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On Ampleness of vector bundles

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Abstract. In this article, we give a necessary and sufficient condition for ampleness of semistable vector bundles with vanishing discriminant on a smooth projective variety *X*. As an application, we show ampleness of some special vector bundles on certain ruled surfaces. We prove similar results for parabolic ampleness.

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1. Introduction

Let *X* be a complex manifold of dimension *n*, and *E* be a holomorphic vector bundle of rank *r* on *X* endowed with a hermitian metric *h*. The hermitian bundle (E, h) determines a unique hermitian connection compatible with the complex structure on *X* and *E*, called as Chern connection, and it is denoted by D_E . This connection D_E in turn gives rise to a curvature tensor, called as Chern curvature tensor and denoted by $\Theta(E, h) \in C^{\infty}(X, \wedge^{1,1}T_X^* \otimes \text{End}(E))$ a End(E)-valued (1,1) form on *X*. If $z_1, z_2, ..., z_n$ are local coordinates on *X*, and if $(e_{\lambda})_{1 \leq \lambda \leq r}$ is a local orthonormal frame on *E*, then one can write

$$i\Theta(E,h) = \sum_{1 \le j,k \le n, 1 \le \lambda, \mu \le r} c_{jk\lambda\mu} \mathrm{d} z_j \wedge \mathrm{d} \overline{z}_k \otimes e_\lambda^* \otimes e_\mu,$$

where $c_i k \lambda \mu = c_k j \mu \lambda$. One looks at the associated quadratic form on $S = T_X \otimes E$ as follows:

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) = \left\langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \right\rangle_h = \sum_{1 \le j, k \le n, 1 \le \lambda, \mu \le r} c_j k \lambda \mu \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu$$

The hermitian bundle (E, h) is said to be Griffiths positive if at any point $z \in X$, we have $\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0$ for all $0 \neq \xi \in T_{X,z}$ and for all $0 \neq v \in E_z$.

A holomorphic vector bundle *E* on a complex projective manifold is called ample in the sense of Hartshorne if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample. i.e. there exists a smooth hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ with positive curvature.

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It is always true that if a hermitian holomorphic vector bundle (E, h) on a complex projective manifold X is Griffiths positive, then E is ample in the sense of Hartshorne. A famous conjecture of Griffiths asks whether ample bundles in the sense of Hartshorne admit Griffiths positively curved metrics. Also it is well known that if E is ample, then det(E) is ample. However, ampleness of det(E) does not ensure ampleness of E in general.

For a vector bundle of rank r on a complex manifold X, the characteristic class

$$c_2(\operatorname{End}(E)) = 2rc_2(E) - (r-1)c_1^2(E) \in H^4(X, \mathbb{Q})$$

is called the discriminant of *E*, denoted by $\Delta(E)$.

In Section 3, we prove the following.

Theorem 1. Let *X* be a projective variety of dimension *n* and (E, h) be a hermitian holomorphic bundle of rank *r* on *X*. Further assume that *E* is a semistable vector bundle with $\Delta(E) = 0$. Then the following are equivalent:

- (i) (E, h) is Griffiths positive.
- (ii) *E* is ample in the sense of Hartshorne.
- (iii) det(E) is ample.

The Nakai–Moishezon criterion for ampleness says that a line bundle *L* on a projective variety *X* is ample if and only if $L^{\dim Y} \cdot Y > 0$ for every positive dimensional subvarieties *Y* of *X*. Mumford gave an example of a non-ample line bundle on a ruled surface whose intersection with every curve is positive (see [14, Chapter 1]). Therefore, in general, it is not sufficient to check the condition only for curves in Nakai–Moishezon criterion. However, in some special cases, to check ampleness of a line bundle *L* on *X*, it is enough to check that $L \cdot C > 0$ for every irreducible curve $C \subset X$ (e.g., on abelian varieties [21], on flag bundles [7]). One must also note that for a globally generated vector bundle *E* on *X*, *E* is ample if and only if it's restriction to every curve $C \subset X$ is ample. This follows easily from Gieseker's Lemma (see [15, Proposition 6.1.7]). In general, there is no straight forward way to check ampleness of a given vector bundle on a projective variety *X*. In [12], it is proved that an equivariant vector bundle on a toric variety *X* is ample if and only if its restriction to finitely many invariant curves in *X* are ample. Similar result holds for torus equivariant vector bundles on certain homogenous variety (see [6]). In [1], a sufficient condition is given to check ampleness of a vector bundle of rank 2 on some specific smooth surfaces with Picard rank 1.

We recall from [11, Chapter 5] that a vector bundle W of rank 2 on a smooth projective curve C is said to be normalized if $H^0(W) \neq 0$, but $H^0(W \otimes L) = 0$ for all line bundle L on C with deg(L) < 0. We notice that a normalized bundle W is semistable if and only if deg $(W) \ge 0$. An important consequence of Theorem 1 is the following.

Corollary 2. Let $\rho: X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface defined by a normalized rank 2 bundle on a smooth curve C such that $\mu_{\min}(W) = \deg(W)$. Let E be a semistable vector bundle of rank r on X with discriminant $\Delta(E) = 0$. Then, E is ample if and only if $E|_{\sigma}$ and $E|_f$ are ample, where σ is the smooth section of ρ such that $\mathcal{O}_X(\sigma) \cong \mathcal{O}_{\mathbb{P}(W)}(1)$ and f is a fibre of ρ .

The above Corollary 2 implies the following:

Corollary 3. Let $\rho : X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface on a smooth curve C defined by a normalized rank 2 bundle W on C with $\mu_{\min}(W) = \deg(W)$, and E be a vector bundle on C. Then the vector bundle $E = \rho^*(V) \otimes \mathcal{O}_{\mathbb{P}(W)}(m)$ is ample on X if and only if m > 0 and $\mu_{\min}(E) > -m \deg(W)$.

We also prove similar result for parabolic ampleness in Section 4.

2. Preliminaries

2.1. Harder-Narasimhan Filtration

A non-zero torsion-free coherent sheaf \mathcal{G} on X is said to be H-semistable if

$$\mu_{H}(\mathscr{F}) = \frac{c_{1}(\mathscr{F}) \cdot H^{n-1}}{\operatorname{rank}(\mathscr{F})} \le \mu_{H}(\mathscr{G}) = \frac{c_{1}(\mathscr{G}) \cdot H^{n-1}}{\operatorname{rank}(\mathscr{G})}$$

for all subsheaves \mathscr{F} of \mathscr{G} . For every vector bundle *E* on *X*, there is a unique filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_k = E$$

of subbundles of *E*, called the Harder–Narasimhan filtration of *E*, such that E_i/E_{i-1} is *H*-semistable torsion free sheaf for each $i \in \{1, 2, ..., k\}$ and $\mu_H(E_i/E_{i-1}) > \mu_H(E_{i+1}/E_i)$ for each $i \in \{1, 2, ..., k-1\}$. We define $Q_k := E_k/E_{k-1}$ and $\mu_{\min}(E) := \mu_H(Q_k) = \mu_H(E_k/E_{k-1})$.

Let $N_1(X)_{\mathbb{R}}$ be the set of all numerical equivalence classes of real one cycles on *X*. Inside $N_1(X)_{\mathbb{R}}$, the closure of the convex cone generated by effective one cycles is called the closed cone of curves and it is denoted by $\overline{NE}(X)$. By Theorem 1.4.29 of [14], a divisor *D* is ample if and only if $D \cdot \gamma > 0$ for all $\gamma \in \overline{NE}(X) - \{0\}$.

3. Main result and applications

We begin this section by proving our main result.

Proof of Theorem 1. (i) \implies (ii). See Theorem 6.1.25 in [15] for a proof.

(ii) \implies (iii). See Corollary 5.3 in [10] for a proof.

(iii) \implies (i). There exists a filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{l-1} \subsetneq E_l = E$$

such that on each $G_j = E_j/E_{j-1}$, there exists a hermitian metric h_j on G_j for which the curvature tensor is equal to $\frac{1}{r}\gamma \otimes \text{Id}_{G_j}$ where γ is (1,1)-form representing the first Chern class $c_1(E)$ (see [19]). Since det(*E*) is ample, each (G_j, h_j) is Griffiths positive. As extension of two Griffiths positive bundles is again Griffiths positive, we have inductively each E_i is Griffiths positive and thus *E* is also Griffiths positive.

Remark 4. Theorem 1 can be thought of as a generalization of Gieseker's ampleness criterion for semistable vector bundles on smooth curves (see [13, Theorem 3.2.7]). However, the condition about vanishing discriminant is not essential for both *V* and det(*V*) to be ample. For example, consider the tangent bundle $T_{\mathbb{P}^2}$. Then $T_{\mathbb{P}^2}$ sits in the following exact sequence:

 $0 \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow \mathscr{O}_{\mathbb{P}^2}(1)^{\oplus 3} \longrightarrow T_{\mathbb{P}^2} \longrightarrow 0.$

Hence, $T_{\mathbb{P}^2}$ being quotient of an ample bundle is ample and $\det(T_{\mathbb{P}^2}) \cong \mathcal{O}_{\mathbb{P}^2}(3)$ is also ample. But $T_{\mathbb{P}^2}$ is semistable with $\Delta(T_{\mathbb{P}^2}) \neq 0$.

Remark 5. Note that for a vector bundle *E* on a smooth projective curve *C*, we have $\triangle(E) = 0$. Hence our result Theorem 1 is analogus to the result in [22]. Also one can compare our result with the results in [16] and [20].

A vector bundle *V* on an abelian variety *X* is called weakly-translation invariant (semihomogeneous in the sense of Mukai) if for every closed point $x \in X$, there is a line bundle L_x on *X* depending on *x* such that $T_x^*(V) \simeq V \otimes L_x$ for all $x \in X$, where T_x is the translation morphism given by $x \in X$. **Corollary 6.** A semi-homogeneous vector bundle E of rank r on an abelian variety X is ample if and only if det(*E*) is ample if and only if det(*E*) \cdot *C* > 0 for all irreducible curve C in X.

Proof. Mukai proved that *E* is Gieseker semistable (see [13, Chapter 1] for definition) with respect to some polarization and it has projective Chern classes zero, i.e., if c(E) is the total Chern class, then $c(E) = \{1 + c_1(E)/r\}^r$ (see [18, Theorem 5.8, p. 260], [18, Proposition 6.13, p. 266]; also see [17, p. 2]). Gieseker semistablity implies slope semistability (see [13]). So, in particular, we have E is slope semistable with $\Delta(E) = 2rc_2(E) - (r-1)c_1^2(E) = 0$. Hence, the result follows from Theorem 19 and Proposition 1.4 in [21].

Corollary 7. Let W be a vector bundle of rank m over a smooth complex projective curve C and $\rho: \mathbb{P}(W) \longrightarrow C$ be the projectivisation map. Let E be a semistable vector bundle on $\mathbb{P}(W)$ of rank r with discriminant $\Delta(E) = 0$, and $c_1(E) \equiv x\xi + yf$, where ξ and f are the numerical classes of $\mathscr{O}_{\mathbb{P}(W)}(1)$ and a fibre of ρ respectively. Then, E is ample if and only if x > 0 and $(x \mu_{\min}(W) + y) > 0$.

Proof. We note that by Lemma 2.1 of [9], the nef cone of divisors in $\mathbb{P}(W)$ is given by

$$\operatorname{Nef}(\mathbb{P}(W)) = \{a(\xi - \mu_{\min}(W)f) + bf \mid a, b \in \mathbb{R}_{\geq 0}\}.$$

Applying duality (see [14, Proposition 1.4.28]), we get

$$\overline{\mathrm{NE}}\big(\mathbb{P}(W)\big) = \Big\{a\big(\xi^{m-1} - \big(\mathrm{deg}(W) - \mu_{\min}(W)\big)\xi^{m-2}f\big) + b\xi^{m-2}f\Big| a, b \in \mathbb{R}_{\geq 0}\Big\}.$$

Hence, det(E) is ample if and only if

- $c_1(E) \cdot \{\xi^{m-1} (\deg(W) \mu_{\min}(W))\xi^{m-2}f\} = (x\mu_{\min}(W) + y) > 0$ and $c_1(E) \cdot \xi^{m-2}f = x > 0.$

Therefore, the result follows from the previous Theorem.

Corollary 8. Let $\rho: X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface defined by a normalized rank 2 bundle on a smooth curve C such that $\mu_{\min}(W) = \deg(W)$. Let E be a semistable vector bundle of rank r on X with discriminant $\Delta(E) = 0$. Then, E is ample if and only if $E|_{\sigma}$ and $E|_{f}$ are ample, where σ is the smooth section of ρ such that $\mathcal{O}_X(\sigma) \simeq \mathcal{O}_{\mathbb{P}(W)}(1)$ and f is a fibre of ρ .

Proof. Let $c_1(E) \equiv x\zeta + \gamma f$, where $\zeta = [\sigma] \in N^1(X)$. Note that, by the given hypothesis, both $E|_{\sigma}$ and $E|_f$ are semistable, and hence both are ample if and only if

- $\deg(E|_{\sigma}) = (x\zeta + yf) \cdot \zeta = (x \deg(W) + y) > 0$, and
- $\operatorname{deg}(E|_f) = (x\zeta + \gamma f) \cdot f = x > 0.$

But, in that case, $(x\mu_{\min}(W) + y) = (x \deg(W) + y) > 0$. Therefore, the result follows from the previous corollary.

Remark 9. Let $\rho: X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface on a smooth curve *C* as in Corollary 8. Then, for any semistable vector bundle R on C and any integer m, $E := \rho^*(R) \otimes \mathcal{O}_{\mathbb{P}(W)}(m)$ is a semistable vector bundle with vanishing discriminant. Hence by Corollary 8, any semistable vector bundle V on X of this form $\rho^*(R) \otimes \mathcal{O}_{\mathbb{P}(W)}(m)$ is ample if and only if $E|_{\sigma}$ and $E|_f$ are ample if and only if m > 0 and $\deg(R) > -m \deg(W)$.

For example, we consider the ruled surface $\rho: X = \mathbb{P}(W) \longrightarrow C$ over the elliptic curve C defined by the nonsplit extension $0 \longrightarrow \mathcal{O}_C \longrightarrow W \longrightarrow \mathcal{O}_C \longrightarrow 0$. Then for any semistable bundle *R* on *C* of positive degree, $E := \rho^*(R) \otimes \mathcal{O}_X(m)$ is ample for every positive integer *m*.

Corollary 10. Let $\rho: X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface on a smooth curve C defined by a normalized rank 2 bundle W on C with $\mu_{\min}(W) = \deg(W)$, and V be a vector bundle on C. Then the vector bundle $E = \rho^*(V) \otimes \mathcal{O}_{\mathbb{P}(W)}(m)$ is ample on X if and only if m > 0 and $\mu_{\min}(V) > 0$ $-m \deg(W)$.

Proof. Let

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{k-1} \subsetneq V_k = V$$

be the Harder–Narasimhan filtration of *V*, and $Q_i = V_i/V_{i-1}$ for each *i*. Since ρ is a smooth map, in particular it is flat and hence ρ^* is an exact functor. We also observe that for any ample line bundle *H* on *X*, we have $\mu_H(\rho^*Q_i) = \mu(Q_i)(f \cdot H)$ and $f \cdot H > 0$, where *f* denotes a fiber of ρ . Fix $E_i := \rho^*(V_i) \otimes \mathcal{O}_{\mathbb{P}(E)}(m)$. Then above observation and the uniqueness of Harder Narasimhan filtration imply that

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_k = E$$

is the Harder Narasimhan filtration of *E* with respect to any polarization *H*.

Now, suppose *m* satisfies m > 0 and $\mu_{\min}(V) > -m \deg(W)$. Then by the previous remark, we conclude that each $R_i := \rho^*(Q_i) \otimes \mathcal{O}_{\mathbb{P}(W)}(m)$ is ample. Inductively, each E_i is ample. In particular *E* is also ample.

Conversely, if *V* is ample for some *m*, then $R_k = \rho^*(Q_k) \otimes \mathcal{O}_{\mathbb{P}(W)}(m)$ is ample for each *k*. Thus *m* must satisfy m > 0 and $\mu_{\min}(E) > -m \deg(W)$.

Example 11. Let us consider the ruled surface $\rho : X = \mathbb{P}(W) \longrightarrow C$ over a curve *C* where $W = \mathcal{O}_C \oplus \mathcal{L}$ for some line bundle \mathcal{L} on *C* with $\deg(\mathcal{L}) < 0$. Then for any vector bundle *E* on *C* with $\mu_{\min}(E) > -m \deg(\mathcal{L})$ for some positive integer *m*, the bundle $V = \rho^*(E) \otimes \mathcal{O}_X(m)$ is ample.

Let $\rho : \mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \longrightarrow \mathbb{P}^1$ be a Hirzebruch surface for some $e \ge 0$. Let C_0 be its normalized section such that $\mathcal{O}_{\mathbb{F}_e}(C_0)$ is the tautological bundle on \mathbb{F}_e , and f be a fibre of ρ . We recall the following results from [11, Chapter 5].

Theorem 12. Let $D \sim aC_0 + bf$ be a divisor on \mathbb{F}_e . Then

- (a) If *D* is an irreducible curve $\neq C_0$, *f*, then a > 0 and $b \ge ae$.
- (b) The linear system |D| contains a section of ρ if and only if a = 1 and either b = 0 or $b \ge e$.
- (c) The linear system |D| contains an irreducible non-singular curve if and only if it contains an irreducible curve if and only if a = 0, b = 1 (namely f); or a = 1, b = 0 (namely C_0); or a > 0, b > ae; or e > 0, a > 0, b = ae.
- (d) *D* is very ample if and only if *D* is ample if and only if a > 0 and b > ae.

Lemma 13. Any irreducible curve of \mathbb{F}_e other than the fibers of ρ is linearly equivalent to an effective curve which is a union of sections of the map ρ .

Proof. Let *C* be an irreducible curve in \mathbb{F}_e other than a fibre and the section C_0 . Then $C \sim xC_0 + yf$ for some x > 0 and $y \ge xe$. Let y = mxe + r for some m > 0 and $0 \le r < xe$. Now, $C \sim xC_0 + yf \sim (x-1)(C_0 + ef) + (C_0 + (e(xm - x + 1) + r)f)$. This proves the result.

Proposition 14. Let *C* be an irreducible curve in \mathbb{F}_e and $C \sim C_1 + \cdots + C_r$ where C_i 's are sections of the map ρ . Let *E* be a vector bundle on \mathbb{F}_e such that for any two curves *B* and *B'* in \mathbb{F}_e with $B \sim B'$, we have $E|_B \cong E|_{B'}$. Then $\rho_*(E|_C) \cong \bigoplus_i \rho_*(E|_{C_i})$ as vector bundles on \mathbb{P}^1 .

Proof. We first observe that for any two curves *B* and *B'* in \mathbb{F}_e which are linearly equivalent to each other,

$$\rho_*(E \otimes \mathcal{O}_B) \cong \rho_*(E \otimes \mathcal{O}_{B'}) \quad \text{on } \mathbb{P}^1.$$

In other words, $\rho_*(E|_B) \cong \rho_*(E|_{B'})$ on \mathbb{P}^1 . So, without loss of generality we assume that $C = C_1 + \cdots + C_r$ and $C_i \hookrightarrow C$ be an irreducible component of it. Then, $\mathcal{O}_C \twoheadrightarrow \mathcal{O}_{C_i}$, which induces a sheaf map $\rho_*(E \otimes \mathcal{O}_C) \longrightarrow \rho_*(E \otimes \mathcal{O}_{C_i})$ on \mathbb{P}^1 for all *i*, and hence induces a map $\rho_*(E \otimes \mathcal{O}_C) \longrightarrow \bigoplus_i \rho_*(E \otimes \mathcal{O}_{C_i})$ as well.

We claim that

$$\rho_*(E\otimes \mathcal{O}_C) \longrightarrow \bigoplus_i \rho_*(E\otimes \mathcal{O}_{C_i})$$

is an isomorphism on \mathbb{P}^1 . Indeed, for any $y \in \mathbb{P}^1$,

$$(\rho_*(E\otimes \mathcal{O}_C))_y \cong \bigoplus_{x\in \{C\cap \rho^{-1}(y)\}} E_x.$$

On the other hand,

$$\left(\bigoplus_{i} \rho_*(E \otimes \mathcal{O}_{C_i})\right)_{\mathcal{Y}} \cong \bigoplus_{i} \left(\rho_*(E \otimes \mathcal{O}_{C_i})\right)_{\mathcal{Y}} \cong \bigoplus_{x \in C_i, \rho(x) = \mathcal{Y}} E_x.$$

Hence, the map is isomorphic at the stalk level. This proves our claim and the result.

Any rank two vector bundle E on \mathbb{F}_e has two numerical invariants describing it as an extension in a cannonical manner. The first invariant d_E is defined by the splitting type of E on a general fiber f, i.e., if $E|_f = \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(d')$ and $d \ge d'$, then $d_E = d$. The second invariant $r_E = r =$ $deg(\rho_*(E(-dC_0)))$. See [8] for more information about these numerical invariants d and r. Note that, if E is globally generated, then for a generic fibre f, $E|_f = \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(d')$, where $d \ge d' \ge 0$.

Theorem 15. Let *E* be a globally generated rank two bundle on \mathbb{F}_e with numerical invariants *d* and *r*, and *E* sits in the exact sequence

$$0 \to \mathcal{O}(dC_0 + rf) \to E \to \mathcal{O}(d'C_0 + r'f) \to 0.$$
⁽¹⁾

 \square

Further assume that for any two curves B and B' in \mathbb{F}_e with $B \sim B'$, we have $E|_B \cong E|_{B'}$. Then, E is ample if and only if $E|_f$, $E|_{C_0}$ and $E|_{C_0+nf}$ are ample on a generic fibre f, on C_0 and sections of ρ of the forms $C_0 + nf$ with $d(n-e) + r \leq 0$ respectively.

Proof. Restriction of ample bundle being ample, $E|_C$ is ample for any curve C in \mathbb{F}_e whenever E is ample.

Conversely, let $E|_f$, $E|_{C_0}$ and $E|_{C_0+nf}$ are ample on a generic fibre f, on C_0 and sections of ρ of the forms $C_0 + nf$ with $d(n - e) + r \le 0$ respectively. Now, if

$$E|_f = \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(d'),$$

for a generic fibre *f* of ρ with $d \ge d' \ge 0$, then the ampleness of $E|_f$ implies that d, d' > 0.

Let f' be a fibre among those finitely many fibre which has different splitting type of E than that of a generic fibre. Restricting the exact sequence (1) to f', we get

$$0 \to \mathscr{O}_{\mathbb{P}^1}(d) \to E|_{f'} \to \mathscr{O}_{\mathbb{P}^1}(d') \to 0$$

Hence, $E|_{f'}$ being an extension of two ample line bundle, is also ample.

Let $C \sim C_0 + nf$ be any section of ρ , where either n = 0 or $n \ge e$. Now, restricting the exact sequence (1) to *C*, we get

$$0 \to \mathcal{O}_{\mathbb{P}^1}(d(n-e)+r) \to E|_C \to \mathcal{O}_{\mathbb{P}^1}(d'(n-e)+r') \to 0.$$

As $E|_{C_0}$ is ample on C_0 , and $\mathcal{O}_{\mathbb{P}^1}(-d'e+r')$ being the quotient is also ample. Hence, (-d'e+r') > 0, which implies d'(n-e) + r' > 0 for any $n \ge 1$. Note that, if d(n-e) + r > 0 then $E|_{C_0+nf}$ is ample, as it is then an extension of two ample bundles. If $d(n-e) + r \le 0$ then $E|_{C_0+nf}$ is also ample by the given hypothesis. Therefore, we conclude that restriction of *E* onto each fibre and each section is ample.

Let *C* be any curve of \mathbb{F}_e other than a fibre of ρ , and $C \sim C_1 + \cdots + C_r$ where C_i 's are sections. Now, using Proposition 14, we get that $\rho_*(E|_C)$ is an ample vector bundle on \mathbb{P}^1 .

If *E* is not ample, then by Gieseker's lemma [15, Proposition 6.1.7], there exists an irreducible curve *C* in \mathbb{F}_e other than the fibres and a surjective homomorphism $u: E|_C \to \mathcal{O}_C$. This induces the sujection $\rho_*(E|_C) \to \rho_*(\mathcal{O}_C) \cong \mathcal{O}_{\mathbb{P}^1}$, as well as the injection $\mathcal{O}_{\mathbb{P}^1} \hookrightarrow (\rho_*(E|_C))^*$ which contradicts the fact that $\rho_*(E|_C)$ is an ample bundle on \mathbb{P}^1 . Therefore, *E* is ample. This completes the proof.

4. Remark about Parabolic Ampleness

Let *X* be a connected smooth complex projective variety of dimension *d* and $D \subset X$ be an effective divisor on *X*.

Definition 16. A quasi parabolic structure on a coherent sheaf *E* with respect to *D* is a filtration $by \mathcal{O}_X$ -coherent subsheaves

$$E = \mathscr{F}_1(E) \supset \mathscr{F}_2(E) \supset \cdots \supset \mathscr{F}_l(E) \supset \mathscr{F}_{l+1}(E) = E(-D)$$

where $E(-D) = E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$. The integer *l* is called the length of the filtration.

A parabolic structure is a quasi-parabolic structure, as above, together with a system of parabolic weights $\{\alpha_1, \alpha_2, ..., \alpha_l\}$ such that $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1$, where each α_i is attached to $\mathscr{F}_i(E)$.

We shall denote the parabolic sheaf by $(E, \mathscr{F}_*, \alpha_*)$ or simply by E_* when there is no confusion. For any parabolic sheaf E_* defined as above, for any $t \in \mathbb{R}$, we define the following filtration $\{E_t\}_{t \in \mathbb{R}}$ of coherent sheaves parametrized by \mathbb{R} :

$$E_t = \mathcal{F}_i(E)(-[t]D)$$

where [*t*] is the integral part of *t* and $\alpha_{i-1} < t - [t] \le \alpha_i$ with $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$. Note that, any coherent subsheaf *M* of *E* has an induced parabolic structure such that if $\{M_t\}_{t \in \mathbb{R}}$ is the corresponding filtration then $M_t = E_t \cap M$ for any $t \ge 0$.

The *parabolic degree* of E_* with respect to a fixed ample bundle *L* on *X*, denoted by par_deg(E_*) is defined as follows :

$$\operatorname{par_deg}(E_*) := \int_{-1}^0 \operatorname{deg}(E_t) \, \mathrm{d}t$$

The *parabolic slope* of E_* , denoted by par_ $\mu(E_*)$ is the quotient par_deg(E_*)/rank(E).

Definition 17. The parabolic sheaf E_* is called parabolic semistable (resp. parabolic stable) if for every subsheaf M of E with $0 < \operatorname{rank}(M) < \operatorname{rank}(E)$, and E/M being torsion-free sheaf, the inequality $\operatorname{par}_{\mu}(M_*) \leq \operatorname{par}_{\mu}(E_*)$ (resp. $\operatorname{par}_{\mu}(M_*) < \mu(E_*)$) is satisfied.

Consider the decomposition

$$D = \sum_{i=1}^{n} n_i D_i \tag{2}$$

where any D_i is a reduced irreducible divisor and $n_i \ge 1$. Let

$$f_i: n_i D_i \longrightarrow X$$

denote the inclusion of the subscheme $n_i D_i$. For $1 \le i \le n$, let

$$0 = F_{l_{i+1}}^{i} \subset F_{l_{i}}^{i} \subset F_{l_{i-1}}^{i} \subset \dots \subset F_{1}^{i} = f_{i}^{*} E$$
(3)

Let α_i^i , $1 \le j \le l_i + 1$ be real numbers satisfying

$$1 = \alpha_{l_i+1}^i > \alpha_{l_i}^i > \alpha_{l_i-1}^i > \dots > \alpha_2^i > \alpha_1^i \ge 0.$$

From now on we will always impose the following three conditions on the parabolic bundles E_* that we will consider:

- (a) the parabolic divisor $D = \sum_{i=1}^{n} n_i D_i$ is a normal crossing divisor, i.e., all $n_i = 1$ and D_i are smooth divisors and they intersect transversally.
- (b) all F_i^i on D_i in sequence (3) are subbundles of $f_i^* E$ for every *i*.
- (c) all the weights α_j^i are rational numbers; so $\alpha_j^i = m_j^i / N$, where N is a fixed integer and $m_j^i \in \{0, 1, ..., N-1\}$.

In [2], parabolic tensor product has been defined. The parabolic *m*-fold symmetric product $S^m(E_*)$, is the invariant subsheaf of the *m*-fold parabolic tensor product of E_* for the natural action of the permutation group for the factors of the tensor product. The underlying sheaf of the parabolic sheaf $S^m(E_*)$ will be denoted by $S^m(E_*)_0$. We recall the definition of parabolic ampleness from [3].

Definition 18. The parabolic sheaf E_* is called parabolic ample if for any coherent sheaf F on X there is an integer m_0 such that for any $m \ge m_0$, the tensor product $F \otimes S^m(E_*)_0$ is globally generated.

Parabolic Chern classes $c_i(E_*) \in H^{2i}(X, \mathbb{Q})$ has been introduced in [3]. For a parabolic vector bundle E_* of rank r we define the parabolic discriminant, denoted by $\Delta_{par}(E_*)$ as follows:

$$\Delta_{\text{par}}(E_*) := 2rc_2(E_*) - (r-1)c_1^2(E_*).$$

Theorem 19. Let E_* be a semistable parabolic vector bundle of rank r on a smooth complex projective variety X such that $\Delta_{par}(E_*) = 0$. Then, E_* is parabolic ample if and only if its parabolic first Chern class $c_1(E_*)$ is in the ample cone of X.

Proof. Let $p: Y \longrightarrow X$ be the Kawamata cover, *V* be the corresponding orbifold bundle on *Y* with $c_1(V) = p^*c_1(E_*)$ (see [2] and [4]). So if E_* is ample, then *V* is also ample (see [3]) and thus $c_1(V)$ is also ample. Using the finiteness of the surjective map *p*, we conclude that $c_1(E_*)$ is in the ample cone of *X*.

Conversely, if $c_1(E_*)$ is in the ample cone of *X*, then det(*V*) is also ample. Also, by the given hypothesis, *V* is orbifold semistable and hence semistable (in the usual sense) with $\Delta(V) = p^* \Delta_{par}(E_*) = 0$. Hence *V* is ample and thus E_* is parabolic ample.

Proposition 20. Let $\pi: X \longrightarrow Y$ be a smooth surjective morphism between two smooth connected complex projective varieties X and Y. Let E_* be a parabolic semistable bundle on Y with parabolic divisor $D \subset Y$ and $\Delta_{par}(E_*) = 0$. Then, the pullback bundle $\pi^*(E_*)$ under the map π is also parabolic semistable on X with parabolic divisor $\pi^*(D) \subset X$ and $\Delta_{par}(\pi^*(E_*)) = 0$.

Conversely, if E_* be a parabolic semistable bundle on projective bundle $\pi : X = \mathbb{P}(\mathscr{E}) \longrightarrow Y$ with parabolic divisor $D' = \pi^{-1}(D)$, with $\triangle_{par}(E_*) = 0$ and the parabolic first Chern class $c_1(E_*) = \pi^*(\mathscr{L})$ for some line bundle \mathscr{L} on Y, then there exists a semistable parabolic bundle E'_* on Y with parabolic divisor D and $\triangle_{par}(E'_*) = 0$ such that $E_* = \pi^*(E'_*)$.

Proof. Let $D = \sum_{i=1}^{n} D_i$ be the normal crossing divisor on *Y* and $D' = \pi^*(D)$. Since π is smooth, the pullback divisor *D'* on *X* is also a normal crossing divisor satisfying condition (a).

Let $p: Y' \longrightarrow Y$ be a Kawamata cover with Galois group *G* such that

$$p^*D_i = k_i N(p^*D_i)_{\text{red}}$$

for some positive integers k_i and N. Consider the following fibre product diagram

Then $\tilde{p}: X' \longrightarrow X$ is a Galois cover with the same Galois group *G*. Let *V* be the orbifold bundle on *Y*' associated to the parabolic bundle E_* on *Y*(see [4]). The pullback orbifold bundle $V' := \tilde{\pi}^*(V)$ then corresponds to the parabolic pullback bundle $\pi^*(E_*)$ on *X* with parabolic divisor *D*'.

We note that $\Delta(V) = \Delta_{par}(E_*) = 0$. Since E_* is parabolic semistable, by using the correspondence in [4], *V* is also orbifold semistable, and hence semistable (in the usual sense). Therefore

the pullback bundle V' is also orbifold semistable with $\triangle(V') = 0$, proving that $\pi^*(E_*)$ is parabolic semistable with $\triangle_{par}(\pi^*(E_*)) = 0$.

Conversely, let V be the orbifold bundle on X' associated to the parabolic bundle E_* on X. Then,

$$c_1(V) = \widetilde{p}^* c_1(E_*) = \widetilde{p}^* \pi^*(\mathscr{L}) = \widetilde{\pi}^* p^*(\mathscr{L})$$

Now by the given hypothesis, *V* is orbifold semistable and hence semistable (in the usual sense). Since $\Delta(V) = 0$, by Theorem 1.2 in [5] $V|_f$ is semistable on $f \simeq \mathbb{P}^m$ for every fibre *f* of the map $\tilde{\pi}$ (Here rank(\mathscr{E}) = m + 1) and deg($V|_f$) = 0. This implies $V \simeq \tilde{\pi}^*(W)$ for some orbifold bundle *W* on *Y'* which must be semistable. Let E'_* be the associated semistable parabolic bundle on *Y*. Note that $\Delta(V) = \tilde{\pi}^*(\Delta(W)) = 0$ and $\tilde{\pi}^*$ is injective. Hence $\Delta(W) = 0$. By a similar argument we have $\Delta_{\text{par}}(E'_*) = 0$. Then by the construction of E'_* , the result follows.

Corollary 21. Let W be a vector bundle of rank m over a smooth complex projective curve C and $\rho: X = \mathbb{P}(W) \longrightarrow C$ be the projectivisation map. Let E_* be a semistable vector bundle on X of rank r with parabolic discriminant $\Delta_{par}(E_*) = 0$, and parabolic 1st Chern class $c_1(E_*) \equiv x\xi + yf$, where ξ and f are the numerical classes of $\mathcal{O}_{\mathbb{P}(W)}(1)$ and a fibre of ρ respectively. Then, E_* is ample if and only if x > 0 and $(x\mu_{min}(W) + y) > 0$.

Proof. We note that

$$\overline{\mathrm{NE}}(\mathbb{P}(W)) = \left\{ a\left(\xi^{m-1} - (\mathrm{deg}(W) - \mu_{\min}(W))\xi^{m-2}f\right) + b\xi^{m-2}f \mid a, b \in \mathbb{R}_{\geq 0} \right\}$$

Hence, $c_1(E_*)$ is in the ample cone if and only if

- $c_1(E_*) \cdot \{\xi^{m-1} (\deg(W) \mu_{\min}(W))\xi^{m-2}f\} = (x\mu_{\min}(W) + y) > 0$ and
- $c_1(E_*) \cdot \xi^{m-2} f = x > 0.$

Therefore, the result follows from the previous theorem.

Example 22. Let $\rho : X = \mathbb{P}(W) \longrightarrow Y$ be a projective bundle on a smooth projective variety *Y*. Let $D \subset Y$ be a normal crossing divisor in *Y* and F_* be a semistable parabolic bundle of rank *r* on *Y* with parabolic divisor *D*. Then $\rho^*(F_*)$ is a parabolic semistable bundle on *X* with parabolic divisor $D' = \rho^*(D)$. Let $D' = \sum_{i=1}^n D'_i$ be the decomposition into irreducible components of D'. A parabolic line bundle with parabolic divisor D' is a data of the form $L_* = (L, \{\alpha_1, ..., \alpha_n\})$, where *L* is a line bundle on *X* and each $0 \le \alpha_i < 1$ corresponds to the divisor D'_i . Assume $\alpha_i \in \mathbb{Q}$ for all *i*. Then $E_* = \rho^*(F_*) \otimes L_*$ is parabolic semistable with $\Delta_{\text{par}}(E_*) = 0$. Note that $c_1(L_*) := c_1(L) + \sum_{i=1}^n \alpha_i[D_i]$. One can choose L_* in such a way that $c_1(E_*) = c_1(\rho^*F_*) + rc_1(L_*)$ is in the ample cone of *X*. This way one can produce parabolic ample bundles on *X*.

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References

- [1] A. Beauville, "An ampleness criterion for rank 2 vector bundles on surfaces", *Vietnam J. Math.* **48** (2020), no. 1, p. 125-129.
- [2] I. Biswas, "Chern classes for parabolic bundles", J. Math. Kyoto Univ. 37 (1997), no. 4, p. 597-613.
- [3] _____, "Parabolic ample bundles", Math. Ann. 307 (1997), no. 3, p. 511-529.
- [4] ______, "Parabolic bundles as orbifold bundles", Duke Math. J. 88 (1997), no. 2, p. 305-325.
- [5] I. Biswas, U. Bruzzo, "On semistable principal bundles over a complex projective manifold", *Int. Math. Res. Not.* 2008 (2008), article no. rnn035 (28 pages).
- [6] I. Biswas, K. Hanumanthu, D. S. Nagaraj, "Positivity of vector bundles on homogeneous varieties", Int. J. Math. 31 (2020), no. 12, article no. 2050097 (11 pages).

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- [7] I. Biswas, S. Mehrotra, A. J. Parameswaran, "Nef line bundles on flag bundles on a curve over $\overline{\mathbb{F}_p}$ ", Arch. Math. 101 (2013), no. 2, p. 105-110.
- [8] V. Brînzănescu, "Algebraic 2-vector bundles on ruled surfaces", Ann. Univ. Ferrara, Nuova Ser., Sez. VII 37 (1991), p. 55-64.
- [9] M. Fulger, "The cones of effective cycles on projective bundles over curves", Math. Z. 269 (2011), no. 1-2, p. 449-459.
- [10] R. Hartshorne, "Ample Vector bundles", Publ. Math., Inst. Hautes Étud. Sci. 29 (1966), p. 63-94.
- [11] _____, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer, 1977.
- [12] M. Hering, M. Mustață, S. Payne, "Positivity properties of toric vector bundles", Ann. Inst. Fourier **60** (2010), no. 2, p. 607-640.
- [13] D. Huybrechts, M. Lehn, The Geometry of Moduli Spaces of Sheaves, Cambridge University Press, 2010.
- [14] R. Lazarsfeld, Positivity in algebraic geometry. I. Classical setting: line bundles and linear series, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48, Springer, 2004.
- [15] ——, Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 49, Springer, 204.
- [16] M. Lübke, "Stability of Einstein-Hermitian vector bundles", Manuscr. Math. 42 (1983), p. 245-257.
- [17] V. B. Mehta, M. V. Nori, "Semistable sheaves on homogeneous spaces and abelian varieties", Proc. Indian Acad. Sci., Math. Sci. 93 (1984), no. 1, p. 1-12.
- [18] S. Mukai, "Semi-homogeneous vector bundles on an abelian variety", J. Math. Kyoto Univ. 18 (1978), no. 2, p. 239-272.
- [19] N. Nakayama, "Normalized Tautological divisors of semi-stable vector bundles", in *Free resolutions of coordinate rings of projective varieties and related topics*, Sūrikaisekikenkyūsho Kōkyūroku, vol. 1075, RIMS, Kyoto University, 1999, p. 167-173.
- [20] M. Schneider, A. Tancredi, "Positive vector bundles on complex surfaces", Manuscr. Math. 50 (1985), p. 133-144.
- [21] F. Serrano, "Strictly nef divisors and Fano threefolds", J. Reine Angew. Math. 464 (1995), p. 187-206.
- [22] H. Umemura, "Some results in the theory of vector bundles", Nagoya Math. J. 52 (1973), p. 97-128.