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On the structure of the $h$-fold sumsets

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Abstract. Let $A$ be a set of nonnegative integers. Let $(hA)^{(t)}$ be the set of all integers in the sumset $hA$ that have at least $t$ representations as a sum of $h$ elements of $A$. In this paper, we prove that, if $k \geq 2$, and $A = \{a_0, a_1, \ldots, a_k\}$ is a finite set of integers such that $0 = a_0 < a_1 < \cdots < a_k$ and $\gcd(a_1, a_2, \ldots, a_k) = 1$, then there exist integers $c_t, d_t$ and sets $C_t \subseteq [0, c_t - 2], D_t \subseteq [0, d_t - 2]$ such that $(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - 1 - D_t)$ for all $h \geq \sum_{i=2}^{k} (ta_i - 1) - 1$. This improves a recent result of Nathanson with the bound $h \geq (k-1)(ta_k - 1) + 1$.

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1. Introduction

Let $A$ and $B$ be sets of integers. The sumsets and difference sets are defined by

$A + B = \{a + b : a \in A, b \in B\}, \quad A - B = \{a - b : a \in A, b \in B\}$

respectively. For any integer $t$, we define the sets

$t + A = \{t\} + A, \quad t - A = \{t\} - A.$

For $h \geq 2$, we denote by $hA$ the $h$-fold sumset of $A$, which is the set of all integers $n$ of the form $n = a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, \ldots, a_h$ are elements of $A$ and not necessarily distinct.

In [3, 4], Nathanson proved the following fundamental beautiful theorem on the structure of $h$-fold sumsets.

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Nathanson’s Theorem A. Let \( A = \{a_0, a_1, \ldots, a_k\} \) be a finite set of integers such that
\[
0 = a_0 < a_1 < \cdots < a_k \quad \text{and} \quad \gcd(A) = 1.
\]
Let \( h_1 = (k-1)(a_k-1) + 1 \). There are nonnegative integers \( c_1 \) and \( d_1 \) and finite sets \( C_1 \) and \( D_1 \) with \( C_1 \subseteq [0, c_1-2] \) and \( D_1 \subseteq [0, d_1-2] \) such that
\[
 hA = C_1 \cup [c_1, ha_k - d_1] \cup (ha_k - D_1)
\]
for all \( h \geq h_1 \).

Later, Wu, Chen and Chen [6] improved the lower bound of \( h_1 \) to \( \sum_{i=2}^{k} a_i - k \). Recently, Granville and Walker [2] gave some further results on this topic.

Let \( A \) be a set of integers. For every positive integer \( h \), the \( h \)-fold representation function \( r_{A,h}(n) \) counts the number of representations of \( n \) as the sum of \( h \) elements of \( A \). Thus,
\[
r_{A,h}(n) = \left\{ \sum_{i=1}^{h} a_{j_i} : (a_{j_1}, \ldots, a_{j_h}) \in A^h, n = \sum_{i=1}^{h} a_{j_i}, \text{ and } a_{j_1} \leq \cdots \leq a_{j_h} \right\}.
\]

For every positive integer \( t \), let \( (hA)^{(t)} \) be the set of all integers \( n \) that have at least \( t \) representations as the sum of \( h \) elements of \( A \), that is,
\[
(hA)^{(t)} = \{ n \in \mathbb{Z} : r_{A,h}(n) \geq t \}
\]

Recently, Nathanson [5] found that the sumsets \( (hA)^{(t)} \) have the same structure as the sumset \( hA \) and proved the following theorem.

Nathanson’s Theorem B. Let \( k \geq 2 \), and let \( A = \{a_0, a_1, \ldots, a_k\} \) be a finite set of integers such that
\[
0 = a_0 < a_1 < \cdots < a_k \quad \text{and} \quad \gcd(A) = 1.
\]
For every positive integer \( t \), let \( h_t = (k-1)(ta_k-1) + 1 \). There are nonnegative integers \( c_t \) and \( d_t \), and finite sets \( C_t \) and \( D_t \) with \( C_t \subseteq [0, c_t-2] \) and \( D_t \subseteq [0, d_t-2] \) such that
\[
(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - D_t)
\]
for all \( h \geq h_t \).

In this paper, motivated by the idea of Wu, Chen and Chen [6], we improved the lower bound of \( h \) in Nathanson’s Theorem B.

Theorem 1. Let \( k \geq 2 \), and let \( A = \{a_0, a_1, \ldots, a_k\} \) be a finite set of integers such that
\[
0 = a_0 < a_1 < \cdots < a_k \quad \text{and} \quad \gcd(A) = 1
\]
For every positive integer \( t \), let
\[
h_t = \sum_{i=2}^{k} (ta_i - 1) - 1
\]
There are nonnegative integers \( c_t \) and \( d_t \) and finite sets \( C_t \) and \( D_t \) with
\[
C_t \subseteq [0, c_t-2] \quad \text{and} \quad D_t \subseteq [0, d_t-2]
\]
such that
\[
(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - D_t)
\]
(1)
for all \( h \geq h_t \).

Remark 2. Theorem 1 is optimal.

We shall prove Theorem 1 and Remark 2 in Section 3. In Section 2, we give some lemmas.
2. Some Lemmas

**Lemma 3 (see [5, Lemma 1]).** Let $A$ be a set of integers. For any positive integer $h$ and $t$, we have

$$\binom{hA}{t} + A \subseteq \binom{(h+1)A}{t}.$$ 

**Lemma 4.** Let $k \geq 2$, and let $A = \{a_0, a_1, \ldots, a_k\}$ be a set of integers satisfying $0 = a_0 < a_1 < \cdots < a_k$ and $\gcd(A) = 1$. For every positive integer $t$, let $h_t = \sum_{i=2}^{k}(ta_i-1)$ and $c_t = \sum_{i=1}^{k-1}a_i(ta_i+1-1)$. If $c_t' - a_k < n < c_t'$, then there exist at least $t$ distinct nonnegative $k$-tuples $(x_{1,s}, x_{2,s}, \ldots, x_{k,s}) \ (1 \leq s \leq t)$ satisfying

$$n = x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k$$

and $x_{1,s} + x_{2,s} + \cdots + x_{k,s} \leq h_t$ for $s = 1, 2, \ldots, t$.

**Proof.** Since $\gcd(a_1, \ldots, a_k) = 1$, there exist integers $x_1, \ldots, x_k$ such that

$$n = x_1a_1 + \cdots + x_ka_k.$$

For any positive integer $s$, $[(s-1)a_2, sa_2-1]$ is a complete residue system modulo $a_2$. Hence there exists an integer $q$ such that $x_1 = a_2q + x_{1,s}$ with $(s-1)a_2 \leq x_{1,s} \leq sa_2-1$. This gives

$$n = x_{1,s}a_1 + (a_2q + x_{2,s})a_2 + \cdots + x_{k,s}a_k.$$

Let $x_2 = a_1q + x_2$. Similarly, there exists an integer $q'$ such that $x_2' = a_3q' + x_{2,s}$ with $(s-1)a_3 \leq x_{2,s} \leq sa_3-1$. Now we have

$$n = x_{1,s}a_1 + x_{2,s}x_2 + (a_2q' + x_{3,s})a_3 + \cdots + x_{k,s}a_k.$$

By continuing this process, we obtain

$$n = x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k$$

with $(s-1)a_{i+1} \leq x_{i,s} \leq sa_{i+1}-1$ for $i = 1, \ldots, k-1$ and $x_{k,s}$ is some integer. Hence, for any integer $s \in [1, t]$, we have

$$0 \leq x_{i,s} \leq ta_{i+1}-1.$$

Since $n > c_t' - a_k$, it follows that

$$x_{k,s}a_k = n - (x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k-1,s}a_{k-1})$$

$$\geq n - (ta_2-1)a_1 - \cdots - (ta_k-1)a_{k-1} = n - c_t' > -a_k,$$

and then $x_{k,s} > -1$. Noting that $x_{k,s}$ is an integer, we have $x_{k,s} \geq 0$. By the bound of $x_{i,s}$, the following nonnegative $k$-tuples

$$(x_{1,s}, x_{2,s}, \ldots, x_{k-1,s}, x_{k,s}) \quad (1 \leq s \leq t)$$

are distinct.

Next, we shall prove that $x_{1,s} + x_{2,s} + \cdots + x_{k,s} \leq h_t$ for $s = 1, 2, \ldots, t$.

For any integer $s \in [1, t]$, let $x_{1,s} + x_{2,s} + \cdots + x_{k,s} = u_s$. Since $n < c_t'$, it follows that

$$n = x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k$$

$$= x_{1,s}a_1 + \cdots + x_{k-1,s}a_{k-1} + (u_s - x_{1,s} - x_{2,s} - \cdots - x_{k-1,s}) a_k$$

$$= u_sa_k - x_{1,s}(a_k - a_1) - \cdots - x_{k-1,s}(a_k - a_{k-1})$$

$$\geq u_sa_k - (ta_2-1)(a_k - a_1) - \cdots - (ta_k-1)(a_k - a_{k-1})$$

$$= u_sa_k - (0a_k - (a_k - 1)) a_k + c_t'$$

$$\geq u_sa_k - (h_t + 1) a_k + n.$$

Hence $u_sa_k - (h_t + 1) a_k < 0$, and then $u_s < h_t + 1$. Therefore, $u_s \leq h_t$.

This completes the proof of Lemma 4. \[\square\]
Lemma 5. Let $c'_i$ and $h_t$ be defined in Lemma 4. Then

$$c'_i = \sum_{i=1}^{k-1} a_i (ta_{i+1} - 1) \in ((h_t + 1) A)^{(t)}.$$  

Proof. For $i = 1, 2, \ldots, k-1$, let $p_i = ta_{i+1} - 1$. Then

$$c'_i = (ta_2 - 1) a_1 + \cdots + (ta_k - 1) a_{k-1} = p_1 a_1 + \cdots + p_{k-1} a_{k-1}.$$  

Noting that

$$p_1 + \cdots + p_{k-1} = \sum_{i=2}^{k} (ta_i - 1) = h_t + 1,$$  

we have $c'_i \in (h_t + 1) A$. Moreover, for any integer $r \in [0, t-1]$, we have

$$c'_i = \sum_{i=1}^{k-1} (ta_{i+1} - 1) a_i = \sum_{i=1}^{k-1} ((t-r)a_{i+1} - 1) a_i + r \sum_{i=1}^{k-1} a_i a_{i+1}$$

$$= ((t-r)a_2 - 1) a_1 + \sum_{i=2}^{k-1} ((t-r)a_{i+1} - 1 + ra_{i-1}) a_i + ra_{k-1} a_k,$$

where $p_{1,r} = (t-r)a_2 - 1$, $p_{k,r} = ra_{k-1}$ and $p_{i,r} = (t-r)a_{i+1} - 1 + ra_{i-1}$ $(2 \leq i \leq k-1)$. Hence $p_{i,r} \geq 0$ for all $i \in [1, k]$ and

$$\sum_{i=1}^{k} p_{i,r} = (t-r)a_2 - 1 + (t-r)a_3 - 1 + r a_1 + \cdots + (t-r) a_k - 1 + ra_{k-2} + ra_{k-1}$$

$$= h_t + 1 - r (a_2 + \cdots + a_k) + r(a_1 + \cdots + a_{k-1})$$

$$= h_t + 1 - r (a_k - a_1) \leq h_t + 1.$$  

Thus, $r_{A,h_t+1}(c'_i) \geq t$, and so $c'_i \in ((h_t + 1) A)^{(t)}$.  

Lemma 6. Let $n$ and $a_1$, $a_2$ be positive integers with gcd$(a_1, a_2) = 1$. For any positive integer $t$, if $n > ta_1 a_2 - a_1 - a_2$, then the diophantine equation

$$a_1 x + a_2 y = n$$  

has at least $t$ nonnegative integer solutions. The lower bound of $n$ is also best possible.

Proof. Suppose that $n > ta_1 a_2 - a_1 - a_2$. Let $(x_0, y_0)$ be a solution of the equation (2). Then all the integer solutions of the equation (2) is

$$\begin{cases} 
 x = x_0 + ka_2, \\ y = y_0 - ka_1, \end{cases} \quad k \in \mathbb{Z}.$$  

(3)

In order to have $x \geq 0$ and $y \geq 0$, we only need $x > -1$ and $y > -1$, that is,

$$\frac{-1 - x_0}{a_2} < k < \frac{y_0 + 1}{a_1}.$$  

(4)

Since

$$\frac{y_0 + 1}{a_1} - \frac{-1 - x_0}{a_2} = \frac{a_1 + a_2 + a_1 x_0 + a_2 y_0}{a_1 a_2} = \frac{a_1 + a_2 + n}{a_1 a_2} > \frac{a_1 + a_2 + t a_1 a_2 - a_1 - a_2}{a_1 a_2} = t,$$

there exist at least $t$ integers $k$ such that (4) holds.

Therefore, the equation (2) has at least $t$ nonnegative integer solutions.

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Now suppose that \( l = t a_1 a_2 - a_1 - a_2 \). Then \((ta_2 - 1, -1)\) is a solution of (2). Take \( x_0 = ta_2 - 1 \) and \( y_0 = -1 \). Then (3) becomes

\[
\begin{align*}
x &= ta_2 - 1 - ka_2, \\
y &= -1 + ka_1,
\end{align*}
\]

Since \( x \geq 0 \) and \( y \geq 0 \), it follows that \( 1 \leq k \leq t - 1 \). Hence there exist at most \( t - 1 \) nonnegative integer solutions.

This completes the proof of Lemma 6.

\[\square\]

3. Proofs

**Proof of Theorem 1.** Let \( c'_i = \sum_{i=1}^{k-1} a_i (ta_{i+1} - 1) \). By Lemma 4, there exist smallest integers \( c_t \) and \( d_t \) satisfying

\[
[c'_t - a_k + 1, c'_t - 1] \subseteq [c_t, h_t a_k - d_t] \subseteq (h_t A)^{(t)}.
\]

It follows that \( c_t - 1 \notin (h_t A)^{(t)} \) and \( h_t a_k - d_t + 1 \notin (h_t A)^{(t)} \). Additionally

\[
c_t \leq c'_t - a_k + 1, \quad (5)
\]

\[
c'_t - 1 \leq h_t a_k - d_t. \quad (6)
\]

Define the finite sets \( C_t \) and \( D_t \) by

\[
C_t = (h_t A)^{(t)} \cap [0, c_t - 2]
\]

and

\[
h_t a_k - D_t = (h_t A)^{(t)} \cap [h_t a_k - (d_t - 2), h_t a_k].
\]

Then

\[
(h_t A)^{(t)} = C_t \cup [c_t, h_t a_k - d_t] \cup (h_t a_k - D_t). \quad (7)
\]

Therefore, (1) holds for \( h = h_t \).

Now we prove (1) by induction on \( h \). Suppose that (1) holds for some \( h \geq h_t \). Define

\[
B^{(t)} = C_t \cup [c_t, (h + 1) a_k - d_t] \cup ((h + 1) a_k - D_t).
\]

Firstly we prove that \( B^{(t)} \subseteq ((h + 1) A)^{(t)} \).

Take an arbitrary integer \( b \in B^{(t)} \).

**Case 1:** \( b \in C_t \cup [c_t, h_t a_k - d_t] \). By (7), we have

\[
b \in (h_t A)^{(t)} \subseteq ((h + 1) A)^{(t)}.
\]

**Case 2:** \( b \in [c_t + a_k, (h + 1) a_k - d_t] \cup ((h + 1) a_k - D_t) \). It follows that

\[
b - a_k \in [c_t, h a_k - d_t] \cup (h a_k - D_t) \subseteq (h A)^{(t)}.
\]

Thus, By Lemma 3, \( b \in (h A)^{(t)} + a_k \subseteq ((h + 1) A)^{(t)} \).

**Case 3:** \( h_t a_k - d_t + 1 \leq b \leq c_t + a_k - 1 \). By (5) and (6), we have

\[
c_t + a_k - 1 \leq c'_t \leq h_t a_k - d_t + 1.
\]

Thus \( b = c'_t \). By Lemma 5, we have

\[
b = c'_t \in ((h + 1) A)^{(t)} \subseteq ((h + 1) A)^{(t)}.
\]

Therefore, \( B^{(t)} \subseteq ((h + 1) A)^{(t)} \).

Next we shall prove that \((h + 1) A)^{(t)} \subseteq B^{(t)} \). Take an arbitrary integer \( a \in ((h + 1) A)^{(t)} \).

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Case 1: $a = c'_t$. By (8) and $h \geq h_t$, we have
\[ c_t \leq c'_t \leq h_t a_k - d_t + 1 \leq (h + 1) a_k - d_t. \]
Hence $a = c'_t \in B^{(t)}$.

Case 2: $a \neq c'_t$ and $a \notin (hA)^{(t)}$. Since $a \in ((h + 1)A)^{(t)}$, there exist nonnegative integer $k$-tuples $(x_{1,s}, x_{2,s}, \ldots, x_{k,s})$ ($1 \leq s \leq t$) satisfying
\[ a = x_{1,s} a_1 + x_{2,s} a_2 + \cdots + x_{k,s} a_k \quad \text{and} \quad x_{1,s} + x_{2,s} + \cdots + x_{k,s} \leq h + 1. \]
Furthermore, we can get
\[ 0 \leq x_{i,s} \leq t a_{i+1} - 1, \quad 1 \leq i \leq k - 1, \quad 1 \leq s \leq t. \tag{9} \]
Otherwise, without loss of generality, assume that $x_{1,1} \geq t a_2$, then for $j = 1, 2, \ldots, t$, we have
\[ a = x_{1,1} a_1 + x_{2,1} a_2 + \cdots + x_{k,1} a_k \]
\[ = (x_{1,1} - j a_2) a_1 + (x_{2,1} + j a_1) a_2 + \cdots + x_{k,1} a_k. \]
Noting that for $j = 1, 2, \ldots, t$,
\[ (x_{1,1} - j a_2) + (x_{2,1} + j a_1) + x_{3,1} + \cdots + x_{k,1} = h + 1 - j (a_2 - a_1) < h + 1, \]
we have $a \notin (hA)^{(t)}$, a contradiction. Hence the inequality (9) holds.

By $a \notin (hA)^{(t)}$, there exists $s \in [1, t]$ such that $a = x_{1,s} a_1 + x_{2,s} a_2 + \cdots + x_{k,s} a_k$ and
\[ x_{1,s} + x_{2,s} + \cdots + x_{k,s} = h + 1. \]
By (9), we have
\[ a = x_{1,s} a_1 + x_{2,s} a_2 + \cdots + x_{k,s} a_k \]
\[ = x_{1,s} a_1 + \cdots + x_{k-1,s} a_{k-1} + (h + 1 - x_{1,s} - x_{2,s} - \cdots - x_{k-1,s}) a_k \]
\[ = (h + 1) a_k - x_{1,s} (a_k - a_1) - \cdots - x_{k-1,s} (a_k - a_{k-1}) \]
\[ \geq (h + 1) a_k - (t a_2 - 1) (a_k - a_1) - \cdots - (t a_2 - 1) (a_k - a_{k-1}) \]
\[ = (h + 1) a_k - a_k [(t a_2 - 1) + \cdots + (t a_k - 1)] + a_1 (t a_2 - 1) + \cdots + a_{k-1} (t a_k - 1) \]
\[ = (h + 1) a_k - (h_t + 1) a_k + c'_t \]
\[ \geq c'_t. \]
Since $a \neq c'_t$, it follows that $a \geq c'_t + 1$. By (5), we have $a \geq c'_t + 1 \geq c_t + a_k$.

If $x_{k,s} = 0$ for some $s$ with $1 \leq s \leq t$, by (9), then
\[ a \leq (t a_2 - 1) a_1 + \cdots + (t a_k - 1) a_{k-1} = c'_t, \]
a contradiction.

Hence $x_{k,s} \geq 1$ for all integers $s = 1, 2, \ldots, t$.

Therefore, $a - a_k \in (hA)^{(t)}$ and $a - a_k \geq c_t$. By the induction hypothesis,
\[ a \in a_k + [c_t, h a_k - d_t] \cup (h a_k - D_t) = [c_t + a_k, (h + 1) a_k - d_t] \cup ((h + 1) a_k - D_t) \subseteq B^{(t)}. \]

Case 3: $a \neq c'_t$ and $a \in (hA)^{(t)}$. By the induction hypothesis, we have
\[ (hA)^{(t)} = C_t \cup [c_t, h a_k - d_t] \cup (h a_k - D_t). \]
Since $C_t \cup [c_t, (h + 1) a_k - d_t] \subseteq B^{(t)}$, we can suppose that $a > (h + 1) a_k - d_t$. By $a \in (hA)^{(t)}$, there exist at least $t$ distinct nonnegative $k$-tuples $(x_{1,s}, x_{2,s}, \ldots, x_{k,s})$ ($1 \leq s \leq t$) such that
\[ a = x_{1,s} a_1 + x_{2,s} a_2 + \cdots + x_{k,s} a_k \]
and
\[ x_{1,s} + x_{2,s} + \cdots + x_{k,s} \leq h. \]
As in the proof of Lemma 4, assume that \(0 \leq x_{i,s} \leq ta_{i+1} - 1\) for \(i = 1, 2, \ldots, k - 1\). If \(x_{k,s} \leq 0\), then by (6) we have

\[
a \leq x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k-1,s}a_{k-1}
= c'_t \leq h_t a_k - d_t + 1
\leq (h_t + 1) a_k - d_t \leq (h + 1) a_k - d_t,
\]

which contradicts with \(a > (h + 1) a_k - d_t\). Therefore \(x_{k,s} \geq 1\) and \(a - a_k \in (hA)^{(t)}\). Since \(a > (h + 1) a_k - d_t\), it follows that \(a - a_k \in ha_k - D_t\). Hence

\[
a \in (h + 1) a_k - D_t \subseteq B^{(t)},
\]

and so \(((h + 1) A)^{(t)} \subseteq B^{(t)}\).

This completes the proof of Theorem 1. \(\square\)

Proof of Remark 2. Let \(n \geq 3\) be an integer and \(A = [0, n, n+1]\). By Theorem 1, there exist integers \(c_t, d_t\) and sets \(C_t \subseteq [0, c_t - 2], D_t \subseteq [0, d_t - 2]\) such that

\[
(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - D_t)
\]

for all \(h \geq h_t = t(n + 1) - 2\).

For any integer \(m \geq c_t\), choose an integer \(h' \geq t(n + 1) - 2\) such that \(h' a_k - d_t \geq m\), then we have \(m \in (h' A)^{(t)}\).

Hence, there exist \(t\) nonnegative integer tuples \((u_i, v_i) (1 \leq i \leq t)\) such that \(m = u_i n + v_i (n + 1)\).

On the other hand, there does not exist \(t\) nonnegative integer tuples \((u_i, v_i) (1 \leq i \leq t)\) such that \(c_t - 1 = u_i n + v_i (n + 1)\). Otherwise, if exist, choose \(h > \max_{1 \leq i \leq t | u_i + v_i|}\), then we have \(c_t - 1 \in (hA)^{(t)}\), a contradiction. Hence, by Lemma 6, it follows that \(c_t - 1 = ta_1 a_2 - a_1 - a_2 = tn(n + 1) - n - (n + 1)\), and then \(c_t = tn(n + 1) - 2n\).

Let \(p \in ((h_t - 1) A)^{(t)}\). Then there exist \(t\) nonnegative integer tuples \((u_i, v_i) (1 \leq i \leq t)\) such that \(p = u_i n + v_i (n + 1)\) and \(u_1 > u_2 > \cdots > u_t\) are the maximal \(t\) numbers in all the representations. Hence

\[
p = u_1 n + v_1 (n + 1) = [u_1 - (n + 1)] n + (v_1 + n)(n + 1)
= [u_1 - 2(n + 1)] n + (v_1 + 2n)(n + 1)
= \cdots
= [u_1 - (t - 1)(n + 1)] n + [v_1 + (t - 1)n](n + 1).
\]

It follows that \(u_t = u_1 - (t - 1)(n + 1), v_t = v_1 + (t - 1)n\). Noting that

\[
u_t + v_t < u_{t-1} + v_{t-1} < \cdots < u_1 + v_1 \leq h_t - 1,
\]

we have

\[
u_t + v_t = u_1 - (t - 1)(n + 1) + v_1 + (t - 1)n
= u_1 + v_1 - (t - 1) \leq h_t - 1 - (t - 1) = tn - 2.
\]

Hence, for every \(p \in ((h_t - 1) A)^{(t)}\),

\[
p = u_t n + v_t (n + 1) \leq (u_t + v_t)(n + 1) \leq (tn - 2)(n + 1)
= tn(n + 1) - 2(n + 1) < tn(n + 1) - 2n = c_t.
\]

By (1), it follows that

\[
((h_t - 1) A)^{(t)} \subseteq [0, tn(n + 1) - 2(n + 1)].
\]

Therefore, (1) cannot hold for \(h = h_t - 1\), and so Theorem 1 is optimal. \(\square\)

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References