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# Lipschitz Conditions in Damek-Ricci Spaces 

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#### Abstract

In this paper we extend classical Titchmarsh theorems on the Fourier-Helgason transform of Lipschitz functions to the setting of $L^{p}$-space on Damek-Ricci spaces. As consequences, quantitative RiemannLebesgue estimates are obtained and an integrability result for the Fourier-Helgason transform is developed extending ideas used by Titchmarsh in the one dimensional setting.


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## 1. Introduction

The studies of the convergence and of the rate of decay of Fourier transform/ coefficients are among the most classical problems in Fourier analysis. Starting from the Riemann-Lebesgue theorem relating the integrability of a function on the torus $\mathbb{T}^{1}$ and the convergence of its Fourier coefficients, through the Hausdorff-Young inequality relating the integrability of a function and of its Fourier transform. In this vein, Titchmarsh showed that the decay of Fourier transform can be improved for univariate functions satisfying a Lipschitz condition defined by smoothness. His result reads as follows.

Theorem 1 (cf. [26, Theorem 84]). If f belongs to the Lipschitz class Lip $(\eta, p)$ in the $L^{p}$ norm on the real line, that is

$$
\omega_{p}(f, t)=\|f(\cdot+t)-f(\cdot)\|_{p}=O\left(|t|^{\eta}\right), \quad t \rightarrow 0,
$$

then its Fourier transform $\widehat{f}$ belongs to $L^{\delta}(\mathbb{R})$ for

$$
\frac{p}{p+\eta p-1} \leq \delta \leq \frac{p}{p-1}, \quad 0<\eta \leq 1,1<p \leq 2 .
$$

He also proved in [26, Theorem 85] another reversible form in the $L^{2}$ case, namely:
Theorem 2. Let $0<\eta \leq 1$ and $f \in L^{2}(\mathbb{R})$. Then $f \in \operatorname{Lip}(\eta, 2)$ if and only if

$$
\int_{|\lambda| \geq r}|\widehat{f}(\lambda)|^{2} \mathrm{~d} \lambda=O\left(r^{-2 \eta}\right), \quad r \rightarrow \infty
$$

An extension of these theorems to functions of several variables on $\mathbb{R}^{n}$ and on the torus group $\mathbb{T}^{n}$ was studied by Younis [28,29]. Later, analogous results were given, where considering generalized Fourier transforms: Bessel, Dunkl, Jacobi,...One can cite [8-10, 16]. On the other hand, Younis (in [30, Theorem 5.2]) recently has extended Titchmarsh results to functions on $\mathbb{R}^{d}$, replacing Lipschitz condition $|t|^{\eta}$ with Dini-Lipschitz condition $|t|^{\eta}\left(\log \frac{1}{|t|}\right)^{-\gamma}$. These were inspired from Weiss and Zygmund [27].

A continuous version was studied by Bray and Pinsky [5]. They proved the following estimate:

$$
\begin{equation*}
\left(\int_{\mathbb{R}} \min \left\{1,(\lambda t)^{2 p^{\prime}}\right\}|\widehat{f}(\lambda)|^{p^{\prime}} \mathrm{d} \lambda\right)^{1 / p^{\prime}} \leq c_{p} \Omega_{p}(f, t) \tag{1}
\end{equation*}
$$

where $1<p \leq 2, p^{\prime}=p / p-1$ and the modulus of smoothness $\Omega_{p}(f, t)$ of a function $f \in L^{p}(\mathbb{R})$ is defined by

$$
\Omega_{p}(f, t)=\sup _{0<h<t}\|f(\cdot+h)+f(\cdot-h)-2 f(\cdot)\|_{p}
$$

The significance of this inequality stems from the presence of the minimum function that gives control over the Fourier transform for small and large $\lambda$. Indeed, for $1<p \leq 2$, the inequality may be rewritten

$$
\begin{equation*}
\underbrace{\int_{|\lambda| \geq 1 / t}|\widehat{f}(\lambda)|^{p^{\prime}} \mathrm{d} \lambda}_{\text {large } \lambda}+\underbrace{t^{2 p^{\prime}} \int_{|\lambda|<1 / t} \lambda^{2 p^{\prime}}|\widehat{f}(\lambda)|^{p^{\prime}} \mathrm{d} \lambda}_{\text {small } \lambda} \leq c_{p}^{p^{p^{\prime}} \Omega_{p}^{p^{\prime}}(f, t), ~} \tag{2}
\end{equation*}
$$

As shown in [5], the estimate for large $\lambda$ yields a qualitative Riemann-Lebesgue lemma (i.e. a result of the type Titchmarsh Theorem 2, with Lipschitz or Dini-Lipschitz conditions). On the other hand, from the estimate for small $\lambda$, an integrability result can be achieved as done by Titchmarsh in Theorem 1 (see also [4, Theorem 3.4]).

In our present paper, we investigate among other things the validity of classical Titchmarsh theorems in case of functions of the wider Lipschitz and Dini-Lipschitz class in the context of Damek-Ricci spaces, also known as harmonic NA groups. This generalizes the corresponding result for noncompact rank one symmetric spaces (see [14]). Our current interest in this theme stems from a result of Kumar and al. [20] which is based on the work of Bray and Pinsky [5].

## 2. Preliminaries on Damek-Ricci spaces

A Damek-Ricci space is a one-dimensional extension of a generalized Heisenberg group and a Lie group with the Lie algebra of Iwasawa type. It is a solvable Lie group with a left invariant metric, and is a Riemannian manifolds which includes all rank-one symmetric spaces of the noncompact type; except from these, Damek-Ricci spaces are harmonic manifolds in general non symetric [12]. One of the interesting features of these spaces is that the radial analysis on these spaces behaves similar to the hyperbolic spaces as observed in [1] and therefore it fits into the perfect setting of Jacobi analysis developed by Flensted-Jensen and Koornwinder [18, 19].

In this section, we will explain the notation and gather relevant results on Damek-Ricci spaces. Most of these results can be found in $[1,7,12,25]$. Relevant results for the spherical and Fourier transforms on these spaces can be found in [1-3].

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra equipped with an inner product $\langle\cdot, \cdot\rangle$. Let $\mathfrak{z}$ be the centre of $\mathfrak{n}$ and $\mathfrak{a}$ its orthogonal complement. We say that $\mathfrak{n}$ is an $H$-type algebra if for every $Z \in \mathfrak{z}$ the map $J_{Z}: \mathfrak{a} \rightarrow \mathfrak{a}$ defined by

$$
\left\langle J_{Z} X, Y\right\rangle=\langle Z,[X, Y]\rangle, \quad X, Y \in \mathfrak{a},
$$

satisfies the condition $J_{Z}^{2}=-\|Z\|^{2} I_{\mathfrak{a}}, I_{\mathfrak{a}}$ being the identity operator on $\mathfrak{a}$. $A$ connected and simply connected Lie group $N$ is called an $H$-type group if its Lie algebra is $H$-type. Since $\mathfrak{n}$ is nilpotent, the exponential map is a dffeomorphism and hence we can parametrize the elements
in $N=\exp \mathfrak{n}$ by $(X, Y)$, for $X \in \mathfrak{a}, Z \in \mathfrak{z}$. It follows from the Campbell-Baker-Hausdorff formula that the group law in $N$ is given by

$$
(X, Z)\left(X^{\prime}, Z^{\prime}\right)=\left(X+X^{\prime}, Z+Z^{\prime}+\frac{1}{2}\left[X, X^{\prime}\right]\right), \quad X, X^{\prime} \in \mathfrak{a}, \quad Z, Z^{\prime} \in \mathfrak{z} .
$$

The group $A=\mathbb{R}_{+}^{*}$ acts on an $H$-type group $N$ by nonisotropic dilation: $(X, Y) \mapsto\left(a^{\frac{1}{2}} X, a Z\right)$. Let $S=N A$ be the semidirect product of $N$ and $A$ under the above action. Thus the multiplication in $S$ is given by

$$
(X, Z, a)\left(X^{\prime}, Z^{\prime}, a^{\prime}\right)=\left(X+a^{\frac{1}{2}} X^{\prime}, Z+a Z^{\prime}+\frac{1}{2} a^{\frac{1}{2}}\left[X, X^{\prime}\right], a a^{\prime}\right)
$$

for $X, X^{\prime} \in \mathfrak{a}, Z, Z^{\prime} \in \mathfrak{z}, a, a^{\prime} \in \mathbb{R}_{+}^{*}$. Then $S$ is a solvable, connected and simply connected Lie group having Lie algebra $\mathfrak{s}=\mathfrak{a} \oplus \mathfrak{z} \oplus \mathbb{R}$ with Lie bracket

$$
\left[(X, Z, k),\left(X^{\prime}, Z^{\prime}, k^{\prime}\right)\right]=\left(\frac{1}{2} k X^{\prime}-\frac{1}{2} k^{\prime} X, k Z^{\prime}-k^{\prime} Z+\left[X, X^{\prime}\right], 0\right)
$$

We suppose $\operatorname{dim} \mathfrak{a}=m$ and $\operatorname{dim} \mathfrak{z}=l$. Then $Q=\frac{m}{2}+l$ is called the homogenous dimension of S . For convenience we will use the symbol $\rho$ for $\frac{Q}{2}$ and $d$ for $m+l+1=\operatorname{dim}(\mathfrak{s})$.

The group $S$ is equipped with the left-invariant Riemannian metric induced by

$$
\left\langle(X, Z, k),\left(X^{\prime}, Z^{\prime}, k^{\prime}\right)\right\rangle=\left\langle X, X^{\prime}\right\rangle+\left\langle Z, Z^{\prime}\right\rangle+k k^{\prime}
$$

on $\mathfrak{s}$. The associated left Haar measure on $S$ is given by $a^{-Q-1} \mathrm{~d} X \mathrm{~d} Z \mathrm{~d} a$, where $\mathrm{d} X, \mathrm{~d} Z$ and $\mathrm{d} a$ are the Lebesgue measures on $\mathfrak{a}, \mathfrak{z}$ and $\mathbb{R}_{+}^{*}$ respectively.

To define the Fourier-Helgason transform on $S$ we need to introduce the notion of Poisson kernel [2]. The Poisson kernel $\mathscr{P}: S \times N \rightarrow \mathbb{R}$ is given by

$$
\mathscr{P}\left(n a_{t}, n^{\prime}\right)=P_{a_{t}}\left(n^{\prime-1} n\right),
$$

where

$$
P_{a_{t}}(n)=P_{a_{t}}(X, Z)=C a_{t}^{Q}\left(\left(a_{t}+\frac{|X|^{2}}{4}\right)^{2}+|Z|^{2}\right)^{-Q}
$$

and $a_{t}=e^{t}, t \in \mathbb{R} ; n=(X, Z) \in N$. The value of $C$ is suitably adjusted so that $\int_{N} P_{a}(n) \mathrm{d} n=1$ and $P_{1}(n) \leq 1$. For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel is defined by

$$
\mathscr{P}_{\lambda}(x, n)=\mathscr{P}(x, n)^{\frac{1}{2}-\frac{i \lambda}{Q}} .
$$

It is known [2,24] that for each fixed $x \in S, \mathscr{P}_{\lambda}(x,.) \in L^{p}(N)$ for $1 \leq p \leq \infty$ if $\lambda=i \gamma_{p} \rho$, where $\gamma_{p}=\frac{2}{p}-1$. A very special feature of $\mathscr{P}_{\lambda}(x, n)$ is that it is constant on the hypersurfaces $H_{n, a_{t}}=$ $\left\{n \sigma\left(a_{t} n^{\prime}\right): n^{\prime} \in N\right\}$, where $\sigma$ stands for the geodesic inversion [24].

Let $\Delta_{S}$ be the Laplace-Beltrami operator on $S$. Then for every fixed $n \in N, \mathscr{P}_{\lambda}(x, n)$ is an eigenfunction of $\Delta_{S}$ with eigenvalue $-\left(\lambda^{2}+\frac{Q^{2}}{4}\right)$ (see [2]). For a measurable function $f$ on $S$, the Fourier-Helgason transform is defined as

$$
\tilde{f}(\lambda, n)=\int_{S} f(x) \mathscr{P}_{\lambda}(x, n) \mathrm{d} x,
$$

whenever the integral converge.
It is known that for $f \in C_{c}^{\infty}(S)$ the following Fourier inversion and the Plancherel formula holds [2]:
(1) For $f \in C_{c}^{\infty}(S)$,

$$
f(x)=C \int_{\mathbb{R}} \int_{N} \tilde{f}(\lambda, n) \mathscr{P}_{-\lambda}(x, n)|c(\lambda)|^{-2} \mathrm{~d} \lambda \mathrm{~d} n, \quad \forall x \in S
$$

where

$$
c(\lambda)=\frac{2^{Q-2 i \lambda} \Gamma(2 i \lambda) \Gamma\left(\frac{2 m+l+1}{2}\right)}{\Gamma\left(\frac{Q}{2}+i \lambda\right) \Gamma\left(\frac{m+1}{2}+i \lambda\right)} .
$$

(2) The Fourier transform extends from $C_{c}^{\infty}(S)$ to an isometry from $L^{2}(S)$ onto the space $L^{2}\left(\mathbb{R}_{+} \times N, C|c(\lambda)|^{-2} \mathrm{~d} \lambda \mathrm{~d} n\right)$.
The precise value of the constants $C$ are given in [2]. The following estimates for the function $|c(\lambda)|$ holds:

$$
\begin{equation*}
c^{\prime}|\lambda|^{d-1} \leq|c(\lambda)|^{-2} \leq(1+|\lambda|)^{d-1}, \tag{3}
\end{equation*}
$$

for all $\lambda \in \mathbb{R}$ (e. g. see [24]).
A function $f$ on $S$ is called radial if for all $x, y \in S, f(x)=f(y)$ if $\mu(x, e)=\mu(y, e)$ where $\mu$ is the metric induced by the canonical left invariant Riemannian structure on $S$ and $e$ is the identity element of $S$. Note that radial functions on $S$ can be identified with the functions $f=f(r)$ of the geodesic distance $r=\mu(x, e) \in[0, \infty)$ to the identity. It is clear that $\mu\left(a_{t}, e\right)=|t|$ for $t \in \mathbb{R}$. At times, for any radial function $f$ we use the notation $f\left(a_{t}\right)=f(t)$. For any function space $\mathscr{F}(S)$ on $S$, the subspace of radial functions will be denoted by $\mathscr{F}(S)^{\sharp}$. The elementary spherical function $\phi_{\lambda}(x)$ is defined by

$$
\phi_{\lambda}(x):=\int_{N} \mathscr{P}_{\lambda}(x, n) \mathscr{P}_{-\lambda}(x, n) \mathrm{d} n .
$$

It follows [1,2] that $\phi_{\lambda}$ is a radial eigenfunction of the Laplace-Beltrami operator $\Delta_{S}$ of $S$ with eigenvalue $-\left(\lambda^{2}+\frac{Q^{2}}{2}\right)$ such that $\phi_{\lambda}(x)=\phi_{-\lambda}(x), \phi_{\lambda}(x)=\phi_{\lambda}\left(x^{-1}\right)$ and $\phi_{\lambda}(e)=1$. It is also evident from the fact that, for every fixed $n \in N, \mathscr{P}_{\lambda}(x, n)$ is an eigenfunction of $\Delta_{S}$ with eigenvalue $-\left(\lambda^{2}+\frac{Q^{2}}{2}\right)$, that, for suitable function $f$ on $S$, we have

$$
\widetilde{\Delta_{S}^{l}} f(\lambda, n)=-\left(\lambda^{2}+\frac{Q^{2}}{2}\right)^{l} \widetilde{f}(\lambda, n),
$$

for every natural number $l$ (cf. [2, p. 416]). In [1], the authors showed that the radial part (in geodesic polar coordinates) of the Laplace-Beltrami operator $\Delta_{S}$ given by

$$
\operatorname{rad} \Delta_{S}=\frac{\partial^{2}}{\partial t}+\left(\frac{m+l}{2} \operatorname{coth} \frac{t}{2}+\frac{l}{2} \tanh \frac{t}{2}\right) \frac{\partial}{\partial t},
$$

is (by substituting $r=\frac{t}{2}$ ) equal to $\frac{1}{4} \mathscr{L}_{\alpha, \beta}$ with indices $\alpha=\frac{m+l+2}{2}$ and $\beta=\frac{l-1}{2}$, where $\mathscr{L}_{\alpha, \beta}$ is the Jacobi operator studied by Koornwinder [19] in detail. It is worth noting that we are in the ideal situation of Jacobi analysis with $\alpha>\beta>-\frac{1}{2}$. In fact, the Jacobi functions $\phi_{\lambda}^{\alpha, \beta}$ and elementary spherical functions $\phi_{\lambda}$ are related as [1]: $\phi_{\lambda}(t)=\phi_{2 \lambda}^{\alpha, \beta}\left(\frac{t}{2}\right)$. As consequence of this relation, the following estimates for the elementary spherical functions hold true:

Lemma 3 (cf. [22]). The following inequalities are valid for the spherical functions $\phi_{\lambda}(t)\left(\lambda, t \in \mathbb{R}_{+}\right)$
(i) $\left|\phi_{\lambda}(t)\right| \leq 1$.
(ii) $\left|1-\phi_{\lambda}(t)\right| \leq \frac{t^{2}}{2}\left(\lambda^{2}+\frac{Q^{2}}{4}\right)$.
(iii) There exists a constant c $>0$, depending only on $\lambda$, such that

$$
\left|1-\phi_{\lambda}(t)\right| \geq c,
$$

for $\lambda t \geq 1$.
Lemma 4 (cf. [6]). Let $\alpha>-1 / 2,-1 / 2 \leq \beta \leq \alpha$, and let $0<\gamma_{0}<\rho$, there exists a positive constant $c_{1}=C(\alpha, \beta, \rho)$ such that

$$
\left|1-\phi_{\lambda+i \gamma}(t)\right| \geq c_{1} \min \left\{1,(\lambda t)^{2}\right\}
$$

for all $|\gamma| \leq \gamma_{0}, \lambda \in \mathbb{R}$, and $t>0$.

Let $\sigma_{t}$ be the normalized surface measure of the geodesic sphere of radius $t$. Then $\sigma_{t}$ is a nonnegative radial measure. The spherical mean operator $M_{t}$ on a suitable function space on $S$ is defined by $M_{t} f:=f * \sigma_{t}$. It can be noted that $M_{t} f(x)=\mathscr{R}\left(f^{x}\right)(t)$, where $f^{x}$ denotes the right translation of function $f$ by $x$ and $\mathscr{R}$ is the radialization operator defined, for suitable function $f$, by

$$
\mathscr{R} f(x)=\int_{S_{v}} f(y) \mathrm{d} \sigma_{v}(y),
$$

where $v=r(x)=\mu(C(x), 0)$, here $C$ is the Cayley transform, and $\mathrm{d} \sigma_{v}$ is the normalized surface measure induced by the left invariant Riemannian metric on the geodesic sphere $S_{v}=$ $\{y \in S: \mu(y, e)=v\}$. It is easy to see that $\mathscr{R} f$ is a radial function and for any radial function $f$, $\mathscr{R} f=f$. Consequently, for a radial function $f, M_{t} f$ is the usual translation of $f$ by $t$. In [20], the authors proved that, for a suitable function $f$ on $S, \widetilde{M_{t} f}(\lambda, n)=\phi_{\lambda}\left(a_{t}\right) \widetilde{f}(\lambda, n)$ whenever both make sense. Also, $M_{t} f$ converges to $f$ as $t \rightarrow 0$ i.e., $\mu\left(a_{t}, e\right) \rightarrow 0$. It is also known that $M_{t}$ is a bounded operator on $L^{2}(S)$ with operator norm equal to $\phi_{0}(t)$. In particular, for $f \in L^{2}(S)$, we have $\left\|M_{t} f\right\|_{2} \leq \phi_{0}(t)\|f\|_{2}$. In [20, Theorem 4], the authors proved the following inequality: For $1<p \leq 2, p \leq q \leq p^{\prime}=p /(p-1)$ and $f \in L^{p}(S)$ we have

$$
\begin{equation*}
\int_{\mathbb{R}} \min \left\{1,(\lambda t)^{2 p^{\prime}}\right\}\left(\int_{N}\left|\widetilde{f}\left(\lambda+i \gamma_{q} \rho, n\right)\right|^{q} \mathrm{~d} n\right)^{p^{\prime} / q} \mathrm{~d} \mu(\lambda) \leq C_{p, q}^{p^{\prime}}\left\|M_{t} f-f\right\|_{p}^{p^{\prime}} \tag{4}
\end{equation*}
$$

where $\mathrm{d} \mu(\lambda)=|c(\lambda)|^{-2} \mathrm{~d} \lambda$.

## 3. Lipschitz conditions in Damek-Ricci spaces

In this section, we give the main result of the paper but first we need to define the Lipschitz class.
Definition 5. Let $0<\eta \leq 1$. A function $f \in L^{p}(S)$ is said to be in the Damek-Ricci-Lipschitz class, denoted by $\operatorname{Lip}(\eta, p)$, if it satisfies

$$
\left\|M_{t} f-f\right\|_{p}=O\left(|t|^{\eta}\right), \quad t \rightarrow 0
$$

The following Theorem represents a quantified Riemann-Lebesgue lemma (item (1)), and is an extension of results in one dimension given in Titchmarsh [26].

Theorem 6. Let $1<p \leq 2$ and $p^{\prime}=p /(p-1)$.
(1) If $f \in \operatorname{Lip}(\eta, p), 0<\eta \leq 1$, then

$$
\int_{|\lambda| \geq r} \int_{N}\left|\widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{p^{\prime}} \mathrm{d} n \mathrm{~d} \lambda=O\left(r^{-p^{\prime} \eta-d+1}\right) \text {, as } \quad r \rightarrow \infty ;
$$

(2) when $p=2$ and $0<\eta<1$, the converse statement holds as well.

Proof. (1). The proof of this result is immediate from the estimate (4). Indeed, for $q=p^{\prime}$ we obtain,

$$
\int_{|\lambda| \geq 1 / t} \int_{N}\left|\widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{p^{\prime}} \mathrm{d} n \mathrm{~d} \mu(\lambda) \leq C_{p, p^{\prime}}^{p^{\prime}}\left\|M_{t} f-f\right\|_{p}^{p^{\prime}}
$$

then

$$
\int_{|\lambda| \geq 1 / t} \int_{N}\left|\widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{p^{\prime}} \mathrm{d} n \mathrm{~d} \mu(\lambda)=O\left(|t|^{p^{\prime} \eta}\right)
$$

and by $|c(\lambda)|^{-2}=|\lambda|^{d-1}$, we get

$$
\int_{|\lambda| \geq r} \int_{N}\left|\tilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{p^{\prime}} \mathrm{d} n \mathrm{~d} \lambda=O\left(r^{-p^{\prime} \eta-d+1}\right), \quad \text { as } \quad r \rightarrow \infty
$$

(2). For the converse, when $p=2$, the same proof presented in [14] for noncompact rank one symmetric spaces can be rewritten with minor adjustments as follows. Suppose that

$$
\int_{|\lambda| \geq r} \int_{N}\left|\tilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{2} \mathrm{~d} n \mathrm{~d} \lambda=O\left(r^{-2 \eta-d+1}\right), \quad \text { as } \quad r \rightarrow \infty
$$

and

$$
F(\lambda)=\int_{N}\left|\tilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{2} \mathrm{~d} n
$$

Then, we have

$$
\begin{aligned}
\int_{r \leq|\lambda| \leq 2 r} F(\lambda)|\lambda|^{d-1} \mathrm{~d} \lambda & \leq(2 r)^{d-1} \int_{r \leq|\lambda| \leq 2 r} F(\lambda) \mathrm{d} \lambda \\
& \leq 2^{d-1} r^{d-1} \int_{|\lambda| \geq r} F(\lambda) \mathrm{d} \lambda \\
& \leq c_{2} r^{-2 \eta} .
\end{aligned}
$$

Now,

$$
\begin{aligned}
\int_{|\lambda| \geq r} F(\lambda)|\lambda|^{d-1} \mathrm{~d} \lambda & =\sum_{k=0}^{\infty} \int_{2^{k} r \leq|\lambda| \leq 2^{k+1} r} F(\lambda)|\lambda|^{d-1} \mathrm{~d} \lambda \\
& \leq c_{2} \sum_{k=0}^{\infty} 2^{-2 k \eta} r^{-2 \eta} .
\end{aligned}
$$

Consequently,

$$
\int_{|\lambda| \geq r} F(\lambda)|\lambda|^{d-1} \mathrm{~d} \lambda=O\left(r^{-2 \eta}\right),
$$

and, by $|c(\lambda)|^{-2}=|\lambda|^{d-1}$,

$$
\int_{|\lambda| \geq r} F(\lambda) \mathrm{d} \mu(\lambda)=O\left(r^{-2 \eta}\right) .
$$

According to the Plancherel formula, one has $\left\|M_{t} f-f\right\|_{2}^{2}=I_{1}+I_{2}$, where

$$
I_{1}=\int_{0}^{1 / t}\left|1-\phi_{\lambda+i \gamma_{p^{\prime}} \rho}\left(a_{t}\right)\right|^{2} F(\lambda) \mathrm{d} \mu(\lambda) \quad \text { and } \quad I_{2}=\int_{\frac{1}{t}}^{+\infty}\left|1-\phi_{\lambda+i \gamma_{p^{\prime}} \rho}\left(a_{t}\right)\right|^{2} F(\lambda) \mathrm{d} \mu(\lambda),
$$

estimate the summands $I_{1}$ and $I_{2}$ from above. Firstly, it follows from the inequality $\left|\phi_{\lambda+i \gamma_{p^{\prime}} \rho}\left(a_{t}\right)\right| \leq 1$ that

$$
I_{2} \leq 4 \int_{\frac{1}{t}}^{+\infty} F(\lambda) \mathrm{d} \mu(\lambda)=O\left(t^{2 \eta}\right), \quad \text { as } \quad t \rightarrow 0
$$

To estimate $I_{1}$, we use the inequalities (i) and (ii) of Lemma 3

$$
\begin{aligned}
I_{1} & =\int_{0}^{1 / t}\left|1-\phi_{\lambda+i \gamma_{p^{\prime} \rho} \rho}\left(a_{t}\right)\right| 1-\phi_{\lambda}\left(a_{t}\right) \mid F(\lambda) \mathrm{d} \mu(\lambda) \\
& \leq 2 \int_{0}^{1 / t}\left|1-\phi_{\lambda+i \gamma_{p^{\prime}} \rho}\left(a_{t}\right)\right| F(\lambda) \mathrm{d} \mu(\lambda) \\
& \leq t^{2} \int_{0}^{1 / t}\left(\lambda^{2}+\frac{Q^{2}}{2}\right) F(\lambda) \mathrm{d} \mu(\lambda) .
\end{aligned}
$$

Consider the function $\varphi(r)=\int_{r}^{\infty} F(\lambda) \mathrm{d} \mu(\lambda)$. An integration by parts gives:

$$
\begin{aligned}
\int_{0}^{1 / t}\left(\lambda^{2}+\frac{Q^{2}}{2}\right) F(\lambda) \mathrm{d} \mu(\lambda) & =\int_{0}^{1 / t}-\left(r^{2}+\frac{Q^{2}}{2}\right) \varphi^{\prime}(r) \mathrm{d} r \\
& \leq \int_{0}^{1 / t}-r^{2} \varphi^{\prime}(r) \mathrm{d} r \\
& =-\frac{1}{t^{2}} \varphi\left(\frac{1}{t}\right)+2 \int_{0}^{1 / t} r \varphi(r) \mathrm{d} r \\
& \leq 2 \int_{0}^{1 / t} r \varphi(r) \mathrm{d} r
\end{aligned}
$$

Since $\varphi(r)=O\left(r^{-2 \eta}\right)$, we have $r \varphi(r)=O\left(r^{1-2 \eta}\right)$ and

$$
\int_{0}^{1 / t} r \varphi(r) \mathrm{d} r=O\left(\int_{0}^{1 / t} r^{1-2 \eta} \mathrm{~d} r\right)=O\left(t^{2 \eta-2}\right),(\text { the integral exists since } \eta<1)
$$

so that $I_{1}=O\left(t^{2 \eta}\right)$. Combining the estimates for $I_{1}$ and $I_{2}$ gives

$$
\left\|M_{t} f-f\right\|_{2}=O\left(t^{\eta}\right) \quad \text { as } \quad t \rightarrow 0
$$

and this ends the proof of the theorem.
For $f \in L^{p}(S)$, we define the finite differences of first and higher order as follows:

$$
\begin{aligned}
& \Delta_{t}^{1} f=\Delta_{t} f=\left(I-M_{t}\right) f \\
& \Delta_{t}^{k} f=\Delta_{t}\left(\Delta_{t}^{k-1} f\right)=\left(I-M_{t}\right)^{k} f, \quad k=2,3, \ldots
\end{aligned}
$$

where $I$ is the unit operator in the space $L^{p}(S)$.
Consequently, for each $f \in L^{p}(S)$,

$$
\widetilde{\Delta_{t}^{k} f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)=\left(1-\phi_{\lambda+i \gamma_{p^{\prime}} \rho}\left(a_{t}\right)\right)^{k} \widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)
$$

and, by Plancherel formula, we have

$$
\begin{equation*}
\left\|\Delta_{t}^{k} f\right\|_{2}^{2}=\int_{0}^{+\infty} \int_{N}\left|1-\phi_{\lambda+i \gamma_{p^{\prime}} \rho}\left(a_{t}\right)\right|^{2 k}\left|\widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{2}|c(\lambda)|^{-2} \mathrm{~d} \lambda \mathrm{~d} n \tag{5}
\end{equation*}
$$

By analogy with the proof of Theorem 6, we can establish from formula (5) the following result:
Theorem 7. Let $1<p \leq 2$ and $p^{\prime}=p /(p-1)$.
(1) If $\left\|\Delta_{t}^{k} f\right\|_{2}=O\left(|t|^{\eta}\right), 0<\eta \leq 1$, then

$$
\int_{|\lambda| \geq r} \int_{N}\left|\widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{p^{\prime}} \mathrm{d} n \mathrm{~d} \lambda=O\left(r^{-p^{\prime} \eta-d+1}\right), \quad \text { as } \quad r \rightarrow \infty
$$

(2) when $p=2,0<\eta<1$ and $k=1,2, \ldots$, the converse statement holds as well.

We now state our second main result which extends the integrability Theorem 1 to DamekRicci spaces.

Theorem 8. Let $1<p \leq 2, p^{\prime}=p /(p-1), 0<\eta \leq 1$ and $f \in \operatorname{Lip}(\eta, p)$. Then its transform $\widetilde{f}\left(\cdot+i \gamma_{p^{\prime}} \rho, \cdot\right)$ is in $L^{\delta}(\mathbb{R} \times N)$ with respect to the Plancherel measure $\mathrm{d} n \mathrm{~d} \mu(\lambda)$ for every $\delta$,

$$
\frac{p d}{d(p-1)+p \eta}<\delta \leq p^{\prime}
$$

Proof. Using formula (4), we see that

$$
\begin{equation*}
\int_{|\lambda| \leq 1 / t} \lambda^{2 p^{\prime}} G(\lambda) \mathrm{d} \mu(\lambda)=O\left(|t|^{(\eta-2) p^{\prime}}\right) \tag{6}
\end{equation*}
$$

where

$$
G(\lambda)=\int_{N}\left|\widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{p^{\prime}} \mathrm{d} n
$$

Now, let

$$
\varphi(X)=\int_{1 \leq|\lambda| \leq X} \lambda^{2 \delta} G(\lambda) \mathrm{d} \mu(\lambda)
$$

Applying Hölder's inequality with $\delta \leq p^{\prime}$ for the last estimate one arrives at

$$
\varphi(X) \leq\left(\int_{1 \leq|\lambda| \leq X} \lambda^{2 p^{\prime}} G(\lambda) \mathrm{d} \mu(\lambda)\right)^{\frac{\delta}{p^{\prime}}}\left(\int_{1 \leq|\lambda| \leq X} 1 \mathrm{~d} \mu(\lambda)\right)^{1-\frac{\delta}{p^{\prime}}}
$$

Hence, by using relations (3) and (6), we obtain

$$
\begin{equation*}
\varphi(X)=O\left(X^{(2-\eta) \delta+d\left(1-\frac{\delta}{p^{\prime}}\right)}\right) \tag{7}
\end{equation*}
$$

Remark that

$$
\int_{1 \leq|\lambda| \leq X} G(\lambda) \mathrm{d} \mu(\lambda)=\int_{1 \leq|\lambda| \leq X} \lambda^{-2 \delta} \varphi^{\prime}(\lambda) \mathrm{d} \lambda
$$

Making an integration by parts, we get

$$
\begin{equation*}
\int_{1 \leq|\lambda| \leq X} G(\lambda) \mathrm{d} \mu(\lambda)=X^{-2 \delta} \varphi(X)+2 \delta \int_{1 \leq|\lambda| \leq X} t^{-2 \delta-1} \varphi(t) \mathrm{d} t, \tag{8}
\end{equation*}
$$

From relation (7), we have

$$
\int_{1 \leq|\lambda| \leq X} G(\lambda) \mathrm{d} \mu(\lambda)=O\left(X^{-2 \delta+(2-\eta) \delta+d\left(1-\frac{\delta}{p^{\prime}}\right)}\right)+O\left(\int_{1 \leq|\lambda| \leq X} t^{-2 \delta-1+(2-\eta) \delta+d\left(1-\frac{\delta}{p^{\prime}}\right)} \mathrm{d} t\right)
$$

and this is bounded as $X \rightarrow \infty$ if $-\delta\left(\eta+\frac{d}{p^{\prime}}\right)+d<0$, which gives

$$
\delta>\frac{p d}{d(p-1)+p \eta}
$$

## 4. Dini-Lipschitz conditions in Damek-Ricci spaces

The reader can find analogous results of this section in the references [11, 13-15, 17, 21, 23].
Definition 9. Let $0<\eta \leq 1$ and $\gamma \geq 0$. A function $f \in L^{p}(S)$ is said to be in the Damek-Ricci-DiniLipschitz class, denoted by $\operatorname{DLip}(\eta, \gamma, p)$, if

$$
\left\|M_{t} f-f\right\|_{p}=O\left(|t|^{\eta}\left(\log \frac{1}{|t|}\right)^{-\gamma}\right) \text { as } \quad|t| \rightarrow 0
$$

By using the same tricks of calculation that we have already used to show the previous theorems, we prove the following theorems.

Theorem 10. Let $1<p \leq 2$ and $p^{\prime}=p /(p-1)$.
(1) If f $\in \operatorname{DLip}(\eta, \gamma, p), 0<\eta \leq 1, \gamma \geq 0$, then

$$
\int_{|\lambda| \geq r} \int_{N}\left|\widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{p^{\prime}} \mathrm{d} n \mathrm{~d} \lambda=O\left(r^{-p^{\prime} \eta-d+1}(\log r)^{-p^{\prime} \gamma}\right), \quad \text { as } \quad r \rightarrow \infty
$$

(2) when $p=2, \gamma \geq 0$ and $0<\eta<1$, the converse statement holds as well.

Proof. By proceeding similarly to Theorem 3.2, item (1), we have,

$$
\int_{|\lambda| \geq 1 / t} \int_{N}\left|\widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{p^{\prime}} \mathrm{d} n \mathrm{~d} \mu(\lambda)=O\left(|t|^{p^{\prime} \eta}\left(\log \frac{1}{|t|}\right)^{-p^{\prime} \gamma}\right)
$$

Thus,

$$
\int_{|\lambda| \geq r} \int_{N}\left|\widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right|^{p^{\prime}} \mathrm{d} n \mathrm{~d} \lambda=O\left(r^{-p^{\prime} \eta-d+1}(\log r)^{-p^{\prime} \gamma}\right), \quad \text { as } \quad r \rightarrow \infty
$$

The converse can be done in the same way as in Theorem 6 above and Theorem 8 in [14] for noncompact rank one symmetric spaces. Consider the same notation $\left\|M_{t} f-f\right\|_{2}^{2}=I_{1}+I_{2}$ and $\varphi(r)=O\left(r^{-2 \eta}(\log r)^{-2 \gamma}\right)$. Then, we get

$$
I_{2}=O\left(t^{2 \eta}\left(\log \frac{1}{t}\right)^{-2 \gamma}\right), \quad \text { as } \quad t \rightarrow 0
$$

and,

$$
I_{1}=O\left(t^{2} \int_{0}^{1 / t} r \varphi(r) \mathrm{d} r\right)=O\left(t^{2 \eta}\left(\log \frac{1}{t}\right)^{-2 \gamma}\right), \quad \text { as } \quad t \rightarrow 0
$$

Theorem 11. Let $1<p \leq 2, p^{\prime}=p /(p-1), 0<\eta \leq 1, \gamma \geq 0$ and $f \in \operatorname{Dip}(\eta, \gamma, p)$. Then its transform $\widetilde{f}\left(\cdot+i \gamma_{p^{\prime}} \rho, \cdot\right)$ is in $L^{\delta}(\mathbb{R} \times N)$ with respect to the Plancherel measure $\mathrm{d} n \mathrm{~d} \mu(\lambda)$ for every $\delta$,

$$
\frac{p d}{d(p-1)+p \eta}<\delta \leq p^{\prime}
$$

Proof. As in Theorem 8, we have

$$
\int_{|\lambda| \leq 1 / t} \lambda^{2 p^{\prime}} G(\lambda) \mathrm{d} \mu(\lambda)=O\left(|t|^{(\eta-2) p^{\prime}}\left(\log \frac{1}{|t|}\right)^{-p^{\prime} \gamma}\right)
$$

For $\delta \leq p^{\prime}$, this implies via Hölder's inequality

$$
\varphi(X)=O\left(X^{(2-\eta) \delta+d\left(1-\frac{\delta}{p^{\prime}}\right)}(\log X)^{-\delta \gamma}\right)
$$

This allows us to deduce, by relation (8), that

$$
\int_{1 \leq|\lambda| \leq X} G(\lambda) \mathrm{d} \mu(\lambda)=O\left(X^{-\eta \delta+d\left(1-\frac{\delta}{p^{\prime}}\right)}(\log X)^{-\delta \gamma}\right)+O\left(\int_{1 \leq|\lambda| \leq X} t^{-1-\eta \delta+d\left(1-\frac{\delta}{p^{\prime}}\right)}(\log t)^{-\delta \gamma} \mathrm{d} t\right)
$$

For the right hand side of the last estimate to be bounded as $X$ goes to $\infty$ we must have $-\delta\left(\eta+\frac{d}{p^{\prime}}\right)+d<0$, which gives $\delta>\frac{p d}{d(p-1)+p \eta}$.

This section concludes with the following result:
Theorem 12. Let $\eta>2, \gamma \geq 0$ and $f \in \operatorname{DLip}(\eta, \gamma, 2)$, then $f=0$ a.e.
Proof. Assume that $f \in \operatorname{DLip}(\eta, \gamma, 2)$. Then

$$
\left\|M_{t} f-f\right\|_{2} \leq c_{3}|t|^{\eta}\left(\log \frac{1}{|t|}\right)^{-\gamma}
$$

In view of formula (5), we conclude that

$$
\int_{0}^{+\infty}\left|1-\phi_{\lambda+i \gamma_{p^{\prime}} \rho}\left(a_{t}\right)\right|^{2} F(\lambda) \mathrm{d} \mu(\lambda) \leq c_{3}^{2}|t|^{2 \eta}\left(\log \frac{1}{|t|}\right)^{-2 \gamma}
$$

Thus,

$$
\frac{\int_{0}^{+\infty}\left|1-\phi_{\lambda+i \gamma_{p^{\prime}} \rho}\left(a_{t}\right)\right|^{2} F(\lambda) \mathrm{d} \mu(\lambda)}{|t|^{4}} \leq c_{3}^{2}|t|^{2 \eta-4}\left(\log \frac{1}{|t|}\right)^{-2 \gamma}
$$

Since $\eta>2$, then

$$
\lim _{t \rightarrow 0}|t|^{2 \eta-4}\left(\log \frac{1}{|t|}\right)^{-2 \gamma}=0
$$

Hence,

$$
\lim _{t \rightarrow 0} \int_{0}^{+\infty}\left(\frac{\left|1-\phi_{\lambda+i \gamma_{p^{\prime}} \rho}\left(a_{t}\right)\right|}{\lambda^{2} t^{2}}\right)^{2} \lambda^{4} F(\lambda) \mathrm{d} \mu(\lambda)=0,
$$

and also from Lemma 4 and Fatou theorem, we obtain

$$
\left\|\lambda^{2} \widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)\right\|_{L^{2}\left(\mathbb{R}_{+} \times N\right)}=0
$$

Thereby for all $(\lambda, n) \in \mathbb{R}_{+} \times N, \lambda^{2} \widetilde{f}\left(\lambda+i \gamma_{p^{\prime}} \rho, n\right)=0$. The injectivity of the Fourier-Helgason transform yields to the wanted result.

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## References

[1] J.-P. Anker, E. Damek, C. Yacoub, "Spherical analysis on harmonic AN groups", Ann. Sc. Norm. Super. Pisa, Cl. Sci. 23 (1996), no. 4, p. 643-679.
[2] F. Astengo, R. Camporesi, B. Di Blasio, "The Helgason Fourier transform on a class of nonsymmetric harmonic spaces", Bull. Aust. Math. Soc. 55 (1997), no. 3, p. 405-424.
[3] F. Astengo, B. Di Blasio, "A Paley-Wiener theorem on $N A$ harmonic spaces", Colloq. Math. 80 (1999), no. 2, p. 211-233.
[4] W. O. Bray, "Growth and integrability of Fourier transforms on Euclidean space", J. Fourier Anal. Appl. 20 (2014), no. 6, p. 1234-1256.
[5] W. O. Bray, M. A. Pinsky, "Growth properties of Fourier transforms via moduli of continuity", J. Funct. Anal. 255 (2008), no. 9, p. 2265-2285.
[6] —, "Growth properties of the Fourier transform", Filomat 26 (2012), no. 4, p. 755-760.
[7] M. Cowling, A. Dooley, A. Korányi, F. Ricci, "An approach to symmetric spaces of rank one via groups of Heisenberg type", J. Geom. Anal. 8 (1998), no. 2, p. 199-237.
[8] R. Daher, J. Delgado, M. Ruzhansky, "Titchmarsh theorems for Fourier transforms of Hölder-Lipschitz functions on compact homogeneous manifolds", Monatsh. Math. 189 (2019), no. 1, p. 23-49.
[9] R. Daher, M. El Hamma, "An analog of Titchmarsh's theorem for the generalized Dunkl transform", J. Pseudo-Differ. Oper. Appl. 7 (2016), no. 1, p. 59-65.
[10] R. Daher, M. El Hamma, S. El Ouadih, "An analog of Titchmarsh's theorem for the generalized Fourier-Bessel Transform", Lobachevskii J. Math. 37 (2016), no. 2, p. 114-119.
[11] R. Daher, S. El Ouadih, "Best trigonometric approximation and Dini-Lipschitz classes", J. Pseudo-Differ. Oper. Appl. 9 (2018), no. 4, p. 903-912.
[12] E. Damek, F. Ricci, "Harmonic analysis on solvable extensions of H-type groups", J. Geom. Anal. 2 (1992), no. 3, p. 213-248.
[13] M. El Hamma, R. Daher, "Dini Lipschitz functions for the Dunkl transform in the space $L^{2}\left(\mathbb{R}^{d}, w_{k}(x) d x\right)$ ", Rend. Circ. Mat. Palermo 64 (2015), no. 2, p. 241-249.
[14] S. El Ouadih, R. Daher, "Characterization of Dini-Lipschitz functions for the Helgason Fourier transform on rank one symmetric spaces", Adv. Pure Appl. Math. 7 (2016), no. 4, p. 223-230.
[15] , "Jacobi-Dunkl Dini Lipschitz functions in the space $L^{p}\left(\mathbb{R}, A_{\alpha, \beta}(x) d x\right) "$, Appl. Math. E-Notes 16 (2016), p. 8898.
[16] , "Lipschitz conditions for the generalized discrete Fourier transform associated with the Jacobi operator on $[0, \pi] "$, C. R. Math. Acad. Sci. Paris 355 (2017), no. 3, p. 318-324.
[17] S. Fahlaoui, M. Boujeddaine, M. El Kassimi, "Fourier transforms of Dini-Lipschitz functions on rank 1 symmetric spaces", Mediterr. J. Math. 13 (2016), no. 6, p. 4401-4411.
[18] M. Flensted-Jensen, T. H. Koornwinder, "Jacobi functions: the addition formula and the positivity of the dual convolution structure", Ark. Mat. 17 (1979), p. 139-151.
[19] T. H. Koornwinder, "Jacobi functions and analysis on noncompact semisimple Lie groups", in Special functions: Group theoretical aspects and applications, Mathematics and its Applications, vol. 18, Reidel Publishing Company, 1984, p. 1-85.
[20] P. Kumar, S. K. Ray, R. P. Sarkar, "The role of restriction theorems in harmonic analysis on harmonic $N A$ groups", $J$. Funct. Anal. 258 (2010), no. 7, p. 2453-2482.
[21] S. Negzaoui, "Lipschitz conditions in Laguerre hypergroup", Mediterr. J. Math. 14 (2017), no. 5, article no. 191 (12 pages).
[22] S. S. Platonov, "Approximation of functions in the $L^{2}$ Metric on noncompact rank 1 symmetric spaces", Algebra Anal. 11 (1999), no. 1, p. 244-270.
[23] -, "The Fourier transform of functions satisfying the Lipschitz condition on rank 1 symmetric spaces", Sib. Math. J. 46 (2005), no. 6, p. 1108-1118.
[24] S. K. Ray, R. P. Sarkar, "Fourier and Radon transform on harmonic NA groups", Trans. Am. Math. Soc. 361 (2009), no. 8, p. 4269-4297.
[25] F. Rouvière, "Espaces de Damek-Ricci, géométrie et analyse", in Analyse sur les groupes de Lie et théorie des représentations, Séminaires et Congrès, vol. 7, Société Mathématique de France, 2003, p. 45-100.
[26] E. C. Titchmarsh, Introduction to the theory of Fourier integrals, Clarendon Press, 1937.
[27] M. Weiss, A. Zygmund, "A note on smooth functions", Indag. Math. 62 (1959), p. 52-58.
[28] M. S. Younis, "Fourier transforms in $L^{p}$ spaces", PhD Thesis, Chelsea College (UK), 1970.
[29] , "Fourier transforms of Lipschitz functions on compact groups", PhD Thesis, McMaster University (Canada), 1974.
[30] , "Fourier transforms of Dini-Lipschitz functions", Int. J. Math. Math. Sci. 9 (1986), no. 2, p. 301-312.

