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Unstable spectrum of a Rayleigh–Bénard system with variable viscosity

Olivier Lafitte[*, a, b]

[a] IRL CRM, UMI3457, Centre de recherches Mathématiques, Université de Montréal, Montréal, Canada.

[b] LAGA, UMR7539, Université Sorbonne Paris Nord, 93430 Villetaneuse, France.

E-mail: lafitte@math.univ-paris13.fr (O. Lafitte)

Abstract. This Note studies a Rayleigh–Bénard system in an infinite layer, in the case of temperature-dependent viscosity, with rigid boundary conditions for the velocity at the bottom and free-slip at a top of the layer. It states the linearized problem in the relevant functional operator set-up and identifies, for each nonzero transverse frequency k and Rayleigh number R the (finite) number of modes which are unstable in time. This number is equal to the number of eigenvalues of a particular operator which are smaller than R.

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Version française abrégée

Nous nous intéressons dans cette Note au problème des instabilités de Rayleigh–Bénard, décrites il y a plus d'un siècle par Bénard [1] du côté expérimental et par Rayleigh [13]. Dans un article précédent [12] nous avons étudié la question de l'échange de stabilité (c'est à dire la détermination de la valeur du nombre de Rayleigh R pour laquelle il apparait des instabilités pour le système linéarisé) dans un cas présentant les deux caractéristiques suivantes : la viscosité est une fonction de la température dans la couche où il y a convection, et la vitesse de l'écoulement correspond à une surface inférieure rigide ($u = 0$) et une surface supérieure libre ($u.n = 0$). Ayant linéarisé autour d'un profil de température T_0, solution stationnaire laminaire de l'équation de la chaleur, et ayant noté $\nu(z) := \nu(T_0(z))$ nous avions obtenu un système d'équations différentielles ordinaires et ses conditions aux limites (pour des solutions très régulières). Nous reformulons ce problème dans les espaces fonctionnels adéquats afin de pouvoir obtenir des résultats mathématiques complétant les résultats numériques de [12] (les résultats mathématiques de [12, l'Appendice] n'étant pas écrits pour le problème étudié dans l'article).

Nous supposons

$$(H) \quad \nu \text{ au moins de classe } C^2, \nu(z) \geq \nu_0 > 0, \nu''(z) \geq 0.$$

[*] Corresponding author.

Le système considéré utilise l'approximation dite de Boussinesq, qui suppose que la variation de densité est négligeable, sauf dans les termes où elle est multipliée par g. On note $Pr > 0$ le nombre de Prandtl associé à cet écoulement.

Nous considérons ainsi, après écriture en modes normaux de la solution du système linéarisé autour de l'écoulement $(\vec{v}, T, p) = (0, T_0 - \beta(z + \frac{d}{2}), p_0 - \rho_1 gz[1 + \frac{\alpha\beta d}{2} + \frac{\alpha\beta z}{2}])$ comme suit :

$$\left(\vec{u}, \theta, \delta p\right) = \left(U_x(z), U_y(z), W(z), \Theta(z), \delta P(z)\right) e^{\sigma t + i(k_x x + k_y y)}. \tag{1}$$

Dans [12], après introduction de $k = \sqrt{k_x^2 + k_y^2}$, nous avons obtenu un système d'équations différentielles ordinaires et des conditions aux limites. Nous écrivons dans la présente Note sous la forme d'un système d'équations ordinaires au sens faible :

$$\begin{cases} \frac{\sigma}{Pr} Z & = -\mathscr{L}^b Z \\ \frac{\sigma}{Pr} LW & = Rk^2\Theta - \mathscr{Q}W - k^2 v''(z) W \\ \sigma\Theta + L\Theta & = W. \end{cases} \tag{2}$$

Comme on le verra, les espaces fonctionnels tiendront compte d'une partie des conditions aux limites, les autres étant déduites de la formulation faible.

On ne considèrera dans cette Note que le cas

$$k > 0, R > 0. \tag{3}$$

Dans ce système, les opérateurs $L, \mathscr{L}, \mathscr{L}^b$ sont les opérateurs auto-adjoints coercifs sur $H_0^1([-\frac{1}{2}, \frac{1}{2}])$ (et sur $H^1([-\frac{1}{2}, \frac{1}{2}])$ à trace nulle en $-\frac{1}{2}$ pour $\mathscr{L}^b$) donnés par

$$\begin{cases} \forall\, \phi, \psi \in H_0^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right), \langle L\phi, \psi\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\phi'\psi' + k^2\phi\psi\right) dz, \\ \forall\, \phi, \psi \in H_0^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right), \langle \mathscr{L}\phi, \psi\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(z) \left[\phi'\psi' + k^2\phi\psi\right] dz, \\ \forall\, \phi, \psi \in H^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right), \phi\left(-\frac{1}{2}\right) = \psi\left(-\frac{1}{2}\right) = 0, \langle \mathscr{L}^b\phi, \psi\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(z) \left[\phi'\psi' + k^2\phi\psi\right] dz. \end{cases} \tag{4}$$

Notant $\widetilde{H} = \{W \in H^2([-\frac{1}{2}, \frac{1}{2}]), W(\pm\frac{1}{2}) = 0, W'(-\frac{1}{2}) = 0\}$, sous-espace de Hilbert de $H^2([-\frac{1}{2}, \frac{1}{2}])$ (muni de la norme induite), qui prend en compte le fait que le fond $z = -\frac{1}{2}$ est rigide et la surface est libre, on définit l'opérateur $\mathscr{Q}$ de $\widetilde{H}$ dans $\widetilde{H}'$:

$$\forall\, \phi, \psi \in \widetilde{H} \left\langle \mathscr{Q}\phi, \psi \right\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(z) \left[\phi''\psi'' + 2k^2\phi'\psi' + k^4\phi\psi\right] dz. \tag{5}$$

Les conditions sur v et k impliquent qu'il existe $\alpha > 0$ tel que, pour tout $\phi \in \widetilde{H}$, $\langle \mathscr{Q}\phi, \phi\rangle \geq \alpha ||\phi||_{H^2}^2$. Les opérateurs $\mathscr{Q}$ et $\mathscr{Q} + k^2 v''(z)$. sont coercifs sur $\widetilde{H}$. On introduit aussi l'espace

$$\mathscr{H} := \widetilde{H} \times H_0^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right). \tag{6}$$

Le système (2) est alors posé pour $Z \in H^1([-\frac{1}{2}, \frac{1}{2}]), Z(-\frac{1}{2}) = 0$, $(W, \Theta) \in \mathscr{H}$, $\mathscr{H}$ donné par (6).

Notre Note concerne en particulier la recherche des valeurs propres instables (ou spectre instable) de ce système de Rayleigh–Bénard, comme l'indique la

Definition 1. *On appelle valeur propre instable du problème ou valeur du spectre une valeur de* $\sigma \geq 0$ *telle qu'il existe une solution non triviale* (W, Θ) *dans* $\mathscr{H}$ *du problème* (2). *Pour une telle valeur propre,* $Z = 0$ *et* (2) *se résume au système sur* (W, Θ), *qui sera, par abus de notations, assimilé au système* (2).

Nous obtenons le résultat suivant

Theorem 2. *On suppose k, ν vérifiant les conditions (H) et* (3) *données ci-dessus.*

- *L'opérateur $L^{-\frac{1}{2}}[\mathcal{Q} + k^2\nu'']^{-1}L^{-\frac{1}{2}}$ est autoadjoint compact sur $L^2([-\frac{1}{2}, \frac{1}{2}])$. On appelle $\frac{1}{r_n(0,k)}$ la suite strictement décroissante de ses valeurs propres.*
- *Lorsque $Rk^2 < r_1(0,k)$, le système de Rayleigh–Bénard pour la fréquence transverse k est linéairement stable, et si $Rk^2 \geq r_1(0,k)$, le système* (2) *muni des conditions aux limites décrites ci-dessus admet une solution non triviale $(\sigma_1, Z_1, W_1, \Theta_1)$ avec $\sigma_1 > 0$ et $Z_1 = 0$. La valeur $\frac{r_1(0,k)}{k^2}$ est la valeur dite d'échange de stabilité, et son minimum en k est le nombre de Rayleigh critique R_c du problème.*
- *Lorsque $Rk^2 \geq r_1(0,k)$, il existe un nombre fini N de valeurs de σ vérifiant la Définition 1. Ce nombre N est donné par $r_N(0,k) \leq Rk^2 < r_{N+1}(0,k)$.*

Ce Théorème est une conséquence des Lemmes 5 et 7 démontrés dans la version anglaise.

1. Introduction

This paper is the follow-up of the paper of F. Pla, H. Herrero and O.L [12] where one studied the thermal convection problem with temperature dependent viscosity problem, so called the Rayleigh–Bénard convection problem (after [1] and [13]). This convection problem appears for example in the mantle of the Earth. One could also think that it could also appear in a pond where the bottom is the permafrost and the top is lighted by the sun in summer if the temperature is not monotonous in the fluid (hypothese (H) is not fulfilled in this case). Detailed results on this instability as well as others can be found in [2, 3]. The model is the Navier–Stokes system of equations under a Boussinesq approximation with a viscosity depending on the temperature between two infinite plates (which, by convention of choice of units, are labeled as $z = \pm\frac{1}{2}$). In [12], we reduced the study of the linearized problem in to a system of ODEs through the normal modes method, and solved numerically this system of ODEs with suitable initial conditions and end conditions to obtain an eigenvalue as a root of a suitable determinant for finding the smallest value $R_0(k)$ for which an eigenvalue of the linearized problem is $\sigma = 0$. Very often, on the other side, a variational principle allows the study of the largest positive growth rate of the linearized system (see Guo and Hwang [6], Guo and Tice [7] for example among many works). In [6], the growth rate λ is obtained such that $\frac{gk}{\lambda^2}$ is the maximum value of a certain Rayleigh quotient. In [7], the growth rate λ is obtained through the infimum of a Rayleigh quotient depending on λ.

In hydrodynamics, spectral theory of a celebrated equation (namely the Orr–Sommerfeld equation) gives rise to numerous articles (to just quote three recent ones, mention that Skorokhodov [14] performs a numerical analysis of the spectrum, Tan and Su [15] use a projection on Fourier modes and Grenier, Guo and Nguyen [5] perform a construction of approximate modes and an analysis of the dispersion equation for finding the spectrum). However, even if some ideas are similar, Orr–Sommerfeld equation does not lead to a self-adjoint formulation and does not lead to the analysis of a spectrum for a self-adjoint operator.

In [12, the Appendix], we described, using spectral analysis, a problem close to the one studied in [12] (namely the case where W satisfies $W(\pm\frac{1}{2}) = 0$, that is rigid-rigid conditions $u.n = 0$ at the top and the bottom) studied through an operator on $H^3_{0,0}([-\frac{1}{2}, \frac{1}{2}])$ for which we obtained the complete discrete spectrum. On the other side, the numerical study therein was done for a rigid-free problem and did not use this spectral analysis. Moreover, we did not address in this Appendix the consequences of having the description of the complete discrete spectrum.

Obtaining the spectral result in the rigid-free situation (where the Hilbert space used is less classical) and counting the numero of unstable modes in terms of (R, k) are the aim of the present Note.

In the present paper, we prove that the spectral analysis of all the eigenmodes of the Rayleigh–Bénard problem for a rigid-free boundary condition is a consequence of compactness of a 6^{th} operator (more complicated than the operator studied in [12, the Appendix] because of the Hilbert space $\mathscr{H}^*$ on which it acts). Indeed $\mathscr{H}^*$ depends on σ, because the rigid boundary condition at the bottom expresses, for the vertical component of the velocity W, as $W'(-\frac{1}{2}) = 0$. As in Orr–Sommerfeld equation, the growth rate (or a function of it) does not appear as an eigenvalue of an operator independent on σ, but one aims at finding a non trivial kernel for an operator depending on σ.

This is seemingly new for this problem with a variable viscosity (which prevents to perform a direct spectral analysis using $\sin n\pi(z+\frac{1}{2})$ as a basis of diagonalization of the problem), and gives a criterion to identify the number of positives values of σ (see Théorème 2). Classically, mathematicians and physicists interested by unstabilities aim at finding at least one unstable mode for the linearized system, as it is enough to show that the problem is linearly unstable and do not count them: the present Note gives additional information on growth rates.

Apart from the cases of constant coefficients systems (as it is the case in the classical Rayleigh–Bénard problem with rigid-rigid, rigid-free, or free-free boundaries (as in the works of Drazin and Reid [3], Reid and Harris [2, 8] among other authors) where the eigenmodes can be computed easily using the boundary conditions), not much has been done on the complete set of unstable values. The complete unstable spectrum for the classical Rayleigh–Taylor instability is for example described in a paper with B. Helffer [9] (and references therein). Infinitely many growth rates appear also in the seminal work of Erpenbeck on detonation stability ([4, Section *Analytic treatment*], proved rigorously recently in [10]). One of the reasons of so few results on this sequence of growth rates could be that linear instability relies on *at least one unstable mode* and nonlinear results rely on a bound of any growth rate (by essentially the largest one) hence the sequence of unstable modes is interesting, but not for applications. This spectral analysis is of one of the themes of the ongoing thesis of Tien Tai Nguyen [11].

We rephrase the result of Théorème 2:

Let $R > 0$ be the Reynolds number of the problem, $Pr > 0$ the Prandtl number of the system, and assume that the laminar profile is $(\vec{0}, p_0(z), T_0 + (z+\frac{1}{2})(T_1 - T_0))$ in the infinite layer $-\frac{1}{2} < z < \frac{1}{2}$. Assume the transverse wave number of the perturbation of this profile (for which the viscosity $\upsilon(z) := \nu(T_0 + (z+\frac{1}{2})(T_1 - T_0))$ is $k > 0$, and ν satisfies (H). We are able (as in the case $\nu(T) = \nu$ constant) to *count* the number of positive growth rates σ for a given value of (k, R). The number of positive distinct values of growth rates, for a given value of (k, R), is N such that $r_N(0, k) \leq Rk^2 < r_{N+1}(0, k)$.

This theorem is illustrated in the toy model (described for example in Reid and Harris [8]), that we investigate in Section 3 of this Note in order to illustrate first this result. We recall in the first section (Section 2) how the model was described and obtained in [12], then we write an operator form of the problem of ODEs we adress using a variational formulation (Section 4), and describe the spectrum $\frac{1}{r_n(\sigma,k)}$ of a self-adjoint compact operator from which one shall deduce the growth rates by studying the equations $r_n(\sigma, k) = Rk^2$ (Section 5).

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2. Formulation of the problem

We recall briefly the model (described in [12]) and perform the normal modes analysis. Details are in [12, Section 2, 3].

The governing equations are, on $\mathbb{R}^2 \times [-\frac{d}{2}, \frac{d}{2}]$, under the Boussinesq approximation:

$$\begin{cases} \nabla.\vec{v} = 0 \\ \partial_t \vec{v} + (\vec{v}.\nabla)\,\vec{v} = -\frac{1}{\rho(T)}\left(\nabla p + \rho g \vec{e}_3\right) + \mathrm{div}\left(\nu(T)\left(\nabla\vec{v} + (\nabla\vec{v})^T\right)\right) \\ \partial_t T + \vec{v}.\nabla T = K\Delta T \end{cases} \tag{7}$$

where $\vec{v}$ is the velocity vector field, p is the pressure, T is the temperature, ρ is the density, and $\nu(T)$ the viscosity of the fluid (the Boussinesq approximation has been used here as $\nu(T) = \frac{\mu(T)}{\rho_0}$).

The equation of state is

$$\rho(T) = \rho_1\left(1 - \alpha(T - T_1)\right), \tag{8}$$

and the physical boundary conditions available are

$$\begin{cases} T|_{z=-\frac{d}{2}} = T_0, T|_{z=\frac{d}{2}} = T_1 \\ \vec{v}|_{z=-\frac{d}{2}} = 0 \\ v_z = 0|_{z=\frac{d}{2}} \\ \partial_z v_x|_{z=\frac{d}{2}} = 0, \partial_z v_y|_{z=\frac{d}{2}} = 0. \end{cases} \tag{9}$$

From now on, we consider $d = 1$ through a change of units. After linearization in the neighborhood of the laminar solution, one obtains the system of ODEs on the normal modes (where the operator $\mathscr{L}^*$ here is the differential operator[1] acting on C^2 functions, expressed as $\mathscr{L}^* f(z) = k^2\nu(z)f(z) - \frac{d}{dz}(\nu(z)\frac{df}{dz})$)

$$\begin{cases} i\left(k_x U_x + k_y U_y\right) + W' = 0 \\ \frac{\sigma}{Pr}U_x = -ik_x\delta P - \mathscr{L}^* U_x + ik_x\nu'(z)W \\ \frac{\sigma}{Pr}U_y = -ik_y\delta P - \mathscr{L}^* U_y + ik_y\nu'(z)W \\ \frac{\sigma}{Pr}W = -(\delta P)' - \mathscr{L}^* W + \nu'(z)W' + R\Theta \\ \sigma\Theta = \Theta'' - k^2\Theta + W. \end{cases} \tag{10}$$

This system is written as a classical system of ODEs on regular functions. It was deduced in [12] that the system of ODEs (10) implies the following partly decoupled system, denoting by $Z = ik_xU_y - ik_yU_x$.

$$\begin{cases} \frac{\sigma}{Pr}Z - \frac{d}{dz}\left(\nu(z)\frac{dZ}{dz}\right) + k^2\nu(z)Z = 0 \\ \frac{\sigma}{Pr}\left(k^2W - W''\right) = Rk^2\Theta - \frac{d^2}{dz^2}\left(\nu(z)\frac{d^2W}{dz^2}\right) + 2k^2\frac{d}{dz}\left(\nu(z)\frac{d}{dz}W\right) - k^4\nu(z)W - k^2\nu''(z)W \\ \left(\sigma + k^2 - \frac{d^2}{dz^2}\right)\Theta = W. \end{cases} \tag{11}$$

Our purpose is to state the problem as a problem for $Z \in H^1([-\frac{1}{2}, \frac{1}{2}]), Z(-\frac{1}{2}) = 0$ and $(W, \Theta) \in \mathscr{H}$ to apply spectral results on operators on Hilbert spaces, and to derive properly the additional boundary condition in this weaker set-up. This type of analysis is well known for linear elasticity problems in the domain of strength of materials.

Before this spectral analysis, we treat the toy model of $\nu(z)$ constant, in order to illustrate the third item of Theorem 2.

[1] the notation used is different from $\mathscr{L}$: $\mathscr{L}$ is the extension of $\mathscr{L}^*$ on H^1_0.

3. The toy model where ν is independent on z

We illustrate one of the aims of this short Note in the case of a toy model. This toy model is classical, as mentioned above, and is described in [3, Section 8.3, Section 10]. The eigenmodes are known for the free-free boundary conditions, and for free-rigid boundaries (see [8]). Recall that, in this case, one studies the system, for $(W,\Theta)\in(C^\infty([-\frac{1}{2},\frac{1}{2}]))^2$,

$$\exists\,\sigma\geq 0,\begin{cases}\left(\frac{\sigma}{Pr}L+\nu L^2\right)W=Rk^2\Theta\\ (\sigma+L)\Theta=W\end{cases}$$

with the boundary conditions $\Theta(\pm\frac{1}{2})=0, W(\pm\frac{1}{2})=0, W'(-\frac{1}{2})=0, W''(\frac{1}{2})=0$ (the last condition being the one stated in the physics literature).

This constant coefficient system of ODE yields

$$\begin{aligned}\Theta(z)=&A_1\cos\tau_1\left(z+\frac{1}{2}\right)+A_2\cos\tau_2\left(z+\frac{1}{2}\right)+A_3\cos\tau_3\left(z+\frac{1}{2}\right)\\ &+B_1\sin\tau_1\left(z+\frac{1}{2}\right)+A_2\sin\tau_2\left(z+\frac{1}{2}\right)+A_3\sin\tau_3\left(z+\frac{1}{2}\right),\end{aligned}$$

where

$$\tau_1\left(\sigma,k,Rk^2\right)\in\mathbb{R},\tau_\pm\left(\sigma,k,Rk^2\right)=-\alpha\left(\sigma,k,Rk^2\right)\pm i\beta\left(\sigma,k,Rk^2\right)\in\mathbb{C}$$

are respectively given by $\tau_1^2=X_1-k^2$, $\tau_\pm^2=X_\pm^2-k^2$ where $X_1,X_\pm\in\mathbb{R}$ solve

$$\left(\frac{\sigma}{Pr}X+\nu X^2\right)(\sigma+X)-Rk^2=0. \tag{12}$$

We prove

Proposition 3. *Introduce n_0 such that*

$$\left(k^2+n_0^2\pi^2\right)^3<\frac{Rk^2}{\nu}\leq\left(k^2+(n_0+1)^2\pi^2\right)^3.$$

For all $n\leq n_0$, there exists at least one value of $X_1\in(k^2+n^2\pi^2,k^2+(n+1)^2\pi^2)$, solution of (13)*. An associated growth rate is given by*

$$\sigma=-\frac{1+\theta}{2\theta}X_1+\sqrt{\frac{X_1^3-\frac{k^2R}{\nu}}{\theta X_1}-\left(\frac{1+\theta}{2\theta}\right)^2X_1^2}>0$$

which is an admissible growth rate for the Rayleigh–Bénard problem in the toy case. There is no admissible value of σ coming from the solutions of (13) *greater than $(k^2+n_0^2\pi^2)^3$.*

Proof. As $W=(\sigma+L)\Theta$, the boundary conditions write

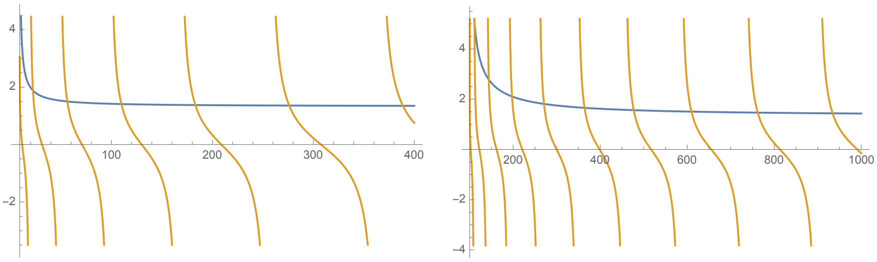
$$\begin{cases}A_1+A_++A_-=0,(\sigma+X_1)A_1+(\sigma+X_+)A_++(\sigma+X_-)A_-=0,\\ (\sigma+X_1)\tau_1^2A_1+(\sigma+X_+)\tau_+^2A_++(\sigma+X_-)\tau_-^2A_-=0\\ (A_1\cos\tau_1+B_1\sin\tau_1)+(A_+\cos\tau_++B_+\sin\tau_+)+(A_-\cos\tau_-+B_-\sin\tau_-)=0,\\ (\sigma+X_1)(A_1\cos\tau_1+B_1\sin\tau_1)+(\sigma+X_+)(A_+\cos\tau_++B_+\sin\tau_+)\\ +(\sigma+X_-)(A_-\cos\tau_-+B_-\sin\tau_-)=0,\\ (\sigma+X_1)\tau_1B_1+(\sigma+X_+)\tau_+B_++(\sigma+X_-)\tau_-B_-=0.\end{cases}$$

A X_1,X_+,X_- are distinct, the eigenvalue equation of the system is

$$(\sigma+X_1)\frac{\tau_1}{\tan\tau_1}\left(\tau_+^2-\tau_-^2\right)+(\sigma+X_+)\frac{\tau_+}{\tan\tau_+}\left(\tau_-^2-\tau_1^2\right)+(\sigma+X_-)\frac{\tau_-}{\tan\tau_-}\left(\tau_1^2-\tau_+^2\right)=0,$$

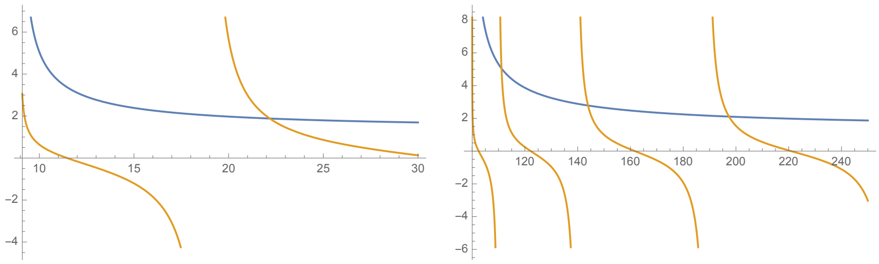
that is

$$(\sigma+X_1)\frac{\tau_1}{\tan\tau_1}\left(\tau_+^2-\tau_-^2\right)+2\Re\left[(\sigma+X_+)\frac{\tau_+}{\tan\tau_+}\left(\tau_-^2-\tau_1^2\right)\right]=0.$$



$k=3, \frac{Rk^2}{\nu}=1, \frac{1}{\nu Pr}=10^{-7}$ $k=10, \frac{Rk^2}{\nu}=1, \frac{1}{\nu Pr}=10^{-7}$

Figure 1. Two cases on R and k, in absciss X_1 (blue: right hand side of (13), yellow: $y=\frac{1}{\tan\sqrt{X_1-k^2}}$).



$k=3, \frac{Rk^2}{\nu}=1, \frac{1}{\nu Pr}=10^{-7}$ $k=10, \frac{Rk^2}{\nu}=1, \frac{1}{\nu Pr}=10^{-7}$

Figure 2. A zoom on the two previous cases.

As one can express $\tau_\pm$ in terms of $\tau_1 := \sqrt{X_1^2-k^2}$, this equation becomes a real transcendental equation, which poles are $\tau_1 = n\pi$. This equation is

$$\frac{1}{\tan\tau_1} = -\frac{2\Re\left[(\sigma+X_+)\frac{\tau_+}{\tan\tau_+}\left(\tau_-^2-\tau_1^2\right)\right]}{\left(\tau_+^2-\tau_-^2\right)\tau_1(\sigma+X_1)}. \tag{13}$$

One shows that there exists a unique solution $\tau_1^n(R,\sigma,k)$ of this equation in $(n\pi,(n+1)\pi)$ for all $n\geq 0$. In particular $\tau_1=\pi$ cannot be a solution of this equation. It is associated with $X_1^n(\sigma,k) = k^2+(\tau_1^n(\sigma,k))^2$, and σ can be recovered through (12). Note that the two solutions of (12) are real, and that they are both strictly negative when $X_1^3 > \frac{Rk^2}{\nu}$, hence for $((n+1)^2\pi^2+k^2)^3 \geq \frac{Rk^2}{\nu}$, the value of σ obtained from $X_1 \geq (n+1)^2\pi^2+k^2 \geq \frac{Rk^2}{\nu}$ is strictly negative. The proposition is proven, and gives rise to a finite number of number of values of σ. □

We have, in addition, for $\sigma=0$, $\nu X_1^3 = Rk^2$. We obtain the sequence $r_n(k) = \frac{\nu(k^2+(\tau_1^n(0,k))^2)^3}{k^2}$ which will be used for the count of growth rates.

Lemma 5 in Section 5 proves that $r_n(\sigma,k)$ is strictly increasing in σ, goes to $+\infty$ when $\sigma\to+\infty$, for each n (and this is not easily deduced from the function appearing in the right hand side of (13)). In this band, $X_1^n(\sigma,k)$ converges to $(n+1)^2\pi^2+k^2$ when $\sigma\to+\infty$. In two cases we graph (Figure 1), using Matematica, the equation (13) for $X_1>k^2$, with

$$\tau_1=\sqrt{X_1-k^2}, \sigma = -\frac{(1+\theta)X_1}{2\theta}+\sqrt{\left(\frac{(1+\theta)X_1}{2\theta}\right)^2+\frac{\frac{Rk^2}{\nu}-X_1^3}{\theta X_1}}. \tag{14}$$

This illustrates the calculation of $X_1^n(\sigma,k)$.

To have a better view at the first modes, we show, in the same cases, a zoom for the first values. Using FindRoot in Matematica:

$$k=3,\ \frac{R}{\nu}=\frac{1}{9}: X_{1,1}=22.1762, X_{1,2}=109.832, X_{1,3}=183.107, X_{1,4}=276.077,$$
$$k=10, \frac{R}{\nu}=\frac{1}{1000}: X_{1,1}=111.136, X_{1,2}=143.931, X_{1,3}=197.367.$$

4. A self-adjoint equation through a variational formulation

The systems of ODEs on W and Θ is, in the Physics literature, implicitely stated with strong derivatives for W and Θ respectively in $C^4([-\frac{1}{2},\frac{1}{2}])$ and $C^2([-\frac{1}{2},\frac{1}{2}])$. Introduce $\phi\in C^4([-\frac{1}{2},\frac{1}{2}])$ and deduce, as it was done in [12], a variational formulation of the problem. Observe that

$$-\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\mathscr{L}^*W'\right)'\phi dz=-\left[\mathscr{L}^*W'\phi\right]_{-\frac{1}{2}}^{\frac{1}{2}}+\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathscr{L}^*W'\phi' dz$$
$$=-\left[\mathscr{L}^*W'\phi\right]_{-\frac{1}{2}}^{\frac{1}{2}}+k^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\nu(z)W'\phi' dz-\left[\nu W''\phi'\right]_{-\frac{1}{2}}^{\frac{1}{2}}+\int_{-\frac{1}{2}}^{\frac{1}{2}}\nu(z)W''\phi'' dz.$$

We insist that, even though

$$k^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\nu(z)W'\phi' dz+\int_{-\frac{1}{2}}^{\frac{1}{2}}\nu(z)W''\phi'' dz$$

is well defined for $W,\phi\in H^2(-\frac{1}{2},\frac{1}{2})$, we have to use caution to define the boundary terms, if needed.

The second equation of (11) implies, all $(W,\phi)\in C^4([-\frac{1}{2},\frac{1}{2}])$ and for $\Theta\in C^0([-\frac{1}{2},\frac{1}{2}])$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\frac{\sigma}{Pr}\left(W'\phi'+k^2W\phi\right)-Rk^2\Theta\phi+k^2\nu''(z)W\phi+\nu(z)\left(W''\phi''+2k^2W'\phi'+k^4W\phi\right)\right]dz$$
$$=\frac{\sigma}{Pr}\left[W'\phi\right]_{-\frac{1}{2}}^{\frac{1}{2}}-\left[\left(\nu W''\right)'\phi-\nu W''\phi'+2k^2\nu W'\phi\right]_{-\frac{1}{2}}^{\frac{1}{2}}. \quad (*)$$

The last equation of (2) yields, for all $(\Theta,\Psi)\in C^2([-\frac{1}{2},\frac{1}{2}])$ and for $W\in C^0([-\frac{1}{2},\frac{1}{2}])$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}W\Psi dz=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left((\sigma+k^2)\Theta\Psi+\Theta'\Psi'\right)dz-\left[\Theta'\Psi\right]_{-\frac{1}{2}}^{\frac{1}{2}}.$$

In [12], for regular solutions, we used $W=W'=\Theta=0$ on $z=-\frac{1}{2}$, $W=\Theta=0$ on $z=\frac{1}{2}$ and added the additional condition $W''=0$ on $z=\frac{1}{2}$. This condition can be defined for a regular (more than H^2) solution, but cannot be written if one wants to use a weak formulation assuming $W\in H^2$. We propose here to adress this question by considering the system on $(W,\Theta)\in\mathscr{H}$: $\forall\ (\phi,\Psi)\in\mathscr{H}$,

$$\begin{aligned}&\int_{-\frac{1}{2}}^{\frac{1}{2}}W\Psi dz=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left((\sigma+k^2)\Theta\Psi+\Theta'\Psi'\right)dz,\\ &Rk^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\Theta\phi dz=\\ &\qquad\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\frac{\sigma}{Pr}\left(W'\phi'+k^2W\phi\right)+\nu(z)\left(k^4W\phi+2k^2W'\phi'+W''\phi''\right)+k^2\nu''(z)W\phi\right]dz.\end{aligned}\tag{15}$$

As in [12, Theorem 4], it can be proven that all possible values of σ are real, and that $\sigma\geq 0$ implies $Z=0$ for $k>0$. System (2) on $(Z=0,W,\Theta)$ is (15) thanks to the definition of $\mathscr{L}^b,\mathscr{L},L$.

Lemma 4. *The system* (15) *(equivalent to* (2)*) is equivalent to the second and third equations of* (11)*, with* $W \in H^4$*,* $\Theta \in H^2$*, supplemented by the boundary conditions* $W = \Theta = 0$ *for* $z = \pm\frac{1}{2}$*,* $W' = 0$ *for* $z = -\frac{1}{2}$*,* $W'' = 0$ *for* $z = \frac{1}{2}$*.*

Proof. We shall omit the mention of the interval $[-\frac{1}{2}, \frac{1}{2}]$ in the Sobolev spaces in this proof.

Assume that there exists $\sigma \geq 0$ such that $(W, \Theta) \in \mathscr{H}$ is a non trivial solution of (2). The relation $(\sigma + L)\Theta = W$, for $\Theta \in H^1_0$ implies $\Theta'' \in L^2$, from which one deduces $\Theta \in H^2$.

Considering $\Psi \in C^\infty_0([-\frac{1}{2}, \frac{1}{2}])$ we obtain the ODE in $\mathscr{D}'$:

$$\frac{\sigma}{Pr}\left(k^2 W - W''\right) = Rk^2\Theta + \left(\nu W''\right)'' - 2k^2\left(\nu W'\right)' + k^4 W - k^2\nu'' W.$$

This equality on W yields $\mathscr{Q}W \in L^2$ thanks to $LW \in L^2$. As

$$\phi \to \int_{-\frac{1}{2}}^{\frac{1}{2}} \nu(z)\left(-2W'\phi' + k^2 W\phi\right) dz$$

can be extended as a linear form on $\phi \in L^2$ thanks to ν and ν' continuous, one is left with

$$-\left(\nu(z)W''\right)'' := g \in L^2.$$

Let us introduce $h \in H^2$ such that $-h'' = g$. One deduces there exists two constants A and B such that, in $\mathscr{D}'$, $\nu(z)W'' = h + Az + B$, which proves that $W'' \in H^2$ (thanks to $\frac{1}{\nu(z)}$ continuous), hence $W \in H^4$.

The ODE in $\mathscr{D}'$ is thus in L^2 and one can replace Θ by its expression in W in the second equality of (15). The integration by parts formulae are valid because $W \in H^4, \phi \in H^2$ and one can use (*). The only boundary terms left in (*), using the boundary conditions for $\phi, W \in \widetilde{H}$ is $\nu(\frac{1}{2})W''(\frac{1}{2})\phi'(\frac{1}{2}) - 0$ (where $W \in H^4$ allows to write the trace of W'' at $\frac{1}{2}$). One has thus

$$\forall\ \Psi \in \widetilde{H}, \nu\left(\frac{1}{2}\right)W''\left(\frac{1}{2}\right)\phi'\left(\frac{1}{2}\right),$$

from which, with a suitable choice of ϕ one deduces the boundary condition

$$W''\left(\frac{1}{2}\right) = 0. \tag{16}$$

The ODE in L^2 on Θ comes directly. Note that the boundary conditions $\Theta(\pm\frac{1}{2}) = 0, W(\pm\frac{1}{2})$, $W'(-\frac{1}{2}) = 0$ are in the space where the variational formulation is written.

We deduce that (W, Θ) is a $H^4 \times H^2$ solution of (11), supplemented with the boundary conditions stated in Lemma 4.

Notice that the equality $(\frac{\sigma}{Pr} + \mathscr{L}^b)Z = 0$ implies, for $Z \in H^1([-\frac{1}{2}, \frac{1}{2}]), Z(-\frac{1}{2}) = 0$, the usual ODE on Z of (11), for $Z \in H^2$, hence the equality $\forall\ \psi \in H^1([-\frac{1}{2}, \frac{1}{2}]), \psi(-\frac{1}{2}) = 0$:

$$\nu\left(\frac{1}{2}\right)Z'\left(\frac{1}{2}\right)\psi\left(\frac{1}{2}\right) = 0,$$

from which one deduces $Z'(\frac{1}{2}) = 0$, hence recovering *from the variational formulation chosen* the boundary conditions $\partial_z U_x = \partial_z U_y = 0$. Coercivity of $\mathscr{L}^b$ on $H^1 \cap \{Z(-\frac{1}{2})\} = 0$ and $\sigma \geq 0$ imply $Z = 0$ (it is similar to Lemma 10 in [12] for other boundary conditions). Note that one did not need Poincaré inequality thanks to $k > 0$.

Let us now consider $(W, \Theta) \in H^4 \times H^2 \cap \mathscr{H}$ solution of (11) satisfying the boundary conditions of Lemma 4. Multiplying by $\phi \in C^\infty_0$, one deduces

$$Rk^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\Theta\phi dz = \int_{-\frac{1}{2}}^{\frac{1}{2}}\Big[\frac{\sigma}{Pr}\left(W'\phi' + k^2 W\phi\right) + \nu(z)\left(k^4 W\phi + 2k^2 W'\phi' + W''\phi''\right) + k^2 nu''(z)W\phi\Big]dz - \nu\left(\frac{1}{2}\right)W''\left(\frac{1}{2}\right)\phi'\left(\frac{1}{2}\right).$$

After using $W''(\frac{1}{2}) = 0$ (the boundary condition of Lemma 4 allowed because $W \in H^4$), the extension of this equality to $\phi \in H^2$, $\phi(\pm\frac{1}{2}) = 0$, $\phi'(-\frac{1}{2}) = 0$ yields the first equality of (15). The second equality of (15) follows similarly. The equivalence is proven. □

This result allows us to state the eigenvalue problem as a spectral problem for an operator on $\mathscr{H}$ (an operator depending on σ for which we want to find 0 in its spectrum), the extra regularity of an eigenmode allowing to write the boundary condition $W''(\frac{1}{2}) = 0$.

5. An eigenvalue formulation

In this section, we identify three eigenvalue formulations of the problem, one of them where W is an eigenvector of an operator denoted by $Z_{k,\sigma}$, another one where Θ is an eigenvector of the operator $Z^T_{k,\sigma}$ (neither operator being self-adjoint). We identify finally a third operator, self adjoint, for which we can perform the spectral analysis. The operator $Z_{k,\sigma}$ is

$$Z_{k,\sigma}\phi = [\sigma + L]\left[\frac{\sigma}{Pr}L + \mathscr{Q} + k^2 v''(z)\right]\phi.$$

We obtain the ODE (in the distributional sense) $Z_{k,\sigma}W = Rk^2 W$.

Similarly, applying $[\frac{\sigma}{Pr}L + \mathscr{Q} + k^2 v''(z)]$ to the equation on Θ, one deduces (again in $\mathscr{D}'$)

$$\left[\frac{\sigma}{Pr}L + \mathscr{Q} + k^2 v''(z)\right](\sigma + L)\Theta = Z^T_{k,\sigma}\Theta = Rk^2\Theta.$$

However, the operator $Z_{k,\sigma}$ is not self-adjoint because $\mathscr{L}$ and $\frac{d}{dz}$ do not commute, hence one needs to use a symmetrization of the problem. It is similar to [12, the Appendix], but the Sobolev spaces chosen in [12] did not lead to the boundary conditions of the present Note (which was, though, the boundary conditions used in the numerical study of [12]). Here, the set of boundary conditions is distinct. This does not allow to use the spaces described in [12, the Appendix] and the space needed is slightly more difficult to describe (see (17) below).

Considering $[\frac{\sigma}{Pr}L + \mathscr{Q} + k^2 v''(z)]W = Rk^2\Theta$, one observes that $[\frac{\sigma}{Pr}L + \mathscr{Q} + k^2 v''(z)]^{-1}$ is applied on Θ. and yields W. We should thus have

$$\Theta = (\sigma + L)^{-\frac{1}{2}} f \text{ and } W = \left[\frac{\sigma}{Pr}L + \mathscr{Q} + k^2 v''\right]^{-1}(\sigma + L)^{-\frac{1}{2}} f.$$

Thus $f = (\sigma + L)^{\frac{1}{2}}\Theta$ and, through $(\sigma + L)\Theta = W$, $W = (\sigma + L)^{\frac{1}{2}} f$. It seems then reasonable to construct a space for which, for f in this space, $(\sigma + L)^{-\frac{1}{2}} f \in H^1_0([-\frac{1}{2}, \frac{1}{2}])$ (function denoted by Θ) and $(\sigma + L)^{\frac{1}{2}} f \in \widetilde{H}$ (function denoted by W).

One can thus introduce

$$\mathscr{H}^* =: \left\{\phi \in H^3\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right), \left((\sigma + L)^{-\frac{1}{2}}\phi\right)\left(\pm\frac{1}{2}\right) = 0, \left((\sigma + L)^{\frac{1}{2}}\phi\right)\left(\pm\frac{1}{2}\right) = 0, \left((\sigma + L)^{\frac{1}{2}}\phi\right)'\left(-\frac{1}{2}\right) = 0\right\}.$$

Note that this space depends on σ through the boundary conditions (*and more precisely through the boundary condition* $((\sigma + L)^{\frac{1}{2}}\phi)'(-\frac{1}{2}) = 0$). Introduce the base of $H^1_0([-\frac{1}{2}, \frac{1}{2}])$ $(e_n)_{n\geq 1}$, $e_n(z) = \sin n\pi(z + \frac{1}{2})$. This base is a base of eigenvectors of $\sigma + L$, the associated eigenvalues are $(\sigma + k^2 + n^2\pi^2) \geq k^2 + \pi^2$.

Consider ϕ in $\mathscr{H}^*$. One has $(\sigma + L)^{\frac{1}{2}}\phi \in H^1_0([-\frac{1}{2}, \frac{1}{2}])$ hence $(\sigma + L)^{\frac{1}{2}}\phi = \sum_{n\geq 1} b_n e_n$, with $\sum_{n\geq 1}(1 + n^2)b_n^2 < +\infty$. We have, successively,

$$\phi = \sum_{n\geq 1}\frac{b_n}{\sqrt{\sigma + k^2 + n^2\pi^2}}e_n, \quad (\sigma + L)^{-\frac{1}{2}}\phi = \sum_{n\geq 1}\frac{b_n}{\sigma + k^2 + n^2\pi^2}e_n.$$

As

$$\phi \in H^3, \sum_{n\geq 1}\frac{(1 + n^2)^3 b_n^2}{\sigma + k^2 + n^2\pi^2} < +\infty.$$

One deduces

$$\left((\sigma+L)^{\frac{1}{2}}\right)'\phi(z)=\sum_{n\geq 1} n\pi b_n\cos n\pi\left(z+\frac{1}{2}\right).$$

Note that

$$n^2\pi^2 b_n^2=\frac{b_n^2\left(1+n^2\right)^3}{\sigma+n^2\pi^2+k^2}.\frac{\left(\sigma+n^2\pi^2+k^2\right)n^2\pi^2}{\left(1+n^2\right)^3},$$

hence $\sum_{n\geq 1} n\pi b_n\cos n\pi(z+\frac{1}{2})$ is normally converging, and its value at $z=-\frac{1}{2}$ is $n\sum_{n\geq 1} nb_n$. Denoting by

$$a_n=\frac{b_n}{\sqrt{\sigma+k^2+n^2\pi^2}},\left((\sigma+L)^{\frac{1}{2}}\right)'f\left(-\frac{1}{2}\right)=0$$

corresponds to

$$\left(\sigma+k^2+\pi^2\right)^{\frac{1}{2}}a_1=-\sum_{n\geq 2} n\left(\sigma+k^2+n^2\pi^2\right)^{\frac{1}{2}}a_n.$$

This induces the idea of showing the equality

$$\begin{aligned}\mathscr{H}^*=\Big\{\Big[z\to\sum_{n\geq s1} a_n\sin n\pi\left(z+\frac{1}{2}\right)\Big],\sum\left(1+n^2\right)^3 a_n^2<+\infty,\left(\sigma+k^2+\pi^2\right)^{\frac{1}{2}}a_1\\ =-\sum_{n\geq 2}\left(\sigma+k^2+n^2\pi^2\right)^{\frac{1}{2}}na_n.\Big\}.\end{aligned}\tag{17}$$

Proof. We have constructed above a Fourier representation of $\mathscr{H}^*$ such that $\subset$ is true in (17).

Conversely, let

$$\begin{aligned}\phi\in\Big\{\Big[z\to\sum_{n\geq 1} a_n\sin n\pi\left(z+\frac{1}{2}\right)\Big],\sum\left(1+n^2\right)^3 a_n^2<+\infty,\left(\sigma+k^2+\pi^2\right)^{\frac{1}{2}}a_1\\ =-\sum_{n\geq 2}\left(\sigma+k^2+n^2\pi^2\right)^{\frac{1}{2}}na_n.\Big\}.\end{aligned}$$

One checks that $(\sigma+L)^{-\frac{1}{2}}\phi(\pm\frac{1}{2})=0$ and $(\sigma+L)^{\frac{1}{2}}\phi(\pm\frac{1}{2})=0$ because the two sums defining $(\sigma+L)^{\frac{1}{2}}\phi,(\sigma+L)^{-\frac{1}{2}}\phi$ are normally convergent. In addition

$$\left((\sigma+L)^{\frac{1}{2}}\phi\right)'=\sum_n\left(\sigma+k^2+n^2\pi^2\right)^{\frac{1}{2}}n\pi a_n\cos n\pi\left(z+\frac{1}{2}\right)$$

converges in L^2 hence

$$\pi a_1=-\sum_{n\geq 2}\left(\sigma+k^2+n^2\pi^2\right)^{\frac{1}{2}}n\pi a_n$$

is possible thanks to the Cauchy–Schwartz inequality. Hence $((\sigma+L)^{\frac{1}{2}}\phi)'(-\frac{1}{2})=0$. This proves that ϕ belongs to $\mathscr{H}^*$. Equality (17) is proven (and gives a simpler expression of $\mathscr{H}^*$). □

Consider $\sigma\in\mathbb{R}_+$, and recall that v satisfies (H). Introduce the operator $R(\sigma,k)$ given by

$$R(\sigma,k)f=(\sigma+L)^{\frac{1}{2}}\left[\frac{\sigma}{Pr}L+\mathscr{Q}+k^2v''\right](\sigma+L)^{\frac{1}{2}}f.$$

We state now the second additional result to [12], which is the key for obtaining Théorème 2:

Lemma 5. *The operator $R(\sigma,k)$ is self-adjoint coercive on $\mathscr{H}^*$, and $R(\sigma,k)^{-1}$ is self-adjoint compact on $L^2([-\frac{1}{2},\frac{1}{2}])$. The spectrum of $R(\sigma,k)^{-1}$ is a decreasing sequence, which is denoted by $(\frac{1}{r_n(\sigma,k)})_{n\geq 1}$.*

Proof. Consider the operator $(\sigma+L)^{-\frac{1}{2}}[\mathcal{Q}+k^2v'']^{-1}(\sigma+L)^{-\frac{1}{2}}$. It is well defined (through Lax–Milgram lemma) from $(\mathcal{H}^*)'$ to $\mathcal{H}^*$. For all $\sigma\geq 0$, $(\sigma+L)^{-\frac{1}{2}}$ sends $L^2([-\frac{1}{2},\frac{1}{2}])$ to $H^1_0([-\frac{1}{2},\frac{1}{2}])$. The operator $[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v'']$ is self-adjoint coercive on $\widetilde{H}$ and sends $\widetilde{H}$ on its dual $(\widetilde{H})'$. Hence $[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v'']^{-1}$ sends $(\widetilde{H})'$ to $\widetilde{H}$, hence it sends $L^2([-\frac{1}{2},\frac{1}{2}])$ to $\widetilde{H}\subset L^2([-\frac{1}{2},\frac{1}{2}])$. As $(\sigma+L)^{-\frac{1}{2}}$ sends $L^2([-\frac{1}{2},\frac{1}{2}])$ to $H^1_0([-\frac{1}{2},\frac{1}{2}])$, $(\sigma+L)^{-\frac{1}{2}}[\mathcal{Q}+k^2v''](\sigma+L)^{-\frac{1}{2}}$ sends $L^2([-\frac{1}{2},\frac{1}{2}])$ to $H^1_0([-\frac{1}{2},\frac{1}{2}])$ and one concludes by using the compactness of the canonical injection from $H^1_0([-\frac{1}{2},\frac{1}{2}])$ to $L^2([-\frac{1}{2},\frac{1}{2}])$. The result on the spectrum of $R(\sigma,k)^{-1}=(\sigma+L)^{-\frac{1}{2}}[\mathcal{Q}+k^2v''](\sigma+L)^{-\frac{1}{2}}$ follows. □

Lemma 6. *It is equivalent to find an eigenvector in $\mathcal{H}^*$ of $(R(\sigma,k))^{-1}$ associated with the eigenvalue $\frac{1}{r_n(\sigma,k)}$ and find a solution $(W,\Theta)\in\mathcal{H}$ of the system*

$$\begin{cases}\left[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v''\right]W=r_n(\sigma,k)\Theta\\ (\sigma+L)\Theta=W.\end{cases}\tag{18}$$

In other words, it is equivalent to find a nontrivial solution $(W,\Theta)\in\mathcal{H}$ of

$$\begin{cases}\left[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v''\right]W=Rk^2\Theta\\ (\sigma+L)\Theta=W\end{cases}$$

and to solve all equations $r_n(\sigma,k)=Rk^2$.

Proof. Let $f_n^{\sigma,k}$ be an eigenvector of norm 1 in L^2 associated with the eigenvalue $r_n(\sigma,k)$, solution of $(R(\sigma,k))^{-1}f_n^{\sigma,k}=\frac{1}{r_n(\sigma,k)}f_n^{\sigma,k}$. The manipulation of $(\sigma+L)^{\frac{1}{2}}f_n^{\sigma,k}$ is not easy, hence we consider the functions $w_n^{\sigma,k}=(\sigma+L)^{\frac{1}{2}}f_n^{\sigma,k}$ and $\theta_n^{\sigma,k}=(\sigma+L)^{-\frac{1}{2}}f_n^{\sigma,k}$. We have $f_n^{\sigma,k}=(\sigma+L)^{-\frac{1}{2}}(\sigma+L)\theta_n^{\sigma,k}$. one obtains

$$(\sigma+L)^{-\frac{1}{2}}\left[\left[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v''\right]^{-1}\theta_n^{\sigma,k}-\frac{1}{r_n(\sigma,k)}(\sigma+L)\theta_n^{\sigma,k}\right]=0$$

hence $r_n(\sigma,k)[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v'']^{-1}\theta_n^{\sigma,k}-(\sigma+L)\theta_n^{\sigma,k}=0$. One deduces $(\sigma+L)\theta_n^{\sigma,k}=0\in\widetilde{H}$ because $\theta_n^{\sigma,k}\in L^2$. One applies $[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v'']$ to this equality and, thanks to $w_n^{\sigma,k}=(\sigma+L)\theta_n^{\sigma,k}\in\widetilde{H}$, one gets the first equality of (18).

In a similar way, as

$$f_n^{\sigma,k}\in\mathcal{H}^*,\left[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v''\right]^{-1}(\sigma+L)^{-\frac{1}{2}}f_n^{\sigma,k}=\frac{1}{r_n(\sigma,k)}(\sigma+L)^{\frac{1}{2}}f_n^{\sigma,k}.$$

From $(\sigma+L)^{\frac{1}{2}}f_n^{\sigma,k}\in\widetilde{H}$ because $f_n^{\sigma,k}\in\mathcal{H}^*$, one can apply $[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v'']$. As

$$H^1_0\ni r_n(\sigma,k)(\sigma+L)^{-\frac{1}{2}}f_n^{\sigma,k}=\left[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v''\right]w_n^{\sigma,k}$$

one can apply $(\sigma+L)$ to this equality, hence $r_n(\sigma,k)(\sigma+L)^{\frac{1}{2}}w_n^{\sigma,k}=r_n(\sigma,k)(\sigma+L)^{\frac{1}{2}}f_n^{\sigma,k}=(\sigma+L)[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v'']w_n^{\sigma,k}$ in H^{-1} and in H^1_0 thanks to the left hand side of this equality. The second equality of (18) is proven.

Conversely, if one considers a solution of (18) it is straightforward to construct an eigenvector of $(R(\sigma,k))^{-1}$ associated with the eigenvalue $\frac{1}{r_n(\sigma,k)}$. Lemma 6 is proven. □

We prove in addition that $w_n^{\sigma,k}$ and $\theta_n^{\sigma,k}$ are solutions of

$$\begin{cases}\left[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v''\right](\sigma+L)\theta_n^{\sigma,k}=r_n(\sigma,k)\theta_n^{\sigma,k}\\ (\sigma+L)\left[\frac{\sigma}{Pr}L+\mathcal{Q}+k^2v''\right]w_n^{\sigma,k}=r_n(\sigma,k)w_n^{\sigma,k}\end{cases}\tag{19}$$

thanks to $w_n^{\sigma,k}\in\widetilde{H}$ through $f_n^{\sigma,k}\in\mathcal{H}^*$ and $\theta_n^{\sigma,k}\in H^1_0$.

Lemma 7. *Assume $k>0$ and $\sigma\geq 0$.*

(1) *The sequence $\frac{1}{r_n(0,k)}$ is the sequence of eigenvalues of $(L^{\frac{1}{2}}[\mathcal{Q}+k^2v'']L^{\frac{1}{2}})^{-1}$,*

(2) *For each n, the function $\frac{1}{r_n(\sigma,k)}$ is strictly decreasing to 0 when $\sigma \to +\infty$ thanks to* $\partial_\sigma r_n(\sigma,k) \geq [\frac{2\sigma}{Pr}(\pi^2+k^2) + \frac{(\pi^2+k^2)^2}{Pr}] + v_0 k^4$.

Item (ii) of Lemma 7 shows uniqueness (if existence) for all $R \geq \frac{r_n(0,k)}{k^2}$ of the solution of $r_n(\sigma,k) = Rk^2$. Using $(r_n(0,k))$, we can count the number of positive solutions of $r_n(\sigma,k) = Rk^2$ for at least one n.

Proof of item (i)**.** (it is a consequence of the continuity of the discrete spectrum)

Let $f_n^{\sigma,k} \in L^2$ be an eigenvector of $(R(\sigma,k))^{-1}$, of norm 1 in L^2, associated with the eigenvalue $\frac{1}{r_n(\sigma,k)}$. For any $f \in L^2$, one has

$$f = \sum \left\langle f, f_n^{\sigma,k} \right\rangle f_n^{\sigma,k}$$

and by considering the limit when $\sigma \to 0$, $f_n^{0,k}$ belonging to $\mathscr{H}^*$ (for $\sigma = 0$), $f = \sum \langle f, f_n^{0,k} \rangle f_n^{0,k}$. Note that $R(0,k) = L^{\frac{1}{2}}[\mathscr{Q} + k^2 v'']L^{\frac{1}{2}}$.

As $R(0,k)f = \sum \langle f, f_n^{0,k}\rangle R(0,k) f_n^{0,k} = \sum \langle f, f_n^{0,k}\rangle r_n(0,k) f_n^{0,k}$, we have a decomposition of $(R(0,k))^{-1} f$ as well on the orthonormal basis $f_n^{0,k}$, which yields the spectrum of $(R(0,k))^{-1}$ as announced. □

Proof of item (ii)**.** Introduce $P_2 = \frac{1}{Pr}L$, $P_1 = \frac{1}{Pr}L^2 + \mathscr{Q} + k^2 v''(z)$, $P_0 = L(\mathscr{Q} + k^2 v''(z))$. System (19) rewrites

$$\begin{cases} \left(\sigma^2 P_2 + \sigma P_1 + P_0\right) w_n^{\sigma,k} = r_n(\sigma,k) w_n^{\sigma,k}, \\ \left(\sigma^2 P_2 + \sigma P_1 + P_0^T\right) \theta_n^{\sigma,k} = r_n(\sigma,k) \theta_n^{\sigma,k}. \end{cases}$$

Differentiating these equalities with respect to σ yield

$$(2\sigma P_2 + P_1)\, w_n^{\sigma,k} + \left(\sigma^2 P_2 + \sigma P_1 + P_0\right) \partial_\sigma w_n^{\sigma,k} = \partial_\sigma r_n(\sigma,k)\, w_n^{\sigma,k} + r_n(\sigma,k)\, \partial_\sigma w_n^{\sigma,k}, \tag{20}$$

$$(2\sigma P_2 + P_1)\, \theta_n^{\sigma,k} + \left(\sigma^2 P_2 + \sigma P_1 + P_0^T\right) \partial_\sigma \theta_n^{\sigma,k} = \partial_\sigma r_n(\sigma,k)\, \theta_n^{\sigma,k} + r_n(\sigma,k) \partial_\sigma \theta_n^{\sigma,k}. \tag{21}$$

One calculates $\langle (20), \theta_n^{\sigma,k}\rangle + \langle (21), w_n^{\sigma,k}\rangle$ and uses

$$\left\langle f_n^{\sigma,k}, f_n^{\sigma,k} \right\rangle = \left\langle w_n^{\sigma,k}, \theta_n^{\sigma,k} \right\rangle = 1, \quad \partial_\sigma \left\langle f_n^{\sigma,k}, f_n^{\sigma,k} \right\rangle = 0 = \left\langle w_n^{\sigma,k}, \partial_\sigma \theta_n^{\sigma,k} \right\rangle + \left\langle \partial_\sigma w_n^{\sigma,k}, \theta_n^{\sigma,k} \right\rangle,$$

from which one obtains

$$\begin{aligned} \left\langle w_n^{\sigma,k}, (2\sigma P_2 + P_1)\, \theta_n^{\sigma,k} \right\rangle &+ \left\langle (2\sigma P_2 + P_1)\, w_n^{\sigma,k}, \theta_n^{\sigma,k} \right\rangle \\ &+ \left\langle Z_{n,\sigma} \partial_\sigma w_n^{\sigma,k}, \theta_n^{\sigma,k} \right\rangle + \left\langle Z_{n,\sigma}^T \partial_\sigma \theta_n^{\sigma,k}, w_n \right\rangle \\ &= 2\partial_\sigma r_n(\sigma,k) \left\langle w_n^{\sigma,k}, \theta_n^{\sigma,k} \right\rangle. \end{aligned}$$

As

$$\begin{aligned} \left\langle Z_{k,\sigma}^T \partial_\sigma \theta_n^{\sigma,k}, w_n \right\rangle &= \left\langle \partial_\sigma \theta_n^{\sigma,k}, Z_{k,\sigma} w_n \right\rangle = r_n(\sigma,k) \left\langle \partial_\sigma \theta_n^{\sigma,k}, w_n \right\rangle, \\ \left\langle Z_{k,\sigma} \partial_\sigma w_n^{\sigma,k}, \theta_n^{\sigma,k} \right\rangle &= \left\langle \partial_\sigma w_n^{\sigma,k}, Z_{k,\sigma}^T \theta_n^{\sigma,k} \right\rangle = r_n(\sigma,k) \left\langle \partial_\sigma w_n^{\sigma,k}, \theta_n^{\sigma,k} \right\rangle, \end{aligned}$$

one gets finally

$$\left\langle w_n^{\sigma,k}, (2\sigma P_2 + P_1)\, \theta_n^{\sigma,k} \right\rangle + \left\langle (2\sigma P_2 + P_1)\, w_n^{\sigma,k}, \theta_n^{\sigma,k} \right\rangle = 2\partial_\sigma r_n(\sigma,k). \tag{22}$$

The terms containing only L commute with $(\sigma + L)^{\frac{1}{2}}$ and are diagonal on the Fourier basis chosen, hence their contribution to (22) is in $< w_n^{\sigma,k}, (2\sigma P_2 + P_1)\theta_n^{\sigma,k} >$. It yields, for each mode

$$\left[\frac{2\sigma}{Pr}\left(n^2\pi^2 + k^2\right) + \frac{\left(n^2\pi^2 + k^2\right)^2}{Pr}\right] \geq \left[\frac{2\sigma}{Pr}\left(\pi^2 + k^2\right) + \frac{\left(\pi^2 + k^2\right)^2}{Pr}\right].$$

The term $\langle (\mathscr{Q} + k^2 v''(z))\theta_n^{\sigma,k}, w_n^{\sigma,k}\rangle$ rewrites $\langle (\sigma + L)^{\frac{1}{2}}(\mathscr{Q} + k^2 v''(z).)(\sigma + L)^{-\frac{1}{2}} f_n^{\sigma,k}, f_n^{\sigma,k}\rangle$. An eigenvalue μ of $(\sigma + L)^{\frac{1}{2}}(\mathscr{Q} + k^2 v''(z).)(\sigma + L)^{-\frac{1}{2}}$ is also an eigenvalue of the conjugate operator

$(\mathcal{Q}+k^2v''(z).)$, all eigenvalues of $(\mathcal{Q}+k^2v''(z).)$ are thus (thanks to $v''(z)\geq 0$ and $v(z)\geq v_0>0$), greater than v_0k^4. This yields

$$\left\langle w_n^{\sigma,k},(2\sigma P_2+P_1)\theta_n^{\sigma,k}\right\rangle \geq \left[\frac{2\sigma}{Pr}\left(\pi^2+k^2\right)+\frac{\left(\pi^2+k^2\right)^2}{Pr}\right]+v_0k^4.$$

Inequality of (ii) on $\partial_\sigma r_n(\sigma,k)$ is obtained. Each eigenvalue satisfies $r_n(\sigma,k)\to+\infty$ when $\sigma\to+\infty$. □

Proving Théorème 2 relies on studying each equation $r_n(\sigma,k)=Rk^2$.

One proved that (2) has a non trivial solution with $\sigma\geq 0$ if and only if there exists n,σ such that $r_n(\sigma,k)=Rk^2$. Thanks to Lemma 7, (ii), the function $\sigma\to r_n(\sigma,k)$ is a continuous increasing function from $\mathbb{R}_+$ onto $[r_n(0,k),+\infty)$. Hence, when $R<k^{-2}r_1(0,k)$, there is no n such that $r_n(\sigma,k)=Rk^2$ for $\sigma\geq 0$.

For each $R\geq k^{-2}r_1(0,k)$, denote by N the integer such that $\frac{1}{r_{N+1}(0,k)}<\frac{1}{Rk^2}\leq\frac{1}{r_N(0,k)}$. As, for all $m\geq N+1$, $\frac{1}{r_m(\sigma,k)}\leq\frac{1}{r_m(0,k)}<\frac{1}{Rk^2}$, there is no $\sigma\geq 0$ such that $\frac{1}{r_m(\sigma,k)}=\frac{1}{Rk^2}$. For $n\leq N$, the function $\frac{1}{r_n(\sigma,k)}$ is decreasing from $\frac{1}{r_n(0,k)}$ to $0=\lim_{\sigma\to+\infty}$, hence there exists a unique value of σ such that $\frac{1}{r_n(\sigma,k)}=\frac{1}{Rk^2}$. Counting, there exists exactly N positive values of σ for which system (2) has a non trivial solution in $\mathcal{H}$ with $Z=0$. This proves the third item of Théorème 2.

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