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Projective bundles and blowing ups

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Abstract. We study the blowing up $\widetilde{X}$ of a smooth projective variety X along a smooth center B that is equipped with a projective bundle structure over a variety Z. If B is a point, then X is a projective space. If the Picard number $\rho(X)$ is 1, then $\dim Z$ has a lower bound $\dim X - \dim B - 1$. Moreover, when $\dim Z$ is $\dim X - \dim B - 1$, X is a projective space and B is a linear subspace in X. If X is a projective space $\mathbb{P}_n$ and B is a curve, then either n is 3 and B is a twisted cubic curve or n is an arbitrary integer and B is a line in $\mathbb{P}_n$. If X is a quadric Q_n and B is a curve, then n is 3 and B is a line in Q_3.

Résumé. Nous étudions l'éclatement $\widetilde{X}$ d'une variété projective lisse X le long d'un centre lisse B, munie d'une structure de fbré projectif. Si B est un point, X est un espace projectif. Si le nombre de Picard $\rho(X)$ est 1, alors $\dim Z$ a une borne inférieure $\dim X - \dim B - 1$. De plus, lorsque $\dim Z$ est $\dim X - \dim B - 1$, X est un espace projectif et B est un sous-espace linéaire dans X. Si X est l'espace projectif $\mathbb{P}_n$ et B est une courbe, ou n est égale à 3 et B est une courbe cubique tordue, ou n est un entier arbitraire et B est une ligne droite dans $\mathbb{P}_n$. Si X est une quadrique et B est une courbe, alors n est égale à 3 et B est une ligne droite dans Q_3.

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1. Introduction

Let L be a linear subspace of dimension l in a projective space $\mathbb{P}_n$. We consider the rational map $\pi_L : \mathbb{P}_n \dashrightarrow \Gamma$ given by the linear projection from L to Γ, where Γ is a linear subspace of dimension $n-l-1$ disjoint from L. Then the blowing up $\mathrm{Bl_L}(\mathbb{P}_n)$ of $\mathbb{P}_n$ along L is the graph of π_L. More precisely, $\mathrm{Bl_L}(\mathbb{P}_n) \subseteq \mathbb{P}_n \times \Gamma$ is the closed subvariety of $\mathbb{P}_n \times \Gamma$ defined as $\mathrm{Bl_L}(\mathbb{P}_n) = \{(p,q) \in \mathbb{P}_n \times \Gamma | p \in \langle L, q\rangle\}$, where $\langle L, q\rangle$ is the linear subspace generated by L and q. The projection $p_2 : \mathrm{Bl_L}(\mathbb{P}_n) \to \Gamma$ to the second factor is a projective bundle. Actually, for any point q in Γ, the fiber $p_2^{-1}(q)$ is the linear space $\langle L, q\rangle$. So any blowing up of a projective space along a linear subspace is equipped with a projective bundle structure. It is interesting to find more examples of blowing ups that are equipped with projective bundle structures. In [1, Section 4] and [2], there are examples of blowing ups of projective spaces along non-linear subvarieties that are equipped with projective bundle structures.

In this article, we aim to prove some classification results about (X,B)(where X and B are projective smooth varieties and B is a subvariety of X) such that the blowing up $\mathrm{Bl}_B(X)$ is a projective bundle over some variety Z. In Proposition 1, we show that if B is a point, then X is a projective space. In Theorem 4, we show that if the Picard number $\rho(X)$ is 1 and $\dim Z$ is $\dim X - \dim B - 1$, then X is a projective space and B is a linear subspace in X. Finally, in Theorem 6, we show that if X is $\mathbb{P}_n$ and B is a curve, then either n is 3 and B is a twisted cubic curve or n is an arbitrary positive integer and B is a line in $\mathbb{P}_n$; if X is a quadric $Q_n (n \geq 3)$ and B is a curve, then n is 3 and B is a line in Q_3.

Convention. A complex variety is an irreducible integral scheme of finite type defined over $\mathbb{C}$. In this article, all varieties considered are complex projective varieties.

2. Main results

Setup. We assume that X is a smooth projective variety of dimension n and we denote the blowing up $\widetilde{X}$ of X along a closed smooth subvariety B by $\varphi : \widetilde{X} \to X$. Assume that $\pi : \widetilde{X} \to Z$ is a projective bundle over a variety Z. We summarize all the morphisms in a diagram as follows:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\pi} & Z \\ {\scriptstyle\varphi}\downarrow & & \\ X & & \end{array}$$

We introduce some notations here: we let E be the exceptional divisor of φ, let F_1 be a line in a fiber $\pi^{-1}(z)$ (we have $\pi^{-1}(z) \simeq \mathbb{P}_{n-\dim Z}$) and let F_2 be a line in a fiber $\varphi|_E^{-1}(b)$(we have $\varphi|_E^{-1}(b) \simeq \mathbb{P}_{n-\dim B-1}$).

In the remaining part of this section, we keep the assumptions in the Setup and we keep using the notations there.

We firstly consider the blowing up at a point that is equipped with a projective bundle structure.

Proposition 1. *If B is a point, then X is a projective space $\mathbb{P}_n$.*

Proof. We consider the morphism $\pi|_E : E \to Z$. Since E is a projective space $\mathbb{P}_{n-1}$, the morphism $\pi|_E : E \to Z$ is quasi-finite. Otherwise, $\pi|_E$ will contract E to a single point. So $\dim Z$ is $n-1$ and hence $\pi|_E$ is surjective. Note that Z is smooth and projective, by the Lazarsfeld's theorem in [5], Z is a projective space $\mathbb{P}_{n-1}$. By [4, Lemma V.3.7.8], X is a projective space. □

In the remaining part of this section, we keep an additional assumption that the Picard number $\rho(X)$ of X is 1.

Now let us prove a lemma about the lower bound of $\dim Z$.

Lemma 2. *The morphism $\pi|_E : E \to Z$ is surjective and the varieties Z, $\widetilde{X}$, X are Fano varieties. Moreover, $\dim Z$ is at least $n - \dim B - 1$. If $\dim Z$ is $n - \dim B - 1$, then Z is a projective space.*

Proof. Note that E is a divisor in $\widetilde{X}$. The codimension $\operatorname{codim}(\pi(E), Z)$ is at most 1. If $\operatorname{codim}(\pi(E), Z)$ is 1, the divisor E is the pull-back of $\pi(E)$. Since the Picard number of Z is 1, E is a nef divisor, which contradicts to the fact that E is covered by negative curves. Then $\pi|_E$ is surjective. Now Z is a uniruled variety whose Picard number is 1, so Z is Fano.

To prove $\widetilde{X}$ is Fano, we have the canonical bundle formulas:

$$-K_{\widetilde{X}} = \varphi^*(-K_X) - (n - \dim B - 1)E = \pi^*(-K_Z - \det \mathscr{E}) + \mathscr{O}_\pi(n - \dim Z + 1)$$

where $\mathscr{O}_\pi(1)$ is the tautological line bundle of π. Since the Picard number of $\widetilde{X}$ is 2, the cone $\overline{NE(\widetilde{X})}$ is generated by two extremal rays as $\mathbb{R}_{\geq 0}[F_1] + \mathbb{R}_{\geq 0}[F_2]$, where $[F_i] (i = 1, 2)$ are the numerically equivalent classes of F_i. By calculating the intersection numbers of $-K_{\widetilde{X}}$ with F_1 and F_2, we deduce that $\widetilde{X}$ is Fano. Since X is a uniruled variety whose Picard number is 1, X is Fano.

Let $Y \simeq \mathbb{P}_{n-\dim B-1}$ be a fiber of $\varphi|_E : E \to B$. The morphism $\pi|_Y : Y \to Z$ is quasi-finite onto its image. Otherwise, there would be a curve C_Y contracted by π. But C_Y is numerically equivalent to a positive multiple of a line in Y, then the extremal ray of $\overline{NE(\widetilde{X})}$ spanned by $[F_2]$ is contracted by π, which is impossible. So $\dim Z$ is at least $n-\dim B-1$. Moreover, if $\dim Z$ is $n-\dim B-1$, $\pi|_Y$ is surjective. By the Lazarsfeld's theorem in [5], Z is a projective space. □

Now we prove a simple lemma about the intersection numbers, which is very useful in the proof of our main results.

Lemma 3. *Let H_1 be the pull-back of the ample generator H_Z of* $\mathrm{Pic}(Z)$ *and let H_2 be the pull-back of the ample generator H_X of* $\mathrm{Pic}(X)$. *We denote by a, b, c, d the intersection numbers $H_1 \cdot F_2$, $\mathscr{O}_\pi(1) \cdot F_2$, $H_2 \cdot F_1$, $E \cdot F_1$ in the following diagram of intersection numbers:*

	H_1	$\mathscr{O}_\pi(1)$	H_2	E
F_1	0	1	c	d
F_2	a	b	0	-1

Then a equals to c and a divides $1+bd$.

Proof. The torsion free abelian group $\mathrm{Pic}(\widetilde{X})$ has two bases $(H_1, \mathscr{O}_\pi(1))$ and (H_2, E) which satisfy the relation $\begin{pmatrix} H_1 \\ \mathscr{O}_\pi(1) \end{pmatrix} = \begin{pmatrix} \frac{da}{c} & -a \\ \frac{1+bd}{c} & -b \end{pmatrix} \cdot \begin{pmatrix} H_2 \\ E \end{pmatrix}$.

The matrix $A = \begin{pmatrix} \frac{da}{c} & -a \\ \frac{1+bd}{c} & -b \end{pmatrix}$ is an element of $SL_2(\mathbb{Z})$, so c divides $1+bd$ and the determinant $\det A$ $(= \frac{a}{c})$ is 1. □

In the remaining part of this article, we keep using the notations in Lemma 3.

If $\dim Z$ is $n-\dim B-1$, we have the following classification.

Theorem 4. *Let* $\dim B$ *be m. If* $\dim Z$ *is $n-m-1$, then X is a projective space $\mathbb{P}_n$, $B \simeq \mathbb{P}_m$ $(\subseteq \mathbb{P}_n)$ is a linear subspace in $\mathbb{P}_n$ and $\pi \circ \varphi^{-1} : \mathbb{P}_n \dashrightarrow \mathbb{P}_{n-m-1}$ is the linear projection from B.*

Proof. Let R_z be the fiber $\pi^{-1}(z)$ $(\simeq \mathbb{P}_{m+1})$ of some general point $z \in Z$. Then the intersection $Y_z = E \cap R_z$ is a hypersurface of degree d in R_z. We claim that d must equal to 1 (the proof of this claim mainly follows from [7, Lemma 2.1]). Actually, suppose that d is at least 2, then for a general line l in R_z, the intersection $l \cap Y_z$ consists of at least two distinct points y_1 and y_2. Since $\varphi|_{R_z}$ is quasi-finite onto its image, $\varphi(Y_z)$ is B. So we can assume that $b_i = \varphi(y_i)$ $(i = 1,2)$ are distinct points in B. By varying R_z and Y_z, we can construct a one-dimensional family of lines $\{l_t\}_{t\in C}$ in $\widetilde{X}$ such that the intersection of every l_t with each $\varphi^{-1}(b_i)$ is not empty. Then the surface $S = \cup_{t\in C} l_t$ is a ruled surface. Let $\varphi^{-1}(b_i) \cap S$ be C_i. The curves C_i $(i = 1,2)$ satisfy $C_1 \cap C_2 = \emptyset$ and $\varphi(C_i) = b_i$. By the construction of S, S is not contained in E. So $\varphi|_S$ is a birational morphism. Hence C_i $(i = 1,2)$ are exceptional curves, which is impossible.

Now d is 1. Note that E doesn't contain any fiber of π. Otherwise, there will be a morphism $\varphi|_{R_z} : R_z \to B$ where $R_z \simeq \mathbb{P}_{m+1}$ is some fiber of π. Then φ contracts R_z to a point, which is impossible. Hence for any $z \in Z$, the intersection $E \cap R_z = Y_z \simeq \mathbb{P}_m$ is a linear subspace in R_z. Since $\varphi(Y_z)$ is B, the variety B is a projective space. Then E has two projective bundle structures over projective spaces. So by [6, Theorem A], the morphism $(\varphi|_E, \pi|_E) : E \to B \times Z$ is an isomorphism. Hence $\varphi|_Y : Y \to B$ is an isomorphism. So the intersection number $H_2 \cdot F_1 (= H_1 \cdot F_2)$ equals to 1. Suppose that $\pi : \widetilde{X} \to Z$ is given by $|\alpha H_2 - \beta E|$. We have equalities $H_1 \cdot F_2 = (\alpha H_2 - \beta E) \cdot F_2 = 1$ and $H_1 \cdot F_1 = (\alpha H_2 - \beta E) \cdot F_1 = 0$. So both α and β equal to 1. Note the identities $m+2 = -K_{\widetilde{X}} \cdot F_2 = i_X - (n-m-1)$ (where i_X is the index of X). We deduce that i_X is $n+1$. By [3, Corollary of

Theorem 1.1], X is a projective space. Since $\dim H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(H_2-E)) = \dim H^0(X,I_B(1))$ is at least $n-m$, the subvariety B is a linear subspace in $X \simeq \mathbb{P}_n$. □

When B is a smooth curve, we have the following criterion.

Proposition 5. *If B is a smooth curve and n is at least 3, the following conditions are equivalent:*

(1) *π maps F_2 birationally to a line in Z,*
(2) *φ maps F_1 birationally to a line in X,*
(3) *(X,B) is one of the following cases:*
   (a) *($\mathbb{P}_n$, line) ($\pi\circ\varphi^{-1}:\mathbb{P}_n \dashrightarrow \mathbb{P}_{n-2}$ is the linear projection from B).*
   (b) *($\mathbb{P}_3$, twisted cubic curve) ($\pi\circ\varphi^{-1}:\mathbb{P}_3 \dashrightarrow \mathbb{P}_2$ is given by sections of $|\mathcal{O}_{\mathbb{P}_3}(2)|$ vanishing along B).*
   (c) *(Q_3, line) ($\pi\circ\varphi^{-1}:Q_3 \dashrightarrow \mathbb{P}_2$ is the linear projection from B).*

Proof. By Lemma 3, conditions (1) and (2) are equivalent. It is obvious that condition (3) implies conditions (1) and (2).

Now suppose condition (1) or (2) holds. By Lemma 2, $\dim Z$ is at least $n-2$. If $\dim Z$ is $n-2$, then by Theorem 4, (X,B) is case (a).

If $\dim Z$ is $n-1$, then π is a $\mathbb{P}_1$-bundle. So we have identities $-K_{\widetilde{X}}\cdot F_1 = i_X - d(n-2) = 2$. Then i_X is $2+d(n-2)$, which is at most $n+1$. Hence there are only two possibilities: when d is 2, n is 3 and i_X is 4; when d is 1, i_X is n.

If d is 2, then it is easy to see that X is $\mathbb{P}_3$ and Z is $\mathbb{P}_2$. Note that π is given by the linear system $|2H_2-E|$. Then $\dim H^0(\mathbb{P}_3, I_B(2))$ is at least 3. We claim that B is not a plane curve. Otherwise, there exists a plane L in $\mathbb{P}_3$ containing B and the strict transform $\widetilde{L}$ of L is numerically equivalent to H_2-kE for some positive integer k. So we have equalities $1 = H_2\cdot F_1 = \widetilde{L}\cdot F_1 + kE\cdot F_1 = \widetilde{L}\cdot F_1 + 2k$, which is impossible. If $\deg B$ is 4, then B is a complete intersection of two quadrics and $\dim H^0(\mathbb{P}_3, I_B(2))$ is 2. So $\deg B$ is 3.

If d is 1, then i_X is n, hence by [3, Corollary of Theorem 2.1], X is a quadric Q_n. We consider the morphism $\pi|_E : E \to Z$. Since d is 1, $\pi|_E$ is a birational morphism. Note that π is given by the linear system $|H_2-E|$. So $\dim H^0(Q_n, I_B(1))$ is at least n, which implies that B is a line. The exceptional divisor E of φ is isomorphic to $\mathbb{P}_{\mathbb{P}_1}(\mathcal{O}\oplus\mathcal{O}(1)^{\oplus(n-2)})$. So the birational morphism $\pi|_E$ contracts the minimal section of $\varphi|_E$ corresponding to the surjection $\mathcal{O}\oplus\mathcal{O}(1)^{\oplus(n-2)} \to \mathcal{O} \to 0$. Since Z is smooth, the exceptional locus of $\pi|_E$ should be a divisor, which implies the equality $n-2=1$. So X is Q_3 and B is a line in Q_3. □

Now let us prove another main result of this section.

Theorem 6. *Assume that X is $\mathbb{P}_n$ and B is a curve, then either n is 3 and B is a twisted cubic curve or n is an arbitrary integer and B is a line in $\mathbb{P}_n$. Assume that X is Q_n and B is a curve, then n is 3 and B is a line in Q_3.*

Proof. If X is $\mathbb{P}_n$, there are equalities: $1 = H_2^n = \varphi^*(H_X^n) = (-bH_1 + a\mathcal{O}_\pi(1))^n = (-b)^n H_1^n + a\big(\sum_{k=0}^{n-1} a^{n-k-1} C_n^k (-bH_1)^k \cdot \mathcal{O}_\pi(1)^{n-k}\big)$. Since H_1^n vanishes, the integer a divides 1, hence a is 1. Then by Proposition 5, either n is 3 and B is a twisted cubic curve or n is an arbitrary integer and B is a line in $\mathbb{P}_n$.

If X is Q_n, then there are equalities: $2 = H_2^n = \varphi^*(H_X^n) = (-bH_1 + a\mathcal{O}_\pi(1))^n = a\big(\sum_{k=0}^{n-1} a^{n-k-1} C_n^k (-bH_1)^k \cdot \mathcal{O}_\pi(1)^{n-k}\big)$. Suppose that a is 2, then $\sum_{k=0}^{n-1} a^{n-k-1} C_n^k (-bH_1)^k \cdot \mathcal{O}_\pi(1)^k$ is 1. Assume that there is a vector bundle $\mathcal{E}$ such that $\pi : \widetilde{X} \to Z$ is the projectization of $\mathcal{E}$. If $\mathrm{rk}\,\mathcal{E}$ is 2, then by the canonical bundle formulas:

$$-K_{\widetilde{X}} = \varphi^*(-K_X) - (n-2)E = \pi^*(-K_Z - \det\mathcal{E}) + \mathrm{rk}(\mathcal{E})\cdot\mathcal{O}_\pi(1),$$

we have $(i_X H_2 - (n-2)E)\cdot F_2 = ((i_Z - \deg\mathcal{E})H_1 + 2\mathcal{O}_\pi(1))\cdot F_2$. By Lemma 3, n equals to $2(i_Z - \deg\mathcal{E} + b + 1)$, hence n is an even number. Then $\sum_{k=0}^{n-1} a^{n-k-1} C_n^k (-bH_1)^k \cdot \mathcal{O}_\pi(1)^{n-k}$ is $n(-bH_1)^{n-1}\cdot$

$\mathscr{O}_\pi(1)+a\sum_{k=0}^{n-2}a^{n-k-2}C_n^k(-bH_1)^k\cdot\mathscr{O}_\pi(1)^{n-k}$, which is an even number. So rk$\mathscr{E}$ is 3. By Theorem 4, X should be a projective space, which is impossible. So a is 1. By Proposition 5, X is Q_3 and B is a line in Q_3. □

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