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The linear $\mathfrak{n}(1|N)$–invariant differential operators and $\mathfrak{n}(1|N)$–relative cohomology

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Abstract. Over the $(1, N)$-dimensional supercircle $S^{1|N}$, we classify $\mathfrak{n}(1|N)$-invariant linear differential operators acting on the superspaces of weighted densities on $S^{1|N}$, where $\mathfrak{n}(1|N)$ is the Heisenberg Lie superalgebra. This result allows us to compute the first differential $\mathfrak{n}(1|N)$-relative cohomology of the Lie superalgebra $\mathcal{X}(N)$ of contact vector fields with coefficients in the superspace of weighted densities. For $N = 0, 1, 2$, we investigate the first $\mathfrak{n}(1|N)$-relative cohomology space associated with the embedding of $\mathcal{X}(N)$ in the superspace of the supercommutative algebra $\mathcal{FP}(N)$ of pseudodifferential symbols on $S^{1|N}$ and in the Lie superalgebra $\mathcal{ΨDO}(S^{1|N})$ of superpseudodifferential operators with smooth coefficients. We explicitly give 1-cocycles spanning these cohomology spaces.

Résumé. Sur le super cercle $(1, N)$-dimensionnel $S^{1|N}$, nous classifions les opérateurs différentiels linéaires $\mathfrak{n}(1|N)$-invariants agissant sur les densités tensorielles sur $S^{1|N}$, où $\mathfrak{n}(1|N)$ est la superalgèbre de Lie de Heisenberg. Ce résultat permet de calculer le premier espace de cohomologie différentiels $\mathfrak{n}(1|N)$-relative de la superalgèbre de Lie des champs de vecteurs de contact $\mathcal{X}(N)$ à coefficients dans le superspace des densités tensorielles. Pour $N = 0, 1, 2$, nous étudions le premier espace de cohomologie $\mathfrak{n}(1|N)$-relative de $\mathcal{X}(N)$ dans le superspace de l’algèbre supercommutative $\mathcal{FP}(N)$ des symboles pseudodifférentiels sur $S^{1|N}$ et dans la superalgèbre de Lie $\mathcal{ΨDO}(S^{1|N})$ des opérateurs superpseudodifférentiels. Nous donnons explicitement les 1-cocycles engendrant ces espaces de cohomologie.


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1. Introduction

Let \( \text{Vect}(S^1) \) is the Lie algebra of smooth vector fields on the circle \( S^1 \). Consider the 1-parameter deformation of the \( \text{Vect}(S^1) \)-action on \( C^\infty_c(S^1) \):

\[
L^\lambda_{\frac{dX}{dx}}(f) = Xf' + \lambda X' f,
\]

where \( X,f \in C^\infty_c(S^1) \) and \( X' := \frac{dX}{dx} \). Denote by \( F_\lambda \) the \( \text{Vect}(S^1) \)-module structure on \( C^\infty_c(S^1) \) defined by \( L^\lambda \) for a fixed \( \lambda \). Geometrically, \( F_\lambda = \{ f \frac{d}{dx} | f \in C^\infty_c(S^1) \} \) is the space of weighted densities of weight \( \lambda \in \mathbb{R} \). The space \( F_\lambda \) coincides with the space of vector fields, functions and differential 1-forms for \( \lambda = -1, 0 \) and 1, respectively.

Denote by \( D_{\lambda,\mu} := \text{Hom}_{\text{diff}}(F_\lambda, F_\mu) \) the \( \text{Vect}(S^1) \)-module of linear differential operators with the natural \( \text{Vect}(S^1) \)-action denoted \( L^\lambda_X(A) \). Each module \( D_{\lambda,\mu} \) has a natural filtration by the order of differential operators; the graded module \( F_{\lambda,\mu} := \text{gr} D_{\lambda,\mu} \) is called the space of symbols. The quotient-module \( D_{\lambda,\mu}^k / D_{\lambda,\mu}^{k-1} \) is isomorphic to the module of weighted densities \( F_{\mu-\lambda-k} \), the isomorphism is provided by the principal symbol map \( \sigma_r \) defined by:

\[
A = \sum_{i=0}^{k} a_i(x) \left( \frac{\partial}{\partial x} \right)^i \mapsto \sigma_r(A) = a_k(x)(dx)^{\mu-\lambda-k},
\]

We study the classification of \( n(1|N) \)-invariant linear differential operators on \( S^{1|N} \) acting in the spaces \( \mathcal{S}^N_\lambda \). Ovsienko and Roger [11] calculated the space \( H^1(\text{Vect}(S^1), \mathcal{P}\mathcal{D}\mathcal{Q}(S^1)) \), where \( \text{Vect}(S^1) \) is the Lie algebra of smooth vector fields on the circle \( S^1 \) and \( \mathcal{P}\mathcal{D}\mathcal{Q}(S^1) \) is the space of pseudodifferential operators. The action is given by the natural embedding of \( \text{Vect}(S^1) \) in \( \mathcal{P}\mathcal{D}\mathcal{Q}(S^1) \). They used the results of D. B. Fuks [5] on the cohomology of \( \text{Vect}(S^1) \) with coefficients in tensor densities to determine the cohomology with coefficients in the graded module \( \text{Grad}(\mathcal{P}\mathcal{D}\mathcal{Q}(S^1)) \), namely \( H^1(\text{Vect}(S^1), \text{Grad}^p(\mathcal{P}\mathcal{D}\mathcal{Q}(S^1))) \); here \( \text{Grad}^p(\mathcal{P}\mathcal{D}\mathcal{Q}(S^1)) \) is isomorphic, as \( \text{Vect}(S^1) \)-module, to the space of tensor densities \( F_p \) of degree \( p \) on \( S^1 \). To compute \( H^1(\text{Vect}(S^1), \mathcal{P}\mathcal{D}\mathcal{Q}(S^1)) \), V. Ovsienko and C. Roger applied the theory of spectral sequences to a filtered module over a Lie algebra.

In this paper we consider the superspace \( S^{1|N} \) equipped with the contact structure determined by a 1-form \( \alpha_N \), and the Lie superalgebra \( \mathcal{K}(N) \) of contact vector fields on \( S^{1|N} \). We introduce the \( \mathcal{K}(N) \)-module \( \mathcal{S}^N_\lambda \) of \( \lambda \)-densities on \( S^{1|N} \) and the \( \mathcal{K}(N) \)-module of linear differential operators, \( \mathcal{D}^N_{\lambda,\mu} := \text{Hom}_{\text{diff}}(\mathcal{S}^N_\lambda, \mathcal{S}^N_\mu) \), which are super analogues of the spaces \( F_{\lambda,\mu} \) and \( D_{\lambda,\mu} \), respectively. We classify all \( n(1|N) \)-invariant linear differential operators on \( S^{1|N} \) acting in the spaces \( \mathcal{S}^N_\lambda \). We use the result to compute \( H^1_{\text{diff}}(\mathcal{K}(N), n(1|N), \mathcal{S}^N_\lambda) \). We show that, the non-zero cohomology only appear for resonant values of weights. Moreover, we give explicit bases of these cohomology spaces. For \( N = 0,1,2 \), we follow again the same methods by V. Ovsienko and C. Roger [11] to compute the \( n(1|N) \)-relative cohomology \( H^1(\mathcal{K}(N), n(1|N), \mathcal{P}\mathcal{D}\mathcal{Q}(S^{1|N})) \), where \( n(1|N) \) is the Heisenberg Lie superalgebra, and \( \mathcal{P}\mathcal{D}\mathcal{Q}(S^{1|N}) \) is the space of super pseudodifferential operators on \( S^{1|N} \). Moreover, we give explicit bases of these cohomology spaces.

2. Definitions and notations

In this section, we recall the main definitions and facts related to the geometry of the superspace \( S^{1|N} \); for more details, see [6, 7, 8, 9, 10].
2.1. The Lie superalgebra of contact vector fields on $S^{1|N}$

We define the supercircle $S^{1|N}$ in terms of its superalgebra of functions, denoted by $C^\infty_C(S^{1|N})$ and consisting of elements of the form:

$$F = \sum_{s=0}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_s \leq N} f_{i_1i_2\ldots i_s}(x) \theta_{i_1} \ldots \theta_{i_s},$$

where $f_{i_1i_2\ldots i_s} \in C^\infty_C(S^1)$, and where $x$ is the even indeterminate, $\theta_1, \ldots, \theta_N$ are the odd indeterminates, i.e., $\theta_i \theta_j = -\theta_j \theta_i$. Consider the standard contact structure given by the following 1-form:

$$\alpha_N = dx + \sum_{i=1}^{N} \theta_i d\theta_i.$$

On the space $C^\infty_C(S^{1|N})$, we consider the contact bracket

$$\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{p(F)} \sum_{i=1}^{N} \overline{\eta}_i(F) \cdot \overline{\eta}_i(G),$$

where $\overline{\eta}_i = \frac{\partial}{\partial \theta_i} - \theta_i \frac{\partial}{\partial x}$ and $p(F)$ is the parity of $F$. Let $\text{Vect}_C(S^{1|N})$ be the superspace of vector fields on $S^{1|N}$:

$$\text{Vect}_C(S^{1|N}) = \left\{ F_0 \partial_x + \sum_{i=1}^{N} F_i \partial_i \mid F_i \in C^\infty_C(S^{1|N}) \right\},$$

where $\partial_i = \frac{\partial}{\partial \theta_i}$ and $\partial_x = \frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(N)$ of contact vector fields on $S^{1|N}$:

$$\mathcal{K}(N) = \left\{ X \in \text{Vect}_C(S^{1|N}) \mid \text{there exists } F \in C^\infty_C(S^{1|N}) \text{ such that } \mathcal{L}_X(\alpha_N) = F \alpha_N \right\}.$$

The Lie superalgebra $\mathcal{K}(N)$ is spanned by the fields of the form:

$$X_F = F \partial_x - \frac{1}{2}(-1)^{p(F)} \sum_{i=1}^{N} \overline{\eta}_i(F) \overline{\eta}_i, \text{ where } F \in C^\infty_C(S^{1|N}).$$

In particular, we have $\mathcal{K}(0) = \text{Vect}_C(S^1)$. The bracket in $\mathcal{K}(N)$ can be written as:

$$[X_F, X_G] = X_{[F,G]}.$$

The Lie superalgebra $\mathcal{K}(N - 1)$ can be realized as a subalgebra of $\mathcal{K}(N)$:

$$\mathcal{K}(N - 1) = \{ X_F \in \mathcal{K}(N) \mid \partial_N F = 0 \}.$$

Note also that, for any $i \in \{1, 2, \ldots, N\}$, $\mathcal{K}(N - 1)$ is isomorphic to

$$\mathcal{K}(N - 1)^i = \{ X_F \in \mathcal{K}(N) \mid \partial_i F = 0 \}.$$

2.2. The Heisenberg subalgebra $\mathfrak{n}(1|N)$

The Heisenberg Lie superalgebra $\mathfrak{n}(1|N)$ can be realized as a subalgebra of $\mathcal{K}(N)$:

$$\mathfrak{n}(1|N) = \text{Span} \{ X_1, X_0 \}, \quad 1 \leq i \leq N.$$

We easily see that $\mathfrak{n}(1|N - 1)$ is a subalgebra of $\mathfrak{n}(1|N)$:

$$\mathfrak{n}(1|N - 1) = \{ X_F \in \mathfrak{n}(1|N - 1) \mid \partial_N F = 0 \}.$$

Note also that, for any $i \in \{1, 2, \ldots, N - 1\}$, $\mathfrak{n}(1|N - 1)$ is isomorphic to

$$\mathfrak{n}(1|N - 1)^i = \{ X_F \in \mathfrak{n}(1|N - 1) \mid \partial_i F = 0 \}.$$
2.3. Modules of weighted densities

For every contact vector field $X_F$, define a one-parameter family of first-order differential operators on $C^\infty_S(S^{1|N})$:

$$\mathcal{L}^A_{X_F} = X_F + \lambda F', \quad \lambda \in \mathbb{C}. $$

We easily check that

$$[\mathcal{L}^A_{X_F}, \mathcal{L}^A_{X_G}] = \mathcal{L}^A_{X_{[F,G]}}.$$  

We thus obtain a one-parameter family of $\mathcal{K}(N)$-modules on $C^\infty_S(S^{1|N})$ that we denote $\mathfrak{F}\lambda_N$, the space of all weighted densities on $S^{1|N}$ of weight $\lambda$ with respect to $\alpha_N$:

$$\mathfrak{F}\lambda_N = \left\{ F\alpha_N \mid F \in C^\infty(S^{1|N}) \right\}.$$

2.4. Differential operators on weighted densities

A differential operator on $S^{1|N}$ is an operator on $C^\infty_S(S^{1|N})$ of the form:

$$A = \sum_{k=0}^{M} \sum_{\epsilon=(\epsilon_1,\ldots,\epsilon_N)} a_{k,\epsilon}(x,\theta) \partial^{\epsilon_1} \cdots \partial^{\epsilon_N}; \quad \epsilon_i = 0,1; \quad M \in \mathbb{N}. $$

Of course any differential operator defines a linear mapping $F\alpha_N \mapsto (AF)\alpha_N$, thus the space of differential operators becomes a family of $\mathcal{K}(N)$-modules $\mathcal{D}_{\lambda,\mu}$ for the natural action:

$$X_F \cdot A = \Lambda^\mu_{X_F} \circ A - (-1)^{p(A)} \Lambda^\mu_{X^\ast_F} A \circ \Lambda^\lambda_{X_F}. $$

Every differential operator $A \in \mathcal{D}_{\lambda,\mu}$ can be expressed in the form

$$A(F\alpha_N) = \sum_{\epsilon=(\epsilon_1,\ldots,\epsilon_N)} a_{\epsilon}(x,\theta) \bar{\eta}^{\epsilon_1}_1 \cdots \bar{\eta}^{\epsilon_N}_N (F)\alpha_N, $$

where the coefficients $a_{\epsilon}(x,\theta)$ are arbitrary functions.

**Lemma 1** ([2]). As a $\mathcal{K}(N-1)$-module, we have

$$\mathcal{D}_{\lambda,\mu}^N \cong \mathcal{D}_{\lambda,\mu}^{N-1} \oplus \mathcal{D}_{\lambda,\mu}^{N-1} \oplus \Pi \mathcal{D}_{\lambda,\mu}^{N-1} \oplus \mathcal{D}_{\lambda,\mu}^{N-1},$$

where $\Pi$ is the change of parity operator.

2.5. Pseudodifferential operators on $S^{1|N}$

Let $T^* S^{1|N}$ be the cotangent bundle on $S^{1|N}$ with local coordinates $(x,\theta_1,\ldots,\theta_N,\xi,\bar{\theta}_1,\ldots,\bar{\theta}_N)$, where $p(\bar{\theta}_i) = 1$. The superspace of the supercommutative algebra $\mathcal{S}(N)$ of pseudodifferential symbols on $S^{1|N}$ with its natural multiplication is spanned by the series

$$\mathcal{S}(N) = \left\{ \sum_{k=-M}^{\infty} \sum_{\epsilon=(\epsilon_1,\ldots,\epsilon_N)} a_{k,\epsilon}(x,\theta) \xi^{-k} \bar{\theta}_1^{\epsilon_1} \cdots \bar{\theta}_N^{\epsilon_N} \right\} \left\{ a_{k,\epsilon} \in C^\infty(S^{1|N}); \quad \epsilon_i = 0,1; \quad M \in \mathbb{N} \right\}.$$

This space has a structure of the Poisson Lie superalgebra given by the following bracket:

$$\{A, B\} = \partial_x A \partial_\xi B - \partial_\xi A \partial_x B - (-1)^{p(A)} \sum_{i=1}^{N} \left[ \partial_i A \partial_{\bar{\theta}_i} B + \partial_{\bar{\theta}_i} A \partial_i B \right],$$

where $\partial_x = \frac{\partial}{\partial x}$, $\partial_\xi = \frac{\partial}{\partial \xi}$, $\partial_i = \frac{\partial}{\partial \theta_i}$, and $\partial_{\bar{\theta}_i} = \frac{\partial}{\partial \bar{\theta}_i}$. Of course $\mathcal{S}(0)$ is the classical space of symbols, usually denoted $\mathcal{S}$.  

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The associative superalgebra of pseudodifferential operators \( \mathcal{D}(S^{1|N}) \) on \( S^{1|N} \) has the same underlying vector space as \( \mathcal{D}(N) \), but the multiplication is now defined by the following rule:

\[
A \circ B = \sum_{a \geq 0, v_i > 0} \frac{(-1)^{p(A)+1}}{a!} \left( \partial^n_{\zeta} \partial^v_i A \right) \left( \partial^n_{\bar{\zeta}} \partial^v_i B \right).
\]

Denote by \( \mathcal{D}(S^{1|N})_{SL} \) the Lie superalgebra with the same superspace as \( \mathcal{D}(S^{1|N}) \) and the supercommutator defined on homogeneous elements by:

\[
[A, B] = A \circ B - (-1)^{p(A)p(B)} B \circ A.
\]

In particular, we have \( \mathcal{D}(S^{1|0}) = \mathcal{D}(S^1) \).

3. The structure of \( \mathcal{D}(N) \) as a \( \mathcal{K}(N) \)-module

The natural embedding of \( \mathcal{K}(N) \) into \( \mathcal{D}(N) \) defined by

\[
\pi(X_F) = F \xi - \frac{(-1)^{p(F)}}{2} \sum_{i=1}^{N} \eta_i(F) \zeta_i,
\]

where \( \zeta_i = \hat{\theta}_i - \theta_i \xi \),

induces a \( \mathcal{K}(N) \)-module structure on \( \mathcal{D}(N) \).

Setting \( \deg x = \deg \theta_i = 0 \), \( \deg \xi = \deg \hat{\theta}_i = 1 \) for all \( i \), we endow the Poisson superalgebra \( \mathcal{D}(N) \) with a \( \mathbb{Z} \)-grading:

\[
\mathcal{D}(N) = \bigoplus_{n \in \mathbb{Z}} \mathcal{D}_n(N),
\]

where \( \mathcal{D}_n = (\mathcal{D}_{n<0}) \oplus \prod_{n \geq 0} \mathcal{D}_n \) and

\[
\mathcal{D}_n(N) = \{ F \xi^n + G_1 \xi^{n-1} \hat{\theta}_1 + G_2 \xi^{n-2} \hat{\theta}_2 + \ldots + H_{1,2} \xi^{-n} \hat{\theta}_1 \hat{\theta}_2 + \ldots | F, G, H_{1,2} \in C^\infty(S^{1|N}) \}
\]

is the homogeneous subspace of degree \(-n\).

Note that each element of \( \mathcal{D}(S^{1|N}) \) can be expressed as

\[
A = \sum_{k \in \mathbb{Z}} (F_k + G^1_k \xi^{-1} \hat{\theta}_1 + \ldots + H^{1,2}_k \xi^{-2} \hat{\theta}_1 \hat{\theta}_2 + \ldots) \xi^k,
\]

where \( F_k, G^i_k, H^{i,j}_k \in C^\infty(S^{1|N}) \). We define the order of \( A \) to be

\[
\text{ord}(A) = \sup \left\{ k \left| F_k \neq 0 \text{ or } G^i_k \neq 0 \text{ or } H^{i,j}_k \neq 0 \right. \right\}.
\]

This definition of order equips \( \mathcal{D}(S^{1|N}) \) with a decreasing filtration as follows: set

\[
\mathcal{F}_n = \{ A \in \mathcal{D}(S^{1|N}) \left| \text{ord}(A) \leq -n \right. \},
\]

where \( n \in \mathbb{Z} \). So we have

\[
\cdots \subset \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \ldots
\]

This filtration is compatible with the multiplication and the super Poisson bracket, that is, for \( A \in \mathcal{F}_n \) and \( B \in \mathcal{F}_p \), one has \( AB \in \mathcal{F}_{n+p} \) and \( \{ A, B \} \in \mathcal{F}_{n+p-1} \). This filtration makes \( \mathcal{D}(S^{1|N}) \) an associative filtered superalgebra. Moreover, this filtration is compatible with the natural \( \mathcal{K}(N) \)-action on \( \mathcal{D}(S^{1|N}) \). Indeed,

\[
X_F(A) = \{ X_F, A \} \in \mathcal{F}_n \text{ for any } X_F \in \mathcal{K}(N) \text{ and } A \in \mathcal{F}_n.
\]

The induced \( \mathcal{K}(N) \)-module structure on the quotient \( \mathcal{F}_n/\mathcal{F}_{n+1} \) is isomorphic to that of the \( \mathcal{K}(N) \)-module \( \mathcal{D}_n(N) \). Therefore,

\[
\mathcal{D}(N) \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n/\mathcal{F}_{n+1}.
\]
4. $n(1|N)$–invariant linear differential operators

Now, we describe the spaces of $n(1|N)$-invariant linear differential operators $\mathfrak{g}_\lambda^N \rightarrow \mathfrak{h}_\mu^N$ for $N \in \mathbb{N}$. Our main result of this section is the following:

**Theorem 2.** Let $\mathcal{N}_{\lambda,\mu}^N : \mathfrak{g}_\lambda^N \rightarrow \mathfrak{h}_\mu^N$, $(\alpha_N^\lambda) \rightarrow \mathcal{N}_{\lambda,\mu}^N(\alpha_N^\lambda)$ be a non-zero $\mathcal{N}(1|N)$-invariant linear differential operator. Then, up to a scalar factor, the map $\mathcal{N}_{\lambda,\mu}^N$ is given by:

$$\mathcal{N}_{\lambda,\mu}^N(F) = \begin{cases} \sum_{k \geq 0} \gamma_k F^{(k)}, & \text{for } N \in \mathbb{N} \\ \sum_{k \geq 0} \gamma_k \eta_1 \eta_2 \cdots \eta_N(F^{(k)}), & \text{for } N \geq 1, \end{cases}$$

where $\gamma_k \in \mathbb{R}$.

**Proof. (i).** For $N = 0$, the generic form of any such a differential operator is

$$\mathcal{N}_{\lambda,0}^0 : \mathfrak{h}_\lambda^0 \rightarrow \mathfrak{g}_\mu^0, Fd\alpha^\lambda \rightarrow \sum_{i=0}^m \gamma_i F^{(i)} d\alpha^\mu,$$

where $\gamma_i \in C^\infty(S^1)$ are arbitrary functions and $F^{(i)}$ stands for $\frac{d^i F}{dx^i}$. The invariance property with respect to the vector field $X = \frac{d}{dx}$ implies that $\frac{d\gamma_i}{dx} = 0$.

(ii). By induction on $N$. For $N = 1$, let $\mathcal{N}_{\lambda,\mu}^1 : \mathfrak{h}_\lambda^1 \rightarrow \mathfrak{g}_\mu^1$ be an $n(1|1)$-invariant linear differential operator. The $n(1|1)$-invariance of $\mathcal{N}_{\lambda,\mu}$ is equivalent to invariance with respect just to the subalgebra $n(1|0)$ and the vector fields $X_{\theta_1}$. Using the fact that, as $\text{vect}(S^1)$-modules,

$$\mathfrak{g}_\lambda^1 \approx \mathfrak{h}_\lambda^0 \oplus \Pi \left( \mathfrak{g}_\lambda^0 \right),$$

we can deduce, by induction hypothesis, the restriction of $\mathcal{N}_{\lambda,\mu}$ to each component of the right-hand side of (3). The invariance of $\mathcal{N}_{\lambda,\mu}$ with respect $X_{\theta_1}$ determine thus completely the space of $n(1|1)$-invariant linear differential operator $\mathfrak{g}_\lambda^1 \rightarrow \mathfrak{g}_\mu^1$.

Now, assume that the result holds for $N > 1$. Observe that the $n(1|N)$-invariance of any linear differential operators $\mathcal{N}_{\lambda,\mu}^N : \mathfrak{g}_\lambda^N \rightarrow \mathfrak{h}_\mu^N$ is equivalent to invariance with respect just to the subalgebras $n(1|N-1)$ and $n(1|N-1)^i, i = 1, \ldots, N-1$, and that $\mathcal{N}_{\lambda,\mu}^N$ is decomposed into four $n(1|N-1)$-invariant maps:

$$\Pi^i \left( \mathfrak{g}_\lambda^{N-1} \right) \longrightarrow \Pi^j \left( \mathfrak{h}_\mu^{N-1} \right), \quad i, j = 0, 1.$$ (4)

Thus, by induction assumption, we exhibit the $n(1|N-1)$-invariant linear differential operators $\mathfrak{g}_\lambda^N \rightarrow \mathfrak{h}_\mu^N$. More precisely, any $n(1|N-1)$-invariant binary differential operators $\mathcal{N}_{\lambda,\mu}^N : \mathfrak{g}_\lambda^N \rightarrow \mathfrak{h}_\mu^N$ can be expressed as:

$$\mathcal{N}_{\lambda,\mu}^N(F) = \Xi_{\lambda,\mu} (1 - \theta_N \partial_{\theta_N}) (\mathcal{N}_{\lambda,\mu}^{N-1}) - \Theta_{\lambda,\mu} (-1)^{p(F)} \partial_{\theta_N} (\mathcal{N}_{\lambda,\mu}^{N-1}) \theta_N,$$

$$\mathcal{N}_{\lambda,\mu}^N(F) = (-1)^{p(F)} \Omega_{\lambda,\mu} (1 - \theta_i \partial_{\theta_i}) (\mathcal{N}_{\lambda,\mu}^{N-1}) \theta_N + \Gamma_{\lambda,\mu} (\partial_{\theta_i} (\mathcal{N}_{\lambda,\mu}^{N-1})),
$$

where the coefficients $\Omega_{\lambda,\mu}, \Gamma_{\lambda,\mu}, \Xi_{\lambda,\mu}$ and $\Theta_{\lambda,\mu}$ are, a priori, arbitrary constants, but the invariance of $\mathcal{N}_{\lambda,\mu}^N$ with respect $n(1|N-1)^i, i = 1, \ldots, N-1$, shows that

$$\Gamma_{\lambda,\mu} = 0, \quad \Xi_{\lambda,\lambda+k} = \Theta_{\lambda,\lambda+k}.
$$

Therefore, we easily check that $\mathcal{N}_{\lambda,\mu}^N$ is expressed as in Theorem 2. This completes the proof of Theorem 2. □
5. Cohomology

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [4]). Let \( g = g_0 \oplus g_1 \) be a Lie superalgebra acting on a superspace \( V = V_0 \oplus V_1 \) and let \( h \) be a subalgebra of \( g \). (If \( h \) is omitted it assumed to be \( \{0\} \)). The space of \( h \)-relative \( n \)-cochains of \( g \) with values in \( V \) is the \( g \)-module

\[
C^n(g, h; V) := \text{Hom}_h(\Lambda^n(g/h); V).
\]

The coboundary operator \( \delta \) is a linear map satisfying \( \delta \circ \delta = 0 \). The kernel of \( \delta_n \), denoted \( Z^n(g, h; V) \), is the space of \( h \)-relative \( n \)-cocycles, among them, the elements in the range of \( \delta_{n-1} \) are called \( h \)-relative \( n \)-coboundaries. We denote \( B^n(g, h; V) \) the space of \( n \)-coboundaries.

By definition, the \( n \)-th \( h \)-relative cohomology space is the quotient space

\[
H^n(g, h; V) = Z^n(g, h; V) / B^n(g, h; V).
\]

5.1. The spaces \( H^1_{\text{diff}}(\mathcal{K}(N), n(1|N), \mathbb{F}_A^N) \)

In this subsection, we will compute the first differential cohomology spaces \( H^1_{\text{diff}}(\mathcal{K}(N), n(1|N), \mathbb{F}_A^N) \). Our main result is the following:

**Theorem 3.** The space \( H^1_{\text{diff}}(\mathcal{K}(N), n(1|N), \mathbb{F}_A^N) \) has the following structure:

\[
H^1_{\text{diff}}(\mathcal{K}(N), n(1|N), \mathbb{F}_A^N) = \begin{cases} 
\mathbb{R}^2 & \text{if } N = 2 \text{ and } \lambda = 0 \\
\mathbb{R} & \text{if } \begin{cases} 
N = 0 \text{ and } \lambda = 0, 1, 2 \\
N = 1 \text{ and } \lambda = 0, \frac{1}{2}, \frac{3}{2} \\
N = 2 \text{ and } \lambda = 1 \\
N = 3 \text{ and } \lambda = 0, \frac{1}{2} \\
N \geq 4 \text{ and } \lambda = 0 
\end{cases} \\
0 & \text{otherwise.}
\end{cases}
\]

The following 1-cocycles \( \Upsilon^N_\lambda \) span the corresponding cohomology spaces:

\[
\begin{align*}
\Upsilon^0_0(X_F) &= F'; & N \in \mathbb{N}, & \Upsilon^1_0(X_F) &= \tilde{\eta}_1(F') \alpha_1^2, \\
\Upsilon^0_1(X_F) &= F'' dx^1, & \Upsilon^2_0(X_F) &= \tilde{\eta}_1 \tilde{\eta}_2(F) \alpha_2, \\
\Upsilon^0_2(X_F) &= F'' dx^2, & \Upsilon^2_1(X_F) &= \tilde{\eta}_1 \tilde{\eta}_2(F') \alpha_2, \\
\Upsilon^1_1(X_F) &= \tilde{\eta}_1(F') \alpha_1^1, & \Upsilon^3_1(X_F) &= \tilde{\eta}_1 \tilde{\eta}_2 \tilde{\eta}_3(F) \alpha_3^1.
\end{align*}
\]

The proof of Theorem 3 will be the subject of subsection 5.2. In fact, we need first the following classical fact:

**Lemma 4** ([3]). Any 1-cocycle \( \Upsilon \) on \( \mathcal{K}(N) \) vanishing on \( n(1|N) \), with values in \( \mathbb{F}_A^N \), the linear differential operator \( \mathcal{N} : \mathcal{K}(N) \to \mathbb{F}_A^N \) defined by

\[
\mathcal{N}(X) = \Upsilon(X),
\]

is \( n(1|N) \)-invariant.
5.2. Proof of the Theorem 3

Let $Y_{1,\mu}^N$ be a 1-cocycle on $\mathcal{K}(N)$ vanishing on $n(1|N)$, with values in $\mathfrak{g}_\mu^N$. By Lemma 4, up to a scalar factor, $Y_{1,\mu}^N$ is a linear differential operator $n(1|N)$-invariant $\mathcal{N}_{1,\mu}^N : \mathfrak{g}_1 \to \mathfrak{g}_\mu^N$. Thus, by Theorem 2, we get the explicit formulae for $\mathcal{N}_{1,\mu}^N$:

For $N = 0$, \[ \mathcal{N}_{1,\mu}^0(X_F) = \sum_{k \geq 0} Y_k F^{(k)} \, dx^\mu \]
For $N \geq 1$, \[ \mathcal{N}_{1,\mu}^N(X_F) = \sum_{k \geq 0} K_k F^{(k)} \, \alpha_N^\mu \]

Now let us check if each of the maps $\mathcal{N}_{1,\mu}^N$ are 1-cocycles. If the maps $\mathcal{N}_{1,\mu}^N$ are 1-cocycles one has to check the 1–cocycle relation. It reads as follows:

\[ \delta(\mathcal{N}_{1,\mu}^N) = (-1)^{p(X)p(Y)} \sum_X \mathcal{Y} (\mathcal{N}_{1,\mu}^N(Y)) \cdot (-1)^{p(Y)(p(X)+p(\mathcal{N}_{1,\mu}^N))} \sum_Y (\mathcal{N}_{1,\mu}^N(X)) - \mathcal{N}_{1,\mu}^N([X,Y]) = 0, \]

where $X, Y \in \mathcal{K}(N)$. By direct computation, we can see that only the operators $\mathcal{N}_{1,\mu}^N = Y_{1,\mu}^N$ expressed as in (5) are 1-cocycles vanishing on $n(1|N)$.

Finally, we study the non-triviality of these 1-cocycles $\mathcal{N}_{1,\mu}^N$. For instance, assume that the 1-cocycle $\mathcal{N}_{1,2}^0$ is trivial, then there exists a density $\varphi(x) dx^2 \in \mathfrak{g}_2^0$ such that

\[ \mathcal{N}_{1,2}^0(X_F) = \int_{X_F} \varphi(x) dx^2 \]

The coefficient of $F''''$ is zero in the expression of the coboundary and the coefficient of $F'''$ is 1 in the expression of 1-cocycle $\mathcal{N}_{1,2}^0$. Thus, the relation (6) implies $1 = 0$ which is absurd. With the same arguments, we prove the non-triviality of 1-cocycles $\mathcal{N}_{1,0}^N, \mathcal{N}_{1,1}^N, \mathcal{N}_{1,2}^N, \mathcal{N}_{1,1/2}^1, \mathcal{N}_{1,1/2}^1, \mathcal{N}_{1,0}^2, \mathcal{N}_{1,1}^2, \mathcal{N}_{1,1}^3$. Therefore, we easily check that $Y_{1,\mu}^N$ is expressed as in (5). This completes the proof of Theorem 3.

6. $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{D}^\mu_n(N))$ and $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{D}^\Psi\mathcal{O}(S^{1|N}))$

6.1. The space $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{D}^\mu_n(N))$

The space $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{D}^\mu_n(N))$ inherits the grading (3) of $\mathcal{D}^\mu_n(N)$, so it suffices to compute it in each degree. The main result of this section for $N = 0, 1, 2$ is the following.

**Theorem 5.** The space $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{D}^\mu_n(N))$ has the following structure:

\[ H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{D}^\mu_n(N)) \simeq \begin{cases} \mathbb{R}^N & \text{if } N = 2 \text{ and } n = 1 \\
\mathbb{R}^N & \text{if } N = 0 \text{ and } n = 0, 1, 2 \\
\mathbb{R} & \text{if } N = 1 \text{ and } n = 1 \\
\mathbb{R}^N & \text{if } N = 2 \text{ and } n = -1 \\
\mathbb{R}^N & \text{if } N = 1 \text{ and } n = 0 \\
\mathbb{R}^N & \text{if } N = 2 \text{ and } n = 0 \\
0 & \text{otherwise.} \end{cases} \]
The following 1-cocycles $\chi^N$ span the corresponding cohomology spaces:

$$\begin{align*}
\chi^0_N &= F', \text{ for } N = 0, 2, \\
\chi^1_0 &= F''\xi^{-1}, \\
\chi^2_0 &= F'\xi^{-2}, \\
\chi^1_1 &= (1 + (-1)^{p(F)}F')F''\xi^{-1}\xi_1, \\
\chi^2_1 &= F''\xi^{-2}\xi_1, \\
\chi^2_2 &= (1 + (-1)^{p(F)}F')F''\xi^{-2}\xi_1, \\
\chi^2_2 &= (1 + (-1)^{p(F)}F')F''\xi^{-2}\xi_1.
\end{align*}$$

The following 1-cocycles $\gamma^i_\lambda$ span the corresponding cohomology spaces:

$$\begin{align*}
\gamma^i_0 &= F', \\
\gamma^i_1 &= \eta_1(F''), \\
\gamma^i_2 &= \eta_1(F'), \\
\gamma^i_3 &= (-1)^{p(F)}\eta_{3-i}(F')\theta_i, \\
\gamma^i_{-1} &= (-1)^{p(F)}\eta_{3-i}(F'')\theta_i, \\
\gamma^i_{-2} &= F'\theta_i.
\end{align*}$$

Proof. The case where $N = 0$. In this case, we can see that the map $\phi : F^m \rightarrow F^{p(F)}$ defined by $\phi(Fdx^n) = F\xi^{-n}$ provides us with an isomorphism of $Vect(S^1)^{\text{-modules}}$. So, we can deduce the structure of $H^n_{\text{diff}}(Vect(S^1), n(1|0); \mathcal{P}_n)$ from $H^n_{\text{diff}}(Vect(S^1), n(1|0); \mathcal{P}_n)$ given in Theorem 3.

The case where $N = 1$. In this case, as a $\mathcal{K}(1)$-module, we have

$$\mathcal{P}_n(1) = \mathcal{P}_n^1 \oplus \mathcal{P}_n^2,$$

where

$$\mathcal{P}_n^1 = \{1 + (-1)^{p(F)}F\xi^{-n} + \eta_1(F)\xi^{-n-1}\xi_1, F \in C^\infty(S^{1|1})\},$$

$$\mathcal{P}_n^2 = \{F\xi^{-n-1}\xi_1 - 2\theta_1 F\xi^{-n}, F \in C^\infty(S^{1|1})\}.$$

The natural maps

$$\varphi_1 : \mathfrak{J}_n^1 \rightarrow \mathcal{P}_n^1$$

$$\varphi_2 : \Pi\left(\mathfrak{J}_{n+\frac{1}{2}}^1\right) \rightarrow \mathcal{P}_n^2$$

provide us with isomorphisms of $\mathcal{K}(1)$-modules. Hence, as $\mathcal{K}(1)$-modules, we have $\mathcal{P}_n(1) \simeq \mathfrak{J}_n^1 \oplus \Pi(\mathfrak{J}_{n+\frac{1}{2}}^1)$. This isomorphism induces the following isomorphism between cohomology spaces:

$$H^n_{\text{diff}}(\mathcal{K}(1), n(1|1); \mathcal{P}_n(1)) \simeq H^n_{\text{diff}}(\mathcal{K}(1), n(1|1); \mathfrak{J}_n^1) \oplus H^n_{\text{diff}}(\mathcal{K}(1), n(1|1); \Pi\left(\mathfrak{J}_{n+\frac{1}{2}}^1\right)).$$

We deduce from this isomorphism and Theorem 3, the 1-cocycles (8).

The case where $N = 2$. To prove Theorem 5 in this case, we need first the following:

Proposition 6. The space $H^i_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i, \mathfrak{J}_\lambda^i)$ has the following structure:

$$H^i_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i, \mathfrak{J}_\lambda^i) = \begin{cases} \mathbb{R}^2 & \text{if } \lambda = 0 \\ \mathbb{R} & \text{if } \lambda = -\frac{1}{2}, 1, \frac{3}{2} \\ 0 & \text{otherwise.} \end{cases}$$

The following 1-cocycles $\gamma^i_\lambda$ span the corresponding cohomology spaces:

$$\begin{align*}
\gamma^i_0 &= F', \\
\gamma^i_1 &= \eta_1(F''), \\
\gamma^i_2 &= \eta_1(F'), \\
\gamma^i_3 &= (-1)^{p(F)}\eta_{3-i}(F')\theta_i, \\
\gamma^i_{-1} &= (-1)^{p(F)}\eta_{3-i}(F'')\theta_i, \\
\gamma^i_{-2} &= F'\theta_i.
\end{align*}$$
Proof of Proposition 6. Let \( F\alpha_2^1 = (f_0 + f_1\theta_1 + f_2\theta_2 + f_3\theta_1\theta_2)\alpha_2^1 \in \Phi\alpha_2^1 \). The map
\[
\Phi : \tilde{\Phi}_\lambda^1 \rightarrow \tilde{\Phi}_\lambda^1, \quad \Phi(\tilde{\Phi}_\lambda^1) = (1 - \theta_1\partial_\theta_1)(F)\tilde{\Phi}_\lambda^1, \quad \Phi(\tilde{\Phi}_\lambda^1) = \left( -1 \right)^{\beta(F)+1} \partial_\theta_1(F)\tilde{\Phi}_\lambda^1 \),
\]
provides us with an isomorphism of \( \mathcal{K}(1)^1 \)-modules. This map induces the following isomorphism between cohomology spaces:
\[
H^1_{\text{diff}}(\mathcal{K}(1)^1, n(1)|1; \tilde{\Phi}_\lambda^1) = H^1_{\text{diff}}(\mathcal{K}(1)^1, n(1)|1; \tilde{\Phi}_\lambda^1) \oplus H^1_{\text{diff}}(\mathcal{K}(1)^1, n(1)|1; \tilde{\Phi}_\lambda^1).
\]
Of course, we can deduce the structure of \( \mathcal{K}(1)^1, n(1)|1; \tilde{\Phi}_\lambda^1 \) from
\[
H^1_{\text{diff}}(\mathcal{K}(1)^1, n(1)|1; \tilde{\Phi}_\lambda^1) \quad \text{from} \quad H^1_{\text{diff}}(\mathcal{K}(1)^1, n(1)|1; \tilde{\Phi}_\lambda^1).
\]
Indeed, to any \( Y \in H^1_{\text{diff}}(\mathcal{K}(1)^1, n(1)|1; \tilde{\Phi}_\lambda^1) \) corresponds \( \tilde{Y} \in H^1_{\text{diff}}(\mathcal{K}(1)^1, n(1)|1; \tilde{\Phi}_\lambda^1) \) where \( \tilde{Y}(X_F) = \Omega(\sigma \circ Y(X_F)) \) or \( \sigma(F) = (-1)^{\beta(F)}F \). Obviously, \( \tilde{Y} \) is a coboundary if and only if \( Y \) is a coboundary. We deduce from isomorphism (10) and formula (5), the 1-cocycles (9).

Lemma 7. For \( n \in \mathbb{Z} \), any element of \( Z^1(\mathcal{K}(2), n|1); \Phi \mathcal{P}_n(2) \) is a \( n(1)|2 \)-relative coboundary over \( \mathcal{K}(2) \) if and only if its restriction to the subalgebra \( \mathcal{K}(1)^1 \) is a \( n(1)|1 \)-relative coboundary for \( i = 1 \) and 2.

Proof of Lemma 7. It is easy to see that if \( C \) is a \( n(1)|2 \)-relative coboundary over \( \mathcal{K}(2) \), then \( \mathcal{E}(\mathcal{K}(1)^1) \) is a \( n(1)|1 \)-relative coboundary of \( \mathcal{K}(1)^1 \). Now, assume that \( \mathcal{E}(\mathcal{K}(1)^1) \) is a \( n(1)|1 \)-relative coboundary of \( \mathcal{K}(1)^1 \) for \( i = 1 \) and 2. Using the condition of a 1-cocycle, we prove that there exists an element \( n(1)|1 \)-invariant \( G \in \Phi \mathcal{P}_n(2) \) such that
\[ \mathcal{E}(X_{f_0+f_1\theta_1}) = \{ \rho_0(X_{f_0+f_1\theta_1}), G \} \quad \text{for any} \quad f_0, f_1 \in C_\infty^\infty(S^1), \quad i = 1, 2 \]
\[ \mathcal{E}(X_{f_2+f_3\theta_2}) = \{ \rho_2(X_{f_2+f_3\theta_2}), G \} \quad \text{for any} \quad f_1, f_2 \in C_\infty^\infty(S^1). \]
We deduce that \( \mathcal{E}(X_F) = \{ \rho_0(X_F), G \} \), for any \( F \in C_\infty^\infty(S^{1|2}) \), and therefore \( \mathcal{E} \) is a \( n(1)|2 \)-relative coboundary of \( \mathcal{K}(2) \).

We also need the following:

Proposition 8 (11). (1) As a \( \mathcal{K}(1)^1 \)-module, \( i = 1, 2 \), we have
\[ \Phi \mathcal{P}_n(2) = \tilde{\mathcal{P}}_n^2 = \tilde{\mathcal{P}}_{n+1}^2 = \tilde{\mathcal{P}}_{n+1}^2, \quad \text{for} \quad n = 0, -1. \]
\[ \Phi \mathcal{P}_n(2) = \tilde{\mathcal{P}}_{n+1}^2, \quad \text{for} \quad n = 0, -1. \]

(2) For \( n \neq 0, -1 \):
(a) The following subspace of \( \Phi \mathcal{P}_n(2) \):
\[ \Phi \mathcal{P}_{n,i} = \{ B^{(n,i)}_F = F\theta_1\tilde{\eta}_i\xi^{n-1} + \theta_2(\frac{1}{2}\tilde{\eta}_i - \tilde{\eta}_i)F\xi^{n-2} | F \in C_\infty^\infty(S^{1|2}) \} \]
is a \( \mathcal{K}(1)^1 \)-module, \( i = 1, 2 \), isomorphic to \( \tilde{\mathcal{P}}_{n+1}^2 \).
(b) As a \( \mathcal{K}(1)^1 \)-module we have
\[ \Phi \mathcal{P}_{n,i} = \tilde{\mathcal{P}}_{n+1}^2, \quad \text{for} \quad n = 0, -1. \]
\[ \Phi \mathcal{P}_{n,i} = \tilde{\mathcal{P}}_{n+1}^2, \quad \text{for} \quad n = 0, -1. \]
Moreover, in [1] it was proved that the natural maps

\[
\psi_{n,0}^i : \mathfrak{D}^2_n \rightarrow \mathcal{D}(n,0,i), \quad \psi_{n,1}^i : \mathfrak{D}^2_{n+1} \rightarrow \mathcal{D}(n,1,i)
\]

provide us with isomorphisms of \( \mathcal{K}(1) \)-modules.

Now, according to Lemma 7, the restriction of any nontrivial \( n(1|2) \)-relative 1-cocycle of \( \mathcal{K}(2) \) with coefficients in \( \mathcal{D}(1) \) to \( \mathcal{K}(1)^i \) is a nontrivial \( n(1|1)^i \)-relative 1-cocycle. Using Proposition 6 and Propositions 8, we obtain:

\[
H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{D}(1)) \approx \begin{cases}
\mathbb{R}^4 & \text{if } n = -1 \\
\mathbb{R}^5 & \text{if } n = 0 \\
\mathbb{R}^3 & \text{if } n = 1 \\
0 & \text{otherwise}.
\end{cases}
\]  

In the case \( n = -1 \), the space \( H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{D}(1)) \) is spanned by the following 1-cocyles:

\[
\begin{align*}
C_{-1}^{1,i}(X_F) &= \psi_{-1,1}^i \circ \gamma_0^i(X_F), \\
C_{-1}^{3,i}(X_F) &= \psi_{-1,1}^i \circ \Pi \left( \gamma_{-\frac{1}{2}}^i(X_F) \right), \\
C_{-1}^{2,i}(X_F) &= \psi_{-1,1}^i \circ \tilde{\gamma}_0^i(X_F), \\
C_{-1}^{4,i}(X_F) &= \psi_{-1,1}^i \circ \Pi \left( \gamma_{-\frac{1}{2}}^i(X_F) \right).
\end{align*}
\]

In the case \( n = 0 \), the space \( H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{D}(1)) \) is spanned by the following 1-cocyles:

\[
\begin{align*}
C_0^{1,i}(X_F) &= \psi_{0,0}^i \circ \gamma_0^i(X_F), \\
C_0^{3,i}(X_F) &= \psi_{0,0}^i \circ \Pi \left( \gamma_{\frac{1}{2}}^i(X_F) \right), \\
C_0^{2,i}(X_F) &= \psi_{0,0}^i \circ \tilde{\gamma}_0^i(X_F), \\
C_0^{4,i}(X_F) &= \psi_{0,0}^i \circ \Pi \left( \gamma_{\frac{1}{2}}^i(X_F) \right).
\end{align*}
\]

In the case \( n = 1 \), the space \( H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{D}(1)) \) is spanned by the following 1-cocycles:

\[
\begin{align*}
C_1^{1,i}(X_F) &= \psi_{1,0}^i \circ \gamma_1^i(X_F), \\
C_1^{2,i}(X_F) &= \psi_{1,\frac{1}{2}}^i \circ \Pi \left( \gamma_{\frac{1}{2}}^i(X_F) \right), \\
C_1^{3,i}(X_F) &= \tilde{\gamma}_{1,\frac{1}{2}}^i \circ \Pi \left( \gamma_{\frac{1}{2}}^i(X_F) \right), \\
C_1^{4,i}(X_F) &= \psi_{1,\frac{1}{2}}^i \circ \Pi \left( \gamma_{\frac{1}{2}}^i(X_F) \right),
\end{align*}
\]

where the cocycles \( \gamma_0^i, \tilde{\gamma}_0^i, \gamma_\frac{1}{2}^i, \gamma_{-\frac{1}{2}}^i, \gamma_1^i \) and \( \gamma_{\frac{1}{2}}^i \) are defined by the formulae (9) and \( \psi_{n,j}, \tilde{\psi}_{n,j} \) are as in (14).

Now, note that any nontrivial \( n(1|2) \)-relative 1-cocycle of \( \mathcal{K}(2) \) with coefficients in \( \mathcal{D}(1) \) should retain the following general form \( Y = P_1 + P_2 + P_3 + P_4 \), where

\[
\begin{align*}
Y^1 & : \text{vect}(1) \rightarrow \mathcal{D}(2), \\
Y^2, Y^3 & : \mathcal{D}_{-\frac{1}{2}} \rightarrow \mathcal{D}(2), \\
Y^4 & : \mathcal{D}_0 \rightarrow \mathcal{D}(2),
\end{align*}
\]

are linear maps. The space \( H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{D}(1)), i = 1,2 \), determines the linear maps \( Y^1, Y^2 \) and \( Y^3 \). The 1-cocycle conditions determine \( Y^4 \). More precisely, we get:
For \( n = -1 \), the space \( H^1_{\text{diff}}(\mathcal{K}(2), n|2), \mathcal{S}\mathcal{P}_{-1}(2) \) is generated by the nontrivial \( n|2 \)-relative cocycles \( \tilde{\chi}^2_1 \) and \( \tilde{\chi}^2_1 \), corresponding to the \( n|1 \)-relative cocycles \( C^2_{i-1} \) and \( C^3_{i-1} \), respectively, via their restrictions to \( \mathcal{K}(1) \).

For \( n = 0 \), the space \( H^1_{\text{diff}}(\mathcal{K}(2), n|2), \mathcal{S}\mathcal{P}_0(2) \) is generated by the nontrivial \( n|2 \)-relative cocycles \( \chi_1^2, \chi_0^2, \chi_1^2, \chi_0^2 \) and \( \chi_2^1 \), corresponding to the \( n|1 \)-relative cocycles \( C^1_0, C^2_0, C^3_0, C^4_0 \) and \( C^5_0 \), respectively, via their restrictions to \( \mathcal{K}(1) \).

For \( n = 1 \), the space \( H^1_{\text{diff}}(\mathcal{K}(2), n|2), \mathcal{S}\mathcal{P}_1(2) \) is generated by the nontrivial \( n|2 \)-relative cocycles \( \chi^1_1 \), corresponding to the \( n|1 \)-relative cocycles \( C^1_1 \), via their restrictions to \( \mathcal{K}(1) \).

Theorem 5 is proved.

6.2. The spectral sequence for a filtered module over a Lie (super)algebra

The reader should refer to [12], for the details of the homological algebra used to construct spectral sequences. We will merely quote the results for a filtered module \( M \) with decreasing filtration \( \{M_n\}_{n \in \mathbb{Z}} \) over a Lie (super)algebra \( \mathfrak{g} \) so that \( M_{n+1} \subset M_n, \bigcup_{n \in \mathbb{Z}} M_n = M \) and \( \mathfrak{g} M_n \subset M_n \).

Consider the natural filtration induced on the space of cochains by setting:

\[ F^n(C^*(\mathfrak{g}, M)) = C^*(\mathfrak{g}, M_n), \]

then we have:

\[ dF^n(C^*(\mathfrak{g}, M)) \subset F^n(C^*(\mathfrak{g}, M)) \text{ (i.e., the filtration is preserved by } d); \]

\[ F^{n+1}(C^*(\mathfrak{g}, M)) \subset F^n(C^*(\mathfrak{g}, M)) \text{ (i.e. the filtration is decreasing).} \]

Then there is a spectral sequence \( (E_r^{*,*}, d_r) \) for \( r \in \mathbb{N} \) with \( d_r \) of degree \( (r, 1-r) \) and

\[ E_1^{p,q} = F^p(C^{p+q}(\mathfrak{g}, M))/F^{p+1}(C^{p+q}(\mathfrak{g}, M)) \quad \text{and} \quad E_1^{q,p} = H^{p+q}(\mathfrak{g}, \text{Grad}^p(M)). \]

To simplify the notations, we have to replace \( E_0(C^*(\mathfrak{g}, M)) \) by \( F^nC^* \). We define

\[ Z_r^{p,q} = F^pC^{p+q} \cap d^{-1}(F^{p+r}C^{p+q+1}), \]

\[ B_r^{p,q} = F^pC^{p+q} \cap d(F^{p-r}C^{p+q-1}), \]

\[ E_r^{p,q} = Z_r^{p,q}/(Z_r^{p+1,q-1} + B_r^{p,q}). \]

The differential \( d \) maps \( Z_r^{p,q} \) into \( Z_{r+1}^{p+r,q-r+1} \), and hence includes a homomorphism

\[ d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \]

The spectral sequence converges to \( H^*(C, d) \), that is

\[ E_\infty^{p,q} \cong F^pH^{p+q}(C, d)/F^{p+1}H^{p+q}(C, d), \]

where \( F^pH^*(C, d) \) is the image of the map \( H^*(F^pC, d) \rightarrow H^*(C, d) \) induced by the inclusion \( F^pC \rightarrow C \).

6.3. Computing \( H^1_{\text{diff}}(\mathcal{K}(N), n|1N), \mathcal{S}\mathcal{P}\mathcal{O}(S^{1|N}) \)

Since the cohomology space \( H^1_{\text{diff}}(\mathcal{K}(N), n|1N), \mathcal{S}\mathcal{P}\mathcal{O}(S^{1|N}) \) is upper bounded by cohomology space \( H^1_{\text{diff}}(\mathcal{K}(N), n|1N), \mathcal{S}\mathcal{P}\mathcal{O}(S^{1|N}) \), we can check the behavior of the cocycles with values in \( \mathcal{S}\mathcal{P}\mathcal{O}(S^{1|N}) \) under the successive differentials of the spectral sequence. More precisely we consider a cocycle with values in \( \mathcal{S}\mathcal{P}\mathcal{O}(N) \); but we compute its boundary as it was in \( \mathcal{S}\mathcal{P}\mathcal{O}(S^{1|N}) \) for \( N = 0, 1, 2 \), and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one. We iterate this procedure, we establish a recurrence formula between successive terms. A straightforward computations leads to the following result:
Theorem 9. The space $H^1_{\text{diff}}(\mathcal{H}(N), n(1|N); \mathcal{D}(S^{1|N}))$ has the following structure:

$$H^1_{\text{diff}}(\mathcal{H}(N), n(1|N); \mathcal{D}(S^{1|N})) \approx \begin{cases} \mathbb{R}^3 & \text{if } N = 0, 1 \\ \mathbb{R}^8 & \text{if } N = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The following 1-cocycles $\Xi_i^N$ span the corresponding cohomology spaces:

$$\Xi_1^N(X_F) = F', \text{ for } N = 0, 1, 2, \quad \Xi_2^N(X_F) = \eta_1 \eta_2(F),$$

$$\Xi_2^N(X_F) = F'\xi^{-1}\zeta_1\zeta_2,$$

$$\Xi_3^N(X_F) = \eta_1 \eta_2(F)\xi^{-1}\zeta_1\zeta_2,$$

$$\Xi_0^N(X_F) = \sum_{n=2}^\infty \left(-1\right)^n \left(\frac{n-2}{n}\right) \frac{\left(-1\right)^{p(F)} \left(\overline{\eta}_1(F(n))\xi^{-n}\overline{\eta}_1 - \frac{n-3}{n+1} F^{(n)}(x)\xi^{-n}\right)}{n+1},$$

$$\Xi_1^N(X_F) = \sum_{n=2}^\infty \left(-1\right)^n \left(\frac{n-1}{n}\right) \frac{\left(-1\right)^{p(F)} \left(\overline{\eta}_1(F(n))\xi^{-n}\overline{\eta}_1 - \frac{n-1}{n+1} F^{(n)}(x)\xi^{-n}\right)}{n+1},$$

$$\Xi_2^N(X_F) = \sum_{n=2}^\infty \left(-1\right)^n \left(\frac{n}{n+1}\right) \left(\eta_1(F(n+1))\zeta_1 + \eta_2(F(n+1))\zeta_2\right)\xi^{-n-1} + \sum_{n=0}^\infty \frac{2(-1)^n}{n+2} F^{(n+2)}(x)\xi^{-n} - \mathcal{H}(N),$$

$$\Xi_3^N(X_F) = \sum_{n=0}^\infty \left(-1\right)^n \eta_1 \eta_2(F(n+1))\xi^{-n-2}\zeta_1\zeta_2$$

$$+ \sum_{n=1}^\infty \left(-1\right)^n \left(\frac{n}{n+1}\right) \left(\eta_1(F(n+1))\xi^{-n-1}\zeta_1 + \eta_2(F(n+1))\zeta_2\right)\xi^{-n-1} + \sum_{n=1}^\infty \frac{n}{n+2} F^{(n+2)}(x)\xi^{-n-1},$$

$$\Xi_4^N(X_F) = \sum_{n=1}^\infty \left(-1\right)^{p(F)+n} \frac{n}{n+2} F^{(n+2)}(x)\xi^{-n-2}\zeta_1\zeta_2$$

$$+ \sum_{n=1}^\infty \left(-1\right)^n \left(\frac{n}{n+1}\right) \left(\eta_1(F(n+1))\xi^{-n-1}\zeta_2 + \eta_2(F(n+1))\zeta_1\right)\xi^{-n-1} + \sum_{n=1}^\infty \frac{2(-1)^n}{n+2} F^{(n+2)}(x)\xi^{-n}.$$

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References


