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A non-hyperelliptic curve with torsion Ceresa class

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Abstract. We exhibit a non-hyperelliptic curve C of genus 3 such that the class of the Ceresa cycle [C] − [−C] in the intermediate Jacobian of JC is torsion.

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1. Introduction

Let C be a complex curve of genus g ≥ 3, and p a point of C. We embed C into its Jacobian J by the Abel–Jacobi map x → [x] − [p]. The Ceresa cycle z_p(C) is the cycle [C] − [(−1)^g C] in the Chow group CH^1(J)_{hom} of homologically trivial 1-cycles. The Ceresa class c_p(C) is the image of z_p(C) in the intermediate Jacobian J_1(J) parameterizing 1-cycles under the Abel–Jacobi map CH^1(J)_{hom} → J_1(J).

When C is general, z_p(C) is not algebraically trivial [2]. On the other hand, if C is hyperelliptic z_p(C) is algebraically trivial – in fact it is zero if one chooses for p a Weierstrass point. Not much is known besides these two extreme cases. There are few curves for which z_p(C) is known to be not algebraically trivial: Fermat curves of degree ≤ 1000 [4], and the Klein quartic [5]. An essential ingredient of these results is the fact that c_p(C) is not a torsion class.

It is an open question whether there are non-hyperelliptic curves with z_p(C) algebraically trivial. As observed in [3, Remark 2.4], this condition is equivalent to a number of interesting properties: in particular the existence of a multiplicative Chow–Künneth decomposition modulo algebraic equivalence, or the fact that the class [C] ∈ CH_1(J) ⊗ Q is algebraically equivalent to the minimal class θ^{-1} / (g−1)!, where θ ∈ CH^1(J) is the class of the principal polarization.

In this note we exhibit a curve C of genus 3 with the weaker property that the Ceresa class c_p(C) is torsion (under the Bloch–Beilinson conjectures, this actually implies the algebraic triviality of z_p(C) up to torsion). The construction is very simple: the curve C has an automorphism σ which
fixes a point \( p \), and therefore preserves \( c_p(C) \); we just have to check that the fixed point set of \( \sigma \) acting on \( J_1(J) \) is finite.

A similar example, based on a much more sophisticated approach, appears in [1, Remark 3.6].

2. The result

**Proposition 1.** Let \( C \subset \mathbb{P}^2 \) be the genus 3 curve defined by \( X^4 + XZ^3 + Y^3 Z = 0 \), and let \( p = (0, 0, 1) \). The Ceresa class \( c_p(C) \) is torsion.

**Proof.** Let \( \omega \) be a primitive 9th root of unity. We consider the automorphism \( \sigma \) of \( C \) defined by \( \sigma(X, Y, Z) = (X, \omega^2 Y, \omega^3 Z) \). We have \( \sigma(p) = p \); therefore \( \sigma \) preserves the Ceresa cycle \( \bar{z}_p(C) \), and also its class \( c_p(C) \) in \( J := \bar{J}_1(J) \).

Thus it suffices to prove that \( \sigma \) has finitely many fixed points on \( \bar{J} \); equivalently, that the eigenvalues of \( \sigma \) acting on the tangent space \( T_0(\bar{J}) \) are \( \neq 1 \).

Now \( T_0(\bar{J}) \) is identified with \( H^{0,3}(J) \oplus H^{1,2}(J) = \Lambda^3 V^* \oplus (\Lambda^2 V^* \otimes V) \), where \( V = H^{1,0}(J) = H^0(C, K_C) \). We first compute the eigenvalues of \( \sigma \) on \( V \). The elements of \( V \) are of the form \( L \cdot X d Z - Z d X Y^2 Z \), with \( L \in H^0(\mathbb{P}^2, \mathcal{O}_\mathbb{P}(1)) \); it follows that the eigenvalues of \( \sigma \) on \( V \) are \( \omega^5, \omega^7, \omega^8 \). Therefore the eigenvalue on \( \Lambda^3 V^* \) is \( \omega^7 \), and the eigenvalues on \( \Lambda^2 V^* \) are \( \omega^3, \omega^5, \omega^6 \). Thus each product of an eigenvalue on \( \Lambda^2 V^* \) and one on \( V \) is \( \neq 1 \), hence the Proposition. \( \square \)

**References**


