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Gillespie's questions and Grothendieck duality

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Abstract. Gillespie posed two questions in [Front. Math. China 12 (2017) 97-115], one of which states that "for what rings R do we have $\mathrm{K(AC)} = \mathrm{K}(R\text{-Inj})$?". We give an answer to such a question. As applications, we obtain a new homological approach that unifies some well-known conditions of rings such that Krause's recollement holds, and give an example to show that there exists a Gorenstein injective module which is not Gorenstein AC-injective. We also improve Neeman's angle of view to the Grothendieck duality for derived categories of modules from the case of left Noether and right coherent rings such that all flat left modules have finite projective dimension to the case of left and right coherent rings.

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1. Introduction

Throughout this article, all rings R are assumed to be associative rings with identity and all modules are unitary. By an "R-module" or just a "module" we always mean a left R-module, and hence by a "complex" we mean a complex of left R-modules; by "Noether" (resp. "coherent") we mean left Noether (resp. left coherent); by global dimension (resp. "global Gorenstein AC-injective dimension", "global Gorenstein injective dimension") of R we mean left global dimension (resp. left global Gorenstein AC-injective dimension, left global Gorenstein injective dimension) of R. When we wish to refer to a right module (resp. a right coherent ring), we will use the full name or an R^{op}-module (resp. R^{op} is coherent) and so on. Here R^{op} denotes the oppositive ring of R.

In 2005, Krause proved in [27, Theorem 1.1] that, if R is Noether, then both $\mathrm{K}(R\text{-Inj})$ and $\mathrm{K}_{\mathrm{ac}}(R\text{-Inj})$, the homotopy category of injective R-modules and its homotopy subcategory consisting of acyclic complexes of injective R-modules, are compactly generated. Furthermore, there

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is a recollement of compactly generated triangulated categories (the recollement is known as "Krause's recollement")

$$
\begin{tikzcd}
\mathrm{K}_{\mathrm{ac}}(R\text{-}\mathrm{Inj}) \arrow[rr] & & \mathrm{K}(R\text{-}\mathrm{Inj}) \arrow[ll, bend right] \arrow[ll, bend left] \arrow[rr] & & \mathbf{D}(R). \arrow[ll, bend right] \arrow[ll, bend left]
\end{tikzcd}
$$

Ten years later, Šťovíček [35, Theorem 7.7] improved the results to the case of coherent rings. In order to extend Krause's recollement to an arbitrary ring R, Gillespie [21, Corollary 5.9] explained how $\mathrm{K}(R\text{-}\mathrm{Inj})$ and $\mathrm{K}_{\mathrm{ac}}(R\text{-}\mathrm{Inj})$ can be replaced by K(AC-Inj) and K_{ac}(AC-Inj), the homotopy category of all AC-injective R-complexes and its homotopy subcategory consisting of all acyclic AC-injective R-complexes, to get the same beautiful results (see Definition 2.4 for the notion of AC-injective complexes). For a coherent ring R, Šťovíček showed in [35, Corollary 6.13] that K(AC-Inj) = $\mathrm{K}(R\text{-}\mathrm{Inj})$. Gillespie then in [21, Introduction] raised the following questions involving such an equality:

Question 1. *For what rings R do we have* K(AC-Inj) = $\mathrm{K}(R\text{-}\mathrm{Inj})$*?*

Question 2. *Does* K(AC-Inj) = $\mathrm{K}(R\text{-}\mathrm{Inj})$ *characterize that R is coherent?*

Enochs, Jenda and Torrecillas [9, 11] introduced Gorenstein projective, injective and flat modules for any ring and then established Gorenstein homological algebra. Such a relative homological algebra has been developed rapidly during the past several years and become a rich theory. We refer to [1, 2, 4–7, 10, 12, 14, 17, 18, 23, 30, 40] for more details about Gorenstein homological modules. Particularly, in order to give a similarity between Gorenstein rings and Ding-Chen rings, Ding, Mao and Li as well as Gillespie [6, 17, 30] introduced Ding projective and Ding injective modules and then Gillespie [17] established two hereditary abelian model structures with respect to Ding modules; in order that Gorenstein homological algebra should work for any ring, Bravo, Gillespie and Hovey [4] introduced the notions Gorenstein AC-projective and Gorestein AC-injective modules and then established two new hereditary abelian model structures with respect to such Gorenstein AC-modules.

It is well-known that a very natural way to study homological algebra is extending the homological theory on the category of modules to the one of complexes of modules, and that it is an important question to establish relationships between a complex and all modules as its components. Based on these viewpoints, Gorenstein homological complexes, Ding homological complexes and Gorenstein AC-homological complexes have been given a description in [3, 15, 20, 28, 29, 37–39, 41]. Let R be a Noether ring. Liu and Zhang proved in [28] that Gorenstein injective complexes are exactly the complexes consisting of Gorenstein injective modules. This improves the corresponding result in [15] from Gorenstein rings to Noether rings. Using different approaches, Liu, Yang and Yang [37, 41] further improved the result to an arbitrary ring. Recently, Gillespie [20] showed that the behavior for Ding injective complexes holds over any Ding-Chen ring. The result was improved to coherent rings by Yang and Estrada [38]. Notice that Gorenstein AC-injective complexes and Ding injective complexes coincide whenever the ring is coherent. Thus, it is natural to ask

Question 3. *For what rings do we have Gorenstein AC-injective complexes are exactly complexes consisting of Gorenstein AC-injective modules?*

By introducing the notions of Gorenstein n-coherent rings (here $n \in \mathbb{N} \cup \{\infty\}$), Wang, Liu and Yang [36] gave a negative answer to Question 2 above. One of the goals of the present manuscript is to show that Questions 1 and 3 are equivalent as follows.

Theorem 4 (=Corollary 22). *Let R be a ring. Then the following are equivalent:*

(1) K(AC-Inj) = $\mathrm{K}(R\text{-}\mathrm{Inj})$.
(2) K_{ac}(AC-Inj) = $\mathrm{K}_{\mathrm{ac}}(R\text{-}\mathrm{Inj})$.

(3) *For any (acyclic) R-complex X, X is Gorenstein AC-injective if and only if each X_m is a Gorenstein AC-injective R-module.*

We find that Theorem 4 has some interesting applications. Firstly, by the definitions, it is trivial that any Gorenstein AC-injective module (resp. any Gorenstein AC-injective complex) is always Gorenstein injective. But using Theorem 4, we obtain some examples to show that the converse is not true in general (see Examples 24 and 25).

Secondly, we get the following corollary, which unifies some well-known conditions of rings such that Krause's recollement holds.

Corollary 5 (=Corollaries 23 and 26). *Let R be a ring such that any acyclic complex of injective modules is Gorenstein AC-injective or totally AC-acyclic. Then Krause's recollement holds; in particular, both* $\mathrm{K}(R\text{-Inj})$ *and* $\mathrm{K}_{\mathrm{ac}}(R\text{-Inj})$ *are compactly generated.*

We denote by $\mathscr{G}$ (resp. $\mathscr{GI}$) the subcategory of all Gorenstein AC-injective (resp. all Gorenstein injective) R-modules for a ring R. Then $\mathscr{G}$ (resp. $\mathscr{GI}$), together with all short exact sequences in $\mathscr{G}$ (resp. $\mathscr{GI}$), forms a Frobenius category with projective-injective objects all injective R-modules. It follows that the stable category $\underline{\mathscr{G}}$ (resp. $\underline{\mathscr{GI}}$) is a triangulated category. The third application of Theorem 4 as below concerns when such stable categories are compactly generated.

Corollary 6 (=Corollary 27). *Let R be a ring such that any acyclic complex of injective R-modules is totally AC-acyclic. Then $\underline{\mathscr{G}} = \underline{\mathscr{GI}}$ are compactly generated.*

Grothendieck duality is a classical subject which can go back 1958s. Roughly speaking, it is a statement concerning the existence of a right adjoint to the "direct image with compact support" functor between derived categories of sheaves or modules. For a ring R, besides using dualizing complexes, over the years many people investigated Grothendieck duality for derived categories of R-modules by providing the following insights:

(GD1) There is a triangulated equivalence $\mathbf{D}^b(R^{op}\text{-mod})^{op} \simeq \mathbf{D}^b(R\text{-mod})$, where the right category is the bounded derived category of finitely presented R-modules and the left one is the oppositive category of the bounded derived category of finitely presented right R-modules;

(GD2) $\mathrm{K}(R\text{-Proj})$ and $\mathrm{K}(R\text{-Inj})$ are compactly generated and triangulated equivalent, where $\mathrm{K}(R\text{-Proj})$ denotes the homotopy category of projective R-modules.

Building from the works of Krause and Iyengar [24, 27], as well as Jørgensen [25], Neeman [34] gave an new point that (GD2) implies (GD1) provided that R is a Noether and right coherent ring such that all flat R-modules have finite projective dimension. The second goal of this paper is to improve Neeman's angle of view to the case of left and right coherent rings and to investigate the relationship between (GD1) (and/or (GD2)) and the finiteness of global Gorenstain (AC-)injective dimension.

The following result provides another counterexample of Question 2 and shows that (GD2) happens if the global Gorenstain AC-injective dimension of the ring, that is, the supremum of Gorenstain AC-injective dimensions of all modules, is finite.

Theorem 7 (=Theorem 29 and Example 34).

(1) *Let R be a ring with finite global Gorenstain AC-injective dimension. Then* $\mathrm{K}(\text{AC-Inj}) = \mathrm{K}(R\text{-Inj})$ *and* (GD2) *hold.*

(2) *The class of rings with finite global Gorenstain AC-injective dimension includes strictly the one of rings with finite global dimension and the one of coherent rings with finite global Gorenstein injective dimension.*

Let us denote by $\mathbf{D}^b(R\text{-tmod})$ (resp. $\mathbf{D}^b(R^{op}\text{-tmod})^{op}$) the bounded derived category of R-modules of type FP_∞ (resp. the oppositive category of the bounded derived category of right

R-modules of type FP_∞); by $\mathrm{K}(R\text{-}\mathrm{Proj})^c$ (resp. $\mathrm{K}(R\text{-}\mathrm{Inj})^c$) the full triangulated subcategory of $\mathrm{K}(R\text{-}\mathrm{Proj})$ (resp. $\mathrm{K}(R\text{-}\mathrm{Inj})$) consisting of all compact objects. The following result can be viewed as a continuation of [25, Theorem 3.2] and [27, Proposition 2.3].

Theorem 8 (=Proposition 30). *Let R be a ring.*

(1) *If* $\mathrm{K}(R\text{-}\mathrm{Proj})$ *is compactly generated, then there is a triangulated equivalence* $\mathrm{K}(R\text{-}\mathrm{Proj})^c \simeq \mathbf{D}^b(R^{op}\text{-tmod})^{op}$.
(2) *If* $\mathrm{K}(\mathrm{AC}\text{-}\mathrm{Inj}) = \mathrm{K}(R\text{-}\mathrm{Inj})$, *then there exists a triangulated equivalence* $\mathrm{K}(R\text{-}\mathrm{Inj})^c \simeq \mathbf{D}^b(R\text{-tmod})$.

As applications of the preceding two theorems, we have the following corollary, the second result of which improves Neeman's angle of view, i.e., (GD2) implies (GD1), from the case of left Noether and right coherent rings such that all flat left modules have finite projective dimension to the case of left and right coherent rings.

Corollary 9 (=Corollaries 31 and 33).

(1) *Let R be a ring with* $\mathrm{K}(R\text{-}\mathrm{Proj})$ *compactly generated and with the equality* $\mathrm{K}(\mathrm{AC}\text{-}\mathrm{Inj}) = \mathrm{K}(R\text{-}\mathrm{Inj})$. *Then the triangulated equivalence* $\mathrm{K}(R\text{-}\mathrm{Proj}) \simeq \mathrm{K}(R\text{-}\mathrm{Inj})$ *implies another one* $\mathbf{D}^b(R^{op}\text{-tmod})^{op} \simeq \mathbf{D}^b(R\text{-tmod})$. *In particular, the two triangulated equivalences hold provided that R is of finite global Gorenstein AC-injective dimension.*
(2) *Let R be a left and right coherent ring. Then the triangulated equivalence* $\mathrm{K}(R\text{-}\mathrm{Proj}) \simeq \mathrm{K}(R\text{-}\mathrm{Inj})$ *implies another one* $\mathbf{D}^b(R^{op}\text{-mod})^{op} \simeq \mathbf{D}^b(R\text{-mod})$. *In particular, the two triangulated equivalences hold provided that R is of finite global Gorenstein injective dimension.*

2. Preliminaries

In this section we recall some notions which will be used in the article.

Let R be a ring. Denote by R-Mod (resp. R^{op}-Mod) the categories of all R-modules (resp. all right R-modules); by $\mathrm{Ch}(R)$ (resp. $\mathrm{Ch}(R^{op})$) the category of complexes of R-modules (resp. R^{op}-modules); by $\mathbf{D}(R)$ the derived category of all R-modules; by R-Proj, R-Inj and R-Flat the subcategory of R-Mod consisting of all projective, injective and flat R-modules respectively; by $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ the projective, injective and flat dimension of an R-module respectively.

2.1. *The basics of Complexes*

Following [38, Preliminaries], we denote by $\mathrm{Ext}^k_{\mathrm{Ch}(R)}(\cdot,\cdot)$ the right derived functors of $\mathrm{Hom}_{\mathrm{Ch}(R)}(\cdot,\cdot)$.

Given a complex $X = \cdots \to X_{m+1} \xrightarrow{d_{m+1}} X_m \xrightarrow{d_m} X_{m-1} \to \cdots$, we denote by $\mathrm{Z}_k(X)$ (resp. $\mathrm{H}_k(X)$) the module $\mathrm{Ker}\, d_k$ (resp. $\mathrm{Ker}\, d_k / \mathrm{Im}\, d_{k+1}$) for any integer k, and X is said to be *acyclic* (or *exact*) if all $H_k(X) = 0$. X is *bounded* if it is bounded above and below, where we say that X is *bounded above* (resp. *bounded below*) if $X_m = 0$ holds for $m \gg 0$ (resp. $m \ll 0$); for integers n and m, the *hard right-truncation* of X at n (resp. *the hard left-truncation* of X at m) is denoted by $X_{\geq n}$ (resp. $X_{\leq m}$) are defined as the following bounded-below complex and bounded-above complex, respectively:

$$X_{\geq n} = \cdots \to X_{n+1} \xrightarrow{d^X_{n+1}} X_n \to 0 \quad \text{and} \quad X_{\leq m} = 0 \to X_m \xrightarrow{d^X_m} X_{m-1} \to \cdots.$$

Given $X, Y \in \mathrm{Ch}(R)$, the *homomorphism complex* of X and Y, denoted by $\mathscr{H}\mathrm{om}(X,Y)$, is defined as the $\mathbb{Z}$-complex $\cdots \longrightarrow \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(X_k, Y_{k-m}) \xrightarrow{d_m} \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(X_k, Y_{k-m-1}) \longrightarrow \cdots$, where d_m is given by $d_m((f_k)_{k\in\mathbb{Z}}) = (d^Y_{k-m} f_k - (-1)^m f_{k-1} d^X_k)_{k\in\mathbb{Z}}$, for all $(f_k)_{k\in\mathbb{Z}} \in \mathscr{H}\mathrm{om}(X,Y)_m$.

A morphism $f : X \to Y$ is said to be *null-homotopic* if there exists a family $(h_m : X_m \to Y_{m+1})_{m\in\mathbb{Z}}$ of R-homomorphisms such that $f_m = d^Y_{m+1} h_m + h_{m-1} d^X_m$ for all $m \in \mathbb{Z}$.

We refer to readers to see more details in [38, Preliminaries].

2.2. *Cotorsion pairs.*

Let $\mathscr{A}$ be an abelian category and $\mathscr{X},\mathscr{Y}$ subcategories of $\mathscr{A}$. A pair $(\mathscr{X},\mathscr{Y})$ is called a *cotorsion pair* if $\mathscr{X}^{\perp} = \mathscr{Y}$ and $\mathscr{Y} = {}^{\perp}\mathscr{X}$. Here $\mathscr{X}^{\perp} = \{A \in \mathscr{A} \mid \mathrm{Ext}^1_{\mathscr{A}}(X, A) = 0, \forall\ X \in \mathscr{X}\}$, and similarly we can define ${}^{\perp}\mathscr{X}$. A cotorsion pair $(\mathscr{X},\mathscr{Y})$ is said to be *hereditary* if $\mathrm{Ext}^n_{\mathscr{A}}(X, Y) = 0$ for all $X \in \mathscr{X}$, $Y \in \mathscr{Y}$ and $n \geq 1$. A cotorsion pair $(\mathscr{X},\mathscr{Y})$ is called *complete* if for any object $A \in \mathscr{A}$, there are exact sequences $0 \to Y \to X \to A \to 0$ and $0 \to A \to Y' \to X' \to 0$ respectively with $X, X' \in \mathscr{X}$ and $Y, Y' \in \mathscr{Y}$. Let $\mathscr{A}$ be an abelian category with enough injectives. Recall that a subcategory $\mathscr{X}$ of $\mathscr{A}$ is *injective coresolving* if $\mathscr{X}$ contains all injective objects and is closed under extensions as well as cokernels of monic morphisms of $\mathscr{A}$. Recall that a class $\mathscr{X}$ of objects in $\mathscr{A}$ is *thick* if $\mathscr{X}$ is closed under direct summands and such that if any two terms in a short exact sequence are in $\mathscr{X}$, then so is the third. According to [18, Proposition 3.6], a cotorsion pair $(\mathscr{X},\mathscr{Y})$ is called *injective* if $\mathscr{X}$ is thick and $\mathscr{X} \cap \mathscr{Y}$ equals the class of all injective objects.

Gillespie introduced in [16] the following definitions, which extend the notions of dg-injective complexes (recall that a complex C is *dg-injective* if each C_m is injective and each morphism $f : E \to C$ is null-homotopic whenever E is acyclic).

Definition 10 (cf. [16, Definition 3.3]). *Let $(\mathscr{X},\mathscr{Y})$ be a cotorsion pair in R-*Mod *and C a complex.*

(1) *C is called an $\mathscr{X}$* complex *if it is exact and $\mathrm{Z}_i(C) \in \mathscr{X}$ for each $i \in \mathbb{Z}$.*
(2) *C is called a $\mathscr{Y}$* complex *if it is exact and $\mathrm{Z}_i(C) \in \mathscr{Y}$ for each $i \in \mathbb{Z}$.*
(3) *C is called a* dg-$\mathscr{X}$ complex *if $C_i \in \mathscr{X}$ for each $i \in \mathbb{Z}$, and every morphism $f : C \to Y$ is null-homotopic whenever Y is a $\mathscr{Y}$ complex.*
(4) *C is called a* dg-$\mathscr{Y}$ complex *if $C_i \in \mathscr{Y}$ for each $i \in \mathbb{Z}$, and every morphism $f : X \to C$ is null-homotopic whenever X is an $\mathscr{X}$ complex.*

2.3. *Gorenstein AC-injective modules and complexes*

According to [4], recall that an R-module M is of *type FP_∞* if there is an exact sequence of R-modules $\cdots \to P_1 \to P_0 \to M \to 0$ where each P_i is finitely generated free; a module E is *absolutely clean* if $\mathrm{Ext}^1_R(N, E) = 0$ for all modules N of type FP_∞; an R-module M is *Gorenstein AC-injective* [4] (resp. *Gorenstein injective* [10]) if there exists an exact complex of injective R-modules $\cdots \to I_1 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ which remains exact after applying the functor $\mathrm{Hom}_R(A,\cdot)$ for each absolutely clean (resp. injective) R-module A such that $M \cong \mathrm{Im}(I_0 \to I_{-1})$.

The *Gorenstein AC-injective dimension* of R-module M, AC-$\mathrm{Gid}_R(M)$, is defined by declaring that AC-$\mathrm{Gid}_R(M) \leq m$ if and only if there is an exact sequence $0 \to M \to G_0 \to \cdots \to G_m \to 0$ with each G_i Gorenstein AC-injective. The definition of Gorenstein injective dimension, $\mathrm{Gid}_R(M)$, can be defined similarly.

Definition 11 (cf. [3, Definition 2.1 and Proposition 2.2]). *An R-complex T is of* type FP_∞ *if T is bounded and each component T_m is an R-module of type* FP_∞.

Definition 12 (cf. [3, Definition 2.4 and Proposition 2.6]). *An R-complex A is* absolutely clean *if $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(T, A) = 0$ for all R-complexes T of type* FP_∞*, or equivalently, if A is exact and each $\mathrm{Z}_i(A)$ is an absolutely clean R-module.*

Definition 13 (cf. [21, Definition 5.1]). *An R-complex X is* AC-injective *if all its components X_m are an injective R-module and every morphism $A \to X$ is null-homotopic whenever A is an absolutely clean R-complex.*

Definition 14 (cf. [3] and [15]). *An R-complex G is* Gorenstein AC-injective *(resp.* Gorenstein injective*) if there exists an exact sequence of injective R-complexes $\cdots \to I_1 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ which remains exact after applying the functor $\mathrm{Hom}_{\mathrm{Ch}(R)}(A, \cdot)$ for each absolutely clean (resp. injective) R-complex A such that $G \cong \mathrm{Im}(I_0 \to I_{-1})$.*

2.4. *Compactly-generatedness of triangulated categories*

Let $\mathscr{T}$ be a triangulated category with small coproducts. Recall that an object C of $\mathscr{T}$ is *compact* if for each collection $\{Y_j \mid j \in J\}$ of objects of $\mathscr{T}$, the canonical morphism

$$\coprod_{j\in J} \mathrm{Hom}_{\mathscr{T}}(C, Y_j) \to \mathrm{Hom}_{\mathscr{T}}(C, \coprod_{j\in J} Y_j)$$

is an isomorphism. The category $\mathscr{T}$ is *compact generated* if there exists a small set $\mathscr{S} \subseteq \mathscr{T}$ of compact objects such that for each $0 \neq Y \in \mathscr{T}$ there is a morphism $0 \neq f : \Sigma^m S \to Y$ for some $S \in \mathscr{S}$ and $m \in \mathbb{Z}$, where Σ denotes the autofunctor of $\mathscr{T}$.

3. The equivalence between Question 1 and Question 3

In this section, we prove that Question 1 and Question 3, from the introduction, are equivalent (see Corollary 22). As an application, we obtain some examples to show that there exist non-Gorenstein-AC-injective Gorenstein injective modules and Gorenstein injective complexes of modules (see Examples 24 and 25), and give some new homological conditions of rings such that Krause's recollement holds (see see Corollaries 22 and 26).

We start with the following result, which is cited from [3, Theorem 3.2], and is useful in the sequel.

Theorem 15. *Let X be an R-complex. Then X is Gorenstein AC-injective if and only if all its components X_m are Gorenstein AC-injective R-modules, and $\mathscr{H}\mathrm{om}(A, X)$ is acyclic for all absolutely clean complexes A. Equivalently, each X_m is Gorenstein AC-injective and each morphism of complexes $f : A \to X$ with A absolutely clean is null-homotopic.*

We consider some immediate applications of Theorem 15.

Lemma 16. *The following hold:*

(1) *Any dg-injective (and hence, any injective) R-complex is Gorenstein AC-injective.*
(2) *The subcategory of all Gorenstein AC-injective R-complexes is injectively coresolving.*

Proof. (1). Clearly holds.

(2). Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of R-complexes with X Gorenstein AC-injective. On the one hand, for any $m \in \mathbb{Z}$, we have a short exact sequence $0 \to X_m \to Y_m \to Z_m \to 0$ of R-modules. Besides, each X_m is Gorenstein AC-injective by Theorem 15. It follows that Y_m is Gorenstein AC-injective if and only if so is Z_m since the subcategory of all Gorenstein AC-injective R-modules is injectively coresolving. On the other hand, for any absolutely clean R-complex A, we have a short exact sequence $0 \to \mathscr{H}\mathrm{om}(A, X) \to \mathscr{H}\mathrm{om}(A, Y) \to \mathscr{H}\mathrm{om}(A, Z) \to 0$ of $\mathbb{Z}$-complexes. Moreover, $\mathscr{H}\mathrm{om}(A, X)$ is acyclic by Theorem 15. It follows that $\mathscr{H}\mathrm{om}(A, Y)$ is acyclic if and only if so is $\mathscr{H}\mathrm{om}(A, Z)$. Thus, Y is Gorenstein AC-injective if and only if so is Z again by Theorem 15. □

In what follows, we denote by $\mathscr{G}$ the subcategory of R-Mod consisting of all Gorenstein AC-injective R-modules, and by $\mathscr{W} = {}^{\perp}\mathscr{G}$. According to [18, Fact 10.1], $(\mathscr{W}, \mathscr{G})$ is an injective cotorsion pair on R-Mod. Thus, we have the notations $\mathrm{dg}\mathscr{G}$, $\widetilde{\mathscr{G}}$, $\mathrm{dg}\mathscr{W}$ and $\widetilde{\mathscr{W}}$.

Lemma 17. *Let X be an R-complex. If X is in* $\mathrm{dg}\widetilde{\mathscr{G}}$ *or* $\widetilde{\mathscr{G}}$*, or it is a bounded above R-complex with all its components X_m in $\mathscr{G}$, then X is Gorenstein AC-injective.*

Proof. Note that an R-complex X is in $\widetilde{\mathscr{G}}$ if and only if X is acyclic and in $\mathrm{dg}\widetilde{\mathscr{G}}$ by [16, Theorem 3.12], and that any bounded above complex with all its components X_m Gorenstein AC-injective is always in $\mathrm{dg}\widetilde{\mathscr{G}}$ by [16, Lemma 3.4]. Hence, we need only to see the case where X is in $\mathrm{dg}\widetilde{\mathscr{G}}$. By the definition, all morphisms of R-complexes $f : W \to X$ with $W \in \widetilde{\mathscr{W}}$ are null-homotopic. But any absolutely clean R-complex is always in $\widetilde{\mathscr{W}}$. Thus, X is Gorenstein AC-injective by Theorem 15. □

Let X be a complex with all components X_m a Gorenstein AC-injective R-module. Then the previous lemma says that X is Gorenstein AC-injective whenever X is bounded above. By the following two results we consider the case for X to be bounded below.

Lemma 18. *Let X be a bounded below complex with all its components X_m Gorenstein AC-injective. Then there exists a short exact sequence $0 \to K \to E \to X \to 0$ of R-complexes with the following:*

(1) *E is a bounded below complex with each its component E_m an injective R-module.*
(2) *K is an acyclic and bounded below complex with each its component K_m a Gorenstein AC-injective R-module.*
(3) *K is $\mathrm{Hom}_R(M,\cdot)$-exact for all absolutely clean R-modules M.*

Proof. The proof is similar to the one in [38, Lemma 3.1], which describe any bounded below complex as a direct limit of its subcomplexes by the hard left-truncation of it, and use the induction on its degree of bounded below. Besides the induction technology and seeking for appropriate morphisms of complexes to construct the mapping cone, the key points of the proof are as follows:

(a) For any Ding injective R-module D, there exists a "left injective resolution" from the definition which is $\mathrm{Hom}_R(M,\cdot)$-exact for all FP-injective R-module M, meanwhile, any kernel of such "left injective resolution" is still Ding injective.
(b) $\mathrm{Ext}_R^{k\geq 1}(M,D) = 0$ for all Ding injective modules D and all FP-injective modules M.
(c) The class of all Ding injective modules is injective coresolving and closed under finite direct sums.
(d) The class of all FP-injective modules is closed under (arbitrary) direct sums.

Note that (a)–(d) all have a version for "Gorenstein AC-injective" by replacing "FP-injective" and "Ding injective" with "absolutely clean" and "Gorenstein AC-injective" respectively (one can see [4] for the detail). So, one can get the corresponding proof for the case of "Gorenstein AC-injective". □

Lemma 19. *Let X be a bounded below R-complex with all its components X_m Gorenstein AC-injective. Then there exists a short exact sequence $0 \to K \to E \to X \to 0$ of R complexes with K acyclic and Gorenstein AC-injective and all components of E injective.*

Proof. We consider the short exact sequence $0 \to K \to E \to X \to 0$ established in Lemma 18. Let A be any absolutely clean R-complex. Then the $\mathbb{Z}$-complex $\mathrm{Hom}_R(A_m,K)$ is acyclic for all $m \in \mathbb{Z}$ by the condition (3) in Lemma 18, also, the abelian group $\mathrm{Hom}_R(A_m,K_m)$ is zero for all $m \ll 0$ since K is bounded below. So any morphism $A \to K$ is null-homotopic by [8, Lemma 2.3]. Thus, Theorem 15 yields that K is Gorenstein AC-injective since each K_m is a Gorenstein AC-injective R-module. □

According to [41, Proposition 2.8], an R-complex X is Gorenstein injective if and only if all its components X_m are Gorenstein injective R-modules. Now we consider the version for Gorenstein AC-injective R-complexes, in other words, it is an answer to Question 3 from the introduction.

Lemma 20. *The following conditions are equivalent for any ring R:*

(1) *Every bounded below complex of Gorenstein AC-injective modules is Gorenstein AC-injective.*
(2) *Every bounded below complex of injective modules is Gorenstein AC-injective.*
(3) *Every complex of Gorenstein AC-injective modules is Gorenstein AC-injective.*
(4) *Every complex of injective modules is Gorenstein AC-injective.*
(5) *Every acyclic complex of Gorenstein AC-injective modules is Gorenstein AC-injective.*
(6) *Every acyclic complex of injective modules is Gorenstein AC-injective.*

Proof. It is trivial that (3) $\Rightarrow$ (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) $\Rightarrow$ (6) and (3) $\Rightarrow$ (5) $\Rightarrow$ (6).

(2)$\Rightarrow$(3). Let X be a complex of Gorenstein AC-injective R-modules. Consider the short exact sequence of R-complexes $0 \to X_{\leq 0} \to X \to X_{\geq 1} \to 0$. Then both $X_{\leq 0}$ and $X_{\geq 1}$ admit all components to be Gorenstein AC-injective. Since $X_{\leq 0}$ is bounded above, it is Gorenstein AC-injective by Lemma 17. Applying Lemma 16 (2), we need enough to show that $X_{\geq 1}$ is Gorenstein AC-injective. Notice that $X_{\geq 1}$ is bounded-below, Lemma 19 yields that there is a short exact sequence of R-complexes $0 \to K \to E \to X_{\geq 1} \to 0$ with K Gorenstein AC-injective and E admitting all components to be injective. By (2) E is a Gorenstein AC-injective R-complex, so is $X_{\geq 1}$ again by Lemma 16 (2). □

(6)$\Rightarrow$(4). Let X be a complex of injective R-modules. Consider the short exact sequence $0 \to I \to E \to X \to 0$ of R-complexes with I dg-injective and E acyclic. Then E has all components to be an injective R-module since so do I and X. Hence, by (6) E is a Gorenstein AC-injective R-complex, so is X by Lemma 16 (2) because I is a Gorenstein AC-injective R-complex by Lemma 16 (1). □

It is now in a position to give our main result.

Theorem 21. *The following conditions are equivalent for any ring R:*

(1) *Every complex of injective R-modules is AC-injective.*
(2) *Every acyclic R-complex of injective R-modules is AC-injective.*
(3) *Every complex of Gorenstein AC-injective R-modules is Gorenstein AC-injective.*
(4) *Every complex of injective R-modules is Gorenstein AC-injective.*
(5) *Every acyclic complex of injective R-modules is Gorenstein AC-injective.*

Proof. (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (5) follow from Theorem 15; (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) holds by Lemma 20. □

We denote by K(AC-Inj) the homotopy category of all AC-injective R-complexes and by $\mathrm{K_{ac}}$(AC-Inj) its homotopy subcategory consisting of all exact AC-injective R-complexes; by K(R-Inj) the homotopy category of injective R-modules and by $\mathrm{K_{ac}}$(R-Inj) its homotopy subcategory consisting of exact complexes of injective R-modules.

In order to extend Krause's recollement to an arbitrary ring R, Gillespie [21, Corollary 5.9] explained how K(R-Inj) and $\mathrm{K_{ac}}$(R-Inj) can be replaced by K(AC-Inj) and $\mathrm{K_{ac}}$(AC-Inj) to get the same results. For a coherent ring R, the work of Šťovíček [35, Corollary 6.13] shows that K(AC-Inj) = K(R-Inj). Gillespie [21, Remark, p. 109] gave an example to show that the equality may not hold for a non-coherent ring, and then in [21, Introduction] raised the following questions involving such an equality:

- For what rings R do we have K(AC-Inj) = K(R-Inj)?
- Does K(AC-Inj) = K(R-Inj) characterize that$R$ is coherent?

Recall that an R-complex C is *contractible* if the identity endomorphism $C \to C$ is null-homotopic. Note that such a complex is always exact and such a complex of injective modules is exactly injective complexes. These facts help us to get the next corollary, which provides an answer to the second question above.

Corollary 22. *The following conditions are equivalent for any ring R:*

(1) $\mathrm{K}(\text{AC-Inj}) = \mathrm{K}(R\text{-Inj})$.
(2) $\mathrm{K}_{\mathrm{ac}}(\text{AC-Inj}) = \mathrm{K}_{\mathrm{ac}}(R\text{-Inj})$.
(3) *For any (acyclic) complex X, X is Gorenstein AC-injective if and only if each X_m is a Gorenstein AC-injective R-module.*

Proof. We use the notation $\mathrm{Ch}(R\text{-Inj})$ to denote the subcategory of $\mathrm{Ch}(R)$ consisting of all complexes of injective R-modules, similarly for the notations $\mathrm{Ch}_{\mathrm{ac}}(R\text{-Inj})$, $\mathrm{Ch}(\text{AC-Inj})$ and $\mathrm{Ch}_{\mathrm{ac}}(\text{AC-Inj})$. According to Theorem 21, we know that (3) and the following two conditions are equivalent:

(4) $\mathrm{Ch}(\text{AC-Inj}) = \mathrm{Ch}(R\text{-Inj})$.
(5) $\mathrm{Ch}_{\mathrm{ac}}(\text{AC-Inj}) = \mathrm{Ch}_{\mathrm{ac}}(R\text{-Inj})$.

Note that (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (2) are trivial.

(1)$\Rightarrow$(4). Suppose that $\mathrm{K}(\text{AC-Inj}) = \mathrm{K}(R\text{-Inj})$. It is trivial that $\mathrm{Ch}(\text{AC-Inj}) \subseteq \mathrm{Ch}(R\text{-Inj})$. Conversely, let $X \in \mathrm{Ch}(R\text{-Inj})$. Then there exists a contractible complex Y of injective R-modules such that $X \oplus Y$ is in $\mathrm{Ch}(\text{AC-Inj})$ since $\mathrm{K}(\text{AC-Inj}) = \mathrm{K}(R\text{-Inj})$. Notice that contractible complexes of injective R-modules are exactly injective R-complexes, and so they are AC-injective. We conclude that X is in $\mathrm{Ch}(\text{AC-Inj})$.

(2)$\Rightarrow$(5). It is similar to (1) $\Rightarrow$ (4) since any contractible complex is always exact. □

As mentioned above, by replacing $\mathrm{K}(R\text{-Inj})$ and $\mathrm{K}_{\mathrm{ac}}(R\text{-Inj})$ with $\mathrm{K}(\text{AC-Inj})$ and $\mathrm{K}_{\mathrm{ac}}(\text{AC-Inj})$ respectively, Gillespie [21, Corollary 5.9] extend Krause's recollement to an arbitrary ring R. Together with this result and Corollary 22, we have

Corollary 23. *If one of the equivalent conditions in Corollary 22 is satisfied, then Krause's recollement holds over R; in particular,* $\mathrm{K}(R\text{-Inj})$ *and* $\mathrm{K}_{\mathrm{ac}}(R\text{-Inj})$ *are compactly generated.*

By the definition, it is trivial that every Gorenstein AC-injective R-module (resp. R-complex) is always Gorenstein injective. In the following, we give some examples to show that the converse may be not true in general.

Example 24. Let R be the ring $K[x_1, x_2, \ldots]/(x_i x_j)_{i,j\geq 1}$, where K is a field. Then [21, Remark, p. 108] shows that there is a complex X consisting of injective R-modules which is not AC-injective. Then by Theorem 21, X is not Gorenstein AC-injective. However, according to [41, Proposition 2.8], X is always a Gorenstein injective R-complex.

Example 25. Let R be the ring as in Example 24 and consider the ring $S = R[x]/(x^2)$. By virtue of [22, Section 5.5], we know that there exists an isomorphism between the categories of S-modules and R-complexes, which restricts to an isomorphism between the subcategories of Gorenstein AC-injective S-modules and Gorenstein AC-injective R-complexes (resp. between the subcategories of Gorenstein injective S-modules and Gorenstein injective R-complexes). Thus, Example 24 shows that there is a Gorenstein injective S-module which is not Gorenstein AC-injective.

Recall that an acyclic R-complex I of injective R-modules is *totally acyclic* (resp. *totally AC-acyclic*) if $\mathrm{Hom}_R(M, I)$ remains acyclic for any injective (resp. absolutely clean) R-module M. We denote by $\mathrm{K}_{\mathrm{tac}}(R\text{-Inj})$ (resp. $\mathrm{K}_{\text{AC-tac}}(R\text{-Inj})$) the homotopy subcategory of $\mathrm{K}(R\text{-Inj})$ consisting of all totally acyclic (resp. totally AC-acyclic) complexes of injective R-modules; by $\mathrm{dw}\widetilde{\mathscr{G}}$ (resp. $\mathrm{ac}\widetilde{\mathscr{G}}$) the subcategory of $\mathrm{Ch}(R)$ consisting of all (resp. acyclic) complexes with all components X_m to be Gorenstein AC-injective. By virtue of [13, Theorem 3.4], we know that every acyclic complex of injective R-modules is totally AC-acyclic if and only if $\mathrm{dw}\widetilde{\mathscr{G}} = \mathrm{dg}\widetilde{\mathscr{G}}$ if and only if $\mathrm{ac}\widetilde{\mathscr{G}} = \widetilde{\mathscr{G}}$. This result helps us to get the following proposition.

Proposition 26. *The following conditions are equivalent for any ring R:*

(1) $\mathrm{K}_{\mathrm{ac}}(R\text{-Inj}) = \mathrm{K}_{\mathrm{AC\text{-}tac}}(R\text{-Inj})$.
(2) $\mathrm{dw}\widetilde{\mathscr{G}} = \mathrm{dg}\widetilde{\mathscr{G}}$
(3) $\mathrm{ac}\widetilde{\mathscr{G}} = \widetilde{\mathscr{G}}$

If one of the equivalent conditions are satisfied, then the equality $\mathrm{K}(\mathrm{AC\text{-}Inj}) = \mathrm{K}(R\text{-Inj})$ *and Krause's recollement hold over R; in particular,* $\mathrm{K}(R\text{-Inj})$ *and* $\mathrm{K}_{\mathrm{ac}}(R\text{-Inj}) = \mathrm{K}_{\mathrm{tac}}(R\text{-Inj}) = \mathrm{K}_{\mathrm{AC\text{-}tac}}(R\text{-Inj})$ *are compactly generated.*

Proof. It is easy to see that (1)–(3) are equivalent and $\mathrm{K}(\mathrm{AC\text{-}Inj}) = \mathrm{K}(R\text{-Inj})$ by [13, Theorem 3.4] and the proof in Corollary 22. Now suppose that (1)–(3) holds true. Then the equalities $\mathrm{K}_{\mathrm{ac}}(R\text{-Inj}) = \mathrm{K}_{\mathrm{tac}}(R\text{-Inj}) = \mathrm{K}_{\mathrm{AC\text{-}tac}}(R\text{-Inj})$ hold since the implications $\mathrm{K}_{\mathrm{ac}}(R\text{-Inj}) \supseteq \mathrm{K}_{\mathrm{tac}}(R\text{-Inj}) \supseteq \mathrm{K}_{\mathrm{AC\text{-}tac}}(R\text{-Inj})$ hold. At the same time, any acyclic R-complex X is Gorenstein AC-injective if and only if each X_m is a Gorenstein AC-injective R-module. Thus the last assertions then follows from Corollary 23. □

Let R be any ring. Note that, by the definitions, the subcategory $\mathscr{G}$ (resp. $\mathscr{G}\mathscr{I}$, the subcategory of R-Mod consisting of all Gorenstein injective R-modules) in R-Mod consists of all modules as some cycle of a totally AC-acyclic (resp. totally acyclic) R-complex. Furthermore, [27, Proposition 7.2] yields that there is a triangulated equivalence $\mathrm{K}_{\mathrm{tac}}(R\text{-Inj}) \simeq \underline{\mathscr{G}\mathscr{I}}$. One can obtain another triangulated equivalence $\mathrm{K}_{\mathrm{AC\text{-}tac}}(R\text{-Inj}) \simeq \underline{\mathscr{G}}$ by using the similar method in [27, Proposition 7.2]. Here $\underline{\mathscr{G}\mathscr{I}}$ (resp. $\underline{\mathscr{G}}$) denotes the stable category with respect to Gorenstein injective (resp. Gorenstein AC-injective) modules. Hence by Corollary 26, we obtain

Corollary 27. *Let R be a ring such that any acyclic complex of injective modules is totally AC-acyclic. Then $\underline{\mathscr{G}} = \underline{\mathscr{G}\mathscr{I}}$ are compactly generated.*

4. Counterexamples to Question 2 and Grothendieck duality

In this section, we study some homological properties of rings with finite global Gorenstein AC-injective dimension (see Lemma 28). As application, we give a new counterexample to Question 2 from the introduction (see Theorem 29 and Example 34), and establish a extended Grothendieck duality theorem (see Proposition 30 and Corollaries 31 and 33).

Lemma 28. *Let R be a ring with* $\sup\{\mathrm{AC\text{-}Gid}_R(M) \mid M \text{ is an } R\text{-module}\}$ *finite. Then the following are equivalent for any R-module N:*

(1) *N is a Gorenstein AC-injective R-module.*
(2) *N is a Gorenstein injective R-module.*
(3) *There exists an acyclic complex of injective R-modules $I = \cdots \to I_1 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ such that $N \cong \mathrm{Im}(I_0 \to I_{-1})$.*

In particular, any acyclic complex of injective R-modules are totally AC-acyclic.

Proof. We need only to prove (3) ⇒ (1) since (1) ⇒ (2) ⇒ (3) are trivial. To this end, we let A be any absolutely clean R-module and we prove that $\mathrm{Hom}_R(A, I)$ is acyclic, or equivalently, all $\mathrm{H}_k(\mathrm{Hom}_R(A, I)) = 0$. For each R-module M, one has $\mathrm{AC\text{-}Gid}_R(M) \leq m < \infty$ for some nonnegative integer m by assumption. So there is an exact sequence of R-modules

$$0 \to M \to G_0 \to \cdots \to G_m \to 0$$

with each G_i Gorenstein AC-injective. Thus $\mathrm{Ext}_R^{m+1}(A, M) \cong \mathrm{Ext}_R^1(A, G_m) = 0$. This yields that A satisfies $\mathrm{pd}_R(A) \leq m$. We want to prove that $\mathrm{S}_k A$ belongs to ${}^{\perp}\mathscr{X}$ for all $k \in \mathbb{Z}$ by induction on $\mathrm{pd}_R(A)$. Here we denote by $\mathscr{X}$ the class of all acyclic complexes consisting of injective R-modules. Notice first from [18, Proposition 7.2] that $({}^{\perp}\mathscr{X}, \mathscr{X})$ is an injective cotorsion pair on $\mathrm{Ch}(R)$ since the pair (R-Mod, R-Inj) forms an injective cotorsion pair on R-Mod. If $\mathrm{pd}_R(A)=0$, then $\mathrm{S}_k A$ belongs to ${}^{\perp}\mathscr{X}$ by [20, Corollary 3.4 (2)]. Now let $\mathrm{pd}_R(A)>0$. Consider the short exact sequence of R-modules

$$0\to K\to P\to A\to 0$$

with P projective. Then $\mathrm{S}_k P$ belongs to ${}^{\perp}\mathscr{X}$ by what we have proved, at the same time, $\mathrm{pd}_R(K)\leq \mathrm{pd}_R(A)-1$ and so $\mathrm{S}_k K$ belongs to ${}^{\perp}\mathscr{X}$ by induction. We conclude that $\mathrm{S}_k A$ belongs to ${}^{\perp}\mathscr{X}$ since ${}^{\perp}\mathscr{X}$ is thick. Thus we have $\mathrm{H}_k(\mathrm{Hom}_R(A,I))\subseteq \mathrm{Ext}^1_{\mathrm{Ch}(R)}(\mathrm{S}_k A, I)=0$. □

The following theorem is our main result in this section, where K(R-Proj) denotes the homotopy category of projective R-modules.

Theorem 29. *Let R be a ring with* $\sup\{\text{AC-Gid}_R(M)\mid M$ *is an R-module*$\}$ *finite. Then*

(1) *The equality* K(AC-Inj) = K(R-Inj) *and Krause's recollement hold. In particular,* K(R-Inj) *and* $\mathrm{K_{ac}}$(R-Inj) *are compactly generated.*
(2) *There exists a triangulated equivalence* K(R-Proj) $\simeq$ K(R-Inj) *in which two triangulated categories are compactly generated.*

Proof. The assertions in (1) hold by Lemma 28 and Corollary 26.

For the assertions in (2), we note that the existence of the triangulated equivalence is guaranteed by [5, Theorem B] since $\sup\{\text{AC-Gid}_R(M)\mid M \text{ is an } R\text{-module}\}<\infty$ implies that R is *left Gorenstein* in the sense [5]. Moreover, two triangulated categories being compactly generated follows from (1). □

As usual, the notation $\mathbf{D}^b(R\text{-mod})$ (resp. $\mathbf{D}^b(R^{op}\text{-mod})^{op}$) denotes the bounded derived category of finitely presented left R-modules (resp. the oppositive category of the bounded derived category of finitely presented right R-module). Let us denote by $\mathbf{D}^b(R\text{-tmod})$ (resp. $\mathbf{D}^b(R^{op}\text{-tmod})^{op}$) the bounded derived category of left R-modules of type FP_∞ (resp. the oppositive category of the bounded derived category of right R-modules of type FP_∞); by K(R-Proj)^c (resp. K(R-Inj)^c, K(AC-Inj)^c) the full triangulated subcategory of K(R-Proj) (resp. K(R-Proj), K(AC-Inj)) consisting of all compact objects.

Proposition 30. *Let R be a ring.*

(1) *If* K(R-Proj) *is compactly generated, then there exists a triangulated equivalence* K(R-Proj)$^c\simeq \mathbf{D}^b(R^{op}\text{-tmod})^{op}$.
(2) *If* K(AC-Inj) = K(R-Inj), *then there exists a triangulated equivalence* K(R-Inj)$^c\simeq \mathbf{D}^b(R\text{-tmod})$.

Proof. (1). According to [34, Proposition 7.12], we know that an object X in K(R-Proj) is compact if and only if there is a complex Y satisfying

(I) Y is a bounded-below complex of finitely generated projective modules.
(II) $Y^*=\mathrm{Hom}_R(Y,R)$ is also a bounded-below complex.

such that X is isomorphic to Y in K(R-Proj). Let $\mathscr{U}$ be the full subcategory of K(R-Proj) consisting of objects which are finitely built from objects Y as above, using suspensions, distinguished triangles, retractions and homotopy colimits; $\mathscr{V}$ be the full subcategory of K(R^{op}-Proj) consisting of objects which are finitely built from objects Y^* with Y as above. Since Y is a complex of finitely generated projective R-modules, the canonical morphisms of complexes $Y\to Y^*$ and $Y^*\to Y^{***}$ are isomorphisms. It follows that

$$\mathscr{U} \underset{(\cdot)^*}{\overset{(\cdot)^*}{\rightleftarrows}} \mathscr{V}^{op}$$

are quasi-inverse equivalences of triangulated categories. Thus, we are done once we obtain the triangulated equivalences

$$\mathscr{U} = \mathrm{K}(R\text{-}\mathrm{Proj})^c \quad \text{and} \quad \mathscr{V} \simeq \mathbf{D}^b(R^{op}\text{-tmod}).$$

Firstly, since $\mathrm{K}(R\text{-}\mathrm{Proj})^c$ is compactly generated and the category $\mathscr{U}$ consists of objects finitely built from a set of compact generators of $\mathrm{K}(R\text{-}\mathrm{Proj})$, the equality holds by [33, Theorem 2.1.3].

Secondly, for any R^{op}-module N of type FP_∞, by the definition, there is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to N \to 0$$

of R^{op}-modules with each P_i finitely generated free. Consider the R^{op}-complex $P = \cdots \to P_2 \to P_1 \to P_0 \to 0$ and the direct system $\{(P(i), \alpha_{ij}) \mid 0 \le i \le j\}$ in $\mathrm{Ch}(R^{op})$, where

$$P(i) =: \cdots \to 0 \to P_i \to \cdots \to P_1 \to P_0 \to 0$$

and $\alpha_{ij} : X(i) \to X(j)$ is the following natural injection of the form

$$\begin{tikzcd}
P(i) =: \cdots \arrow[r] \arrow[d, "\alpha_{ij}"] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \arrow[r] & P_i \arrow[r] \arrow[d, "="] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "="] & P_0 \arrow[r] \arrow[d, "="] & 0 \\
P(j) =: \cdots \arrow[r] & 0 \arrow[r] & P_j \arrow[r] & \cdots \arrow[r] & P_i \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0
\end{tikzcd}$$

for all $i \le j$. Note that the homotopy colimit of all $P(i)$s is just P. It is a standard way to see that each $P(i)$ has the form $P(i) = Y^*$ with Y satisfying both conditions (I) and (II) above, and so does P. Hence, the category $\mathscr{V}$ consists of the objects finitely built from projective resolutions all R^{op}-modules of type FP_∞, and so it is classical that $\mathscr{V}$ is equivalent to $\mathbf{D}^b(R^{op}\text{-tmod})$.

(2). Let F be an R-module of type FP_∞ and $I = 0 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ the injective resolution of F, that is, each I_i is an R-module and there is an exact sequence of R-modules $0 \to F \to I_{-1} \to I_{-2} \to \cdots$. Then I is a compact object in K(AC-Inj) with an isomorphism $\mathrm{Hom}_{\mathrm{K(AC\text{-}Inj)}}(F, X) \cong \mathrm{Hom}_{\mathrm{K(AC\text{-}Inj)}}(I, X)$ for all $X \in$ K(AC-Inj) by applying [31, Theorem 2.29] to such a homotopy category. By virtue of [21, Theorems 4.4 and 5.2] and their proof, one can see that, for any non-zero object X in K(AC-Inj), there exists an R-module F of type FP_∞ such that $\mathrm{Hom}_{\mathrm{K(AC\text{-}Inj)}}(F, \Sigma^m X) \neq 0$ for some $m \in \mathbb{Z}$. Hence, [32, Lemma 2.2] shows that $\mathrm{K(AC\text{-}Inj)}^c$ equals the thick subcategory of R-modules of type FP_∞. A standard argument shows that the canonical functor K(AC-Inj) $\to \mathbf{D}(R)$ (see [21, Corollary 5.9]) induces a triangulated equivalence

$$\mathrm{K(AC\text{-}Inj)}^c \simeq \mathbf{D}^b(R\text{-tmod}).$$

Thus, the desired triangulated equivalence holds by the equality K(AC-Inj) = $\mathrm{K}(R\text{-}\mathrm{Inj})$. □

Corollary 31. *Let R be a ring such that* $\mathrm{K}(R\text{-Proj})$ *is compactly generated and that the equality* K(AC-Inj) = $\mathrm{K}(R\text{-Inj})$ *holds. Then the triangulated equivalence* $\mathrm{K}(R\text{-Proj}) \simeq \mathrm{K}(R\text{-Inj})$ *implies another one* $\mathbf{D}^b(R^{op}\text{-tmod})^{op} \simeq \mathbf{D}^b(R\text{-tmod})$. *In particular, the two triangulated equivalences hold provided that* $\sup\{\mathrm{AC\text{-}Gid}_R(M) \mid M \text{ is an } R\text{-module}\}$ *is finite.*

Proof. By the assumption $\mathrm{K}(R\text{-}\mathrm{Proj})$ and $\mathrm{K}(R\text{-}\mathrm{Inj})$ are compactly generated. Suppose that there is a triangulated equivalence $\mathrm{K}(R\text{-}\mathrm{Proj}) \simeq \mathrm{K}(R\text{-}\mathrm{Inj})$. Then it restricts to the one $\mathrm{K}(R\text{-}\mathrm{Proj})^c \simeq \mathrm{K}(R\text{-}\mathrm{Inj})^c$. Thus Proposition 30 yields another triangulated equivalence $\mathbf{D}^b(R^{op}\text{-tmod})^{op} \simeq \mathbf{D}^b(R\text{-tmod})$. The last assertion holds by Theorem 29 and what we have proved. □

As we all know, the finiteness of $\sup\{\mathrm{AC\text{-}Gid}_R(M) \mid M \text{ is an } R\text{-module}\}$ implies the finiteness of $\sup\{\mathrm{Gid}_R(M) \mid M \text{ is an } R\text{-module}\}$. The next lemma consider the converse.

Lemma 32. *Let R be a coherent ring. If* $\sup\{\mathrm{Gid}_R(M) \mid M \text{ is an } R\text{-module}\}$ *is finite, then so is* $\sup\{\mathrm{AC\text{-}Gid}_R(M) \mid M \text{ is an } R\text{-module}\}$.

Proof. It suffices to prove $\mathscr{G} = \mathscr{G}\mathscr{I}$. By [4, remarks in p. 17], we need enough to show that any absolutely clean R-module has finite injective dimension. For this, let A be an absolutely clean R-module, and let $\sup\{\mathrm{Gid}_R(M) \mid M \text{ is an } R\text{-module}\} = m < \infty$. Then [4, Corollary 2.9] yields that A is FP-injective since R is left coherent. Hence, there is a pure exact sequence $0 \to A \to I \to C \to 0$ of R-modules with I injective. By [7, Corollary 4.3], one has $\mathrm{fd}_R I \leq m$. It follows that $\mathrm{fd}_R A \leq \mathrm{fd}_R I \leq m$, and hence one has $\mathrm{id}_R A \leq m$ again by [7, Corollary 4.3]. □

According to [19, Theorem 3.21], we know that the two subcategories R^{op}-mod and R^{op}-tmod (resp. R-mod and R-tmod) coincide whenever R is right (resp. left) coherent. Using these facts and Theorem 29, Lemma 32 and Corollary 31, we have

Corollary 33. *Let R be a left and right coherent ring. Then the triangulated equivalence* $\mathrm{K}(R\text{-Proj}) \simeq \mathrm{K}(R\text{-Inj})$ *implies another one* $\mathbf{D}^b(R^{op}\text{-mod})^{op} \simeq \mathbf{D}^b(R\text{-mod})$. *In particular, the two triangulated equivalences hold provided that* $\sup\{\mathrm{Gid}_R(M) \mid M \textit{ is an } R\textit{-module}\}$ *is finite.*

Let $R = \prod_{i=1}^{k} R_i$ be a finite direct product of rings and $M = M_1 \oplus \cdots \oplus M_k, N = N_1 \oplus \cdots \oplus N_k$ be decompositions of R-modules into R_i-modules. Then following from [1, Theorem 3.1], we know that there is a natural isomorphism of abelian groups

$$\mathrm{Hom}_R(M,N) \xrightarrow{\cong} \mathrm{Hom}_{R_1}(M_1,N_1) \oplus \cdots \oplus \mathrm{Hom}_{R_k}(M_k,N_k) \tag{$\natural$}$$

which is given by $\alpha \mapsto \alpha_1 \oplus \cdots \oplus \alpha_k$, where the homomorphism $\alpha_1 \oplus \cdots \oplus \alpha_k$ is defined as $(\alpha_1 \oplus \cdots \oplus \alpha_k)(m_1 \oplus \cdots \oplus m_k) = (\alpha_1 m_1, \ldots, \alpha_k m_k)$. Furthermore, α is injective (resp. surjective) if and only if each α_i is injective (resp. surjective). By such an isomorphism one can easily verify that

Fact 1. A sequence $\mathbf{E} = 0 \to M^1 \to M^2 \to M^3 \to 0$ of R-modules is a short exact sequence if and only if so is each sequence $\mathbf{E}_i = 0 \to M_i^1 \to M_i^2 \to M_i^3 \to 0$ in R_i-Mod, and hence a sequence $\mathbf{M} = \cdots \to M^n \to M^{n-1} \to \cdots \to M^0 \to \cdots$ of R-modules is an exact sequence if and only if so is each $\mathbf{M}_i = \cdots \to M_i^n \to M_i^{n-1} \to \cdots \to M_i^0 \to \cdots$ in R_i-Mod.

Combining Fact 1 with the isomorphism ($\natural$), one can check that

Fact 2. An R-module M is (finitely generated) projective if and only if each M_i is a (finitely generated) projective R_i-module. This induces conventionally that, on the one hand, an R-module M is of type FP_∞ if and only if so is each M_i as R_i-modules; on the other hand, there is a natural isomorphism

$$\mathrm{Ext}^1_R(N,M) \xrightarrow{\cong} \mathrm{Ext}^1_{R_1}(N_1,M_1) \oplus \cdots \oplus \mathrm{Ext}^1_{R_k}(N_k,M_k). \tag{$\dagger$}$$

Using Fact 2, one can prove that

Fact 3. An R-module M is absolutely clean if and only if so is each M_i as R_i-modules.

This is in parallel to:

Fact 4. An R-module M is injective if and only if so is each M_i as R_i-modules.

Here we give a proof for Fact 3 as follows: "only if" part: Let $M = M_1 \oplus \cdots \oplus M_k$ be an absolutely clean R-module. Let T' be any R_i-module of FP_∞, where $1 \leq i \leq k$. Consider the R-module $L = 0_1 \oplus \cdots \oplus 0_{i-1} \oplus T' \oplus 0_{i+1} \oplus \cdots \oplus 0_k$. Then L is an R-module of FP_∞, and so $0 = \mathrm{Ext}^1_R(L,M) \cong \mathrm{Ext}^1_{R_i}(T',M_i)$ (see ($\dagger$)). Thus M_i is absolutely clean. "if"part: Let $M = M_1 \oplus \cdots \oplus M_k$ be an R-module such that each M_i is an absolutely clean R_i-module. Let $T = T_1 \oplus \cdots \oplus T_k$ be any R-module of FP_∞. Then each T_i is an R_i-module of type FP_∞ (see Fact 2), and hence $\mathrm{Ext}^1_{R_i}(T_i,M_i) = 0$ for all $1 \leq i \leq k$. So again by the isomorphism ($\dagger$), we have $\mathrm{Ext}^1_R(T,M) \cong \mathrm{Ext}^1_{R_1}(T_1,M_1) \oplus \mathrm{Ext}^1_{R_2}(T_2,M_2) \oplus \cdots \oplus \mathrm{Ext}^1_{R_1}(T_1,M_k) = 0$. This shows that R-module M is absolutely clean.

Combining with Facts 3, 4 and 1 one can check that $\mathbf{M}$ (as in Fact 1) is an totally AC-acyclic complex of injective R-modules if and only if so is each $\mathbf{M}_i$ in R_i-Mod. In particular, an R-module M is Gorenstein AC-injective if and only if so is each M_i in R_i-Mod, and hence, by the definition of the Gorenstein AC-injective dimension of R-modules, one can check that

$$\text{AC-Gid}_R(M) = \sup\{\text{AC-Gid}_{R_i}(M_i) \mid i = 1, \ldots, k\}. \tag{‡}$$

This is in parallel to the following equality

$$\text{id}_R(M) = \sup\{\text{id}_{R_i}(M_i) \mid i = 1, \ldots, k\}. \tag{♯}$$

Meanwhile, the following fact is followed from [1, Example 3.6 (3)]:

Fact 5. The ring $R = \prod_{i=1}^{k} R_i$ is coherent if and only if each R_i is coherent.

At the end of the paper, we will apply Fact 5 and the equalities (‡) and (♯) to give an example to show the result in Theorem 7 (2) (from the introduction) holds.

Example 34. Let $R = \mathbb{Z}/4\mathbb{Z}$ and $S = D + (x_1, x_2)K[x_1, x_2]$, where D is a Dedekind domain and K its quotient field. Then by [6, Example 2.18 and Remark 3.3] R is a commutative QF ring of infinite global dimension; by [26, Example in p. 128] S is a commutative non-coherent ring of finite global dimension. Consider $R \times S$, the product of rings R and S. Note that $\sup\{\text{AC-Gid}_R(M) \mid M \text{ is an } R\text{-module}\} = 0$ since R is QF, and that $\sup\{\text{AC-Gid}_S(M) \mid M \text{ is an } S\text{-module}\} < \infty$ since global dimension of S is finite. We have $\sup\{\text{AC-Gid}_{R\times S}(M) \mid M \text{ is an } R \times S\text{-module}\} < \infty$ by the equality (‡). However, by applying the equality (♯), the global dimension of $R \times S$ is the value $\sup\{\text{id}_{R\times S}(M) \mid M \text{ is an } R \times S\text{-module}\} = \infty$ since R has infinite global dimension. On the other hand, Fact 5 yields that $R \times S$ is non-coherent since so is S.

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