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Generalized versions of Lipschitz conditions on the modulus of holomorphic functions

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Abstract. In this paper, we establish Lipschitz conditions for the norm of holomorphic mappings between the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ and X, a complex normed space. This extends the work of Djordjević and Pavlović.

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1. Introduction and Preliminaries

Denote by $\mathbb{C}^n$, the n-dimensional complex Hilbert space with the inner product and the norm given by $\langle z, w\rangle := \sum_{j=1}^{n} z_j \overline{w_j}$ and $\|z\| := \sqrt{\langle z, z\rangle}$, where $z, w \in \mathbb{C}^n$, respectively. Write $\mathbb{B}^n := \{z \in \mathbb{C}^n : \|z\| < 1\}$ for the open unit ball in $\mathbb{C}^n$ so that $\mathbb{B}^1 =: \mathbb{D}$ denotes the open unit disk in $\mathbb{C}$. If V and W are two normed spaces and $U \subset V$ is open, then the Fréchet derivative of a holomorphic mapping $f : U \to W$ is defined to be the unique linear map $A = f'(z) : V \to W$ such that

$$f(z+h) = f(z) + f'(z)\cdot h + o(\|h\|^2)$$

for h near the origin of V. The norm of such a map is defined by $\|A\| = \sup_{\|z\|=1} \|Az\|$.

In 1975, Globevnik [6] introduced the notion of uniform c-convexity and proved that L^1-space possesses this property. Namely, a complex normed space X is said to be *uniformly c-convex* if there exists a positive increasing function $\Omega(\delta)$ $(\delta > 0)$ with $\Omega(0^+) = 0$ such that for all $x, y \in X$ and $\delta > 0$ there holds the implication

$$\max_{\substack{|\lambda|\le 1\\ \|x\|=1}} \|x + \lambda y\| \le 1 + \delta \implies \|y\| \le \Omega(\delta).$$

The smallest of the functions Ω is denoted by Ω_X, i.e.,

$$\Omega_X(\delta) := \sup\left\{ \|y\| : \max_{\substack{|\lambda|\le 1\\ \|x\|=1}} \|x + \lambda y\| \le 1 + \delta \right\}.$$

As mentioned in [3], it can be easily seen that

$$\Omega_{\mathbb{C}}(\delta)=\delta \quad \text{and} \quad \Omega_H(\delta)=\sqrt{\delta(2+\delta)},$$

where H is a Hilbert space of dimension at least two.

As in [4], we call a function $\omega:[0,\infty)\to\mathbb{R}$ a *majorant* if ω is continuous, increasing, $\omega(0)=0$, and $t^{-1}\omega(t)$ is nonincreasing on $(0,\infty)$. If, in addition, there is a constant $C(\omega)>0$ such that

$$\int_0^{\delta}\frac{\omega(t)}{t}\,\mathrm{d}t+\delta\int_{\delta}^{\infty}\frac{\omega(t)}{t^2}\,\mathrm{d}t\le C(\omega)\cdot\omega(\delta)$$

whenever $0<\delta<1$, then we say that ω is a *regular majorant.*

Then the space $\mathrm{Lip}(\omega,G,X)$, where G is bounded subset of $\mathbb{C}^n$, is defined to be the set of those functions $g:G\to X$ for which

$$\|g(z)-g(w)\|\le c\cdot\omega(\|z-w\|),$$

where c is a constant. If $\omega(t)=t^{\alpha}$ for some $\alpha\in(0,1]$, then we write $\mathrm{Lip}(\omega,G,X)=\Lambda_{\alpha}(G,X)$. If X is uniformly c-convex, then Ω_X is a majorant (cf. [2]). A majorant ω is said to be a *Dini majorant* if $\int_0^1\frac{\omega(t)}{t}\,\mathrm{d}t<\infty$. For a Dini majorant, we define the majorant $\widetilde{\omega}$ by

$$\widetilde{\omega}(t)=\int_0^t\frac{\omega(x)}{x}\,\mathrm{d}x=\int_0^1\frac{\omega(tx)}{x}\,\mathrm{d}x.$$

A majorant ω is said to be *fast* [5] if

$$\int_0^{\delta}\frac{\omega(t)}{t}\,\mathrm{d}t\le \mathrm{const}\cdot\omega(\delta),\quad 0<\delta<\delta_0,$$

for some $\delta_0>0$. (Of course, if ω is fast, then it is a Dini majorant).

Dyakonov [4] gave some characterizations of the holomorphic functions of class $\Lambda_{\omega}(\mathbb{D},\mathbb{C})$ in terms of their moduli.

Theorem A (cf. [4]). *Let ω be a regular majorant. A function f holomorphic in $\mathbb{D}$ is in $\Lambda_{\omega}(\mathbb{D},\mathbb{C})$ if and only if so is its modulus $|f|$.*

The main ingredient in Dyakonov's proof is a very complicated. However, Pavlovic [8] gave a simple proof of Theorem A. The proof uses only the basic lemmas of [4] and the Schwarz lemma, and is therefore considerably shorter than that of [4]. However, Theorem A does not extend to $\mathbb{C}^k$- valued functions ($k\ge 2$). So we have to consider functions with additional properties (see Theorems 5 and 6).

In [3], Djordjević and Pavlović extended to vector-valued functions of a theorem of Dyakonov [4] on Lipschitz conditions for the modulus of holomorphic functions. Therefore, it is natural for us to extend this result for holomorphic functions on $\mathbb{B}^n$. Very recently, Kalaj [7] established a Schwarz–Pick type inequality for holomorphic mappings between unit balls $\mathbb{B}^n$ and $\mathbb{B}^m$ in the corresponding complex spaces.

Theorem B (cf. [7, Theorem 2.1]). *If f is a holomorphic mapping of the unit ball $\mathbb{B}^n\subset\mathbb{C}^n$ into $\mathbb{B}^m\subset\mathbb{C}^m$, then for $z\in\mathbb{B}^n$ we have*

$$\|f'(z)\|\le\begin{cases}\dfrac{\sqrt{1-\|f(z)\|^2}}{1-\|z\|^2} & \text{for } m\ge 2,\\[2ex] \dfrac{1-\|f(z)\|^2}{1-\|z\|^2} & \text{for } m=1.\end{cases}$$

In [1], Dai and Pan proved the following theorem which establishes a Schwarz–Pick type estimates for gradient of the modulus of holomorphic mappings.

Theorem C (cf. [1, Theorem 1]). *Let $f:\mathbb{B}^n\to\mathbb{B}^m$ be a holomorphic mapping. Then*

$$\big|\nabla\|f\|(z)\big|\le\frac{1-\|f(z)\|^2}{1-\|z\|^2}\quad \text{for } z\in\mathbb{B}^n.$$

For a holomorphic mapping $f:\mathbb{B}^n\to\mathbb{B}^m$, we have

$$\big|\nabla\|f\|(z)\big|=\frac{1}{\|f(z)\|}\left\|\left(\left\langle\frac{\partial f(z)}{\partial z_1},f(z)\right\rangle,\dots,\left\langle\frac{\partial f(z)}{\partial z_n},f(z)\right\rangle\right)\right\|\quad \text{if } f(z)\neq 0. \tag{1}$$

2. The main results

Theorem 1. *Let X be uniformly c-convex and $f:\mathbb{B}^n\to X$ be a holomorphic function satisfying*

$$\big|\|f(z)\|-\|f(w)\|\big|\le c\|z-w\|^\alpha\quad \text{for } z,w\in\mathbb{B}^n, \tag{2}$$

where $c\ge 0$ and $\alpha\in[0,1]$ are constants. Then

$$\|f'(z)\|\le 2K\frac{\Omega_X(cK^{-1}(1-\|z\|)^\alpha)}{1-\|z\|}\quad \text{for } z\in\mathbb{B}^n, \tag{3}$$

where $K=\|f(0)\|+c$. Especially, if $\|f(0)\|=1$, then

$$\|f'(z)\|\le 2(1+c)\frac{\Omega_X(c(1-\|z\|)^\alpha)}{1-\|z\|}\quad \text{for } z\in\mathbb{B}^n. \tag{4}$$

Theorem 2. *Let X be uniformly c-convex such that Ω_X is a Dini majorant and $f:\mathbb{B}^n\to X$ be a holomorphic function such that the function $\|f(z)\|$ belongs to $\Lambda_\alpha(\mathbb{B}^n,\mathbb{R})$ for some $\alpha\in(0,1]$. Then $f\in\mathrm{Lip}(\overline{\omega}_\alpha,\mathbb{B}^n,X)$, where $\overline{\omega}_\alpha(t)=\widetilde{\Omega}_X(t^\alpha)$.*

In particular, the function f is uniformly continuous on $\mathbb{B}^n$ that has a continuous extension to the closed disk.

Corollary 3. *If Ω_X is fast and $f:\mathbb{B}^n\to X$ is a holomorphic function such that the function $\|f(z)\|$ belongs to $\Lambda_\alpha(\mathbb{B}^n,\mathbb{R})$ for some $\alpha\in(0,1]$. Then $f\in\mathrm{Lip}(\omega_\alpha,\mathbb{B}^n,X)$, where $\omega_\alpha(t)=\Omega_X(t^\alpha)$.*

Taking $n=1$ and $X=\mathbb{C}$, we get the following result of Dyakonov [4].

Corollary 4. *If $f:\mathbb{D}\to\mathbb{C}$ is a holomorphic function such that $|f|$ belongs to $\Lambda_\alpha(\mathbb{D},\mathbb{R})$ for some $\alpha\in(0,1]$. Then f belongs to $\Lambda_\alpha(\mathbb{D},\mathbb{C})$.*

Theorem 5. *Let $0<\alpha\le 1$ and $f:\mathbb{B}^n\to\mathbb{C}^m$ be a holomorphic function such that*

$$\|f'(z)\|\|f(z)\|\le K\left\|\left(\left\langle\frac{\partial f(z)}{\partial z_1},f(z)\right\rangle,\dots,\left\langle\frac{\partial f(z)}{\partial z_n},f(z)\right\rangle\right)\right\|\quad \text{for } z\in\mathbb{B}^n, \tag{5}$$

where K is a constant independent of z. Then $f\in\Lambda_\alpha(\mathbb{B}^n,\mathbb{C}^m)$ if and only if $\|f\|\in\Lambda_\alpha(\mathbb{B}^n,\mathbb{R})$.

Theorem 6. *If $f:\mathbb{B}^n\to\mathbb{C}^m$, $m\ge 2$, is holomorphic and if $\|f\|\in\Lambda_\alpha(\mathbb{B}^n,\mathbb{R})$ for some $\alpha\in(0,1]$, then we have $f\in\Lambda_{\alpha/2}(\mathbb{B}^n,\mathbb{C}^m)$.*

The case $n=1$ of Theorems 5 and 6 gives results of Pavlović [9].

3. Proofs of the Theorems

Theorem 1 is a direct consequence of the following lemma.

Lemma 7. *If $f:\mathbb{B}^n\to X$ is a holomorphic function satisfying the condition*

$$\big|\|f(z)\|-\|f(w)\|\big|\le c(1-\|z\|)^\alpha\quad \text{whenever } \|w-z\|\le 1-\|z\|, \tag{6}$$

then there holds (3).

Proof. Fix $z \in \mathbb{B}^n$ with $f(z) \neq 0$, and fix $\beta \in \mathbb{C}^n$ with $\|\beta\| = 1$. Let $L \in X^*$, $\|L\| = 1$, where X^* is the dual of X. Consider the scalar valued function

$$\phi(z) = L \circ f(z),$$

and introduce the following set for the given $z \in \mathbb{B}^n$,

$$D_z := \{w \in \mathbb{C}^n : \|w - z\| < 1 - \|z\|\} \quad \text{and} \quad M_z := \sup\{\|f(w)\| : w \in D_z\}.$$

If $z = 0$ and $M_0 = 1$, then the Schwarz–Pick lemma (see Theorem B) gives

$$|\phi'(0)| \leq 1 - |\phi(0)|^2 \leq 2(1 - |\phi(0)|), \tag{7}$$

which is our inequality in this special case. The general case follows by applying the special case to the function Φ defined by

$$\Phi(\zeta) = \frac{\phi(z + (1 - \|z\|)\beta\zeta)}{M_z} \quad \text{for } \zeta \in \mathbb{B}^n.$$

As

$$\Phi(0) = \frac{L(f(z))}{M_z} \quad \text{and} \quad \Phi'(0) = \frac{(1 - \|z\|)}{M_z} L(f'(z)\beta),$$

we deduce from (7) that

$$(1 - \|z\|)|L(f'(z)\beta/2)| + |L(f(z))| \leq M_z.$$

Hence, for every $\lambda \in \mathbb{D}$, we obtain

$$|\lambda(1 - \|z\|)L(f'(z)\beta/2) + L(f(z))| \leq M_z.$$

Since this holds for every L of norm 1, by taking the supremum over all L with $\|L\| = 1$ and by applying the Hahn–Banach theorem, we get

$$\left\|\lambda \frac{(1 - \|z\|)f'(z)\beta}{2} + f(z)\right\| \leq M_z, \quad \text{i.e.,} \quad \left\|\frac{f(z)}{\|f(z)\|} + \lambda \frac{(1 - \|z\|)f'(z)\beta}{2\|f(z)\|}\right\| \leq \frac{M_z}{\|f(z)\|}.$$

Now denoting

$$x = \frac{f(z)}{\|f(z)\|}, \quad y = \frac{(1 - \|z\|)f'(z)\beta}{2\|f(z)\|} \quad \text{and} \quad \delta = \frac{M_z - \|f(z)\|}{\|f(z)\|},$$

we see from the definition of Ω_X that

$$(1 - \|z\|)\|f'(z)\beta\| \leq 2\|f(z)\|\Omega_X\left(\frac{M_z - \|f(z)\|}{\|f(z)\|}\right).$$

Hence, the last inequality holds for every $\beta \in \mathbb{C}^n$ with $\|\beta\| = 1$, we get

$$(1 - \|z\|)\|f'(z)\| \leq 2\|f(z)\|\Omega_X\left(\frac{M_z - \|f(z)\|}{\|f(z)\|}\right). \tag{8}$$

Therefore by (6) and (8), we obtain that

$$(1 - \|z\|)\|f'(z)\| \leq 2\|f(z)\|\Omega_X\left(\frac{c(1 - \|z\|)^\alpha}{\|f(z)\|}\right).$$

Now (3) follows from the fact that $\Omega_X(t)/t$ is a decreasing function and the inequality $\|f(z)\| \leq K$. The proof is complete. □

Lemma 8. *If a C^1-function $u : \mathbb{B}^n \to \mathbb{R}$ satisfies*

$$\|\nabla u(z)\| \leq \frac{\omega(1 - \|z\|)}{1 - \|z\|} \quad \textit{for } z \in \mathbb{B}^n,$$

where ω is a Dini majorant, then

$$|u(a) - u(b)| \leq 3\,\widetilde{\omega}(\|a - b\|) \quad \textit{for } a, b \in \mathbb{B}^n.$$

Proof. We begin the proof with the following observation: $\omega \le \widetilde{\omega}$. In fact, we let $t_0 \in (0,\infty)$. Since $\frac{\omega(t)}{t}$ is decreasing on $(0,\infty)$, we have

$$\frac{\omega(t_0)}{t_0} \le \frac{\omega(t_0 x)}{t_0 x} \quad \text{for } x \in (0,1].$$

Integrating on both sides of the last inequality from 0 to 1, we obtain by definition of $\widetilde{\omega}$ that $\omega(t_0) \le \widetilde{\omega}(t_0)$.

Let $\|a\| \le \|b\| \le 1$. By Lagrange's mean-value theorem,

$$|u(a) - u(b)| \le \|\nabla u(c)\| \|a - b\|,$$

where $c = (1-\lambda)a + \lambda b$ for some $\lambda \in (0,1)$. Since $\|c\| \le \|b\|$ and $\omega(t)/t$ decreases, we see that

$$\frac{\omega(1-\|c\|)}{1-\|c\|} \le \frac{\omega(1-\|b\|)}{1-\|b\|}$$

and hence,

$$|u(a) - u(b)| \le \omega(\|a-b\|) \le \widetilde{\omega}(\|a-b\|),$$

under the condition $\|a-b\| \le 1 - \|b\|$.

If $1 - \|b\| \le \|a-b\| \le 1 - \|a\|$, then

$$|u(a) - u(b)| \le |u(a) - u(b')| + |u(b') - u(b)|,$$

where $b' = \frac{(1-\delta)b}{\|b\|}$ and $\delta = \|a-b\|$. Using the Lagrange's mean-value theorem as above we get

$$|u(a) - u(b')| \le \frac{\omega(1-\|b'\|)}{1-\|b'\|}\|a - b'\| = \frac{\omega(\delta)}{\delta}\|a - b'\| \le \omega(\delta) \le \widetilde{\omega}(\delta).$$

In the case of $|u(b') - u(b)|$, we have

$$|u(b') - u(b)| \le \int_{\|b'\|}^{\|b\|} \frac{\omega(1-t)}{1-t}\,\mathrm{d}t \le \int_{1-\delta}^{1} \frac{\omega(1-t)}{1-t}\,\mathrm{d}t = \widetilde{\omega}(\delta).$$

Finally, if $\delta > 1 - \|a\|$, we use the inequality

$$|u(a) - u(b)| \le |u(a) - u(a')| + |u(a') - u(b')| + |u(b') - u(b)|,$$

where $a' = \frac{(1-\delta)a}{\|a\|}$, and then proceed in a similar way as above, using the inequality $\|a' - b'\| \le \|a - b\|$. □

Lemma 9 can easily be proved by applying the previous lemma to the functions $\mathrm{Re}\,(L \circ f(z))$ and $\mathrm{Im}\,(L \circ f(z))$, where $L \in X^*$ and $\|L\| = 1$.

Lemma 9. *If f is an X-valued holomorphic function in $\mathbb{B}^n$ and satisfies the condition*

$$\|f'(z)\| \le \frac{\omega(1-\|z\|)}{1-\|z\|} \quad \text{for } z \in \mathbb{B}^n,$$

where ω is a Dini majorant, then $f \in \mathrm{Lip}(\widetilde{\omega}, \mathbb{B}^n, X)$.

Proof of Theorem 2. Let f satisfy the hypotheses of the theorem. Then

$$\|f'(z)/2K\| \le \frac{\omega(1-\|z\|)}{1-\|z\|},$$

by Theorem 1, where $\omega(t) = \Omega_X(cK^{-1}t^{\alpha})$. But a simple calculation shows that $\widetilde{\omega}(t) = \alpha^{-1}\widetilde{\Omega}_X(cK^{-1}t^{\alpha})$ and so we can appeal to Lemma 9 to conclude the proof. □

Proof of Theorem 5. The "only if" part is trivial. Assume that $\|f(z)\| \in \Lambda_\alpha(\mathbb{B}^n, \mathbb{R})$ and we proceed as in Theorem 1. Fix $z \in \mathbb{B}^n$ with $f(z) \ne 0$, and consider the following sets for a given $z \in \mathbb{B}^n$,

$$D_z := \{w \in \mathbb{C}^n : \|w - z\| < 1 - \|z\|\} \quad \text{and} \quad M_z := \sup\{\|f(w)\| : w \in D_z\}.$$

If $z = 0$ and $M_0 = 1$, Theorem C gives

$$|\nabla \|f\|(0)| \le 1 - \|f(0)\|^2 \le 2(1 - \|f(0)\|).$$

Therefore, from (5) and the formula (1), we have that

$$\|f'(0)\| \le 2K(1-\|f(0)\|),$$

which is our inequality in this special case. The general case follows by applying the special case to the function F defined by

$$F(\zeta) = \frac{f(z+\zeta(1-\|z\|))}{M_z} \quad \text{for } \zeta \in \mathbb{B}^n, \tag{9}$$

and obtain

$$\frac{1}{2K}(1-\|z\|)\|f'(z)\| + \|f(z)\| \le M_z \quad \text{for } z \in \mathbb{B}^n. \tag{10}$$

Since $\|f\| \in \Lambda_\alpha(\mathbb{B}^n,\mathbb{R})$, we have

$$\|f(w)\| - \|f(z)\| \le c\|w-z\|^\alpha \le c(1-\|z\|)^\alpha,$$

for $z \in \mathbb{B}^n$ and $w \in D_z$. Taking the supremum over all $w \in D_z$ and then using the inequality (10), we get

$$\|f'(z)\| \le C\frac{\omega(1-\|z\|)}{1-\|z\|},$$

where C is a constant and $\omega(t) = t^\alpha$. The desired conclusion follows from Lemma 9. □

Proof of Theorem 6. Let $z \in \mathbb{B}^n$ and proceed the steps as in the above proof. If $z = 0$ and $M_0 = 1$, then the higher dimensional version of Schwarz–Pick lemma (Theorem C) gives

$$\|f'(0)\| \le \sqrt{1-\|f(0)\|^2} \le \sqrt{2}\sqrt{1-\|f(0)\|},$$

which is our inequality in this special case. The general case follows by applying the special case to the function F defined by (9). Indeed, we obtain

$$(1-\|z\|)\|f'(z)\| \le c\sqrt{M_z - \|f(z)\|}, \tag{11}$$

for some constant c. Since $\|f\| \in \Lambda_\alpha(\mathbb{B}^n,\mathbb{R})$, we have

$$\|f(w)\| - \|f(z)\| \le c\|w-z\|^\alpha \le c(1-\|z\|)^\alpha,$$

for $z \in \mathbb{B}^n$ and $w \in D_z$. Taking the supremum over $w \in D_z$ and then using the inequality (11), we get

$$\|f'(z)\| \le C\frac{\omega(1-\|z\|)}{1-\|z\|},$$

where C is a constant and $\omega(t) = t^{\alpha/2}$. Now the result follows from Lemma 9. □

Remark 10. The index $\alpha/2$ in Theorem 6 is optimal as demonstrated by the following example (see [9]). Consider the function $f : \mathbb{D} \to \mathbb{C}^2$ by $f(z) = (1,(1-z)^{\alpha/2}),\ 0 < \alpha \le 1$. We have

$$\begin{aligned}\Big|\|f(z)\| - \|f(w)\|\Big| &= \Big|\sqrt{\|1-z\|^\alpha+1} - \sqrt{\|1-w\|^\alpha+1}\Big| \\ &\le \Big|\|1-w\|^\alpha - \|1-z\|^\alpha\Big| \le \|z-w\|^\alpha,\end{aligned}$$

while $\|f(1)-f(r)\| = (1-r)^{\alpha/2}, 0 < r < 1$. This shows that the index $\alpha/2$ is optimal.

4. Concluding Remarks

As mentioned in [3], the inequality (4) is in a sense optimal for the case $n = 1$. To see this, let $\omega(t) > 0$ be an arbitrary increasing function on $(0,\infty)$ such that $\omega(0^+) = 0$. We say that a Banach space X has the property $\mathscr{L}(\omega,\alpha)$, if the following holds: *For every* $c \in (0,1)$ *and every analytic function* $f : \mathbb{D} \to X$ *with* $\|f(0)\| = 1$*, the inequality* (2) *implies that*

$$\|f'(\lambda)\| \le \frac{\omega(c(1-|\lambda|)^{\alpha})}{1-|\lambda|} \quad \text{for } \lambda \in \mathbb{D}.$$

It is well-known that, if the Banach space X has the property $\mathscr{L}(\omega,\alpha)$ (see [3, Proposition 10]), then X is uniformly c-convex and $\Omega_X(\delta) \le B\omega(\delta)$ for $0 < \delta < 1$, where B is a constant. This result is to emphasize the fact that $\|f(0)\| = 1$ provides condition for uniformly c-convexity of the Banach space X.

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