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Generalized versions of Lipschitz conditions on the modulus of holomorphic functions

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Abstract. In this paper, we establish Lipschitz conditions for the norm of holomorphic mappings between the unit ball $B^n$ in $\mathbb{C}^n$ and $X$, a complex normed space. This extends the work of Djordjević and Pavlović.

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1. Introduction and Preliminaries

Denote by $\mathbb{C}^n$, the $n$-dimensional complex Hilbert space with the inner product and the norm given by $\langle z, w \rangle := \sum_{j=1}^{n} z_j w_j$ and $\|z\| := \sqrt{\langle z, z \rangle}$, where $z, w \in \mathbb{C}^n$, respectively. Write $B^n := \{ z \in \mathbb{C}^n : \| z \| < 1 \}$ for the open unit ball in $\mathbb{C}^n$ so that $B^1 = \mathbb{D}$ denotes the open unit disk in $\mathbb{C}$. If $V$ and $W$ are two normed spaces and $U \subset V$ is open, then the Fréchet derivative of a holomorphic mapping $f : U \to W$ is defined to be the unique linear map $A = f'(z) : V \to W$ such that

$$f(z + h) = f(z) + f'(z) \cdot h + o(\|h\|^2)$$

for $h$ near the origin of $V$. The norm of such a map is defined by $\| A \| = \sup_{\|z\| = 1} \| Az \|$. 

In 1975, Globevnik [6] introduced the notion of uniform c-convexity and proved that $L^1$-space possesses this property. Namely, a complex normed space $X$ is said to be uniformly c-convex if there exists a positive increasing function $\Omega(\delta)$ ($\delta > 0$) with $\Omega(0^+) = 0$ such that for all $x, y \in X$ and $\delta > 0$ there holds the implication

$$\max_{\|A\| \leq 1} \|x + A y\| \leq 1 + \delta \implies \|y\| \leq \Omega(\delta).$$

The smallest of the functions $\Omega$ is denoted by $\Omega_X$, i.e.,

$$\Omega_X(\delta) := \sup \left\{ \| y \| : \max_{\|A\| \leq 1} \| x + A y\| \leq 1 + \delta \right\}.$$
As mentioned in [3], it can be easily seen that
\[ \Omega_C(\delta) = \delta \quad \text{and} \quad \Omega_H(\delta) = \sqrt{\delta(2+\delta)}, \]
where \( H \) is a Hilbert space of dimension at least two.

As in [4], we call a function \( \omega : [0, \infty) \to \mathbb{R} \) a majorant if \( \omega \) is continuous, increasing, \( \omega(0) = 0 \), and \( t^{-1}\omega(t) \) is nonincreasing on \((0, \infty)\). If, in addition, there is a constant \( C(\omega) > 0 \) such that
\[ \int_0^\delta \frac{\omega(t)}{t} \, dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2} \, dt \leq C(\omega) \cdot \omega(\delta) \]
whenever \( 0 < \delta < 1 \), then we say that \( \omega \) is a regular majorant.

Then the space \( \text{Lip}(\omega, G, X) \), where \( G \) is bounded subset of \( \mathbb{C}^n \), is defined to be the set of those functions \( g : G \to X \) for which
\[ \|g(z) - g(w)\| \leq c \cdot \omega(\|z - w\|), \]
where \( c \) is a constant. If \( \omega(t) = t^\alpha \) for some \( \alpha \in (0, 1] \), then we write \( \text{Lip}(\omega, G, X) = \Lambda_\alpha(G, X) \). If \( X \) is uniformly \( c \)-convex, then \( \Omega_X \) is a majorant (cf. [2]). A majorant \( \omega \) is said to be a Dini majorant if \( \int_0^1 \omega(t) \, dt < \infty \). For a Dini majorant, we define the majorant \( \tilde{\omega} \) by
\[ \tilde{\omega}(t) = \int_0^t \frac{\omega(x)}{x} \, dx = \int_0^1 \frac{\omega(tx)}{x} \, dx. \]
A majorant \( \omega \) is said to be fast [5] if
\[ \int_0^\delta \frac{\omega(t)}{t} \, dt \leq \text{const} \cdot \omega(\delta), \quad 0 < \delta < \delta_0, \]
for some \( \delta_0 > 0 \). (Of course, if \( \omega \) is fast, then it is a Dini majorant).

Dyakonov [4] gave some characterizations of the holomorphic functions of class \( \Lambda_\omega(D, \mathbb{C}) \) in terms of their moduli.

**Theorem A (cf. [4]).** Let \( \omega \) be a regular majorant. A function \( f \) holomorphic in \( D \) is in \( \Lambda_\omega(D, \mathbb{C}) \) if and only if so is its modulus \( |f| \).

The main ingredient in Dyakonov’s proof is a very complicated. However, Pavlovic [8] gave a simple proof of Theorem A. The proof uses only the basic lemmas of [4] and the Schwarz lemma, and is therefore considerably shorter than that of [4]. However, Theorem A does not extend to \( \mathbb{C}^k \)-valued functions (\( k \geq 2 \)). So we have to consider functions with additional properties (see Theorems 5 and 6).

In [3], Djordjević and Pavlović extended to vector-valued functions of a theorem of Dyakonov [4] on Lipschitz conditions for the modulus of holomorphic functions. Therefore, it is natural for us to extend this result for holomorphic functions on \( \mathbb{B}^n \). Very recently, Kalaj [7] established a Schwarz–Pick type inequality for holomorphic mappings between unit balls \( \mathbb{B}^n \) and \( \mathbb{B}^m \) in the corresponding complex spaces.

**Theorem B (cf. [7, Theorem 2.1]).** If \( f \) is a holomorphic mapping of the unit ball \( \mathbb{B}^n \subset \mathbb{C}^n \) into \( \mathbb{B}^m \subset \mathbb{C}^m \), then for \( z \in \mathbb{B}^n \) we have
\[ \|f'(z)\| \leq \begin{cases} \sqrt{1 - \|f(z)\|^2} & \text{for } m \geq 2, \\ \frac{1 - \|z\|^2}{1 - \|f(z)\|^2} & \text{for } m = 1. \end{cases} \]

In [1], Dai and Pan proved the following theorem which establishes a Schwarz–Pick type estimates for gradient of the modulus of holomorphic mappings.
Theorem C (cf. [1, Theorem 1]). Let \( f : \mathbb{B}^n \to \mathbb{B}^n \) be a holomorphic mapping. Then
\[
|\nabla \|f\|(z)| \leq \frac{1 - \|f(z)\|^2}{1 - \|z\|^2} \quad \text{for } z \in \mathbb{B}^n.
\]

For a holomorphic mapping \( f : \mathbb{B}^n \to \mathbb{B}^m \), we have
\[
|\nabla \|f\|(z)| = \frac{1}{\|f(z)\|} \left\| \left( \frac{\partial f(z)}{\partial z_1}, f(z), \ldots, \frac{\partial f(z)}{\partial z_n}, f(z) \right) \right\| \quad \text{if } f(z) \neq 0. \tag{1}
\]

2. The main results

Theorem 1. Let \( X \) be uniformly \( \alpha \)-convex and \( f : \mathbb{B}^n \to X \) be a holomorphic function such that the function \( \|f(z)\| \) belongs to \( \Lambda_\alpha(\mathbb{B}^n, \mathbb{R}) \) for some \( \alpha \in (0, 1] \). Then \( f \in \operatorname{Lip}(\tilde{m}_\alpha, \mathbb{B}^n, X) \), where \( \tilde{m}_\alpha(t) = \Omega_X(t^\alpha) \).

In particular, the function \( f \) is uniformly continuous on \( \mathbb{B}^n \) that has a continuous extension to the closed disk.

Corollary 3. If \( \Omega_X \) is fast and \( f : \mathbb{B}^n \to X \) is a holomorphic function such that the function \( \|f(z)\| \) belongs to \( \Lambda_\alpha(\mathbb{B}^n, \mathbb{R}) \) for some \( \alpha \in (0, 1] \). Then \( f \in \operatorname{Lip}(\tilde{m}_\alpha, \mathbb{B}^n, X) \), where \( \tilde{m}_\alpha(t) = \Omega_X(t^\alpha) \).

Taking \( n = 1 \) and \( X = \mathbb{C} \), we get the following result of Dyakonov [4].

Corollary 4. If \( f : \mathbb{D} \to \mathbb{C} \) is a holomorphic function such that \( |f| \) belongs to \( \Lambda_\alpha(\mathbb{D}, \mathbb{R}) \) for some \( \alpha \in (0, 1] \). Then \( f \) belongs to \( \Lambda_\alpha(\mathbb{D}, \mathbb{C}) \).

Theorem 5. Let \( 0 < \alpha \leq 1 \) and \( f : \mathbb{B}^n \to \mathbb{C}^m \) be a holomorphic function such that
\[
\frac{\|f'(z)\|}{\|f(z)\|} \leq K \left\| \left( \frac{\partial f(z)}{\partial z_1}, f(z), \ldots, \frac{\partial f(z)}{\partial z_n}, f(z) \right) \right\| \quad \text{for } z \in \mathbb{B}^n, \tag{5}
\]
where \( K \) is a constant independent of \( z \). Then \( f \in \Lambda_\alpha(\mathbb{B}^n, \mathbb{C}^m) \) if and only if \( \|f\| \in \Lambda_\alpha(\mathbb{B}^n, \mathbb{R}) \).

Theorem 6. If \( f : \mathbb{B}^n \to \mathbb{C}^m \), \( m \geq 2 \), is holomorphic and if \( \|f\| \in \Lambda_\alpha(\mathbb{B}^n, \mathbb{R}) \) for some \( \alpha \in (0, 1] \), then we have \( f \in \Lambda_{\alpha/2}(\mathbb{B}^n, \mathbb{C}^m) \).

The case \( n = 1 \) of Theorems 5 and 6 gives results of Pavlović [9].

3. Proofs of the Theorems

Theorem 1 is a direct consequence of the following lemma.

Lemma 7. If \( f : \mathbb{B}^n \to X \) is a holomorphic function satisfying the condition
\[
\|f(z)\| - \|f(w)\| \leq c(1 - \|z\|)^\alpha \quad \text{whenever } \|w - z\| \leq 1 - \|z\|, \tag{6}
\]
then there holds (3).
Proof. Fix \( z \in \mathbb{B}^n \) with \( f(z) \neq 0 \), and fix \( \beta \in \mathbb{C}^n \) with \( \|\beta\| = 1 \). Let \( L \in X^\ast, \|L\| = 1 \), where \( X^\ast \) is the dual of \( X \). Consider the scalar valued function

\[
\phi(z) = L \circ f(z),
\]

and introduce the following set for the given \( z \in \mathbb{B}^n \),

\[
D_z := \{ w \in \mathbb{C}^n : \|w - z\| < 1 - \|z\| \} \quad \text{and} \quad M_z := \sup\{\|f(w)\| : w \in D_z\}.
\]

If \( z = 0 \) and \( M_0 = 1 \), then the Schwarz–Pick lemma (see Theorem B) gives

\[
|\phi'(0)| \leq 1 - |\phi(0)|^2 \leq 2(1 - |\phi(0)|),
\]

which is our inequality in this special case. The general case follows by applying the special case to the function \( \Phi \) defined by

\[
\Phi(\zeta) = \frac{\phi(z + (1 - \|z\|)\beta\zeta)}{M_z} \quad \text{for} \ \zeta \in \mathbb{B}^n.
\]

As

\[
\Phi(0) = \frac{L(f(z))}{M_z} \quad \text{and} \quad \Phi'(0) = \frac{1 - \|z\|)}{M_z}L(f'(z)\beta),
\]

we deduce from (7) that

\[
(1 - \|z\|)||L(f'(z)\beta/2)| + |L(f(z))| \leq M_z.
\]

Hence, for every \( \lambda \in \mathbb{D} \), we obtain

\[
|\lambda(1 - \|z\|)||L(f'(z)\beta/2) + L(f(z))| \leq M_z.
\]

Since this holds for every \( L \) of norm 1, by taking the supremum over all \( L \) with \( \|L\| = 1 \) and by applying the Hahn–Banach theorem, we get

\[
\left\| \frac{(1 - \|z\|)}{2} f'(z)\beta + f(z) \right\| \leq M_z, \quad \text{i.e.,} \quad \left\| f(z) \frac{1 - \|z\|}{2f(z)} + \frac{(1 - \|z\|)}{2} f'(z)\beta \right\| \leq \frac{M_z}{\|f(z)\|}.
\]

Now denoting

\[
x = \frac{f(z)}{\|f(z)\|}, \quad y = \frac{(1 - \|z\|)}{2} f'(z)\beta \quad \text{and} \quad \delta = \frac{M_z - \|f(z)\|}{\|f(z)\|},
\]

we see from the definition of \( \Omega_X \) that

\[
(1 - \|z\|)||f'(z)\beta\| \leq 2\|f(z)\|\Omega_X \left( \frac{M_z - \|f(z)\|}{\|f(z)\|} \right).
\]

Hence, the last inequality holds for every \( \beta \in \mathbb{C}^n \) with \( \|\beta\| = 1 \), we get

\[
(1 - \|z\|)||f'(z)\| \leq 2\|f(z)\|\Omega_X \left( \frac{M_z - \|f(z)\|}{\|f(z)\|} \right).
\]

Therefore by (6) and (8), we obtain that

\[
(1 - \|z\|)||f'(z)\| \leq 2\|f(z)\|\Omega_X \left( \frac{c(1 - \|z\|)^a}{\|f(z)\|} \right).
\]

Now (3) follows from the fact that \( \Omega_X(t)/t \) is a decreasing function and the inequality \( \|f(z)\| \leq K \). The proof is complete. \( \square \)

Lemma 8. If a \( C^1 \)-function \( u : \mathbb{B}^n \to \mathbb{R} \) satisfies

\[
\| \nabla u(z) \| \leq \frac{\omega(1 - \|z\|)}{1 - \|z\|} \quad \text{for} \ z \in \mathbb{B}^n,
\]

where \( \omega \) is a Dini majorant, then

\[
|u(a) - u(b)| \leq 3 \tilde{\omega}(\|a - b\|) \quad \text{for} \ a, b \in \mathbb{B}^n.
\]
Lemma 9. If $f$ is an $X$-valued holomorphic function in $\mathbb{B}^n$ and $\operatorname{Im}(a)$ as in Theorem 1. Fix $z \in \mathbb{B}^n$. Finally, if $\delta > 1 - \|a\|$, we use the inequality

$$|u(a) - u(b)| \leq |u(a) - u(a')| + |u(a') - u(b')| + |u(b') - u(b)|,$$

where $a' = \frac{(1-\delta)a}{\|a\|}$, and then proceed in a similar way as above, using the inequality $\|a' - b'\| \leq \|a - b\|$.

Lemma 9 can easily be proved by applying the previous lemma to the functions $\operatorname{Re}(L \circ f(z))$ and $\operatorname{Im}(L \circ f(z))$, where $L \in X^*$ and $\|L\| = 1$.

Lemma 9. If $f$ is an $X$-valued holomorphic function in $\mathbb{B}^n$ and satisfies the condition

$$\|f'(z)\| \leq \frac{\omega(1 - \|z\|)}{1 - \|z\|}$$

for $z \in \mathbb{B}^n$, where $\omega$ is a Dini majorant, then $f \in \operatorname{Lip}(\tilde{\omega}, \mathbb{B}^n, X)$.

Proof of Theorem 2. Let $f$ satisfy the hypotheses of the theorem. Then

$$\|f'(z)/2K\| \leq \frac{\omega(1 - \|z\|)}{1 - \|z\|},$$

by Theorem 1, where $\omega(t) = \Omega_X(cK^{-1}ta)$. But a simple calculation shows that $\tilde{\omega}(t) = a^{-1}\Omega_X(cK^{-1}ta)$ and so we can appeal to Lemma 9 to conclude the proof.

Proof of Theorem 5. The “only if” part is trivial. Assume that $\|f(z)\| \in \Lambda_\omega(\mathbb{B}^n, \mathbb{R})$ and we proceed as in Theorem 1. Fix $z \in \mathbb{B}^n$ with $f(z) \neq 0$, and consider the following sets for a given $z \in \mathbb{B}^n$,

$$D_z := \{w \in C^n : \|w - z\| < 1 - \|z\|\} \quad \text{and} \quad M_z := \sup\{\|f(w)\| : w \in D_z\}.$$

If $z = 0$ and $M_0 = 1$, Theorem C gives

$$|\nabla f(0)|^2 \leq 1 - \|f(0)\|^2 \leq 2(1 - \|f(0)\|).$$
Therefore, from (5) and the formula (1), we have that
\[ \| f'(0) \| \leq 2K(1 - \| f(0) \|), \]
which is our inequality in this special case. The general case follows by applying the special case to the function \( F \) defined by
\[ F(\zeta) = \frac{f(z + \zeta(1 - \| z \|))}{M_z} \quad \text{for} \quad \zeta \in \mathbb{B}^n, \tag{9} \]
and obtain
\[ \frac{1}{2K} (1 - \| z \|) \| f'(z) \| + \| f(z) \| \leq M_z \quad \text{for} \quad z \in \mathbb{B}^n. \tag{10} \]
Since \( \| f \| \in \Lambda_\alpha(\mathbb{B}^n, \mathbb{R}) \), we have
\[ \| f(w) \| - \| f(z) \| \leq c\| w - z \|^\alpha \leq c(1 - \| z \|)^\alpha, \]
for \( z \in \mathbb{B}^n \) and \( w \in D_z \). Taking the supremum over all \( w \in D_z \) and then using the inequality (10), we get
\[ \| f'(z) \| \leq C \frac{\omega(1 - \| z \|)}{1 - \| z \|}, \]
where \( C \) is a constant and \( \omega(t) = t^{\alpha/2} \). The desired conclusion follows from Lemma 9. \( \square \)

**Proof of Theorem 6.** Let \( z \in \mathbb{B}^n \) and proceed the steps as in the above proof. If \( z = 0 \) and \( M_0 = 1 \), then the higher dimensional version of Schwarz–Pick lemma (Theorem C) gives
\[ \| f'(0) \| \leq \sqrt{1 - \| f(0) \|^2} \leq \sqrt{2} \sqrt{1 - \| f(0) \|}, \]
which is our inequality in this special case. The general case follows by applying the special case to the function \( F \) defined by (9). Indeed, we obtain
\[ (1 - \| z \|) \| f'(z) \| \leq c \sqrt{M_z - \| f(z) \|}, \tag{11} \]
for some constant \( c \). Since \( \| f \| \in \Lambda_\alpha(\mathbb{B}^n, \mathbb{R}) \), we have
\[ \| f(w) \| - \| f(z) \| \leq c\| w - z \|^\alpha \leq c(1 - \| z \|)^\alpha, \]
for \( z \in \mathbb{B}^n \) and \( w \in D_z \). Taking the supremum over \( w \in D_z \) and then using the inequality (11), we get
\[ \| f'(z) \| \leq C \frac{\omega(1 - \| z \|)}{1 - \| z \|}, \]
where \( C \) is a constant and \( \omega(t) = t^{\alpha/2} \). Now the result follows from Lemma 9. \( \square \)

**Remark 10.** The index \( \alpha/2 \) in Theorem 6 is optimal as demonstrated by the following example (see [9]). Consider the function \( f : \mathbb{D} \rightarrow \mathbb{C}^2 \) by \( f(z) = (1, (1 - z)^{\alpha/2}) \), \( 0 < \alpha \leq 1 \). We have
\[ \| f(z) \| - \| f(w) \| = \sqrt{\| 1 - z \|^\alpha + 1 - \sqrt{\| 1 - w \|^\alpha + 1}} \leq \| 1 - w \|^\alpha - \| 1 - z \|^\alpha \leq \| z - w \|^\alpha, \]
while \( \| f(1) - f(r) \| = (1 - r)^{\alpha/2}, 0 < r < 1 \). This shows that the index \( \alpha/2 \) is optimal.
4. Concluding Remarks

As mentioned in [3], the inequality (4) is in a sense optimal for the case \( n = 1 \). To see this, let \( \omega(t) > 0 \) be an arbitrary increasing function on \((0, \infty)\) such that \( \omega(0^+) = 0 \). We say that a Banach space \( X \) has the property \( \mathcal{L}(\omega, \alpha) \), if the following holds: For every \( c \in (0, 1) \) and every analytic function \( f : \mathbb{D} \to X \) with \( \| f(0) \| = 1 \), the inequality (2) implies that

\[
\| f'(\lambda) \| \leq \frac{\omega(c(1 - |\lambda|)^{\alpha})}{1 - |\lambda|} \quad \text{for} \quad \lambda \in \mathbb{D}.
\]

It is well-known that, if the Banach space \( X \) has the property \( \mathcal{L}(\omega, \alpha) \) (see [3, Proposition 10]), then \( X \) is uniformly \( c \)-convex and \( \Omega_X(\delta) \leq B\omega(\delta) \) for \( 0 < \delta < 1 \), where \( B \) is a constant. This result is to emphasize the fact that \( \| f(0) \| = 1 \) provides condition for uniformly \( c \)-convexity of the Banach space \( X \).

References