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# On Ampleness of vector bundles 

Snehajit Misra*, ${ }^{*}$ and Nabanita Ray ${ }^{a}$<br>${ }^{a}$ Tata Institute of Fundamental Research (TIFR), Homi Bhabha Road, Colaba, Mumbai 400005, India.<br>E-mails: smisra@math.tifr.res.in (Snehajit Misra), nray@math.tifr.res.in (Nabanita Ray)


#### Abstract

In this article, we give a necessary and sufficient condition for ampleness of semistable vector bundles with vanishing discriminant on a smooth projective variety $X$. As an application, we show ampleness of some special vector bundles on certain ruled surfaces. We prove similar results for parabolic ampleness.


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## 1. Introduction

Let $X$ be a complex manifold of dimension $n$, and $E$ be a holomorphic vector bundle of rank $r$ on $X$ endowed with a hermitian metric $h$. The hermitian bundle ( $E, h$ ) determines a unique hermitian connection compatible with the complex structure on $X$ and $E$, called as Chern connection, and it is denoted by $D_{E}$. This connection $D_{E}$ in turn gives rise to a curvature tensor, called as Chern curvature tensor and denoted by $\Theta(E, h) \in C^{\infty}\left(X, \wedge^{1,1} T_{X}^{*} \otimes \operatorname{End}(E)\right)$ a $\operatorname{End}(E)$ valued ( 1,1 ) form on $X$. If $z_{1}, z_{2}, \ldots, z_{n}$ are local coordinates on $X$, and if $\left(e_{\lambda}\right)_{1 \leq \lambda \leq r}$ is a local orthonormal frame on $E$, then one can write

$$
i \Theta(E, h)=\sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j k \lambda \mu} \mathrm{~d} z_{j} \wedge \mathrm{~d} \bar{z}_{k} \otimes e_{\lambda}^{*} \otimes e_{\mu}
$$

where $c_{j} k \lambda \mu=c_{k} j \mu \lambda$. One looks at the associated quadratic form on $S=T_{X} \otimes E$ as follows:

$$
\widetilde{\Theta}_{E, h}(\xi \otimes v)=\left\langle\Theta_{E, h}(\xi, \bar{\xi}) \cdot v, \nu\right\rangle_{h}=\sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j} k \lambda \mu \xi_{j} \bar{\xi}_{k} \nu_{\lambda} \bar{v}_{\mu} .
$$

The hermitian bundle $(E, h)$ is said to be Griffiths positive if at any point $z \in X$, we have $\widetilde{\Theta}_{E, h}(\xi \otimes v)>0$ for all $0 \neq \xi \in T_{X, z}$ and for all $0 \neq v \in E_{z}$.

A holomorphic vector bundle $E$ on a complex projective manifold is called ample in the sense of Hartshorne if the tautological line bundle $\mathscr{O}_{\mathbb{P}(E)}(1)$ is ample. i.e. there exists a smooth hermitian metric on $\mathscr{O}_{\mathbb{P}(E)}(1)$ with positive curvature.

[^0]It is always true that if a hermitian holomorphic vector bundle ( $E, h$ ) on a complex projective manifold $X$ is Griffiths positive, then $E$ is ample in the sense of Hartshorne. A famous conjecture of Griffiths asks whether ample bundles in the sense of Hartshorne admit Griffiths positively curved metrics. Also it is well known that if $E$ is ample, then $\operatorname{det}(E)$ is ample. However, ampleness of $\operatorname{det}(E)$ does not ensure ampleness of $E$ in general.

For a vector bundle of rank $r$ on a complex manifold $X$, the characteristic class

$$
c_{2}(\operatorname{End}(E))=2 r c_{2}(E)-(r-1) c_{1}^{2}(E) \in H^{4}(X, \mathbb{Q})
$$

is called the discriminant of $E$, denoted by $\triangle(E)$.
In Section 3, we prove the following.
Theorem 1. Let $X$ be a projective variety of dimension $n$ and $(E, h)$ be a hermitian holomorphic bundle of rank $r$ on $X$. Further assume that $E$ is a semistable vector bundle with $\triangle(E)=0$. Then the following are equivalent:
(i) $(E, h)$ is Griffiths positive.
(ii) $E$ is ample in the sense of Hartshorne.
(iii) $\operatorname{det}(E)$ is ample.

The Nakai-Moishezon criterion for ampleness says that a line bundle $L$ on a projective variety $X$ is ample if and only if $L^{\operatorname{dim} Y} \cdot Y>0$ for every positive dimensional subvarieties $Y$ of $X$. Mumford gave an example of a non-ample line bundle on a ruled surface whose intersection with every curve is positive (see [14, Chapter 1]). Therefore, in general, it is not sufficient to check the condition only for curves in Nakai-Moishezon criterion. However, in some special cases, to check ampleness of a line bundle $L$ on $X$, it is enough to check that $L \cdot C>0$ for every irreducible curve $C \subset X$ (e.g., on abelian varieties [21], on flag bundles [7]). One must also note that for a globally generated vector bundle $E$ on $X, E$ is ample if and only if it's restriction to every curve $C \subset X$ is ample. This follows easily from Gieseker's Lemma (see [15, Proposition 6.1.7]). In general, there is no straight forward way to check ampleness of a given vector bundle on a projective variety $X$. In [12], it is proved that an equivariant vector bundle on a toric variety $X$ is ample if and only if its restriction to finitely many invariant curves in $X$ are ample. Similar result holds for torus equivariant vector bundles on certain homogenous variety (see [6]). In [1], a sufficient condition is given to check ampleness of a vector bundle of rank 2 on some specific smooth surfaces with Picard rank 1.

We recall from [11, Chapter 5] that a vector bundle $W$ of rank 2 on a smooth projective curve $C$ is said to be normalized if $H^{0}(W) \neq 0$, but $H^{0}(W \otimes L)=0$ for all line bundle $L$ on $C$ with $\operatorname{deg}(L)<0$. We notice that a normalized bundle $W$ is semistable if and only if $\operatorname{deg}(W) \geq 0$. An important consequence of Theorem 1 is the following.

Corollary 2. Let $\rho: X=\mathbb{P}(W) \longrightarrow C$ be a ruled surface defined by a normalized rank 2 bundle on a smooth curve $C$ such that $\mu_{\min }(W)=\operatorname{deg}(W)$. Let $E$ be a semistable vector bundle of rank $r$ on $X$ with discriminant $\Delta(E)=0$. Then, $E$ is ample if and only if $\left.E\right|_{\sigma}$ and $\left.E\right|_{f}$ are ample, where $\sigma$ is the smooth section of $\rho$ such that $\mathscr{O}_{X}(\sigma) \cong \mathscr{O}_{\mathbb{P}(W)}(1)$ and $f$ is a fibre of $\rho$.

The above Corollary 2 implies the following:
Corollary 3. Let $\rho: X=\mathbb{P}(W) \longrightarrow C$ be a ruled surface on a smooth curve $C$ defined by $a$ normalized rank 2 bundle $W$ on $C$ with $\mu_{\min }(W)=\operatorname{deg}(W)$, and $E$ be a vector bundle on $C$. Then the vector bundle $E=\rho^{*}(V) \otimes \mathscr{O}_{\mathbb{P}(W)}(m)$ is ample on $X$ if and only if $m>0$ and $\mu_{\min }(E)>$ $-m \operatorname{deg}(W)$.

We also prove similar result for parabolic ampleness in Section 4.

## 2. Preliminaries

### 2.1. Harder-Narasimhan Filtration

A non-zero torsion-free coherent sheaf $\mathscr{G}$ on $X$ is said to be $H$-semistable if

$$
\mu_{H}(\mathscr{F})=\frac{c_{1}(\mathscr{F}) \cdot H^{n-1}}{\operatorname{rank}(\mathscr{F})} \leq \mu_{H}(\mathscr{G})=\frac{c_{1}(\mathscr{G}) \cdot H^{n-1}}{\operatorname{rank}(\mathscr{G})}
$$

for all subsheaves $\mathscr{F}$ of $\mathscr{G}$. For every vector bundle $E$ on $X$, there is a unique filtration

$$
0=E_{0} \subsetneq E_{1} \subsetneq E_{2} \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_{k}=E
$$

of subbundles of $E$, called the Harder-Narasimhan filtration of $E$, such that $E_{i} / E_{i-1}$ is H semistable torsion free sheaf for each $i \in\{1,2, \ldots, k\}$ and $\mu_{H}\left(E_{i} / E_{i-1}\right)>\mu_{H}\left(E_{i+1} / E_{i}\right)$ for each $i \in\{1,2, \ldots, k-1\}$. We define $Q_{k}:=E_{k} / E_{k-1}$ and $\mu_{\min }(E):=\mu_{H}\left(Q_{k}\right)=\mu_{H}\left(E_{k} / E_{k-1}\right)$.

Let $N_{1}(X)_{\mathbb{R}}$ be the set of all numerical equivalence classes of real one cycles on $X$. Inside $N_{1}(X)_{\mathbb{R}}$, the closure of the convex cone generated by effective one cycles is called the closed cone of curves and it is denoted by $\overline{\mathrm{NE}}(X)$. By Theorem 1.4.29 of [14], a divisor $D$ is ample if and only if $D \cdot \gamma>0$ for all $\gamma \in \overline{\mathrm{NE}}(X)-\{0\}$.

## 3. Main result and applications

We begin this section by proving our main result.
Proof of Theorem 1. (i) $\Longrightarrow$ (ii). See Theorem 6.1.25 in [15] for a proof.
(ii) $\Longrightarrow$ (iii). See Corollary 5.3 in [10] for a proof.
(iii) $\Longrightarrow$ (i). There exists a filtration

$$
0=E_{0} \subsetneq E_{1} \subsetneq E_{2} \subsetneq \cdots \subsetneq E_{l-1} \subsetneq E_{l}=E
$$

such that on each $G_{j}=E_{j} / E_{j-1}$, there exists a hermitian metric $h_{j}$ on $G_{j}$ for which the curvature tensor is equal to $\frac{1}{r} \gamma \otimes \operatorname{Id}_{G_{j}}$ where $\gamma$ is (1,1)-form representing the first Chern class $c_{1}(E)$ (see [19]). Since $\operatorname{det}(E)$ is ample, each $\left(G_{j}, h_{j}\right)$ is Griffiths positive. As extension of two Griffiths positive bundles is again Griffiths positive, we have inductively each $E_{i}$ is Griffiths positive and thus $E$ is also Griffiths positive.

Remark 4. Theorem 1 can be thought of as a generalization of Gieseker's ampleness criterion for semistable vector bundles on smooth curves (see [13, Theorem 3.2.7]). However, the condition about vanishing discriminant is not essential for both $V$ and $\operatorname{det}(V)$ to be ample. For example, consider the tangent bundle $T_{\mathbb{P}^{2}}$. Then $T_{\mathbb{P}^{2}}$ sits in the following exact sequence:

$$
0 \longrightarrow \mathscr{O}_{\mathbb{P}^{2}} \longrightarrow \mathscr{O}_{\mathbb{P}^{2}}(1)^{\oplus 3} \longrightarrow T_{\mathbb{P}^{2}} \longrightarrow 0 .
$$

Hence, $T_{\mathbb{P}^{2}}$ being quotient of an ample bundle is ample and $\operatorname{det}\left(T_{\mathbb{P}^{2}}\right) \cong \mathscr{O}_{\mathbb{P}^{2}}(3)$ is also ample. But $T_{\mathbb{P}^{2}}$ is semistable with $\Delta\left(T_{\mathbb{P}^{2}}\right) \neq 0$.

Remark 5. Note that for a vector bundle $E$ on a smooth projective curve $C$, we have $\Delta(E)=0$. Hence our result Theorem 1 is analogus to the result in [22]. Also one can compare our result with the results in [16] and [20].

A vector bundle $V$ on an abelian variety $X$ is called weakly-translation invariant (semihomogeneous in the sense of Mukai) if for every closed point $x \in X$, there is a line bundle $L_{x}$ on $X$ depending on $x$ such that $T_{x}^{*}(V) \simeq V \otimes L_{x}$ for all $x \in X$, where $T_{x}$ is the translation morphism given by $x \in X$.

Corollary 6. A semi-homogeneous vector bundle $E$ of rank $r$ on an abelian variety $X$ is ample if and only if $\operatorname{det}(E)$ is ample if and only if $\operatorname{det}(E) \cdot C>0$ for all irreducible curve $C$ in $X$.

Proof. Mukai proved that $E$ is Gieseker semistable (see [13, Chapter 1] for definition) with respect to some polarization and it has projective Chern classes zero, i.e., if $c(E)$ is the total Chern class, then $c(E)=\left\{1+c_{1}(E) / r\right\}^{r}$ (see [18, Theorem 5.8, p. 260], [18, Proposition 6.13, p. 266]; also see [17, p. 2]). Gieseker semistablity implies slope semistability (see [13]). So, in particular, we have $E$ is slope semistable with $\Delta(E)=2 r c_{2}(E)-(r-1) c_{1}^{2}(E)=0$. Hence, the result follows from Theorem 19 and Proposition 1.4 in [21].

Corollary 7. Let $W$ be a vector bundle of rank $m$ over a smooth complex projective curve $C$ and $\rho: \mathbb{P}(W) \longrightarrow C$ be the projectivisation map. Let $E$ be a semistable vector bundle on $\mathbb{P}(W)$ of rank $r$ with discriminant $\triangle(E)=0$, and $c_{1}(E) \equiv x \xi+y f$, where $\xi$ and $f$ are the numerical classes of $\mathscr{O}_{\mathbb{P}(W)}(1)$ and a fibre of $\rho$ respectively. Then, $E$ is ample if and only if $x>0$ and $\left(x \mu_{\min }(W)+y\right)>0$.

Proof. We note that by Lemma 2.1 of [9], the nef cone of divisors in $\mathbb{P}(W)$ is given by

$$
\operatorname{Nef}(\mathbb{P}(W))=\left\{a\left(\xi-\mu_{\min }(W) f\right)+b f \mid a, b \in \mathbb{R}_{\geq 0}\right\}
$$

Applying duality (see [14, Proposition 1.4.28]), we get

$$
\overline{\mathrm{NE}}(\mathbb{P}(W))=\left\{a\left(\xi^{m-1}-\left(\operatorname{deg}(W)-\mu_{\min }(W)\right) \xi^{m-2} f\right)+b \xi^{m-2} f \mid a, b \in \mathbb{R}_{\geq 0}\right\}
$$

Hence, $\operatorname{det}(E)$ is ample if and only if

- $c_{1}(E) \cdot\left\{\xi^{m-1}-\left(\operatorname{deg}(W)-\mu_{\min }(W)\right) \xi^{m-2} f\right\}=\left(x \mu_{\min }(W)+y\right)>0$ and
- $c_{1}(E) \cdot \xi^{m-2} f=x>0$.

Therefore, the result follows from the previous Theorem.
Corollary 8. Let $\rho: X=\mathbb{P}(W) \longrightarrow C$ be a ruled surface defined by a normalized rank 2 bundle on a smooth curve $C$ such that $\mu_{\min }(W)=\operatorname{deg}(W)$. Let $E$ be a semistable vector bundle of rank $r$ on $X$ with discriminant $\triangle(E)=0$. Then, $E$ is ample if and only if $\left.E\right|_{\sigma}$ and $\left.E\right|_{f}$ are ample, where $\sigma$ is the smooth section of $\rho$ such that $\mathscr{O}_{X}(\sigma) \simeq \mathscr{O}_{\mathbb{P}(W)}(1)$ and $f$ is a fibre of $\rho$.

Proof. Let $c_{1}(E) \equiv x \zeta+y f$, where $\zeta=[\sigma] \in N^{1}(X)$. Note that, by the given hypothesis, both $\left.E\right|_{\sigma}$ and $\left.E\right|_{f}$ are semistable, and hence both are ample if and only if

- $\operatorname{deg}\left(\left.E\right|_{\sigma}\right)=(x \zeta+y f) \cdot \zeta=(x \operatorname{deg}(W)+y)>0$, and
- $\operatorname{deg}\left(\left.E\right|_{f}\right)=(x \zeta+y f) \cdot f=x>0$.

But, in that case, $\left(x \mu_{\min }(W)+y\right)=(x \operatorname{deg}(W)+y)>0$. Therefore, the result follows from the previous corollary.

Remark 9. Let $\rho: X=\mathbb{P}(W) \longrightarrow C$ be a ruled surface on a smooth curve $C$ as in Corollary 8. Then, for any semistable vector bundle $R$ on $C$ and any integer $m, E:=\rho^{*}(R) \otimes \mathscr{O}_{\mathbb{P}(W)}(m)$ is a semistable vector bundle with vanishing discriminant. Hence by Corollary 8 , any semistable vector bundle $V$ on $X$ of this form $\rho^{*}(R) \otimes \mathscr{O}_{\mathbb{P}(W)}(m)$ is ample if and only if $\left.E\right|_{\sigma}$ and $\left.E\right|_{f}$ are ample if and only if $m>0$ and $\operatorname{deg}(R)>-m \operatorname{deg}(W)$.

For example, we consider the ruled surface $\rho: X=\mathbb{P}(W) \longrightarrow C$ over the elliptic curve $C$ defined by the nonsplit extension $0 \longrightarrow \mathscr{O}_{C} \longrightarrow W \longrightarrow \mathscr{O}_{C} \longrightarrow 0$. Then for any semistable bundle $R$ on $C$ of positive degree, $E:=\rho^{*}(R) \otimes \mathscr{O}_{X}(m)$ is ample for every positive integer $m$.

Corollary 10. Let $\rho: X=\mathbb{P}(W) \longrightarrow C$ be a ruled surface on a smooth curve $C$ defined by $a$ normalized rank 2 bundle $W$ on $C$ with $\mu_{\min }(W)=\operatorname{deg}(W)$, and $V$ be a vector bundle on $C$. Then the vector bundle $E=\rho^{*}(V) \otimes \mathscr{O}_{\mathbb{P}(W)}(m)$ is ample on $X$ if and only if $m>0$ and $\mu_{\min }(V)>$ $-m \operatorname{deg}(W)$.

Proof. Let

$$
0=V_{0} \subsetneq V_{1} \subsetneq V_{2} \subsetneq \cdots \subsetneq V_{k-1} \subsetneq V_{k}=V
$$

be the Harder-Narasimhan filtration of $V$, and $Q_{i}=V_{i} / V_{i-1}$ for each $i$. Since $\rho$ is a smooth map, in particular it is flat and hence $\rho^{*}$ is an exact functor. We also observe that for any ample line bundle $H$ on $X$, we have $\mu_{H}\left(\rho^{*} Q_{i}\right)=\mu\left(Q_{i}\right)(f \cdot H)$ and $f \cdot H>0$, where $f$ denotes a fiber of $\rho$. Fix $E_{i}:=\rho^{*}\left(V_{i}\right) \otimes \mathscr{O}_{\mathbb{P}(E)}(m)$. Then above observation and the uniqueness of Harder Narasimhan filtration imply that

$$
0=E_{0} \subsetneq E_{1} \subsetneq E_{2} \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_{k}=E
$$

is the Harder Narasimhan filtration of $E$ with respect to any polarization $H$.
Now, suppose $m$ satisfies $m>0$ and $\mu_{\min }(V)>-m \operatorname{deg}(W)$. Then by the previous remark, we conclude that each $R_{i}:=\rho^{*}\left(Q_{i}\right) \otimes \mathscr{O}_{\mathbb{P}(W)}(m)$ is ample. Inductively, each $E_{i}$ is ample. In particular $E$ is also ample.

Conversely, if $V$ is ample for some $m$, then $R_{k}=\rho^{*}\left(Q_{k}\right) \otimes \mathscr{O}_{\mathbb{P}(W)}(m)$ is ample for each $k$. Thus $m$ must satisfy $m>0$ and $\mu_{\min }(E)>-m \operatorname{deg}(W)$.

Example 11. Let us consider the ruled surface $\rho: X=\mathbb{P}(W) \longrightarrow C$ over a curve $C$ where $W=\mathscr{O}_{C} \oplus \mathscr{L}$ for some line bundle $\mathscr{L}$ on $C$ with $\operatorname{deg}(\mathscr{L})<0$. Then for any vector bundle $E$ on $C$ with $\mu_{\min }(E)>-m \operatorname{deg}(\mathscr{L})$ for some positive integer $m$, the bundle $V=\rho^{*}(E) \otimes \mathscr{O}_{X}(m)$ is ample.

Let $\rho: \mathbb{F}_{e}=\mathbb{P}\left(\mathscr{O}_{\mathbb{P}^{1}} \oplus \mathscr{O}_{\mathbb{P}^{1}}(-e)\right) \longrightarrow \mathbb{P}^{1}$ be a Hirzebruch surface for some $e \geq 0$. Let $C_{0}$ be its normalized section such that $\mathscr{O}_{\mathbb{F}_{e}}\left(C_{0}\right)$ is the tautological bundle on $\mathbb{F}_{e}$, and $f$ be a fibre of $\rho$. We recall the following results from [11, Chapter 5].

Theorem 12. Let $D \sim a C_{0}+$ bf be a divisor on $\mathbb{F}_{e}$. Then
(a) If $D$ is an irreducible curve $\neq C_{0}, f$, then $a>0$ and $b \geq a e$.
(b) The linear system $|D|$ contains a section of $\rho$ if and only if $a=1$ and either $b=0$ or $b \geq e$.
(c) The linear system $|D|$ contains an irreducible non-singular curve if and only if it contains an irreducible curve if and only if $a=0, b=1$ (namely $f$ ) ; or $a=1, b=0$ (namely $C_{0}$ ) ; or $a>0, b>a e$; or $e>0, a>0, b=a e$.
(d) $D$ is very ample if and only if $D$ is ample if and only if $a>0$ and $b>a e$.

Lemma 13. Any irreducible curve of $\mathbb{F}_{e}$ other than the fibers of $\rho$ is linearly equivalent to an effective curve which is a union of sections of the map $\rho$.

Proof. Let $C$ be an irreducible curve in $\mathbb{F}_{e}$ other than a fibre and the section $C_{0}$. Then $C \sim x C_{0}+y f$ for some $x>0$ and $y \geq x e$. Let $y=m x e+r$ for some $m>0$ and $0 \leq r<x e$. Now, $C \sim x C_{0}+y f \sim$ $(x-1)\left(C_{0}+e f\right)+\left(C_{0}+(e(x m-x+1)+r) f\right)$. This proves the result.
Proposition 14. Let $C$ be an irreducible curve in $\mathbb{F}_{e}$ and $C \sim C_{1}+\cdots+C_{r}$ where $C_{i}$ 's are sections of the map $\rho$. Let $E$ be a vector bundle on $\mathbb{F}_{e}$ such that for any two curves $B$ and $B^{\prime}$ in $\mathbb{F}_{e}$ with $B \sim B^{\prime}$, we have $\left.\left.E\right|_{B} \cong E\right|_{B^{\prime}}$. Then $\rho_{*}\left(\left.E\right|_{C}\right) \cong \bigoplus_{i} \rho_{*}\left(\left.E\right|_{C_{i}}\right)$ as vector bundles on $\mathbb{P}^{1}$.

Proof. We first observe that for any two curves $B$ and $B^{\prime}$ in $\mathbb{F}_{e}$ which are linearly equivalent to each other,

$$
\rho_{*}\left(E \otimes \mathscr{O}_{B}\right) \cong \rho_{*}\left(E \otimes \mathscr{O}_{B^{\prime}}\right) \quad \text { on } \mathbb{P}^{1}
$$

In other words, $\rho_{*}\left(\left.E\right|_{B}\right) \cong \rho_{*}\left(\left.E\right|_{B^{\prime}}\right)$ on $\mathbb{P}^{1}$. So, without loss of generality we assume that $C=$ $C_{1}+\cdots+C_{r}$ and $C_{i} \hookrightarrow C$ be an irreducible component of it. Then, $\mathscr{O}_{C} \rightarrow \mathscr{O}_{C_{i}}$, which induces a sheaf map $\rho_{*}\left(E \otimes \mathscr{O}_{C}\right) \longrightarrow \rho_{*}\left(E \otimes \mathscr{O}_{C_{i}}\right)$ on $\mathbb{P}^{1}$ for all $i$, and hence induces a map $\rho_{*}\left(E \otimes \mathscr{O}_{C}\right) \longrightarrow$ $\bigoplus_{i} \rho_{*}\left(E \otimes \mathscr{O}_{C_{i}}\right)$ as well.

We claim that

$$
\rho_{*}\left(E \otimes \mathscr{O}_{C}\right) \longrightarrow \bigoplus_{i} \rho_{*}\left(E \otimes \mathscr{O}_{C_{i}}\right)
$$

is an isomorphism on $\mathbb{P}^{1}$. Indeed, for any $y \in \mathbb{P}^{1}$,

$$
\left(\rho_{*}\left(E \otimes \mathscr{O}_{C}\right)\right)_{y} \cong \bigoplus_{x \in\left\{C \cap \rho^{-1}(y)\right\}} E_{x} .
$$

On the other hand,

$$
\left(\bigoplus_{i} \rho_{*}\left(E \otimes \mathscr{O}_{C_{i}}\right)\right)_{y} \cong \bigoplus_{i}\left(\rho_{*}\left(E \otimes \mathscr{O}_{C_{i}}\right)\right)_{y} \cong \bigoplus_{x \in C_{i}, \rho(x)=y} E_{x}
$$

Hence, the map is isomorphic at the stalk level. This proves our claim and the result.
Any rank two vector bundle $E$ on $\mathbb{F}_{e}$ has two numerical invariants describing it as an extension in a cannonical manner. The first invariant $d_{E}$ is defined by the splitting type of $E$ on a general fiber $f$, i.e., if $\left.E\right|_{f}=\mathscr{O}_{\mathbb{P}^{1}}(d) \oplus \mathscr{O}_{\mathbb{P}^{1}}\left(d^{\prime}\right)$ and $d \geq d^{\prime}$, then $d_{E}=d$. The second invariant $r_{E}=r=$ $\operatorname{deg}\left(\rho_{*}\left(E\left(-d C_{0}\right)\right)\right)$. See [8] for more information about these numerical invariants $d$ and $r$. Note that, if $E$ is globally generated, then for a generic fibre $f,\left.E\right|_{f}=\mathscr{O}_{\mathbb{P}^{1}}(d) \oplus \mathscr{O}_{\mathbb{P}^{1}}\left(d^{\prime}\right)$, where $d \geq d^{\prime} \geq 0$.

Theorem 15. Let $E$ be a globally generated rank two bundle on $\mathbb{F}_{e}$ with numerical invariants $d$ and $r$, and $E$ sits in the exact sequence

$$
\begin{equation*}
0 \rightarrow \mathscr{O}\left(d C_{0}+r f\right) \rightarrow E \rightarrow \mathscr{O}\left(d^{\prime} C_{0}+r^{\prime} f\right) \rightarrow 0 \tag{1}
\end{equation*}
$$

Further assume that for any two curves $B$ and $B^{\prime}$ in $\mathbb{F}_{e}$ with $B \sim B^{\prime}$, we have $\left.\left.E\right|_{B} \cong E\right|_{B^{\prime}}$. Then, $E$ is ample if and only if $\left.E\right|_{f},\left.E\right|_{C_{0}}$ and $\left.E\right|_{C_{0}+n f}$ are ample on a generic fibre $f$, on $C_{0}$ and sections of $\rho$ of the forms $C_{0}+n f$ with $d(n-e)+r \leq 0$ respectively.

Proof. Restriction of ample bundle being ample, $\left.E\right|_{C}$ is ample for any curve $C$ in $\mathbb{F}_{e}$ whenever $E$ is ample.

Conversely, let $\left.E\right|_{f},\left.E\right|_{C_{0}}$ and $\left.E\right|_{C_{0}+n f}$ are ample on a generic fibre $f$, on $C_{0}$ and sections of $\rho$ of the forms $C_{0}+n f$ with $d(n-e)+r \leq 0$ respectively. Now, if

$$
\left.E\right|_{f}=\mathscr{O}_{\mathbb{P}^{1}}(d) \oplus \mathscr{O}_{\mathbb{P}^{1}}\left(d^{\prime}\right),
$$

for a generic fibre $f$ of $\rho$ with $d \geq d^{\prime} \geq 0$, then the ampleness of $\left.E\right|_{f}$ implies that $d, d^{\prime}>0$.
Let $f^{\prime}$ be a fibre among those finitely many fibre which has different splitting type of $E$ than that of a generic fibre. Restricting the exact sequence (1) to $f^{\prime}$, we get

$$
\left.0 \rightarrow \mathscr{O}_{\mathbb{P}^{1}}(d) \rightarrow E\right|_{f^{\prime}} \rightarrow \mathscr{O}_{\mathbb{P}^{1}}\left(d^{\prime}\right) \rightarrow 0
$$

Hence, $\left.E\right|_{f^{\prime}}$ being an extension of two ample line bundle, is also ample.
Let $C \sim C_{0}+n f$ be any section of $\rho$, where either $n=0$ or $n \geq e$. Now, restricting the exact sequence (1) to $C$, we get

$$
\left.0 \rightarrow \mathscr{O}_{\mathbb{P}^{1}}(d(n-e)+r) \rightarrow E\right|_{C} \rightarrow \mathscr{O}_{\mathbb{P}^{1}}\left(d^{\prime}(n-e)+r^{\prime}\right) \rightarrow 0
$$

As $\left.E\right|_{C_{0}}$ is ample on $C_{0}$, and $\mathscr{O}_{\mathbb{P}^{1}}\left(-d^{\prime} e+r^{\prime}\right)$ being the quotient is also ample. Hence, $\left(-d^{\prime} e+r^{\prime}\right)>0$, which implies $d^{\prime}(n-e)+r^{\prime}>0$ for any $n \geq 1$. Note that, if $d(n-e)+r>0$ then $\left.E\right|_{C_{0}+n f}$ is ample, as it is then an extension of two ample bundles. If $d(n-e)+r \leq 0$ then $\left.E\right|_{C_{0}+n f}$ is also ample by the given hypothesis. Therefore, we conclude that restriction of $E$ onto each fibre and each section is ample.

Let $C$ be any curve of $\mathbb{F}_{e}$ other than a fibre of $\rho$, and $C \sim C_{1}+\cdots+C_{r}$ where $C_{i}$ 's are sections. Now, using Proposition 14 , we get that $\rho_{*}\left(\left.E\right|_{C}\right)$ is an ample vector bundle on $\mathbb{P}^{1}$.

If $E$ is not ample, then by Gieseker's lemma [15, Proposition 6.1.7], there exists an irreducible curve $C$ in $\mathbb{F}_{e}$ other than the fibres and a surjective homomorphism $u:\left.E\right|_{C} \rightarrow \mathscr{O}_{C}$. This induces the sujection $\rho_{*}\left(\left.E\right|_{C}\right) \rightarrow \rho_{*}\left(\mathscr{O}_{C}\right) \cong \mathscr{O}_{\mathbb{P}^{1}}$, as well as the injection $\mathscr{O}_{\mathbb{P}^{1}} \hookrightarrow\left(\rho_{*}\left(\left.E\right|_{C}\right)\right)^{*}$ which contradicts the fact that $\rho_{*}\left(\left.E\right|_{C}\right)$ is an ample bundle on $\mathbb{P}^{1}$. Therefore, $E$ is ample. This completes the proof.

## 4. Remark about Parabolic Ampleness

Let $X$ be a connected smooth complex projective variety of dimension $d$ and $D \subset X$ be an effective divisor on $X$.

Definition 16. A quasi parabolic structure on a coherent sheaf E with respect to $D$ is a filtration by $\mathscr{O}_{X}$-coherent subsheaves

$$
E=\mathscr{F}_{1}(E) \supset \mathscr{F}_{2}(E) \supset \cdots \supset \mathscr{F}_{l}(E) \supset \mathscr{F}_{l+1}(E)=E(-D)
$$

where $E(-D)=E \otimes_{\mathscr{O}_{X}} \mathscr{O}_{X}(-D)$. The integer $l$ is called the length of the filtration.
A parabolic structure is a quasi-parabolic structure, as above, together with a system of parabolic weights $\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{l}\right\}$ such that $0 \leq \alpha_{1}<\alpha_{2}<\cdots<\alpha_{l-1}<\alpha_{l}<1$, where each $\alpha_{i}$ is attached to $\mathscr{F}_{i}(E)$.

We shall denote the parabolic sheaf by $\left(E, \mathscr{F}_{*}, \alpha_{*}\right)$ or simply by $E_{*}$ when there is no confusion. For any parabolic sheaf $E_{*}$ defined as above, for any $t \in \mathbb{R}$, we define the following filtration $\left\{E_{t}\right\}_{t \in \mathbb{R}}$ of coherent sheaves parametrized by $\mathbb{R}$ :

$$
E_{t}=\mathscr{F}_{i}(E)(-[t] D)
$$

where $[t]$ is the integral part of $t$ and $\alpha_{i-1}<t-[t] \leq \alpha_{i}$ with $\alpha_{0}=\alpha_{l}-1$ and $\alpha_{l+1}=1$. Note that, any coherent subsheaf $M$ of $E$ has an induced parabolic structure such that if $\left\{M_{t}\right\}_{t \in \mathbb{R}}$ is the corresponding filtration then $M_{t}=E_{t} \cap M$ for any $t \geq 0$.

The parabolic degree of $E_{*}$ with respect to a fixed ample bundle $L$ on $X$, denoted by par_deg $\left(E_{*}\right)$ is defined as follows :

$$
\operatorname{par} \_\operatorname{deg}\left(E_{*}\right):=\int_{-1}^{0} \operatorname{deg}\left(E_{t}\right) \mathrm{d} t
$$

The parabolic slope of $E_{*}$, denoted by par_ $\mu\left(E_{*}\right)$ is the quotient par_deg $\left(E_{*}\right) / \operatorname{rank}(E)$.
Definition 17. The parabolic sheaf $E_{*}$ is called parabolic semistable (resp. parabolic stable) if for every subsheaf $M$ of $E$ with $0<\operatorname{rank}(M)<\operatorname{rank}(E)$, and $E / M$ being torsion-free sheaf, the inequality par_ $\mu\left(M_{*}\right) \leq$ par_ $\mu\left(E_{*}\right)$ (resp. par_ $\left.\mu\left(M_{*}\right)<\mu\left(E_{*}\right)\right)$ is satisfied.

Consider the decomposition

$$
\begin{equation*}
D=\sum_{i=1}^{n} n_{i} D_{i} \tag{2}
\end{equation*}
$$

where any $D_{i}$ is a reduced irreducible divisor and $n_{i} \geq 1$. Let

$$
f_{i}: n_{i} D_{i} \longrightarrow X
$$

denote the inclusion of the subscheme $n_{i} D_{i}$. For $1 \leq i \leq n$, let

$$
\begin{equation*}
0=F_{l_{i}+1}^{i} \subset F_{l_{i}}^{i} \subset F_{l_{i}-1}^{i} \subset \cdots \subset F_{1}^{i}=f_{i}^{*} E \tag{3}
\end{equation*}
$$

Let $\alpha_{j}^{i}, 1 \leq j \leq l_{i}+1$ be real numbers satisfying

$$
1=\alpha_{l_{i}+1}^{i}>\alpha_{l_{i}}^{i}>\alpha_{l_{i}-1}^{i}>\cdots>\alpha_{2}^{i}>\alpha_{1}^{i} \geq 0 .
$$

From now on we will always impose the following three conditions on the parabolic bundles $E_{*}$ that we will consider:
(a) the parabolic divisor $D=\sum_{i=1}^{n} n_{i} D_{i}$ is a normal crossing divisor, i.e., all $n_{i}=1$ and $D_{i}$ are smooth divisors and they intersect transversally.
(b) all $F_{j}^{i}$ on $D_{i}$ in sequence (3) are subbundles of $f_{i}^{*} E$ for every $i$.
(c) all the weights $\alpha_{j}^{i}$ are rational numbers; so $\alpha_{j}^{i}=m_{j}^{i} / N$, where $N$ is a fixed integer and $m_{j}^{i} \in\{0,1, \ldots, N-1\}$.

In [2], parabolic tensor product has been defined. The parabolic $m$-fold symmetric product $S^{m}\left(E_{*}\right)$, is the invariant subsheaf of the $m$-fold parabolic tensor product of $E_{*}$ for the natural action of the permutation group for the factors of the tensor product. The underlying sheaf of the parabolic sheaf $S^{m}\left(E_{*}\right)$ will be denoted by $S^{m}\left(E_{*}\right)_{0}$. We recall the definition of parabolic ampleness from [3].

Definition 18. The parabolic sheaf $E_{*}$ is called parabolic ample if for any coherent sheaf $F$ on $X$ there is an integer $m_{0}$ such that for any $m \geq m_{0}$, the tensor product $F \otimes S^{m}\left(E_{*}\right)_{0}$ is globally generated.

Parabolic Chern classes $c_{i}\left(E_{*}\right) \in H^{2 i}(X, \mathbb{Q})$ has been introduced in [3]. For a parabolic vector bundle $E_{*}$ of rank $r$ we define the parabolic discriminant, denoted by $\Delta_{\operatorname{par}}\left(E_{*}\right)$ as follows:

$$
\triangle_{\mathrm{par}}\left(E_{*}\right):=2 r c_{2}\left(E_{*}\right)-(r-1) c_{1}^{2}\left(E_{*}\right)
$$

Theorem 19. Let $E_{*}$ be a semistable parabolic vector bundle of rank $r$ on a smooth complex projective variety $X$ such that $\Delta_{\mathrm{par}}\left(E_{*}\right)=0$. Then, $E_{*}$ is parabolic ample if and only if its parabolic first Chern class $c_{1}\left(E_{*}\right)$ is in the ample cone of $X$.
Proof. Let $p: Y \longrightarrow X$ be the Kawamata cover, $V$ be the corresponding orbifold bundle on $Y$ with $c_{1}(V)=p^{*} c_{1}\left(E_{*}\right)$ (see [2] and [4]). So if $E_{*}$ is ample, then $V$ is also ample (see [3]) and thus $c_{1}(V)$ is also ample. Using the finiteness of the surjective map $p$, we conclude that $c_{1}\left(E_{*}\right)$ is in the ample cone of $X$.

Conversly, if $c_{1}\left(E_{*}\right)$ is in the ample cone of $X$, then $\operatorname{det}(V)$ is also ample. Also, by the given hypothesis, $V$ is orbifold semistable and hence semistable (in the usual sense) with $\Delta(V)=$ $p^{*} \Delta_{\operatorname{par}}\left(E_{*}\right)=0$. Hence $V$ is ample and thus $E_{*}$ is parabolic ample.

Proposition 20. Let $\pi: X \longrightarrow Y$ be a smooth surjective morphism between two smooth connected complex projective varieties $X$ and $Y$. Let $E_{*}$ be a parabolic semistable bundle on $Y$ with parabolic divisor $D \subset Y$ and $\triangle_{\mathrm{par}}\left(E_{*}\right)=0$. Then, the pullback bundle $\pi^{*}\left(E_{*}\right)$ under the map $\pi$ is also parabolic semistable on $X$ with parabolic divisor $\pi^{*}(D) \subset X$ and $\triangle_{\operatorname{par}}\left(\pi^{*}\left(E_{*}\right)\right)=0$.

Conversely, if $E_{*}$ be a parabolic semistable bundle on projective bundle $\pi: X=\mathbb{P}(\mathscr{E}) \longrightarrow Y$ with parabolic divisor $D^{\prime}=\pi^{-1}(D)$, with $\triangle_{\mathrm{par}}\left(E_{*}\right)=0$ and the parabolic first Chern class $c_{1}\left(E_{*}\right)=$ $\pi^{*}(\mathscr{L})$ for some line bundle $\mathscr{L}$ on $Y$, then there exists a semistable parabolic bundle $E_{*}^{\prime}$ on $Y$ with parabolic divisor $D$ and $\triangle_{\mathrm{par}}\left(E_{*}^{\prime}\right)=0$ such that $E_{*}=\pi^{*}\left(E_{*}^{\prime}\right)$.
Proof. Let $D=\sum_{i=1}^{n} D_{i}$ be the normal crossing divisor on $Y$ and $D^{\prime}=\pi^{*}(D)$. Since $\pi$ is smooth, the pullback divisor $D^{\prime}$ on $X$ is also a normal crossing divisor satisfying condition (a).

Let $p: Y^{\prime} \longrightarrow Y$ be a Kawamata cover with Galois group $G$ such that

$$
p^{*} D_{i}=k_{i} N\left(p^{*} D_{i}\right)_{\mathrm{red}}
$$

for some positive integers $k_{i}$ and $N$. Consider the following fibre product diagram


Then $\widetilde{p}: X^{\prime} \longrightarrow X$ is a Galois cover with the same Galois group $G$. Let $V$ be the orbifold bundle on $Y^{\prime}$ associated to the parabolic bundle $E_{*}$ on $Y$ (see [4]). The pullback orbifold bundle $V^{\prime}:=\tilde{\pi}^{*}(V)$ then corresponds to the parabolic pullback bundle $\pi^{*}\left(E_{*}\right)$ on $X$ with parabolic divisor $D^{\prime}$.

We note that $\Delta(V)=\Delta_{\mathrm{par}}\left(E_{*}\right)=0$. Since $E_{*}$ is parabolic semistable, by using the correspondence in [4], $V$ is also orbifold semistable, and hence semistable (in the usual sense). Therefore
the pullback bundle $V^{\prime}$ is also orbifold semistable with $\Delta\left(V^{\prime}\right)=0$, proving that $\pi^{*}\left(E_{*}\right)$ is parabolic semistable with $\triangle_{\operatorname{par}}\left(\pi^{*}\left(E_{*}\right)\right)=0$.

Conversely, let $V$ be the orbifold bundle on $X^{\prime}$ associated to the parabolic bundle $E_{*}$ on $X$. Then,

$$
c_{1}(V)=\tilde{p}^{*} c_{1}\left(E_{*}\right)=\tilde{p}^{*} \pi^{*}(\mathscr{L})=\tilde{\pi}^{*} p^{*}(\mathscr{L})
$$

Now by the given hypothesis, $V$ is orbifold semistable and hence semistable (in the usual sense). Since $\Delta(V)=0$, by Theorem 1.2 in [5] $\left.V\right|_{f}$ is semistable on $f \simeq \mathbb{P}^{m}$ for every fibre $f$ of the map $\widetilde{\pi}$ (Here $\operatorname{rank}(\mathscr{E})=m+1$ ) and $\operatorname{deg}\left(\left.V\right|_{f}\right)=0$. This implies $V \simeq \widetilde{\pi}^{*}(W)$ for some orbifold bundle $W$ on $Y^{\prime}$ which must be semistable. Let $E_{*}^{\prime}$ be the associated semistable parabolic bundle on $Y$. Note that $\Delta(V)=\widetilde{\pi}^{*}(\Delta(W))=0$ and $\widetilde{\pi}^{*}$ is injective. Hence $\Delta(W)=0$. By a similar argument we have $\Delta_{\mathrm{par}}\left(E_{*}^{\prime}\right)=0$. Then by the construction of $E_{*}^{\prime}$, the result follows.
Corollary 21. Let $W$ be a vector bundle of rank $m$ over a smooth complex projective curve $C$ and $\rho: X=\mathbb{P}(W) \longrightarrow C$ be the projectivisation map. Let $E_{*}$ be a semistable vector bundle on $X$ of rank $r$ with parabolic discriminant $\triangle_{\mathrm{par}}\left(E_{*}\right)=0$, and parabolic 1st Chern class $c_{1}\left(E_{*}\right) \equiv x \xi+y f$, where $\xi$ and $f$ are the numerical classes of $\mathscr{O}_{\mathbb{P}(W)}(1)$ and a fibre of $\rho$ respectively. Then, $E_{*}$ is ample if and only if $x>0$ and $\left(x \mu_{\min }(W)+y\right)>0$.
Proof. We note that

$$
\overline{\mathrm{NE}}(\mathbb{P}(W))=\left\{a\left(\xi^{m-1}-\left(\operatorname{deg}(W)-\mu_{\min }(W)\right) \xi^{m-2} f\right)+b \xi^{m-2} f \mid a, b \in \mathbb{R}_{\geq 0}\right\}
$$

Hence, $c_{1}\left(E_{*}\right)$ is in the ample cone if and only if

- $c_{1}\left(E_{*}\right) \cdot\left\{\xi^{m-1}-\left(\operatorname{deg}(W)-\mu_{\min }(W)\right) \xi^{m-2} f\right\}=\left(x \mu_{\min }(W)+y\right)>0$ and
- $c_{1}\left(E_{*}\right) \cdot \xi^{m-2} f=x>0$.

Therefore, the result follows from the previous theorem.
Example 22. Let $\rho: X=\mathbb{P}(W) \longrightarrow Y$ be a projective bundle on a smooth projective variety $Y$. Let $D \subset Y$ be a normal crossing divisor in $Y$ and $F_{*}$ be a semistable parabolic bundle of rank $r$ on $Y$ with parabolic divisor $D$. Then $\rho^{*}\left(F_{*}\right)$ is a parabolic semistable bundle on $X$ with parabolic divisor $D^{\prime}=\rho^{*}(D)$. Let $D^{\prime}=\sum_{i=1}^{n} D_{i}^{\prime}$ be the decomposition into irreducible components of $D^{\prime}$. A parabolic line bundle with parabolic divisor $D^{\prime}$ is a data of the form $L_{*}=\left(L,\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}\right)$, where $L$ is a line bundle on $X$ and each $0 \leq \alpha_{i}<1$ corresponds to the divisor $D_{i}^{\prime}$. Assume $\alpha_{i} \in \mathbb{Q}$ for all $i$. Then $E_{*}=\rho^{*}\left(F_{*}\right) \otimes L_{*}$ is parabolic semistable with $\triangle_{\text {par }}\left(E_{*}\right)=0$. Note that $c_{1}\left(L_{*}\right):=c_{1}(L)+\sum_{i=1}^{n} \alpha_{i}\left[D_{i}\right]$. One can choose $L_{*}$ in such a way that $c_{1}\left(E_{*}\right)=c_{1}\left(\rho^{*} F_{*}\right)+r c_{1}\left(L_{*}\right)$ is in the ample cone of $X$. This way one can produce parabolic ample bundles on $X$.

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[^0]:    * Corresponding author.

