ is odd.} \\ \mathbf{0} & \text{or finite cyclic if } n \text{ is even.} \end{cases}$$

Proof.

(**Step 1**) For simplicity, let us assume that Y has only 1 isolated singularity, say $\{p\}$.
Now let $\mathfrak{h} : \mathbb{C}^{n+2} \to \mathbb{C}$ be a homogeneous polynomial of degree d, defining the $\mathbb{C}$-projective hypersurface

$$Y = \left\{x \in \mathbb{P}_{n+1} \middle| \mathfrak{h}(x) = 0\right\}$$

In view of the Sard theorem, there exist $\epsilon > 0$ and a general homogeneous polynomial of degree d, say h_d, so that for any $s \in \Delta := \{s \in \mathbb{C} \mid 0 \leq |s| < \epsilon\}$, the total space of the pencil

$$\mathcal{M} = \left\{(x, t) \in \mathbb{P}_{n+1} \times \Delta \middle| \mathfrak{h} + sh_d = 0\right\}$$

is a one-parameter smoothing of degree d, for Y.

From the second projection

$$pr_2 : \mathbb{P}_{n+1} \times \Delta \to \Delta$$

let $\pi := pr_2|\mathcal{M} : \mathcal{M} \to \Delta$ be its restriction. Then one can check that

(a) $\pi^{-1}(0) = Y$ and

(b) $Y_s := \pi^{-1}(s)$ is a smooth $\mathbb{C}$-projective hypersurface of degree d, for any $s \neq 0$.

Now by identifying the unique singular point $\{p\}$ with the origin $0 \in \mathbb{C}^{n+1}$, the singularity (Y, 0) can be defined by $\{f = 0\}$ where $f : (\mathbb{B}_r \subset \mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is an analytic function germ with an isolated critical point at $0 \in \mathbb{C}^{n+1}$ and $\mathbb{B}_r$ is a ball centered at 0 with sufficiently small radius r. Then [3, § 3].

(c) For $t \neq 0$, $\Xi := f^{-1}(t) \cap \mathbb{B}_r$ is the Milnor fibre of the isolated singularity germ $(Y, 0)$.

(d) Ξ has the homotopy type of a bouquet of n-spheres $\mathbb{S}_n$ say, $\bigvee^{\mu} \mathbb{S}_n$, where

$$mu := \dim_{\mathbb{C}} \frac{\mathcal{O}_0\left(\mathbb{C}^{n+1}\right)}{J_f},$$

$\mathcal{O}_0(\mathbb{C}^{n+1})$, is the local ring of holomorphic functions at

$$0 \in \mathbb{C}^{n+1} \text{ and } J_f := \left(\partial f/\partial x_0, \ldots.., \partial f/\partial x_n\right)$$

is the Jacobian ideal of the singularity of f.

(e) The ball $\mathbb{B}_r$ has $Y \cap \mathbb{B}_r$ as a deformation retract.

(**Step 2**) Now let r be such a retraction and let i be the inclusion [15, § I.7]

$$f^{-1}(t) \cap \mathbb{B}_r \hookrightarrow \mathbb{B}_r$$

Then the composite $r \circ i$ gives a map

$$f^{-1}(t) \cap \mathbb{B}_r \to Y \cap \mathbb{B}_r$$

which contracts Ξ to $\{p\}$.
Now let $\mathbb{B} \subset \mathscr{M}$ be a sufficiently small ball centered at $\{p\}$. Then $\pi^{-1}(s) \cap \mathbb{B}$ can be identified with the Milnor fibre of the isolated hypersurface singularity germ $(Y, 0)$, for $s \neq 0$.

Notice that such a contraction can be extended [18, Chapter V, § 14, *Exercices (3)* p. 332] to a continuous map

$$\Phi : Y_s \to Y$$

(**Step 3**) Now let $\phi := \Phi | X_s$, $X_s := Y_s \setminus \phi^{-1}(\Gamma)$ and let us consider the following commutative diagram of *integral* homology groups with exact rows

$$\begin{array}{ccccccccc}
\longrightarrow & H_k(\{p\}) & \longrightarrow & H_k(X) & \xrightarrow{\delta_*} & H_k(X, \{p\}) & \longrightarrow & H_{k-1}(\{p\}) & \longrightarrow \\
 & \uparrow & & \uparrow \phi_* & & \simeq \uparrow & & \uparrow & \\
\longrightarrow & H_k(\Xi) & \longrightarrow & H_k(X_s) & \longrightarrow & H_k(X_s, \Xi) & \longrightarrow & H_{k-1}(\Xi) & \longrightarrow
\end{array}$$

Since, for any $k > 1$, δ_* is an isomorphism, we deduce from the above commutative diagram, the following exact sequence

$$H_k(\Xi) \longrightarrow H_k(X_s) \xrightarrow{\phi_*} H_k(X) \longrightarrow H_{k-1}(\Xi) \tag{5}$$

Now it follows from (2)(b), that

$$H_j(\Xi) \simeq 0 \quad \text{for} \quad 1 \leq j \leq n-1.$$

Therefore we infer from (5) that ϕ_* is an isomorphism, provided $2 \leq k \leq n-1$ and our conclusion follows from [11, Theorem 9].

(**Step 4**) Since $H_0(X) \simeq \mathbb{Z} \simeq H_0(X_s)$, we infer from the above commutative diagram, the following exact sequence

$$H_1(\Xi) \longrightarrow H_1(X_s) \xrightarrow{\alpha} H_1(X) \longrightarrow H_0(\Xi) \xrightarrow{\gamma} H_0(\{p\}) \longrightarrow 0 \tag{6}$$

Notice that
(a) In view of (2)(b), α is injective.
(b) Since $H_0(\Xi) \simeq \mathbb{Z} \simeq H_0(\{p\})$, γ is bijective; consequently α is also surjective.
Therefore we infer from (6) that

$$H_1(X) \simeq H_1(X_s) = 0.$$

(**Step 5**) Now let Y with arbitrary isolated singularities $\{p_j\}$. Then exactly as in (**Step 1**), one can exhibit [1, § 3] a family

$$\pi : \mathscr{M} \to \Delta$$

such that
(a) $\pi^{-1}(0) = Y$ and
(b) $Y_s := \pi^{-1}(s)$ for any $s \neq 0$, is a smooth $\mathbb{C}$-projective hypersurface of degree d which is a smooth deformation of Y.

(**Step 6**) Then a construction of the specialization map

$$\Phi : Y_s \longrightarrow Y$$

which contracts each Milnor fibre Ξ_j to p_j, will be proceeded exactly as carried out in detail in [1, § 3]. Finally the same arguments as in (**Step 3**) and (**Step 4**) above will complete our proof. □

By using the Universal coefficient Theorem, we infer from Theorem 14 the following result

Corollary 15. *Let X be as in Theorem 14 and let $\mathscr{X}$ be its associated $\mathbb{C}$-analytic space. Then one has*

(1) $H^i(X, \mathbb{Z}) = 0 \quad if \quad 1 \le i \le n-1.$
(2) $\mathbb{P}ic(\mathscr{X})$ *is trivial.*

Remark 16. Notice that, the *transversal* hypothesis of Γ in Theorem 14 is crucial here, as shown by the following

Example 17 ([11, § 4 p. 213]). Let $Y_2 := \{x^2 + y^2 + z^2 + w^2 = 0\} \subset \mathbb{P}_4\{x : y : z : w : t\}$ be a quadric hypersurface with a single (isolated) singular point $q := (0:0:0:0:1)$ and let $A_2 := Y_2 \cap \{x \neq 0\}$. Then it is clear that $A_2 \simeq \{\zeta^2 + \xi^2 + \nu^2 = -1\} \subset \mathbb{C}^4(\xi, \zeta, \nu, \tau)$ is a non-singular affine algebraic variety, where $\zeta := \frac{y}{x}, \xi := \frac{z}{x}, \nu := \frac{w}{x}$, and $\tau := \frac{t}{x}$. Certainly A_2 is homotopically equivalent to $A_2 \cap \{\tau = 0\}$ which has the same homotopy type as the 2-sphere $\mathbb{S}^2$; consequently, one has

$$\mathbb{P}ic(\mathscr{A}_2) \simeq H^2\left(\mathbb{S}^2, \mathbb{Z}\right) \simeq \mathbb{Z}$$

where $\mathscr{A}_2$ is the Stein 3-fold associated to A_2.

3.1. *Question*

Let X be as in Theorem 14. Does one also have

$$Pic(X) = 0\,?$$

In this direction, we would like to provide a positive answer to this question. i.e. an *algebraic* analogue to our Corollary 15.

Theorem 18 ([17]). *Let X and Y be as in Theorem 14. Then one has*

$$Pic(X) \quad \textit{is trivial.}$$

Consequently, by using Corollary 15, we obtain the following

Corollary 19. *Let $Y \subset \mathbb{P}_{n+1}$ with $n \ge 3$, be a non-singular hypersurface, let $\Gamma \subset Y$ be a transverse hyperplane section and let $X := Y \setminus \Gamma$. Then one has*

$$Pic(X) \quad \textit{and} \quad \mathbb{P}ic(\mathscr{X}) \;\; \textit{are trivial.}$$

4. Proper hyperplane sections

Motivated by Example 17, throughout this section, let us consider the following:

Definition 20. *Let $Y \subset \mathbb{P}_{n+1}$ be an irreducible hypersurface and let $\mathbb{H} \subset \mathbb{P}_{n+1}$ be a non-singular hyperplane. Then*

$$\mathscr{H} := Y \cap \mathbb{H}$$

will be referred to as a proper *hyperplane section, if $\mathbb{C} - \dim_x \mathscr{H} = n-1$, for any $x \in \mathscr{H}$.*

Example 21. Let $Y_2 \subset \mathbb{P}_4(z_0 : z_1 : z_2 : z_3 : z_4)$ be a singular quadric hypersurface defined by $\sum_{i=0}^{3} z_i^2 = 0$ with a single isolated singular point $p := (0:0:0:0:1)$. Let $Cl(Y_2) :=$ the Divisor class group of Y_2. It is known [10, Example 6.5, p. 147] that

$$Cl(Y_2) \simeq H_4\left(Y_2, \mathbb{Z}\right) \simeq \mathbb{Z} \oplus \mathbb{Z} \tag{7}$$

On the other hand

$$Pic(Y_2) \simeq H^2\left(Y_2, \mathbb{Z}\right) \simeq \mathbb{Z} \tag{8}$$

From (7) and (8), we have a well known fact that the local ring $\mathscr{O}_{Y_2, p}$ is not $\mathbb{Q}$-factorial.

Remark 22. In sharp contrast with Example 21, one has the following important result

Theorem 23 ([6, Chapter XI, p. 314]). *Let $Y \subset \mathbb{P}_{n+1}$ be a hypersurface with only isolated singularities $\{p_j\}$. Assume that* $\dim . Y \geq 4$. *Then the local rings $\mathscr{O}_{Y,p_j}$ are factorial for any j (i.e. any* Weil *divisor on Y is also* Cartier*).*

Now we infer from this result, the Universal Coefficient Theorem and the proof of Theorem 18 the following

Theorem 24 ([17]). *Let $Y \subset \mathbb{P}_{n+1}$ be an irreducible hypersurface with only isolated singularities and let $\mathscr{H} \subset Y$ be a proper hyperplane section. Let $X := Y \setminus \mathscr{H}$ and let $\mathscr{X}$ be its associated analytic space. Then*

$$Pic(X) \quad \text{and} \quad \mathbb{P}ic(\mathscr{X}) \text{ are trivial} \tag{9}$$

provided $n \geq 4$.

Remark 25. Example 17 shows that the bound given in this Theorem is quite sharp.

5. The transverse hypersurface sections

Throughout this section, let us consider *exclusively* a *non-singular hypersurface $Y \subset \mathbb{P}_{n+1}$* and its *transverse hypersurface section $\mathscr{H} \subset Y$* i.e. $\mathscr{H} := Y \pitchfork \mathsf{H}_\nu$, for some *non-singular hypersurface* $\mathsf{H}_\nu \subset \mathbb{P}_{n+1}$, of degree $\nu \geq 1$. Then, from seminal works by Kato in [12] and [13], one derives from his far reaching result [13, Theorem 6.3], the following

Theorem 26. *Let $Y \subset \mathbb{P}_{n+1}$ be a non-singular hypersurface with $n \geq 3$, let $\mathscr{H} \subset Y$ be a transverse hypersurface section and let $X := Y \setminus \mathscr{H}$. Then one has*

$$H_i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z}_\nu & \text{if } i \text{ is odd and } 1 \leq i \leq n-1. \\ 0 & \text{if } i \text{ is even and } 2 \leq i \leq n-1. \end{cases}$$

We are now in a position to provide the following result which generalizes Corollary 19.

Theorem 27 ([17]). *Let $Y \subset \mathbb{P}_{n+1}$ be a non-singular hypersurface with $n \geq 3$ and let $\mathscr{H} \subset Y$ be a transverse hypersurface section. Let $X := Y \setminus \mathscr{H}$ and let $\mathscr{X}$ be its associated Stein manifold. Then one has*

$$Pic(X) \simeq \mathbb{P}ic(\mathscr{X}) \simeq \mathbb{Z}_\nu$$

provided $n \geq 3$.

Remark 28. Notice that, dimensionwise, Theorem 27 is *optimal*; indeed besides Example 3 and Proposition 4, let us consider the following:

Example 29. Let $\mathscr{C} \subset \mathbb{P}_2$ be a non-singular *cubic* plane curve (i.e. $g(\mathscr{C}) = 1$) and let $\mathbf{X} = \mathbb{P}_2 \setminus \mathscr{C}$. Then [10, Chapter II, Example 6.5.1] one has

$$Pic(\mathbf{X}) \simeq \mathbb{Z}_3$$

On the other hand, from the following exact sequence of integral cohomology groups of the pair $(\mathbb{P}_2, \mathbf{X})$

$$H^2(\mathbf{X}) \xrightarrow{\lambda} H^3\big(\mathbb{P}_2, \mathbf{X}\big) \longrightarrow H^3(\mathbb{P}_2) \simeq 0$$

since $H^3(\mathbb{P}_2, \mathbf{X}) \simeq H_1(\mathscr{C})$ and *Rank* of $H_1(\mathscr{C}) = 2g(\mathscr{C}) = 2$, we infer from the surjectivity of λ, that

$$\mathbb{P}ic(\mathbf{X}) \simeq H^2(\mathbf{X}, \mathbb{Z}) \neq \mathbb{Z}_3 \simeq Pic(\mathbf{X})$$

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