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# On some properties of the Łojasiewicz exponent 

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#### Abstract

In this note, we investigate the behaviour of the Łojasiewicz exponent under hyperplane sections and its relation to the order of tangency.


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## 1. Introduction and statements of the results

It is well known (see [5,6]) that any pair of closed analytic subsets $X, Y \subset \mathbf{C}^{m}(m \geq 2)$ satisfies socalled Łojasiewicz regular separation property at any point of $X \cap Y$. Precisely, for any $x^{0} \in X \cap Y$ there are constants $c, v>0$ such that for some neighbourhood $U \subset \mathbf{C}^{m}$ of $x^{0}$ we have

$$
\begin{equation*}
\rho(x, X)+\rho(x, Y) \geq c \rho(x, X \cap Y)^{v} \quad \text { for } x \in U, \tag{1}
\end{equation*}
$$

where $\rho$ is the distance induced by the standard Hermitian norm on $\mathbf{C}^{m}$. Note that if $x^{0} \notin$ $\operatorname{int}(X \cap Y)$, where the interior is computed in $\mathbf{C}^{m}$, then necessarily $v \geq 1$ (see [2]). Also, observe that $X$ and $Y$ satisfy (1) with a constant $v \geq 1$ if and only if there exist a neighbourhood $U^{\prime}$ of $x^{0}$ and a constant $c^{\prime}>0$ such that

$$
\begin{equation*}
\rho(x, Y) \geq c^{\prime} \rho(x, X \cap Y)^{v} \quad \text { for } x \in U^{\prime} \cap X \tag{2}
\end{equation*}
$$

(see $[1,2,5]$ ). Any exponent $v$ satisfying the relation (1) for some $U$ and $c>0$ is called a regular separation exponent of $X$ and $Y$ at $x^{0}$. The infimum of such exponents is called the Łojasiewicz exponent of $X$ and $Y$ at $x^{0}$ and is denoted by $\mathscr{L}\left(X, Y ; x^{0}\right)$; it is important to observe that the latter is a regular separation exponent itself (see [14]). The number $\mathscr{L}\left(X, Y ; x^{0}\right)$ is an interesting metric invariant of the pointed pair ( $X, Y ; x^{0}$ ) which have been the subject of vast studies in analytic geometry (see, for instance, the references in [14]).

[^0]The goal of this note is to investigate the behaviour of the Łojasiewicz exponent under hyperplane sections. Precisely we show the following theorem.

Theorem 1. Let $X$ and $Y$ be closed analytic subsets in $\mathbf{C}^{m}$, and let $x^{0} \in X \cap Y$ such that $\mathscr{L}\left(X, Y ; x^{0}\right) \geq 1$. Then for a general hyperplane $H_{0}$ of $\mathbf{C}^{m}$ passing through $x^{0}$ we have

$$
\mathscr{L}\left(X \cap H_{0}, Y \cap H_{0} ; x^{0}\right) \leq \mathscr{L}\left(X, Y ; x^{0}\right) .
$$

This theorem is a consequence of the following result, which is the main part of the present work.

Theorem 2. Let $X$ be a closed analytic subset in $\mathbf{C}^{m}$, and let $x^{0} \in X$. Then for a general hyperplane $H_{0}$ of $\mathbf{C}^{m}$ passing through $x^{0}$, there exist a constant $c>0$ and a neighbourhood $U$ of $x^{0}$ such that for all $x \in U \cap H_{0}$ we have

$$
\rho\left(x, X \cap H_{0}\right) \leq c \rho(x, X) .
$$

Theorems 1 and 2 are proved in Sections 2 and 3 respectively. To conclude this paper, in Section 4, we also briefly discuss the relation between the Łojasiewicz exponent and the order of tangency for pairs of closed analytic submanifolds of $\mathbf{C}^{m}$ with the same dimension.

## 2. Proof of Theorem 1

Without loss of generality, we may assume that $x^{0}$ is the origin $0 \in \mathbf{C}^{m}$. If $v$ is a regular separation exponent for $X$ and $Y$ at 0 , then $v \geq \mathscr{L}(X, Y ; 0) \geq 1$, and by (2), for some $c^{\prime}>0$ we have

$$
\begin{equation*}
\rho(x, Y) \geq c^{\prime} \rho(x, X \cap Y)^{v} \tag{3}
\end{equation*}
$$

for all $x \in X$ near 0 . By Theorem 2, applied to $X \cap Y$, for a general hyperplane $H_{0}$ of $\mathbf{C}^{m}$ there is a constant $c>0$ such that for all $x \in H_{0}$ near 0 we have

$$
c \rho(x, X \cap Y)^{v} \geq \rho\left(x, X \cap Y \cap H_{0}\right)^{v} .
$$

Combined with (3), this gives

$$
\rho\left(x, Y \cap H_{0}\right) \geq \rho(x, Y) \geq c^{\prime} \rho(x, X \cap Y)^{v} \geq\left(c^{\prime} / c\right) \rho\left(x, X \cap Y \cap H_{0}\right)^{v}
$$

for all $x \in X \cap H_{0}$ near 0 , so that $v$ is a regular separation exponent for $X \cap H_{0}$ and $Y \cap H_{0}$ at 0 . Applying this with $v=\mathscr{L}\left(X, Y ; x^{0}\right)$ shows that

$$
\mathscr{L}\left(X \cap H_{0}, Y \cap H_{0} ; x^{0}\right) \leq \mathscr{L}\left(X, Y ; x^{0}\right) .
$$

## 3. Proof of Theorem 2

It strongly relies on the Lipschitz equisingularity theory of complex analytic sets developed in [7] by the second named author. Throughout, we always work with Hermitian orthonormal bases $\left\{e_{1}, \ldots, e_{m}\right\}$ in $\mathbf{C}^{m}$, and the corresponding coordinates $x=\left(x_{1}, \ldots, x_{m}\right)$. As in Section 2, we assume $x^{0}=0$ and we work in a small neighbourhood of it.

Let $\check{\mathbf{P}}^{m-1}$ denote the set of all hyperplanes of $\mathbf{C}^{m}$ through 0 , with its usual structure of manifold. The distance between two elements $H, K \in \check{\mathbf{P}}^{m-1}$ is the angle $\Varangle(H, K)$ between them, that is,

$$
\Varangle(H, K):=\arccos \frac{|\langle\nu, w\rangle|}{|\nu||w|} \in[0, \pi / 2]
$$

where $v$ and $w$ are normal vectors to the hyperplanes $H$ and $K$, respectively, and $\langle\cdot, \cdot\rangle$ is the standard Hermitian product on $\mathbf{C}^{m}$ (see, e.g., [13]).

Step 1. Let

$$
\mathscr{X}:=\left\{(H, x) \in \check{\mathbf{P}}^{m-1} \times \mathbf{C}^{m} \mid x \in H \cap X\right\} .
$$

By [7, Proposition 1.1], in a neighbourhood

$$
\mathscr{U}:=\left\{(H, x) \in \check{\mathbf{P}}^{m-1} \times \mathbf{C}^{m} \mid \Varangle\left(H_{0}, H\right)<a \text { and }|x|<b\right\}
$$

of a generic $\left(H_{0}, 0\right)$, we have that $\mathscr{X}$ is Lipschitz equisingular over $\check{\mathbf{P}}^{m-1} \times\{0\}$. That is, for any $(H, 0) \in \mathscr{U} \cap\left(\check{\mathbf{P}}^{m-1} \times\{0\}\right)$, there is a (germ of) Lipschitz homeomorphism

$$
\varphi:\left(\check{\mathbf{P}}^{m-1} \times \mathbf{C}^{m},(H, 0)\right) \rightarrow\left(\check{\mathbf{P}}^{m-1} \times \mathbf{C}^{m},(H, 0)\right)
$$

(with a Lipschitz inverse) such that $p \circ \varphi=p$ and $\varphi(\mathscr{X})=\check{\mathbf{P}}^{m-1} \times(H \cap X)$ (as germs at $(H, 0)$ ). (Here, $p: \check{\mathbf{P}}^{m-1} \times \mathbf{C}^{m} \rightarrow \check{\mathbf{P}}^{m-1}$ is the standard projection.) Actually, if $h=\left(h_{1}, \ldots, h_{m-1}\right)$ are coordinates in $\check{\mathbf{P}}^{m-1}$ around $H_{0}$ such that

$$
h_{1}\left(H_{0}\right)=\cdots=h_{m-1}\left(H_{0}\right)=0
$$

then, locally near $\left(H_{0}, 0\right)$, the standard "constant" vector fields $\partial_{h_{j}}(1 \leq j \leq m-1)$ on $\check{\mathbf{P}}^{m-1} \times\{0\}$ can be lifted to Lipschitz vector fields $v_{j}$ on $\check{\mathbf{P}}^{m-1} \times \mathbf{C}^{m}$ such that the flows of $v_{j}$ preserve $\mathscr{X}$ (see the proof of [7, Proposition 1.1, p. 10]). So, in particular, $v_{j}$ is a Lipschitz vector field of the form

$$
v_{j}(h, x)=\partial_{h_{j}}(h, x)+\sum_{\ell=1}^{m} w_{j \ell}(h, x) \partial_{x_{\ell}}(h, x)
$$

so that $v_{j}(h, 0)=\partial_{h_{j}}(h, 0)$ and there exists a constant $c^{\prime}>0$ such that

$$
\begin{equation*}
\left|w_{j \ell}(h, x)\right| \leq c^{\prime}|x| \text { near } 0 \tag{4}
\end{equation*}
$$

for all $j, \ell$.
Step 2. Pick a point $y^{0} \in H_{0}$. We want to prove that if $y^{0}$ is sufficiently close to 0 , then

$$
\begin{equation*}
\rho\left(y^{0}, X \cap H_{0}\right) \leq c \rho\left(y^{0}, X\right) \tag{5}
\end{equation*}
$$

for some constant $c>0$ independent of $y^{0}$. Let $y^{1} \in X$ be one of the closest points to $y^{0}$, that is, $\rho\left(y^{0}, X\right)=\left|y^{1}-y^{0}\right|$. If $y^{0} \in X$, then $\rho\left(y^{0}, X \cap H_{0}\right)=\rho\left(y^{0}, X\right)=0$, and the inequality (5) is obviously true. So, hereafter, we assume that $y^{0} \notin X$. Of course, without loss of generality, we may also assume that $\left|y^{0}\right|<b$ and $\left|y^{1}\right|<b$. Choose $H_{1} \in \check{\mathbf{P}}^{m-1}$ such that $y^{1} \in H_{1}$ and $\Varangle\left(H_{0}, H_{1}\right)$ is minimal. If $\Varangle\left(H_{0}, H_{1}\right)=0$ (i.e., if $\left.y^{1} \in H_{0}\right)$, then again $\rho\left(y^{0}, X \cap H_{0}\right)=\rho\left(y^{0}, X\right)$ and (5) is true. From now on, let us assume that $\Varangle\left(H_{0}, H_{1}\right) \neq 0$. Then we have the following lemma.
Lemma 3. If $\left(H_{1}, y^{1}\right) \notin \mathscr{U}$ (i.e., if $\Varangle\left(H_{0}, H_{1}\right) \geq a$ ), then there exists $a^{\prime}>0$ depending only on a such that

$$
\left|y^{1}-y^{0}\right| \geq a^{\prime}\left|y^{0}\right|
$$

In particular, since $0 \in X \cap H_{0}$, if $\left(H_{1}, y^{1}\right) \notin \mathscr{U}$ then we have

$$
\begin{equation*}
\rho\left(y^{0}, X \cap H_{0}\right) \leq\left|y^{0}\right| \leq\left(1 / a^{\prime}\right) \rho\left(y^{0}, X\right) \tag{6}
\end{equation*}
$$

as desired.
Proof of Lemma 3. By a proper choice of the basis $\left\{e_{1}, \ldots, e_{m}\right\}$, we may assume that $H_{0}$ is defined by the equation $x_{m}=0$, so that $e_{m}$ is orthogonal to $H_{0}$. Now, if $x_{m}=\sum_{\ell=1}^{m-1} q_{\ell} x_{\ell}$ is an equation for $H_{1}$, then, clearly, for each $1 \leq \ell \leq m-1$, the vector $E_{\ell}:=e_{\ell}+q_{\ell} e_{m}$ is in $H_{1}$. Thus, if $N=\sum_{\ell=1}^{m-1} u_{\ell} e_{\ell}+u_{m} e_{m}$ is a normal vector to $H_{1}$, then we must have $\left\langle N, E_{\ell}\right\rangle=0$, and hence, $u_{\ell}=-u_{m} \bar{q}_{\ell}$, so that we can take $N:=-\sum_{\ell=1}^{m-1} \bar{q}_{\ell} e_{\ell}+e_{m}$.

Now, saying that $\Varangle\left(H_{0}, H_{1}\right)$ is minimal means that

$$
\cos \Varangle\left(H_{0}, H_{1}\right)=\frac{\left|\left\langle N, e_{m}\right\rangle\right|}{|N|\left|e_{m}\right|}=\frac{1}{\sqrt{1+\sum_{\ell=1}^{m-1}\left|q_{\ell}\right|^{2}}}
$$

is maximal, that is, $\sum_{\ell=1}^{m-1}\left|q_{\ell}\right|^{2}$ is minimal. By adjusting the choice of the basis, we may further assume that $y^{1}=\left(y_{1}^{1}, 0, \ldots, 0, y_{m}^{1}\right)$, so that its orthogonal projection onto $H_{0}$ is $y^{2}:=\left(y_{1}^{1}, 0, \ldots, 0\right)$. As $y^{1} \in H_{1}$, we have $q_{1}=y_{m}^{1} / y_{1}^{1} \neq 0$. Thus, $\sum_{\ell=1}^{m-1}\left|q_{\ell}\right|^{2}$ is minimal if and only if $q_{2}=\cdots=q_{m-1}=0$. So, if $\Varangle\left(H_{0}, H_{1}\right)$ is minimal, then $H_{1}$ is given by the equation $x_{m}=q_{1} x_{1}$.

It follows that if $\Varangle\left(H_{0}, H_{1}\right) \geq a$ (assumption of the lemma), then we must have

$$
\cos \Varangle\left(H_{0}, H_{1}\right)=1 / \sqrt{1+\left|q_{1}\right|^{2}} \leq a_{1},
$$

and hence $\left|q_{1}\right| \geq a_{2}$, for some constants $a_{1}, a_{2}>0$ depending only on $a$. Now, clearly, we may always assume $\left|y^{0}-y^{1}\right|<(1 / 10)\left|y^{0}\right|$. Thus, $\left|y^{2}-y^{0}\right| \leq\left|y^{1}-y^{0}\right|<(1 / 10)\left|y^{0}\right|$, and hence,

$$
\left|y^{2}-0\right|=\left|y_{1}^{1}\right|>(9 / 10)\left|y^{0}\right|
$$

(see Figure 1). It follows that

$$
\left|y^{0}-y^{1}\right| \geq\left|y^{1}-y^{2}\right|=\left|q_{1}\right|\left|y_{1}^{1}\right| \geq a_{2}(9 / 10)\left|y^{0}\right|,
$$

and this completes the proof of Lemma 3.


Figure 1. Hyperplanes $H_{0}$ and $H_{1}$
Step 3. Lemma 3 solves the case where $\left(H_{1}, y^{1}\right) \notin \mathscr{U}$ (see (6)). Now let us look at the case where $\left(H_{1}, y^{1}\right) \in \mathscr{U}$; here comes Lipschitz equisingularity (see Step 1 ). Let $h^{1}=\left(h_{1}^{1}, \ldots, h_{m-1}^{1}\right)$ be the coordinates of $H_{1}$. (Note that $\left|h^{1}\right| \leq d \cdot \Varangle\left(H_{0}, H_{1}\right)$ for some constant $d>0$ independent of $H_{1}$.) Consider the Lipschitz vector field $v$ on $\check{\mathbf{P}}^{m-1} \times \mathbf{C}^{m}$ defined by

$$
\begin{aligned}
v(h, x) & :=-\sum_{j=1}^{m-1} h_{j}^{1} v_{j}(h, x) \\
& =-\sum_{j=1}^{m-1} h_{j}^{1} \partial_{h_{j}}(h, x)+\sum_{\ell=1}^{m}\left(-\sum_{j=1}^{m-1} h_{j}^{1} w_{j, \ell}(h, x)\right) \partial_{x_{\ell}}(h, x),
\end{aligned}
$$

and look at the integral curve $\gamma(t)=(h(t), x(t))$ of $v$ starting at $\left(H_{1}, y^{1}\right)$. So, in particular, we have:

$$
\begin{array}{ll}
\dot{h}_{j}(t)=-h_{j}^{1}, & \dot{x}_{\ell}(t)=-\sum_{j=1}^{m-1} h_{j}^{1} w_{j, \ell}(h, x), \\
h_{j}(0)=h_{j}^{1}, & x_{\ell}(0)=y_{\ell}^{1} .
\end{array}
$$

As the flows of the vector fields $v_{j}$ preserve $\mathscr{X}$ and since $\gamma(0) \in \mathscr{X}$, the curve $\gamma(t)$ lies in $\mathscr{X}$. Moreover, since $h_{j}(t)=h_{j}^{1}(1-t)$, we have $h_{j}(1)=0$ for all $j$, and hence $x(1)$ lies in $H_{0}$. Finally, observe that the length $L_{I}(x)$ of the restriction of the curve $x(t)$ to the compact interval $I=[0,1]$ satisfies

$$
\begin{aligned}
& L_{I}(x):=\int_{0}^{1}|\dot{x}(t)| d t \leq c_{1} \int_{0}^{1} \sum_{j=1}^{m-1}\left(\left|h_{j}^{1}\right| \cdot\left(\sum_{\ell=1}^{m}\left|w_{j, \ell}(\gamma(t))\right|\right)\right) d t \\
& \stackrel{\text { bv(4) }}{\leq} c_{2}\left|h^{1}\right| \int_{0}^{1}|x(t)| d t \leq c_{3}\left|h^{1}\right||x(0)| \leq c_{4}\left|y^{0}-x(0)\right|
\end{aligned}
$$

for some constants $c_{i}>0$ independent of $y^{0}, H_{1}$ and $y^{1}$. The first and third inequalities are clear. The second one follows from the crucial relation (4) (i.e., from Lipschitz equisingularity). To show the last inequality, we may proceed as in the proof of Lemma 3, exchanging the roles of $H_{0}$ and $H_{1}$. Namely, for a new proper choice of the basis, we may assume that $H_{1}$ is defined by $x_{m}=0$ and that $y^{0}=\left(y_{1}^{0}, 0, \ldots, 0, y_{m}^{0}\right)$, so that the orthogonal projection of $y^{0}$ onto $H_{1}$ is $y^{3}:=\left(y_{1}^{0}, 0, \ldots, 0\right)$. As the angle $\Varangle\left(H_{0}, H_{1}\right)$ is minimal, we may suppose that $H_{0}$ is given by an equation of the form $x_{m}=q_{1} x_{1}$. Clearly, we may also assume that $\left|y^{0}-y^{1}\right|<(1 / 10)\left|y^{1}\right|$. Thus $\left|y^{3}-y^{1}\right| \leq\left|y^{0}-y^{1}\right|<(1 / 10)\left|y^{1}\right|$, and hence, $\left|y^{3}-0\right|=\left|y_{1}^{0}\right|>(9 / 10)\left|y^{1}\right|$. It follows that

$$
\left|y^{0}-y^{1}\right| \geq\left|y^{0}-y^{3}\right|=\left|q_{1}\right|\left|y_{1}^{0}\right|>\left|q_{1}\right|(9 / 10)\left|y^{1}\right|
$$

But we have

$$
\left|h^{1}\right| \leq d \cdot \Varangle\left(H_{0}, H_{1}\right)=d \cdot \arccos \left(1 / \sqrt{1+\left|q_{1}\right|^{2}}\right) \leq d^{\prime}\left|q_{1}\right|,
$$

where the constants $d, d^{\prime}>0$ are independent of $y^{0}, H_{1}$ and $y^{1}$. It follows that

$$
\left|y^{0}-y^{1}\right| \geq \frac{9}{10 d^{\prime}}\left|h^{1}\right|\left|y^{1}\right|
$$

as desired (remind that $\left.y^{1}=x(0)\right)$. Now, by the estimate of the length $L_{I}(x)$ given above, we have

$$
\begin{aligned}
\rho\left(y^{0}, X \cap H_{0}\right) & \leq\left|y^{0}-x(1)\right| \leq\left|y^{0}-x(0)\right|+|x(0)-x(1)| \leq\left|y^{0}-x(0)\right|+L_{I}(x) \\
& \leq\left(1+c_{4}\right)\left|y^{0}-x(0)\right|=\left(1+c_{4}\right) \rho\left(y^{0}, X\right)
\end{aligned}
$$

and this completes the proof of Theorem 2.
Remark 4. Note that the proof of Theorem 2 (and hence of Theorem 1) given above only depends on the Lipschitz equisingularity theory of complex analytic sets developed in [7] by the second named author. Real versions of this theory for the semi-analytic and subanalytic categories were addressed by A. Parusiński in [9-12] while the case of sets definable in a polynomially bounded o-minimal structure was obtained by Nguyen Nhan and G. Valette in [8]. Theorems 1 and 2 must then be true in these categories as well.

## 4. Remark on the Lojasiewicz exponent and the order of tangency

To conclude this paper, we give a lower bound for the Łojasiewicz exponent $\mathscr{L}\left(X, Y ; x^{0}\right)$ of two $p$-dimensional closed analytic submanifolds $X$ and $Y$ of $\mathbf{C}^{m}$ at $x^{0} \in X \cap Y$ in terms of the order of tangency of $X$ and $Y$ at $x^{0}$.

Following [3, 4], we say that the order of tangency between $X$ and $Y$ at $x^{0}$ is greater than or equal to an integer $k$ if there exist parametrizations (i.e., biholomorphisms onto their images)

$$
q:\left(U, u^{0}\right) \rightarrow\left(X, x^{0}\right) \quad \text { and } \quad q^{\prime}:\left(U, u^{0}\right) \rightarrow\left(Y, x^{0}\right)
$$

where $U \ni u^{0}$ is an open subset of $\mathbf{C}^{p}$, such that

$$
\begin{equation*}
q(u)-q^{\prime}(u)=o\left(\left|u-u^{0}\right|^{k}\right) \tag{7}
\end{equation*}
$$

when $U \ni u \rightarrow u^{0}$. The order of tangency between $X$ and $Y$ at $x^{0}$ (denoted by $s\left(X, Y ; x^{0}\right)$ ) is the supremum of all such integers $k$.

Observation 5. Let $X$ and $Y$ be p-dimensional closed analytic submanifolds of $\mathbf{C}^{m}$, and let $x^{0} \in X \cap Y$. Suppose that $s\left(X, Y ; x^{0}\right)$ is finite. If $\mathscr{L}\left(X, Y ; x^{0}\right) \geq 1$, then

$$
s\left(X, Y ; x^{0}\right) \leq \mathscr{L}\left(X, Y ; x^{0}\right)-1
$$

Proof. Put $s:=s\left(X, Y ; x^{0}\right), \mathscr{L}:=\mathscr{L}\left(X, Y ; x^{0}\right)$, and for this proof write $\mathbf{C}^{m}=\mathbf{C}_{x}^{p} \times \mathbf{C}_{y}^{m-p}$ where $x=\left(x_{1}, \ldots, x_{p}\right)$ and $y=\left(x_{p+1}, \ldots, x_{m}\right)$. As above, we assume that $x^{0}$ is the origin $0 \in \mathbf{C}^{m}$. In a neighbourhood of 0 , the analytic submanifold $X$ is given by $y=f(x)$ for some analytic function

$$
f=\left(f_{1}, \ldots, f_{m-p}\right):\left(\mathbf{C}_{x}^{p}, 0\right) \rightarrow\left(\mathbf{C}_{y}^{m-p}, 0\right)
$$

Similarly, $Y$ is also the graph of an analytic function $g$, and without loss of generality, we may assume that $g=0$. Now, let $s^{\prime}$ be the smallest integer $k$ for which there exists a multi-index $\alpha=\left(\alpha_{1}, \ldots, \alpha_{p}\right)$ such that $|\alpha|=\alpha_{1}+\cdots+\alpha_{p}=k$ and $D^{\alpha}(f-g)(0) \neq 0$. Clearly, $s=s^{\prime}-1$. Each component $f_{i}$ has the Taylor expansion

$$
f_{i}(x)=F_{i}(x)+o\left(|x|^{r_{i}}\right)
$$

where $F_{i}$ is a homogeneous polynomial of degree $r_{i}$. Of course, we may assume $r_{1} \leq r_{i}$ for all $i$, so that $r_{1}=s^{\prime}$. Consider the standard projection

$$
\pi: \mathbf{C}_{x}^{p} \times \mathbf{C}_{y}^{m-p} \rightarrow \mathbf{C}_{x}^{p}
$$

and look at the hypersurface $\pi(X \cap Y)=\left\{x \in \mathbf{C}_{x}^{p} ; f(x)=0\right\}$ of $\mathbf{C}_{x}^{p}$. It is easy to see that if $L$ is a line through 0 which is not in the tangent cone of $\pi(X \cap Y)$ at 0 , then

$$
\rho(x, \pi(X \cap Y)) \sim|x|
$$

for $x \in L$ near $0 .{ }^{1}$ Now, if $F_{1} \neq 0$ on $L$, then for any $x \in L$ near 0 , we also have

$$
\left|f_{1}(x)\right| \sim|x|^{r_{1}}=|x|^{s^{\prime}} \quad \text { and } \quad\left|f_{i}(x)\right| \leq a|x|^{r_{i}} \leq a|x|^{s^{\prime}}
$$

for some constant $a>0$. It follows that for any $(x, y) \in \pi^{-1}(L) \cap X=\{(x, y) ; x \in L$ and $y=f(x)\}$ near 0 , we have

$$
\rho((x, y), Y)=|f(x)| \sim|x|^{s^{\prime}} \quad \text { and } \quad \rho((x, y), X \cap Y) \sim|x|
$$

Now, the Łojasiewicz exponent $\mathscr{L}$ satisfies $\rho((x, y), Y) \geq c \rho((x, y), X \cap Y)^{\mathscr{L}}$, that is, $|x|^{s^{\prime}} \geq c|x|^{\mathscr{L}}$ for some constant $c>0$. Thus $s^{\prime} \leq \mathscr{L}$, and hence, $s=s^{\prime}-1 \leq \mathscr{L}-1$.
Remark 6. We may also investigate the relationship between $s:=s\left(X, Y ; x^{0}\right)$ and $\mathscr{L}:=$ $\mathscr{L}\left(X, Y ; x^{0}\right)$ using Theorem 1 but this second approach only gives the inequality $s<\mathscr{L}$. However, for completeness, let us briefly explain the argument. First, we consider the special case where $x^{0}$ is an isolated point of $X \cap Y$. In this case, there exists a constant $c^{\prime}>0$ such that

$$
\rho(x, Y) \geq c^{\prime} \rho(x, X \cap Y)^{\mathscr{L}}=c^{\prime}\left|x-x^{0}\right|^{\mathscr{L}} \quad \text { for } x \in X \text { near } x^{0}
$$

or equivalently, $\rho(q(u), Y) \geq c^{\prime}\left|q(u)-q\left(u^{0}\right)\right|^{\mathscr{L}}$ for $u$ near $u^{0}$. Since $q$ is locally bi-Lipschitz, there exists a constant $c^{\prime \prime}>0$ such that

$$
c^{\prime}\left|q(u)-q\left(u^{0}\right)\right|^{\mathscr{L}} \geq c^{\prime \prime}\left|u-u^{0}\right|^{\mathscr{L}} \quad \text { for } u \text { near } u^{0}
$$

Now, by (7), we have

$$
\rho(q(u), Y) \leq\left|q(u)-q^{\prime}(u)\right|<c^{\prime \prime}\left|u-u^{0}\right|^{s} \quad \text { for } u \text { near } u^{0} .
$$

Combining these relations gives

$$
c^{\prime \prime}\left|u-u^{0}\right|^{\mathscr{L}} \leq \rho(q(u), Y)<c^{\prime \prime}\left|u-u^{0}\right|^{s} \quad \text { for } u \text { near } u^{0}
$$

and hence $s<\mathscr{L}$.
The general case (i.e., $\operatorname{dim} X \cap Y=n>0$ ) follows from the 0 -dimensional case and Theorem 1. Indeed, take $n$ general hyperplanes $H_{1}, \ldots, H_{n}$ in $\mathbf{C}^{m}$ passing through $x^{0}$, so that $X \cap Y \cap H_{1} \cap \cdots \cap$ $H_{n}$ is an isolated intersection. Let $s_{i}$ (respectively, $\mathscr{L}_{i}$ ) denote the order of tangency (respectively, the Łojasiewicz exponent) of $X \cap H_{1} \cap \cdots \cap H_{i}$ and $Y \cap H_{1} \cap \cdots \cap H_{i}$ at $x^{0}$. Clearly, (7) implies

[^1]$s_{i} \leq s_{i+1}$ while Theorem 1 shows $\mathscr{L}_{i} \geq \mathscr{L}_{i+1}$. (Note that since $\operatorname{int}\left(X \cap Y \cap H_{1} \cap \cdots \cap H_{i}\right)=\varnothing$, we have $\mathscr{L}_{i} \geq 1$, so that Theorem 1 applies.) Now the relation $s<\mathscr{L}$ follows from the inequality $s_{n}<\mathscr{L}_{n}$ (0-dimensional case).

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[^1]:    ${ }^{1}$ As usual, the expression $\varphi(x) \sim \psi(x)$ for $x \in E$ near 0 means that there exist constants $c, c^{\prime}>0$ such that $c \psi(x)$ $\leq \varphi(x) \leq c^{\prime} \psi(x)$ for all $x \in E$ near 0 .

