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Sergey A. Denisov

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# Subharmonic functions in scattering theory 

Sergey A. Denisov ${ }^{a}$

${ }^{a}$ University of Wisconsin-Madison Mathematics Department 480 Lincoln Dr., Madison, WI, 53706, USA
E-mail: denissov@wisc.edu


#### Abstract

We present a method that uses the properties of subharmonic functions to control spatial asymptotics of Green's kernel of multidimensional Schrödinger operator with rough potential. 2020 Mathematics Subject Classification. 35P25, 81Q10. Funding. The work was supported by NSF DMS-1764245 and NSF DMS-2054465. Manuscript received 30th March 2021, accepted 21st April 2021.


## 1. Introduction

In this note, we look at the multidimensional Schrödinger operator $H: H=-\Delta+V, x \in \mathbb{R}^{3}$ in the case when the real-valued potential $V \in L^{\infty}\left(\mathbb{R}^{3}\right)$ does not satisfy the classical assumptions of the scattering theory [14]. Our motivation is to present a method based on the elementary properties of subharmonic functions which provides control on spatial asymptotics of Green's function for $H$ in the situation when the standard tools of perturbation theory do not work. In the classical situation, when $V$ is short-range, i.e., $|V(x)|<C(|x|+1)^{-\gamma}, \gamma>1$ the results of Kato [9] and Agmon [1] provide existence of wave operators in Schrödinger dynamics and the positive spectrum of $H$ is purely absolutely continuous. The case when $V$ is long-range, e.g., $\left|D^{j} V\right|<C(|x|+1)^{-\gamma-j}, \gamma>0.5,0 \leq j \leq j_{0}$ was addressed in the works of Hörmander [7,8] and, again, the positive spectrum was shown to be purely a.c. (see also Saito [12] for results with even weaker conditions on $V$ ). Following the significant progress in one-dimensional scattering theory (see [13] for the survey), the focus shifted to understanding the spectrum of multidimensional operator with rough potentials, e.g., $V$ satisfying only an upper bound $|V|<C(|x|+1)^{-\gamma}$ with $\gamma \in(0.5,1]$. For that case, the existence of a.c. spectrum is not known, but some results were obtained $[4,5,11]$ if $V$ is assumed to oscillate additionally. We notice that for potentials so rough the singular spectrum can coexist with the absolutely continuous one and the standard tools of scattering theory, such as absorption principle, has no chance to work. The alternative method rooted in complex analysis rather than harmonic analysis is based on the study of spatial asymptotics of Green's function. Here is its outline. Given $z \in \mathbb{C}^{+}$and $F \in L^{2}\left(\mathbb{R}^{3}\right)$, we define $R_{z} F=(H-z)^{-1} F$ and Green's function $G(x, y, z)$ as $\left(R_{z} F\right)(x)=\int_{\mathbb{R}^{3}} G(x, y, z) F(y) \mathrm{d} y$. For the free Schrödinger equation when $V=0$, the formula for Green's function is known:

$$
G_{0}\left(x, y, k^{2}\right)=e^{i k|x-y|}(4 \pi|x-y|)^{-1}, k=\sqrt{z}, k \in \mathbb{C}^{+} .
$$

Let $F$ have compact support and $F \not \equiv 0$. If one defines $U_{F}(x, k)=R_{k^{2}} F$, then the central quantity of interest is an amplitude $A_{F}(k, \theta)$ defined by

$$
\begin{equation*}
A_{F}(k, \theta)=\lim _{|x| \rightarrow \infty, x| | x \mid=\theta \in \mathbb{S}^{2}}|x| e^{-i k|x|} U_{F}(x, k) . \tag{1}
\end{equation*}
$$

If one formally takes $F=\delta_{0}$, then $A_{F}$ tells how the perturbed Green's function deviates from the unperturbed one. Suppose $\sigma_{F}(E)$ denotes the spectral measure of $F$ relative to $H$. The importance of the amplitude in the study of the spectral type is explained by the following Lemma (see [3, formula (4.2)]).
Lemma 1. Suppose $V$ and $F$ are compactly supported, then

$$
\begin{equation*}
\sigma_{F}^{\prime}\left(k^{2}\right)=\pi^{-1} k\left\|A_{F}(k, \theta)\right\|_{L^{2}\left(\mathbb{S}^{2}\right)}^{2}, \quad k \in \mathbb{R}^{+} . \tag{2}
\end{equation*}
$$

This identity is remarkable for two reasons. First, by the Spectral Theorem, we always have equality

$$
\begin{equation*}
\int_{\mathbb{R}} \mathrm{d} \sigma_{F}(E)=\|F\|^{2} \tag{3}
\end{equation*}
$$

Second, the function $h(k):=\log \left\|A_{F}(k, \theta)\right\|_{L^{2}\left(\mathbb{S}^{2}\right)}$ is subharmonic in $k \in\{\mathcal{S} k>0, \Re k>0\}$. Thus, if we can obtain
(A) rough upper bounds for $h$, e.g., $h(k) \leq c(\Im k)^{-\alpha}$ for some $\alpha>0$,
(B) some lower bound away from the real line, e.g., $h\left(k_{0}\right)>\delta, \Im k_{0}=\epsilon$ with some positive $\epsilon$ and $\delta$
then the mean-value inequality for subharmonic functions written for a suitable isosceles trapezoid $\mathscr{T} \subset \mathbb{C}^{+}$with the base $I \subset \mathbb{R}^{+}$gives

$$
\begin{equation*}
\int_{\partial \mathscr{T}} h(\xi) \mathrm{d} \omega_{k_{0}} \geq h\left(k_{0}\right) . \tag{4}
\end{equation*}
$$

The symbol $\omega_{k_{0}}$ denotes the harmonic measure with the reference point $k_{0} \in \mathscr{T}$. In particular, (A) and (B) provide the lower estimate for $h$ on interval I. Combined with (3) and Chebyshev's inequality, this gives the following bound

$$
\begin{equation*}
\left|\left\{k \in I:\left\|A_{F}(k, \theta)\right\|_{L^{2}\left(\mathbb{S}^{2}\right)}^{2} \leq e^{-\Lambda}\right\}\right|+\left|\left\{k \in I:\left\|A_{F}(k, \theta)\right\|_{L^{2}\left(\mathbb{S}^{2}\right)}^{2} \geq e^{\Lambda}\right\}\right|<C(I, c, \alpha, \epsilon, \delta) \Lambda^{-1} \tag{5}
\end{equation*}
$$

for every $\Lambda>1$. Thanks to (2), we also have control on the spectral measure $\sigma_{F}$ :

$$
\left|\left\{k \in I: \sigma_{F}^{\prime}\left(k^{2}\right) \leq e^{-\Lambda}\right\}\right|+\left|\left\{k \in I: \sigma_{F}^{\prime}\left(k^{2}\right) \geq e^{\Lambda}\right\}\right|<C(I, c, \alpha, \epsilon, \delta) \Lambda^{-1}
$$

These bounds indicate that "scattering happens" for most positive energies $E=k^{2}$. At the same time, they are also consistent with the possibility that some energies carry the singular part of $\sigma_{F}$.

In the next section, we use that strategy to study potentials $V$ supported on large annuli. In [2], this technique was applied to show the presence of a.c. spectral type for 1d Schrödinger with operator-valued rough potential. We expect this approach to be useful for other problems, e.g., spectral theory of difference operators on graphs.

## Notation.

- Given an interval $I \subset \mathbb{R}^{+}$and positive $\alpha$, the symbol $R_{I, \alpha}$ denotes the rectangle $I \times(0, \alpha)$ in complex plane; $c_{I}$ is the center of $I$.
- For the function $g \in L^{2}\left(\mathbb{S}^{2}\right)$, the symbol $\|g\|$ stands for its $L^{2}\left(\mathbb{S}^{2}\right)$ norm. $B_{r}(x)$ is the ball with radius $r$ centered at $x$.
- Symbol $B$ is Laplace-Beltrami operator on the unit sphere. It is nonpositive in $L^{2}\left(\mathbb{S}^{2}\right)$. Given a set $E \subset \mathbb{R}^{3}$, the symbol $\chi_{E}$ denotes the characteristic function of $E$.
- We define the angular component of the gradient (in spherical coordinates) as

$$
\nabla_{\tau} V=\nabla V-\frac{x}{|x|}\left\langle\frac{x}{|x|}, \nabla V\right\rangle .
$$

## 2. An example: $V$ supported on the large annulus

We are looking for conditions on $V$ that provide estimates (A) and (B) discussed in Introduction. Suppose supp $V \subset\{x: T / 2<|x|<T\}, F=\chi_{B_{1}(0)}, A_{F}$ is defined as in (1), and $T$ is a large parameter.
Theorem 2. Consider $H=-\Delta+V, x \in \mathbb{R}^{3}$ where non-negative real-valued potential $V$ satisfies

$$
\begin{equation*}
\operatorname{supp} V \subset\{x: T / 2<|x|<T\}, \quad|V| \leq C T^{-\gamma}, \quad\left|\nabla_{\tau} V\right| \leq C T^{-\gamma-\delta}, \quad \delta, \gamma>0 . \tag{6}
\end{equation*}
$$

If $\gamma \in\left(\frac{2}{3}, 1\right]$ and $\gamma+\frac{\delta}{2}>1$, then

$$
\sup _{T>1} \int_{I}\left|\log \left\|A_{F}(k, \theta)\right\|\right| \mathrm{d} k<\infty
$$

for every segment $I \subset \mathbb{R}^{+},|I| \sim 1$.
Operator $H$ with potential $V$ that satisfies stronger assumptions on angular gradient has been studied in [10].

Remark 3. As a consequence, we get

$$
\begin{equation*}
\left|\left\{k \in I:\left\|A_{F}(k, \theta)\right\|_{L^{2}\left(\mathbb{S}^{2}\right)}^{2} \leq e^{-\Lambda}\right\}\right|+\left|\left\{k \in I:\left\|A_{F}(k, \theta)\right\|_{L^{2}\left(\mathbb{S}^{2}\right)}^{2} \geq e^{\Lambda}\right\}\right|<C(\gamma, \delta, I) \Lambda^{-1} \tag{7}
\end{equation*}
$$

for every $\Lambda>1$ and $T>1$.
To get bounds (A) and (B), we will use Schrödinger equation itself. Recall that the operator $H$ can be written as

$$
H=-\partial_{r r}^{2}-\frac{B}{r^{2}}+V(r, \theta)
$$

in the spherical coordinates $(r, \theta) \in \mathbb{R}^{+} \times \mathbb{S}^{2}$ and the corresponding map $F(x) \rightarrow f(r, \theta)=r F(r, \theta)$ satisfies $\|F\|_{L^{2}\left(\mathbb{R}^{3}\right)}=\|f\|_{L^{2}\left(\mathbb{R}^{+}\right)}$where $f$ is considered as $L^{2}\left(\mathbb{S}^{2}\right)$-valued function in $r$. If $f$ represents $F$ and $u$ represents $U_{F}=R_{k^{2}} F$, we have an equation for $u$ :

$$
-u^{\prime \prime}-\frac{B u}{r^{2}}+V u=k^{2} u+f, \quad r>0 .
$$

Similarly, if $a$ represents $e^{-i k|x|} U_{F}$, then $a=u e^{-i k r}$ and

$$
\begin{equation*}
2 i k a^{\prime}=-a^{\prime \prime}-\frac{B a}{r^{2}}+V a-f e^{-i k r} \tag{8}
\end{equation*}
$$

or

$$
\begin{equation*}
a^{\prime}=(2 i k)^{-1}\left(-a^{\prime \prime}-\frac{B a}{r^{2}}+V a-f e^{-i k r}\right) . \tag{9}
\end{equation*}
$$

Recall that supp $f \subset[0,1]$. Then, the last equation gives the following identity if one takes the inner product and integrates the real part of resulting equation.
Lemma 4. Let $1<r_{1}<r_{2}$ and $k: \Re k>0, \Im k>0$, then

$$
\begin{equation*}
\left\|a\left(r_{2}, k\right)\right\|^{2}+\frac{\Im k}{|k|^{2}} \int_{r_{1}}^{r_{2}}\left(\left\|a^{\prime}\right\|^{2}-\frac{\langle B a, a\rangle}{\rho^{2}}\right) \mathrm{d} \rho+\frac{\Im k}{|k|^{2}} \int_{r_{1}}^{r_{2}}\langle V a, a\rangle \mathrm{d} \rho=\left\|a\left(r_{1}, k\right)\right\|^{2}+Q\left(r_{2}\right)-Q\left(r_{1}\right) \tag{10}
\end{equation*}
$$

where

$$
Q(r):=\frac{i}{2 k}\left\langle a^{\prime}(r, k), a(r, k)\right\rangle-\frac{i}{2 \bar{k}}\left\langle a(r, k), a^{\prime}(r, k)\right\rangle .
$$

If $V$ satisfies

$$
V \geq 0, \quad \operatorname{supp} V \in[T / 2, T], \quad\|V\|_{L^{\infty}\left(\mathbb{S}^{2}\right)}<\infty
$$

all terms in the left-hand side of (10) are non-negative and we get the following apriori estimate after setting $r_{2}=\infty$ and applying Cauchy-Schwartz to $Q\left(r_{1}\right)$ :

$$
\begin{equation*}
\|a(\infty, k)\|^{2}<C_{I}\left(\left\|a\left(r_{1}, k\right)\right\|^{2}+\left\|a^{\prime}\left(r_{1}, k\right)\right\|^{2}\right) \tag{11}
\end{equation*}
$$

provided that $k \in R_{I, 1}$ and $I \subset \mathbb{R}^{+}$. The Spectral Theorem implies that $\|U\|_{L^{2}\left(\mathbb{R}^{3}\right)} \leq C_{I}(\Im k)^{-1}$. Thus, from equation $H U=k^{2} U+F$, we get $\|\Delta U\|_{L^{2}\left(\mathbb{R}^{3}\right)}<C_{I}(\Im k)^{-1}$. So, the theorem about
traces of functions in Sobolev class implies $\|a(2, k)\|^{2}+\left\|a^{\prime}(2, k)\right\|^{2}<C_{I}(\Im k)^{-1}$ and (11) yields $\|a(\infty, k)\|^{2}<C_{I}(\Im k)^{-1}$ if we set $r_{1}=2$. Thus,

$$
\begin{equation*}
h=\log \|a(\infty, k)\|<C_{1}+C_{2}|\log \Im k|, \quad k \in R_{I, 1} \tag{12}
\end{equation*}
$$

with some positive $T$-independent $C_{1}$ and $C_{2}$. This is a rough bound (A) from the Introduction that we are looking for and it does not require any smallness assumption on $V$ at all. Our next immediate goal is to obtain better control on $h$ away from the real line, i.e., an estimate (B). The idea is to use (10) with $r_{2}=\infty, r_{1}=T / 4$, and $k \in R_{I, 1}$ to write

$$
\begin{align*}
& \|a(\infty, k)\|^{2}-\|a(T / 4, k)\|^{2} \mid \\
& \qquad \leq C(I)\left(|Q(T / 4)|+(\Im k) T^{-\gamma} \int_{T / 2}^{T}\|a\|^{2} \mathrm{~d} \rho+(\Im k) \int_{T / 4}^{\infty}\left(\left\|a^{\prime}\right\|^{2}-\frac{\langle B a, a\rangle}{\rho^{2}}\right) \mathrm{d} \rho\right) \tag{13}
\end{align*}
$$

Here, we choose $r_{1}=T / 4$ to guarantee enough separation from the support of $V$. Our goal is to show that the right-hand side is small if $k$ is separated from the real axis. If

$$
\begin{equation*}
\|V\|_{L^{\infty}\left(\mathbb{S}^{2}\right)} \leq C T^{-\gamma}, \quad k \in R_{I, 1}, \quad \Im k>c T^{-\gamma} \tag{14}
\end{equation*}
$$

with suitable $T$-independent constant $c$, the standard perturbation theory based on CombesThomas estimates (see, e.g., [6]) yields

$$
\begin{equation*}
\|a(T / 4, k)\|^{2} \sim 1, \quad\left\|a^{\prime}(T / 4, k)\right\|^{2} \leq C_{I} e^{-T^{k}}, \quad\langle B a(T / 4, k), a(T / 4, k)\rangle / T^{2} \leq C_{I} e^{-T^{k}}, \quad \kappa>0 . \tag{15}
\end{equation*}
$$

Hence, $|Q(T / 4)|<C_{I} e^{-T^{\kappa}}$. In what follows, we will assume that (14) holds. We need to control the other three terms in the right-hand side of (13). Taking the inner product with $a^{\prime}$ in (8) and integrating the real part from $r$ to $\infty$ gives the following lemma.

Lemma 5. Let $\Re k>0, \Im k>0$, and $r>1$, then

$$
\begin{aligned}
& (4 \Im k) \int_{r}^{\infty}\left\langle a^{\prime}, a^{\prime}\right\rangle \mathrm{d} \rho+\left\|a^{\prime}(r, k)\right\|^{2}-2 \int_{r}^{\infty} \frac{\langle B a(\rho, k), a(\rho, k)\rangle}{\rho^{3}} \mathrm{~d} \rho \\
& \quad=-\frac{\langle B a(r, k), a(r, k)\rangle}{r^{2}}-2 \Re\left(\int_{r}^{\infty}\left\langle V a, a^{\prime}\right\rangle \mathrm{d} \rho\right) .
\end{aligned}
$$

Since the terms in the left-hand side are all non-negative, we obtain another apriori estimate for $k \in R_{I, 1}, \Im k>C T^{-\gamma}$ :

$$
\begin{equation*}
(\Im k) \int_{T / 4}^{\infty}\left\|a^{\prime}(\rho, k)\right\|^{2} \mathrm{~d} \rho \leq C_{I}\left(e^{-T^{\kappa}}+T^{-\gamma}\left(\int_{T / 4}^{T}\|a\|^{2} \mathrm{~d} r\right)^{1 / 2}\left(\int_{T / 4}^{\infty}\left\|a^{\prime}\right\|^{2} \mathrm{~d} r\right)^{1 / 2}\right) \tag{16}
\end{equation*}
$$

Taking $r_{1}=T / 4$ and integrating (10) in $r_{2}$ from $T / 4$ to $T$, we apply Cauchy-Schwartz and (15) to get

$$
\int_{T / 4}^{T}\|a\|^{2} \mathrm{~d} r \leq C_{I}\left(T+\left(\int_{T / 4}^{T}\|a\|^{2} \mathrm{~d} r\right)^{1 / 2}\left(\int_{T / 4}^{T}\left\|a^{\prime}\right\|^{2} \mathrm{~d} r\right)^{1 / 2}\right)
$$

for $k \in R_{I, 1}, \Im k>C T^{-\gamma}$. This implies

$$
\begin{equation*}
\int_{T / 4}^{T}\|a\|^{2} \mathrm{~d} r \leq C_{I}\left(T+\int_{T / 4}^{T}\left\|a^{\prime}\right\|^{2} \mathrm{~d} r\right) \leq C_{I}\left(T+\int_{T / 4}^{\infty}\left\|a^{\prime}\right\|^{2} \mathrm{~d} r\right) \tag{17}
\end{equation*}
$$

which, being substituted into (16), yields

$$
\int_{T / 4}^{\infty}\left\|a^{\prime}\right\|^{2} \mathrm{~d} r<C_{I}\left((\Im k)^{-1} e^{-T^{\kappa}}+(\Im k)^{-2} T^{1-2 \gamma}\right)
$$

and, from (17),

$$
\begin{equation*}
T^{-1} \int_{T / 4}^{T}\|a\|^{2} \mathrm{~d} r \leq C_{I} \tag{18}
\end{equation*}
$$

These two last bounds control two additional terms in the right-hand side of (13) in the regime when $k \in R_{I, 1}$ and $\Im k>C T^{-\gamma}$. Thus, we are only left with estimating

$$
\int_{T / 4}^{\infty}-\frac{\langle B a, a\rangle}{\rho^{2}} \mathrm{~d} \rho
$$

in (13). To this end, we use an extra smoothness of $V$, i.e., the last condition in (6). Taking the gradient of (9) in the angular variables, we rewrite (6) as

$$
\begin{equation*}
\left(\nabla_{\theta} a\right)^{\prime}=(2 i k)^{-1}\left(-\left(\nabla_{\theta} a\right)^{\prime \prime}-\frac{B \nabla_{\theta} a}{r^{2}}+V \nabla_{\theta} a+a \nabla_{\theta} V\right) \tag{19}
\end{equation*}
$$

This equation has the same structure as equation for $a$ except for the term $a \nabla_{\theta} V$. The last condition on $V$ in (6) can be written as $\left\|\nabla_{\theta} V\right\|_{L^{\infty}\left(\mathbb{S}^{2}\right)} \leq C T^{1-\gamma-\delta}$. Repeating the previous arguments and using the bound (18), we obtain

$$
\int_{T / 4}^{T}-\frac{\langle B a, a\rangle}{\rho^{2}} \mathrm{~d} \rho \sim T^{-2} \int_{T / 4}^{T}\left\|\nabla_{\theta} a\right\|^{2} \mathrm{~d} \rho \leq C_{I}\left((\Im k)^{-2} T^{1-2 \gamma-2 \delta}+T^{3-2 \gamma-2 \delta}\right)
$$

For $r>T$, one has $V=0$ and the bound

$$
\int_{T / 4}^{\infty}-\frac{\langle B a, a\rangle}{\rho^{2}} \mathrm{~d} \rho=\int_{T / 4}^{\infty} \frac{\left\|\nabla_{\theta} a\right\|^{2}}{\rho^{2}} \mathrm{~d} \rho \leq C_{I}\left((\Im k)^{-2} T^{1-2 \gamma-2 \delta}+T^{3-2 \gamma-2 \delta}\right)
$$

follows by the similar reasoning. In (13), we apply the obtained estimates to control the righthand side. This gives

$$
\left|\|a(\infty, k)\|^{2}-\|a(a, k)\|^{2}\right| \leq C_{I}\left(T^{-1}+(\Im k)\left(T^{1-\gamma}+T^{3-2 \gamma-2 \delta}\right)+(\Im k)^{-1} T^{1-2 \gamma}\right)
$$

Given (6), we can always assume $T^{3-2 \gamma-2 \delta}>T^{1-\gamma}$ and find a small positive $\epsilon$ such that the set $P_{I, T}$ defined by $P_{I, T}=\left\{k \in \mathbb{C}^{+}: \Re k \in I, T^{\epsilon}\left(T^{1-2 \gamma}\right)<\Im k<T^{-\epsilon}\left(T^{2 \gamma+2 \delta-3}\right)\right\}$ is nonempty. Hence, if $k \in P_{I, T}$, then

$$
\begin{equation*}
\|a(\infty, k)\|^{2}=\|a(T / 4, k)\|^{2}+O\left(T^{-\epsilon_{1}}\right) \tag{20}
\end{equation*}
$$

with some $\epsilon_{1}>0$ and thus $\|a(\infty, k)\|^{2} \sim 1$ there. This establishes the required bound (B) and we are ready to prove Theorem 2.

Proof of Theorem 2. For every $I \subset \mathbb{R}^{+}$, consider $R_{I, \delta_{T}}$ with $\delta_{T}=T^{-\epsilon}\left(T^{2 \gamma+2 \delta-3}\right)$ and write meanvalue inequality for subharmonic function $h:=\log \|a(\infty, k)\|$ :

$$
\int_{\partial R_{I, \delta_{T}}} h \mathrm{~d} \omega_{k_{0}} \geq h\left(k_{0}\right)
$$

where $k_{0}=x+i \delta_{T} / 2, x \in I^{\prime} \subset I$ and the subinterval $I^{\prime}$ is chosen such that $c_{I^{\prime}}=c_{I},\left|I^{\prime}\right|>c|I|$ with fixed $T$-independent $c$.


Figure 1. Rectangle $R_{I, \delta_{T}}$
If $\Im k>\delta_{T} / 2$ and $k \in \partial R_{I, \delta_{T}}$, we have $\|a(\infty, k)\| \sim 1$ by (20). On the other hand, (12) guarantees that $\int_{\Gamma}|h| \mathrm{d} \omega_{k_{0}}<C$ uniformly in $T$ where $\Gamma$ are the legs of $R_{I, \delta_{T}}$, i.e., $\Gamma=\left\{k \in \partial R_{I, \delta_{T}}, 0<\Im k<\delta_{T}\right\}$.

In the end, if we use the standard estimates on the harmonic measure of the rectangle and integrate in $x \in I^{\prime}$, we obtain the bound $\int_{I^{\prime}} h(k) \mathrm{d} k>C$ uniformly in $T$. The formula (3) yields $\int_{I}\|a(\infty, k)\|^{2} \mathrm{~d} k<C$ uniformly in $T$ and the trivial estimate $\max \{0, \log t\} \leq t, t>0$ gives

$$
\sup _{T>1} \int_{I^{\prime}}\|\log \| a(\infty, k) \| \mid \mathrm{d} k<\infty .
$$

For an arbitrary $I^{\prime}$, we can find the corresponding $I \subset \mathbb{R}^{+}$so the proof is finished.

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