



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Piotr Graczyk and Patrice Sawyer

Sharp Estimates of Radial Dunkl and Heat Kernels in the Complex Case
 A_n

Volume 359, issue 4 (2021), p. 427-437

Published online: 27 May 2021

<https://doi.org/10.5802/crmath.188>



This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569



Harmonic analysis, Representation theory / *Analyse harmonique, Théorie des représentations*

Sharp Estimates of Radial Dunkl and Heat Kernels in the Complex Case A_n

Piotr Graczyk^a and Patrice Sawyer^{*, b}

^a LAREMA, UFR Sciences, Université d'Angers, 2 bd Lavoisier, 49045 Angers cedex 01, France

^b Department of Mathematics and Computer Science, Laurentian University, Sudbury, ON Canada P3E 2C6

E-mails: graczyk@univ-angers.fr (P. Graczyk), psawyer@laurentian.ca (P. Sawyer)

Abstract. In this article, we consider the radial Dunkl geometric case $k = 1$ corresponding to flat Riemannian symmetric spaces in the complex case and we prove exact estimates for the positive valued Dunkl kernel and for the radial heat kernel.

Résumé. Dans cet article, nous considérons le cas géométrique radial de Dunkl $k = 1$ correspondant aux espaces symétriques riemanniens plats dans le cas complexe et nous prouvons des estimations exactes pour le noyau de Dunkl à valeur positive et pour le noyau de chaleur radial.

2020 Mathematics Subject Classification. 33C67, 43A90, 53C35.

Funding. The authors thank Laurentian University, Sudbury and the DéfiMaths programme of the Région des Pays de la Loire for their financial support.

Manuscript received 22nd December 2020, revised 10th February 2021, accepted 11th February 2021.

1. Introduction and notation

Finding good estimates of Dunkl heat kernels is a challenging and important subject, developed recently in [1]. Dunkl analysis has been developed intensely in recent years (refer to [7]). Optimal Dunkl heat kernel estimates have important consequences in harmonic analysis (e.g. Hardy spaces, maximal inequalities, see for instance [1]), potential theory (e.g. Green function estimates, Martin boundary description, see [2]) and spectral theory (see [3]). Recall that the Dunkl heat kernel is the probability transition function of the Dunkl stochastic process (refer to [5] and [7]). Consequently, Dunkl heat kernel estimates have also stochastic applications for the Dunkl process. Establishing estimates of the heat kernels is equivalent to estimating the Dunkl kernel as demonstrated by equation (3) below.

* Corresponding author.

In this paper we prove exact estimates in the W -radial Dunkl geometric case of multiplicity $k = 1$, corresponding to Cartan motion groups and flat Riemannian symmetric spaces with the ambient complex group G , the Weyl group W and the root system A_n .

We study for the first time the non-centered heat kernel, denoted $p_t^W(X, Y)$, on Riemannian symmetric spaces and we provide its sharp estimates. Exact estimates were obtained in [2] in the centered case $Y = 0$ for all Riemannian symmetric spaces.

We provide exact estimates for the spherical functions $\psi_\lambda(X)$ in the two variables X, λ when λ is real and, consequently, for the heat kernel $p_t^W(X, Y)$ in the three variables t, X, Y .

We recall here some basic terminology and facts about symmetric spaces associated to Cartan motion groups.

Let G be a semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of G . We recall the definition of the Cartan motion group and the flat symmetric space associated with the semisimple Lie group G with maximal compact subgroup K . The Cartan motion group is the semi-direct product $G_0 = K \ltimes \mathfrak{p}$ where the multiplication is defined by $(k_1, X_1) \cdot (k_2, X_2) = (k_1 k_2, \text{Ad}(k_1)(X_2) + X_1)$. The associated flat symmetric space is then $M = \mathfrak{p} \simeq G_0/K$ (the action of G_0 on \mathfrak{p} is given by $(k, X) \cdot Y = \text{Ad}(k)(Y) + X$).

The spherical functions for the symmetric space M are then given by

$$\psi_\lambda(X) = \int_K e^{\lambda(\text{Ad}(k)(X))} dk$$

where λ is a complex linear functional on $\mathfrak{a} \subset \mathfrak{p}$, a Cartan subalgebra of the Lie algebra of G . To extend λ to $X \in \text{Ad}(K)\mathfrak{a} = \mathfrak{p}$, one uses $\lambda(X) = \lambda(\pi_{\mathfrak{a}}(X))$ where $\pi_{\mathfrak{a}}$ is the orthogonal projection with respect to the Killing form (denoted throughout this paper by $\langle \cdot, \cdot \rangle$). Note that in [9–11], λ is replaced by $i\lambda$.

Throughout this paper, we usually assume that G is a semisimple complex Lie group. The complex root systems are respectively A_n for $n \geq 1$ (where \mathfrak{p} consists of the $n \times n$ hermitian matrices with trace 0), B_n for $n \geq 2$ (where $\mathfrak{p} = i\mathfrak{so}(2n+1)$), C_n for $n \geq 3$ (where $\mathfrak{p} = i\mathfrak{sp}(n)$) and D_n for $n \geq 4$ (where $\mathfrak{p} = i\mathfrak{so}(2n)$) for the classical cases and the exceptional root systems E_6, E_7, E_8, F_4 and G_2 .

The radial heat kernel is considered with respect to the invariant measure $\mu(dY) = \pi^2(Y) dY$ on M , where $\pi(Y) = \prod_{\alpha > 0} \alpha(Y)$.

Note also that in the curved case $M_0 = G/K$, the spherical functions for the symmetric space M_0 are then given by

$$\phi_\lambda(e^X) = \int_K e^{(\lambda - \rho)H(e^X k)} dk$$

where ρ is the half-sum of the roots counted with their multiplicities and $H(g)$ is the abelian component in the Iwasawa decomposition of g : $g = k e^{H(g)} n$.

The authors are thankful for the helpful suggestions by the anonymous referee.

2. Estimates of spherical functions and of the heat kernel

We will be developing a sharp estimate for the spherical function $\psi_\lambda(X)$. We introduce the following useful convention. We will write

$$f(t, X, \lambda) \asymp g(t, X, \lambda)$$

in a given domain of f and g if there exist constants $C_1 > 0$ and $C_2 > 0$ independent of t, X and λ such that $C_1 f(t, X, \lambda) \leq g(t, X, \lambda) \leq C_2 f(t, X, \lambda)$ in the domain of consideration.

We conjecture the following global estimate for the spherical function in the complex case.

Conjecture 1. *On flat Riemannian symmetric spaces with complex group G we have,*

$$\psi_\lambda(X) \asymp \frac{e^{\langle \lambda, X \rangle}}{\prod_{\alpha > 0} (1 + \alpha(\lambda)\alpha(X))}, \quad \lambda \in \bar{\mathfrak{a}}^+, X \in \bar{\mathfrak{a}}^+.$$

Remark 2. Recall that, denoting $\delta(X) = \prod_{\alpha > 0} \sinh^2 \alpha(X)$, we have

$$\phi_\lambda(e^X) = \frac{\pi(X)}{\delta^{1/2}(X)} \psi_\lambda(X). \tag{1}$$

Since

$$\delta^{1/2}(X) \asymp \frac{e^{\rho(X)} \pi(X)}{\prod_{\alpha > 0} (1 + \alpha(X))}$$

in the complex case, Conjecture 1 therefore becomes

$$\phi_\lambda(e^X) \asymp e^{(\lambda-\rho)(X)} \prod_{\alpha > 0} \frac{1 + \alpha(X)}{1 + \alpha(\lambda)\alpha(X)} \tag{2}$$

in the curved complex case.

Let us compare the estimate (2) we conjecture for ϕ_λ with the one obtained in [12, Theorem 3.4], cf. also [17, Remark 3.1]. The estimates in [12] apply in all the generality of hypergeometric functions of Heckman and Opdam. The authors show that there exist constants $C_1(\lambda) > 0$, $C_2(\lambda) > 0$ such that

$$C_1(\lambda) e^{(\lambda-\rho)(X)} \prod_{\substack{\alpha > 0, \\ \alpha(\lambda) = 0}} (1 + \alpha(X)) \leq \phi_\lambda(e^X) \leq C_2(\lambda) e^{(\lambda-\rho)(X)} \prod_{\substack{\alpha > 0, \\ \alpha(\lambda) = 0}} (1 + \alpha(X)).$$

Given (1), corresponding estimates clearly also hold in the flat case for $\psi_\lambda(X)$. The interest of our result, in the case A_n , lies in the fact that our estimate is universal in both λ and X .

The results of [12, 17] and our estimates in the A_n case strongly suggest that the Conjecture 1 is true for any complex root system.

Note that asymptotics of $\psi_\lambda(tX)$ when λ and X are singular and $t \rightarrow \infty$ were proven in [8] for all classical complex root systems and the systems F_4 and G_2 .

Consider the relationship between the Dunkl kernel $E_k(X, Y)$ and the Dunkl heat kernel $p_t(X, Y)$, as given in [13, Lemma 4.5]

$$p_t(X, Y) = \frac{1}{2^{\gamma+d/2} c_k} t^{-\frac{d}{2}-\gamma} e^{-\frac{|X|^2-|Y|^2}{4t}} E_k\left(X, \frac{Y}{2t}\right), \tag{3}$$

where γ is the number of positive roots and the constant c_k is the Macdonald–Mehta–Selberg integral. The formula (3) remains true for the W -invariant kernels p_t^W and E^W . In the geometric cases $k = \frac{1}{2}, 1$ and 2 , by [4], the W -invariant formula (3) translates in a similar relationship between the spherical function ψ_λ and the heat kernel $p_t^W(X, Y)$:

$$p_t^W(X, Y) = \frac{1}{2^{\gamma+d/2} c_k} t^{-\frac{d}{2}-\gamma} e^{-\frac{|X|^2-|Y|^2}{4t}} \psi_X\left(\frac{Y}{2t}\right). \tag{4}$$

A simple direct proof of (4) for $k = 1$ is given in [8, Remark 2.9].

Equation (4) and Conjecture 1 bring us to an equivalent conjecture for the heat kernel $p_t^W(X, Y)$.

Conjecture 3. *We have*

$$p_t^W(X, Y) \asymp t^{-\frac{d}{2}} \frac{e^{-\frac{|X-Y|^2}{4t}}}{\prod_{\alpha > 0} (t + \alpha(X)\alpha(Y))}.$$

Consider also the relationship between the heat kernel $p_t^W(X, Y)$ and the heat kernel $\tilde{p}_t^W(X, Y)$ in the curved case. We have

$$\tilde{p}_t^W(X, Y) = e^{-|\rho|^2 t} \frac{\pi(X)\pi(Y)}{\delta^{1/2}(X)\delta^{1/2}(Y)} p_t^W(X, Y). \tag{5}$$

This relation follows directly from the fact that the respective radial Laplacians and radial measures are $\pi^{-1}L_\alpha \circ \pi$ and $\pi(X) dX$ in the flat case and $\delta^{-1/2}(L_\alpha - |\rho|^2) \circ \delta^{1/2}$ and $\delta(X) dX$ in the curved case (L_α stands for the Euclidean Laplacian on \mathfrak{a}).

In the curved complex case, Conjecture 3 becomes

$$\tilde{p}_t^W(X, Y) \asymp e^{-|\rho|^2 t} t^{-\frac{d}{2}} e^{-\rho(X+Y)} \prod_{\alpha>0} \frac{(1 + \alpha(X))(1 + \alpha(Y))}{(t + \alpha(X)\alpha(Y))} e^{-\frac{|X-Y|^2}{4t}}.$$

Remark 4. In [6], sharp estimates of W -invariant Poisson and Newton kernels in the complex Dunkl case were obtained, by exploiting the method of construction of these W -invariant kernels by alternating sums. When a root system Σ acts in \mathbf{R}^d , the sharp estimates of [6] have the common form

$$\mathcal{K}^W(X, Y) \asymp \frac{\mathcal{K}^{\mathbf{R}^d}(X, Y)}{\prod_{\alpha>0} (|X - Y|^2 + \alpha(X)\alpha(Y))}, \quad X, Y \in \bar{\mathfrak{a}}^+, \tag{6}$$

where $\mathcal{K}^W(X, Y)$ is the W -invariant kernel in Dunkl setting and $\mathcal{K}^{\mathbf{R}^d}(X, Y)$ is the classical kernel on \mathbf{R}^d . Let us observe a common pattern in the appearance of the classical kernels $\mathcal{K}^{\mathbf{R}^d}$ and of products of roots $\alpha(X)\alpha(Y)$ in formulas (6) and of the Fourier kernel $e^{\langle \lambda, X \rangle}$ and the classical Gaussian heat kernel and of products $\alpha(\lambda)\alpha(X)$ in the estimates given in Conjecture 1 and Conjecture 3.

2.1. Proof of Conjecture 1 in some cases

We start with a practical result.

Proposition 5. *Let $\alpha_1, \dots, \alpha_r$ be the simple roots and let A_{α_i} be such that $\langle X, A_{\alpha_i} \rangle = \alpha_i(X)$ for $X \in \mathfrak{a}$. Suppose $X \in \mathfrak{a}^+$ and $w \in W \setminus \{id\}$. Then we have*

$$Y - wY = \sum_{i=1}^r 2 \frac{a_i^w(Y)}{|\alpha_i|^2} A_{\alpha_i} \tag{7}$$

where a_i^w is a linear combination of simple roots with non-negative integer coefficients for each i .

Proof. Refer to [6, Proposition 3.5]. □

Remark 6. Note that $a_i^w(Y)/|\alpha_i|^2$ is bounded by $C \max_k |\alpha_k(Y)|$ where C is a constant depending only on $w \in W$ and, ultimately, on W .

Corollary 7. *Let $Y \in \bar{\mathfrak{a}}^+$ and $w \in W$. Consider the decomposition (7) of $Y - wY$. If $a_k^w(Y) \neq 0$ then α_k appears in a_k^w , i.e. $a_k^w = \sum_{i=1}^r n_i \alpha_i$ with $n_k > 0$.*

Proof. Refer to [6, Corollary 3.10]. □

As a prelude to the next Proposition, we give here a description of the Abel transform $\mathcal{A}(f)$ of a function f on \mathfrak{a} in the case of a flat symmetric space (in the case of the curved symmetric space refer for example to [15]). The dual of the Abel transform $\mathcal{A}^*(f)$ can be defined as follows

$$\mathcal{A}^*(f)(X) = \int_K f(\pi_\alpha(\text{Ad}(k)(X))) dk$$

where, as before, π_α is the orthogonal projection with respect to the Killing form. We can then define (in analogy with the curved case)

$$\int_\alpha f(X) \mathcal{A}(g)(X) dX = \int_\alpha \mathcal{A}^*(f)(X) g(X) w(X) dX$$

where $w(X) = \prod_{\alpha>0} |\alpha(X)|^{m_\alpha}$ and m_α is the multiplicity of α . From [9, Chap IV, Theorem 10.2], we can deduce that the support of the dual Abel transform is $C(X)$, the convex hull of $W \cdot X$. The set $C(X)$ has non-empty interior as long as not all roots are 0 on X . From there, we conclude that we can write

$$\mathcal{A}^*(f)(X) = \int_{C(X)} f(Y) K(X, Y) dY$$

and that $K(X, Y) dY$ is a probability measure supported by $C(X)$.

Proposition 8. *Let $\delta > 0$. Suppose $\alpha_i(\lambda) \alpha_j(X) \leq \delta$ for all i, j . Then $\psi_\lambda(X) \asymp e^{\lambda(X)}$ (the constants involved only depend on δ).*

Proof. Notice that

$$e^{w_{min} \lambda(X)} \leq \psi_\lambda(X) = \int_{C(X)} e^{\lambda(Y)} K(X, Y) dY \leq e^{\lambda(X)} \tag{8}$$

where w_{min} is the element of the Weyl group giving the minimum value of $w \lambda(X)$. Now, using Proposition 5 and Remark 6 with $Y = \lambda$, we see that for any $w \in W$

$$\begin{aligned} e^{\lambda(X)} &\geq e^{w \lambda(X)} = e^{\langle w\lambda - \lambda, X \rangle} e^{\langle \lambda, X \rangle} = \prod_{i=1}^r e^{-2 \frac{a_i^w(\lambda)}{|\alpha_i|^2} \alpha_i(X)} e^{\langle \lambda, X \rangle} \\ &\geq \prod_{i=1}^r e^{-2C(\max_k \alpha_k(\lambda)) \alpha_i(X)} e^{\langle \lambda, X \rangle} \geq \prod_{i=1}^r e^{-2C\delta} e^{\langle \lambda, X \rangle}. \end{aligned} \quad \square$$

Remark 9. This case and this method apply for any radial Dunkl case; it suffices to replace $K(X, Y) dY$ by the so-called Rösler measure $\mu_X(dY)$ in the integral in (8), see [14].

Proposition 10. *A spherical function $\psi_\lambda(X)$ on M is given by the formula*

$$\psi_\lambda(X) = \frac{\pi(\rho)}{2\gamma\pi(\lambda)\pi(X)} \sum_{w \in W} \epsilon(w) e^{\langle w\lambda, X \rangle}, \tag{9}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha = \sum_{\alpha \in \Sigma^+} \alpha$ and $\gamma = |\Sigma^+|$ is the number of positive roots (refer to [9, Chap. IV, Proposition 4.8 and Chap. II, Theorem 5.35]).

Proposition 11. *Suppose $\alpha(\lambda) \alpha(X) \geq (\log |W|)/2$ for all $\alpha > 0$. Then*

$$\psi_\lambda(X) \asymp \frac{e^{\lambda(X)}}{\pi(\lambda)\pi(X)}.$$

We are assuming here that $|\alpha_i| \geq 1$ for each i .

Proof. Suppose $w \in W$ is not the identity. In that case, $a_i^w(\lambda)$ is not equal to 0 for some i . By Proposition 5 with $y = \lambda$ and Corollary 7, $\lambda(X) - w \lambda(X) \geq 2 a_i^w(\lambda) \alpha_i(X) / |\alpha_i|^2 \geq 2 \alpha_i(\lambda) \alpha_i(X) \geq \log |W|$. Each term $e^{\langle w\lambda, X \rangle}$ in the alternating sum (9) corresponding to $w \neq \text{id}$ is bounded by $e^{-\log |W|} e^{\lambda(X)} = e^{\lambda(X)} / |W|$. Hence, since only half the terms in the sum are negative,

$$|W| e^{\lambda(X)} \geq \sum_{w \in W} \epsilon(w) e^{\langle w\lambda, X \rangle} \geq e^{\lambda(X)} - \frac{|W|}{2} e^{\lambda(X)} / |W| = \frac{1}{2} e^{\lambda(X)}. \quad \square$$

3. The conjecture in the case of the root system A_n

We will prove the conjecture in the case of the root system of type A .

Theorem 12. *In the case of the root system of type A_n in the complex case, we have*

$$\psi_\lambda(e^X) = \frac{e^{(\lambda, X)}}{\prod_{i < j} (1 + (\lambda_i - \lambda_j)(x_i - x_j))}, \quad \lambda, X \in \bar{\mathfrak{a}}^+ \tag{10}$$

where $X = [x_1, \dots, x_{n+1}]$ with $x_i \geq x_{i+1}$ and $\lambda = [\lambda_1, \dots, \lambda_{n+1}]$ with $\lambda_i \geq \lambda_{i+1}$.

Corollary 13.

$$\begin{aligned} \phi_\lambda(e^X) &= e^{(\lambda - \rho)(X)} \prod_{i < j} \frac{1 + x_i - x_j}{1 + (x_i - x_j)(\lambda_i - \lambda_j)}, \\ p_t^W(X, Y) &= t^{-\frac{d}{2}} \frac{e^{-\frac{|X-Y|^2}{4t}}}{\prod_{i < j} (t + (x_i - x_j)(y_i - y_j))}, \\ \tilde{p}_t^W(X, Y) &= e^{-|\rho|^2 t} t^{-\frac{d}{2}} e^{-\rho(X+Y)} \prod_{i < j} \frac{(1 + x_i - x_j)(1 + y_i - y_j)}{(t + (x_i - x_j)(y_i - y_j))} e^{-\frac{|X-Y|^2}{4t}}. \end{aligned}$$

We recall (refer to [16, Theorem 4.1]) the following iterative formula for the spherical functions of type A in the complex case. Here the Cartan subalgebra \mathfrak{a} for the root system A_{n-1} is isomorphic to \mathbf{R}^n . For $\lambda, X \in \bar{\mathfrak{a}}^+ \subset \mathbf{R}^n$, we have

$$\begin{aligned} \psi_\lambda(e^X) &= e^{\lambda(X)} \text{ if } n = 1 \\ \text{and } \psi_\lambda(e^X) &= (n-1)! e^{\lambda_n \sum_{k=1}^n x_k} \left(\prod_{i < j} (x_i - x_j) \right)^{-1} \int_{x_n}^{x_{n-1}} \cdots \int_{x_2}^{x_1} \psi_{\lambda_0}(e^Y) \\ &\quad \prod_{i < j < n} (y_i - y_j) dy_1 \cdots dy_{n-1} \end{aligned} \tag{11}$$

where $\lambda_0(U) = \sum_{k=1}^{n-1} (\lambda_k - \lambda_n) u_k$.

Remark 14. Formula (11) represents the action of the root system A_{n-1} on \mathbf{R}^n . If we assume

$$\sum_{k=1}^n x_k = 0 = \sum_{k=1}^n \lambda_k,$$

we have then the action of the root system A_{n-1} on \mathbf{R}^{n-1} . We can also consider the action of A_{n-1} on any \mathbf{R}^m with $m \geq n-1$ by considering formula (9) and deciding on which entries x_k , the Weyl group $W = S_n$ acts. These considerations do not affect the conclusion of Theorem 12.

3.1. Approximate factorization for A_n

Before proving the conjecture in the case A_n , we will prove an interesting “factorization”.

Proposition 15. *For $n \geq 1$, consider the root system A_n on \mathbf{R}^{n+1} . Let $\lambda, X \in \bar{\mathfrak{a}}^+ \subset \mathbf{R}^{n+1}$, $X' = [x_1, \dots, x_n]$ so that $X = [X', x_{n+1}]$. Define*

$$\begin{aligned} I^{(n)} = I^{(n)}(\lambda; X) &= \int_{x_{n+1}}^{x_n} \int_{x_n}^{x_{n-1}} \cdots \int_{x_3}^{x_2} \int_{x_2}^{x_1} e^{-\lambda_0(X'-Y)} \\ &\quad \prod_{i < j < n} \frac{(y_i - y_j)(\lambda_i - \lambda_j)}{1 + (y_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \cdots dy_n. \end{aligned}$$

Then the following approximate factorization holds

$$I^{(n)} \simeq \prod_{k=1}^n I_k^{(n)} \tag{12}$$

where

$$I_1^{(n)} = \int_{x_2}^{x_1} e^{-(\lambda_1 - \lambda_{n+1})(x_1 - y_1)} dy_1$$

and

$$I_k^{(n)} = \int_{x_{k+1}}^{x_k} e^{-(\lambda_k - \lambda_{n+1})(x_k - y_k)} \prod_{j=1}^{k-1} \frac{(x_j - y_k)(\lambda_j - \lambda_k)}{1 + (x_j - y_k)(\lambda_j - \lambda_k)} dy_k \text{ for } 1 < k \leq n.$$

Proof. Since $u/(1 + u)$ is an increasing function, we clearly have

$$I^{(n)} \leq \int_{x_{n+1}}^{x_n} \int_{x_n}^{x_{n-1}} \dots \int_{x_3}^{x_2} \int_{x_2}^{x_1} e^{-\lambda_0(X' - Y)} \prod_{i < j < n} \frac{(x_i - y_j)(\lambda_i - \lambda_j)}{1 + (x_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_n.$$

On the other hand,

$$\begin{aligned} I^{(n)} &\geq \int_{(x_n + x_{n+1})/2}^{x_n} \int_{(x_{n-1} + x_n)/2}^{x_{n-1}} \dots \int_{(x_2 + x_3)/2}^{x_2} \int_{(x_1 + x_2)/2}^{x_1} e^{-\lambda_0(X' - Y)} \\ &\quad \prod_{i < j < n} \frac{(y_i - y_j)(\lambda_i - \lambda_j)}{1 + (y_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_n \\ &\geq \int_{(x_n + x_{n+1})/2}^{x_n} \int_{(x_{n-1} + x_n)/2}^{x_{n-1}} \dots \int_{(x_2 + x_3)/2}^{x_2} \int_{(x_1 + x_2)/2}^{x_1} e^{-\lambda_0(X' - Y)} \\ &\quad \prod_{i < j < n} \frac{((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)}{1 + ((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_n \\ &\simeq \int_{(x_n + x_{n+1})/2}^{x_n} \int_{(x_{n-1} + x_n)/2}^{x_{n-1}} \dots \int_{(x_2 + x_3)/2}^{x_2} \int_{(x_1 + x_2)/2}^{x_1} e^{-\lambda_0(X' - Y)} \\ &\quad \prod_{i < j < n} \frac{(x_i - y_j)(\lambda_i - \lambda_j)}{1 + (x_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_n = \\ &= \prod_{k=1}^n \int_{(x_k + x_{k+1})/2}^{x_k} e^{-(\lambda_k - \lambda_{n+1})(x_k - y_k)} \prod_{j=1}^{k-1} \frac{(x_j - y_k)(\lambda_j - \lambda_k)}{1 + (x_j - y_k)(\lambda_j - \lambda_k)} dy_k = \prod_{k=1}^n A_k^{(n)} \end{aligned}$$

since

$$\frac{((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)}{1 + ((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)} \leq \frac{(x_i - y_j)(\lambda_i - \lambda_j)}{1 + (x_i - y_j)(\lambda_i - \lambda_j)}$$

while

$$\frac{((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)}{1 + ((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)} \geq \frac{(x_i - y_j)/2 (\lambda_i - \lambda_j)}{1 + (x_i - y_j)/2 (\lambda_i - \lambda_j)} \geq \frac{1}{2} \frac{(x_i - y_j)(\lambda_i - \lambda_j)}{1 + (x_i - y_j)(\lambda_i - \lambda_j)}.$$

Now, let

$$B_k^{(n)} = \int_{x_{k+1}}^{(x_k + x_{k+1})/2} e^{-(\lambda_k - \lambda_{n+1})(x_k - y_k)} \prod_{j=1}^{k-1} \frac{(x_j - y_k)(\lambda_j - \lambda_k)}{1 + (x_j - y_k)(\lambda_j - \lambda_k)} dy_k$$

and note that $I_k^{(n)} = A_k^{(n)} + B_k^{(n)}$.

Now, using the change of variable $2w = x_k - y_k$, we have

$$\begin{aligned} B_k^{(n)} &= 2 \int_{(x_k - x_{k+1})/4}^{(x_k - x_{k+1})/2} e^{-2(\lambda_k - \lambda_{n+1})w} \prod_{j=1}^{k-1} \frac{(x_j - x_k + 2w)(\lambda_j - \lambda_k)}{1 + (x_j - x_k + 2w)(\lambda_j - \lambda_k)} dw \\ &\leq 4 \int_{(x_k - x_{k+1})/4}^{(x_k - x_{k+1})/2} e^{-2(\lambda_k - \lambda_{n+1})w} \prod_{j=1}^{k-1} \frac{(x_j - x_k + w)(\lambda_j - \lambda_k)}{1 + (x_j - x_k + w)(\lambda_j - \lambda_k)} dw \\ &\leq 4 \int_0^{(x_k - x_{k+1})/2} e^{-(\lambda_k - \lambda_{n+1})w} \prod_{j=1}^{k-1} \frac{(x_j - x_k + w)(\lambda_j - \lambda_k)}{1 + (x_j - x_k + w)(\lambda_j - \lambda_k)} dw = 4A_k^{(n)}, \end{aligned}$$

where the last equality comes from the change of variable $w = x_k - y_k$ in the expression for $A_k^{(n)}$. Therefore $I_k^{(n)} = A_k^{(n)} + B_k^{(n)} \leq 5A_k^{(n)}$. The result follows. \square

The next Proposition 16 gives an inductive way of estimating $I^{(n+1)}$, knowing $I^{(n)}$ and $I^{(n-1)}$.

Proposition 16. Consider the root system A_{n+1} on \mathbf{R}^{n+2} . Let $\lambda, X \in \bar{\mathfrak{a}}^+ \subset \mathbf{R}^{n+2}$. Assume $\alpha_1(X) \geq \alpha_{n+1}(X)$. Then

$$\begin{aligned} I^{(n+1)}(\lambda; X) &= I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; x_1, \dots, x_{n+1}) \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} \\ &\quad \frac{I^{(n)}(\lambda_2, \dots, \lambda_{n+1}, \lambda_{n+2}; x_2, \dots, x_{n+2})}{I^{(n-1)}(\lambda_2, \dots, \lambda_n, \lambda_{n+2}; x_2, \dots, x_{n+1})}. \end{aligned}$$

Proof. We start with an outline of the proof.

- (i) $I^{(n+1)}$ is estimated by a product of $n + 1$ factors $I_k^{(n+1)}(\lambda; X)$.
- (ii) The product of the first n factors $I_1^{(n+1)}(\lambda; X), \dots, I_n^{(n+1)}(\lambda; X)$ gives an estimate of the term $I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; X')$ by Proposition 15.
- (iii) In the last factor $I_{n+1}^{(n+1)}(\lambda; X)$, we “draw off” one term from under the integral, using the additional hypothesis $\alpha_1(X) \geq \alpha_{n+1}(X)$. The remaining integral corresponds to $I_n^{(n)}(\lambda_2, \dots, \lambda_{n+2}; x_2, \dots, x_{n+2})$.
- (iv) The last factor $I_n^{(n)}$ of $I^{(n)}$ is estimated by $I^{(n)}/I^{(n-1)}$, up to a change of variables (we re-use the idea of (ii)).

Since $x_{n+2} \leq y_{n+1} \leq x_{n+1}$ and $x_{n+1} - x_{n+2} \leq x_1 - x_2$, we get $x_1 - x_{n+1} \leq x_1 - y_{n+1} \leq x_1 - x_{n+2} \leq 2(x_1 - x_{n+1})$ and we have

$$\begin{aligned} I_{n+1}^{(n+1)} &= \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1} - \lambda_{n+2})(x_{n+1} - y_{n+1})} \frac{(x_1 - y_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - y_{n+1})(\lambda_1 - \lambda_{n+1})} \\ &\quad \prod_{j=2}^n \frac{(x_j - y_{n+1})(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})(\lambda_j - \lambda_{n+1})} (\lambda_j - \lambda_{n+1}) dy_{n+1} \\ &\asymp \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1} - \lambda_{n+2})(x_{n+1} - y_{n+1})} \\ &\quad \prod_{j=2}^n \frac{(x_j - y_{n+1})(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})(\lambda_j - \lambda_{n+1})} dy_{n+1}. \end{aligned}$$

Hence, noting that $I_1^{(n+1)}(\lambda; X) \cdots I_n^{(n+1)}(\lambda; X) = I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; X')$, we have

$$\begin{aligned} I^{(n+1)}(\lambda; X) &= I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; X') \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} \\ &\quad \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1} - \lambda_{n+2})(x_{n+1} - y_{n+1})} \prod_{j=2}^n \frac{(x_j - y_{n+1})(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})(\lambda_j - \lambda_{n+1})} dy_{n+1}. \end{aligned}$$

Finally,

$$\begin{aligned} & \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1}-\lambda_{n+2})(x_{n+1}-y_{n+1})} \prod_{j=2}^n \frac{(x_j - y_{n+1})(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})(\lambda_j - \lambda_{n+1})} dy_{n+1} \\ &= \frac{\prod_{k=1}^n \int_{x_{k+2}}^{x_{k+1}} e^{-(\lambda_{k+1}-\lambda_{n+2})(x_{k+1}-y_{k+1})} \prod_{j=1}^{k-1} \frac{(x_{j+1} - y_{k+1})(\lambda_{j+1} - \lambda_{k+1})}{1 + (x_{j+1} - y_{k+1})(\lambda_{j+1} - \lambda_{k+1})} dy_{k+1}}{\prod_{k=1}^{n-1} \int_{x_{k+2}}^{x_{k+1}} e^{-(\lambda_{k+1}-\lambda_{n+2})(x_{k+1}-y_{k+1})} \prod_{j=1}^{k-1} \frac{(x_{j+1} - y_{k+1})(\lambda_{j+1} - \lambda_{k+1})}{1 + (x_{j+1} - y_{k+1})(\lambda_{j+1} - \lambda_{k+1})} dy_{k+1}} \\ &= \frac{I^{(n)}(\lambda_2, \dots, \lambda_{n+1}, \lambda_{n+2}; x_2, \dots, x_{n+2})}{I^{(n-1)}(\lambda_2, \dots, \lambda_n, \lambda_{n+2}; x_2, \dots, x_{n+1})}. \quad \square \end{aligned}$$

Remark 17. When $n = 1$, the result of Proposition 16 remains valid if we set $I^{(0)} = 1$.

We now prove our main result.

Proof of Theorem 12. We use induction on the rank. In the case of A_1 , we have

$$\begin{aligned} \psi_\lambda(e^X) &= e^{\lambda_2(x_1+x_2)} (x_1 - x_2)^{-1} \int_{x_2}^{x_1} e^{(\lambda_1-\lambda_2)y} dy \\ &= e^{\lambda_2(x_1+x_2)} (x_1 - x_2)^{-1} \frac{e^{(\lambda_1-\lambda_2)x_1} - e^{(\lambda_1-\lambda_2)x_2}}{\lambda_1 - \lambda_2} \\ &= e^{\lambda_1 x_1 + \lambda_2 x_2} \frac{1 - e^{-(\lambda_1-\lambda_2)(x_1-x_2)}}{(\lambda_1 - \lambda_2)(x_1 - x_2)} \simeq e^{\lambda_1 x_1 + \lambda_2 x_2} \frac{1}{1 + (\lambda_1 - \lambda_2)(x_1 - x_2)} \end{aligned}$$

since $1 - e^{-u} \simeq u/(1 + u)$ for $u \geq 0$.

Assume that the result is true for A_r , $1 \leq r \leq n$, $n \geq 1$. Using (11) and the induction hypothesis, we have for $r = 1, \dots, n + 1$ and λ, X in positive Weyl chamber in \mathbf{R}^{r+1}

$$\begin{aligned} & \pi(X) \pi(\lambda') e^{-\lambda(X)} \psi_\lambda([x_1, \dots, x_r, x_{r+1}]) \\ &= r! \pi(\lambda') e^{-\lambda(X)} e^{\lambda_{r+1} \sum_{k=1}^{r+1} x_k} \int_{x_{r+1}}^{x_r} \dots \int_{x_2}^{x_1} \psi_{\lambda_0}(e^Y) \prod_{i < j < r+1} (y_i - y_j) dy_1 \dots dy_r \\ &\simeq \int_{x_{r+1}}^{x_r} \int_{x_r}^{x_{r-1}} \dots \int_{x_3}^{x_2} \int_{x_2}^{x_1} e^{-\lambda_0(X'-Y)} \prod_{i < j < r+1} \frac{(y_i - y_j)(\lambda_i - \lambda_j)}{1 + (y_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_r \end{aligned}$$

where $X' = \text{diag}[x_1, \dots, x_r]$ and $\lambda' = [\lambda_1, \dots, \lambda_r]$. Using the notation introduced in Proposition 15, we have

$$\pi(X) \pi(\lambda') e^{-\lambda(X)} \psi_{[\lambda_1, \dots, \lambda_{r+1}]}([x_1, \dots, x_{r+1}]) = r! I^{(r)}(\lambda_1, \dots, \lambda_{r+1}, x_1, \dots, x_{r+1}).$$

Still using the induction hypothesis, we have

$$\begin{aligned} \pi(X) \pi(\lambda') e^{-\lambda(X)} \psi_{[\lambda_1, \dots, \lambda_{r+1}]}([x_1, \dots, x_{r+1}]) &= r! I^{(r)}(\lambda_1, \dots, \lambda_{r+1}; x_1, \dots, x_{r+1}) \\ &\simeq \frac{\pi(X) \pi(\lambda')}{\prod_{i < j \leq r+1} (1 + (\lambda_i - \lambda_j)(x_i - x_j))} \end{aligned} \tag{13}$$

for $r = 1, \dots, n$.

It remains to show that (13) holds for $r = n + 1$, i.e. that

$$I^{(n+1)}(\lambda_1, \dots, \lambda_{n+2}; x_1, \dots, x_{n+2}) \simeq \frac{\pi(X) \pi(\lambda')}{\prod_{i < j \leq n+2} (1 + (\lambda_i - \lambda_j)(x_i - x_j))}.$$

It is sufficient to prove the last formula under the hypothesis that $\alpha_1(X) \geq \alpha_{n+1}(X)$ since the case $\alpha_1(X) \leq \alpha_{n+1}(X)$ is symmetric. Now, according to Proposition 16 and (13),

$$\begin{aligned}
 I^{(n+1)}(\lambda; X) &= \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; x_1, \dots, x_{n+1}) \\
 &\quad I^{(n)}(\lambda_2, \dots, \lambda_{n+1}, \lambda_{n+2}; x_2, \dots, x_{n+2}) \left(I^{(n-1)}(\lambda_2, \dots, \lambda_n, \lambda_{n+2}; x_2, \dots, x_{n+1}) \right)^{-1} \\
 &= \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} \\
 &\quad \frac{\prod_{i < j \leq n+1} (x_i - x_j) \prod_{i < j < n+1} (\lambda_i - \lambda_j)}{\prod_{i < j \leq n} (1 + (x_i - x_j)(\lambda_i - \lambda_j)) \prod_{i=1}^n (1 + (x_i - x_{n+1})(\lambda_i - \lambda_{n+2}))} \\
 &\quad \frac{\prod_{1 < i < j \leq n+2} (x_i - x_j) \prod_{1 < i < j \leq n+1} (\lambda_i - \lambda_j)}{\prod_{1 < i < j \leq n+2} (1 + (x_i - x_j)(\lambda_i - \lambda_j))} \\
 &\quad \frac{\prod_{1 < i < j \leq n} (1 + (x_i - x_j)(\lambda_i - \lambda_j)) \prod_{i=2}^n (1 + (x_i - x_{n+1})(\lambda_i - \lambda_{n+2}))}{\prod_{1 < i < j \leq n+1} (x_i - x_j) \prod_{1 < i < j < n+1} (\lambda_i - \lambda_j)} \\
 &= \frac{x_1 - x_{n+1}}{x_1 - x_{n+2}} \frac{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_2)}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_2)} \frac{\prod_{i < j \leq n+2} (x_i - x_j) \prod_{i < j \leq n+1} (\lambda_i - \lambda_j)}{\prod_{i < j \leq n+2} (1 + (\lambda_i - \lambda_j)(x_i - x_j))}.
 \end{aligned}$$

The result follows since $x_1 - x_{n+1} \asymp x_1 - x_{n+2}$ given that $x_1 - x_2 \geq x_{n+1} - x_{n+2}$. □

4. Comparison with the estimates of Anker et al. in [1]. Conjecture for Dunkl setting

In [1, Theorem 4.1 p. 2372 and Theorem 4.4, p. 2377] the following estimates were proven for the heat kernel $p_t(X, Y)$ in the Dunkl setting on \mathbf{R}^n . There exist positive constants c_1, c_2, C_1 and C_2 such that for all $X, Y \in \mathfrak{a}^+$

$$\frac{C_1 e^{-c_1 |X-Y|^2/t}}{\min\{w(B(X, \sqrt{t})), w(B(Y, \sqrt{t}))\}} \leq p_t(X, Y) \leq \frac{C_2 e^{-c_2 |X-Y|^2/t}}{\max\{w(B(X, \sqrt{t})), w(B(Y, \sqrt{t}))\}} \tag{14}$$

where w is the W -invariant reference measure (in our paper $w = \pi(X)^2 dX$) and the w -volume of a ball satisfies the estimate ([1, p. 2365])

$$w(B(X, r)) \asymp r^n \prod_{\alpha > 0} (r + \alpha(X))^{2k(\alpha)}.$$

The same estimates follow for $p_t^W(X, Y)$. Our sharp estimates in Corollary 13 for $k(\alpha) = 1$ in the W -radial case A_n suggest that $c_1 = c_2 = 1/4$ in (14) and that products of terms $(t + \alpha(X)\alpha(Y))^{k(\alpha)}$ are natural in place of separate terms $w(B(X, \sqrt{t}))$ and $w(B(Y, \sqrt{t}))$. On the other hand, estimates (14) and in Corollary 13 suggest that the following conjecture is true in the Dunkl setting.

Conjecture 18. *The Weyl-invariant heat kernel for a root system Σ in \mathbf{R}^d satisfies the following estimates*

$$p_t^W(X, Y) \asymp t^{-\frac{d}{2}} \frac{e^{-\frac{|X-Y|^2}{4t}}}{\prod_{\alpha > 0} (t + \alpha(X)\alpha(Y))^{k(\alpha)}}. \tag{15}$$

Formula (3) then implies that the W -invariant Dunkl kernel satisfies the estimate

$$E_k^W(X, Y) \asymp \frac{e^{\lambda(X)}}{\prod_{\alpha>0} (1 + \alpha(X) \alpha(\lambda))^{k(\alpha)}}.$$

References

- [1] J.-P. Anker, J. Dziubański, A. Hejna, “Harmonic Functions, Conjugate Harmonic Functions and the Hardy Space H^1 in the Rational Dunkl Setting”, *J. Fourier Anal. Appl.* **25** (2019), no. 5, p. 2356-2418.
- [2] J.-P. Anker, L. Ji, “Heat kernel and Green function estimates on noncompact symmetric spaces”, *Geom. Funct. Anal.* **9** (1999), no. 6, p. 1035-1091.
- [3] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, 1989.
- [4] M. De Jeu, “Paley–Wiener theorems for the Dunkl transform”, *Trans. Am. Math. Soc.* **358** (2006), no. 10, p. 4225-4250.
- [5] L. Gallardo, M. Yor, “A chaotic representation property of the multidimensional Dunkl processes”, *Ann. Probab.* **34** (2006), no. 4, p. 1530-1549.
- [6] P. Graczyk, T. Luks, P. Sawyer, “Potential kernels for radial Dunkl Laplacians”, to appear in the *Canadian Journal of Mathematics* (2021), <https://arxiv.org/abs/1910.03105>, 2019.
- [7] P. Graczyk, M. Rösler, M. Yor, *Harmonic and Stochastic Analysis of Dunkl Processes*, Travaux en Cours, vol. 71, Hermann, 2008.
- [8] P. Graczyk, P. Sawyer, “Integral Kernels on Complex Symmetric Spaces and for the Dyson Brownian Motion”, to appear in the *Mathematische Nachrichten* (2021), <https://arxiv.org/abs/2012.10946>, 2020.
- [9] S. Helgason, *Groups and geometric analysis: integral geometry, invariant differential operators, and spherical functions*, Mathematical Surveys and Monographs, vol. 83, American Mathematical Society, 2000, corrected reprint of the 1984 original edition.
- [10] ———, *Differential geometry and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, 2001, reprint with corrections of the 1978 original edition.
- [11] ———, “The Bounded Spherical Functions on the Cartan motion group”, <https://arxiv.org/abs/1503.07598>, 2015.
- [12] E. K. Narayanan, A. Pasquale, S. Pusti, “Asymptotics of Harish–Chandra expansions, bounded hypergeometric functions associated with root systems, and applications”, *Adv. Math.* **252** (2014), p. 227-259.
- [13] M. Rösler, “Generalized Hermite polynomials and the heat equation for Dunkl operators”, *Commun. Math. Phys.* **192** (1998), no. 3, p. 519-542.
- [14] ———, “Positivity of Dunkl’s intertwining operator”, *Duke Math. J.* **98** (1999), no. 3, p. 445-463.
- [15] P. Sawyer, “The Abel transform on symmetric spaces of noncompact type”, in *Lie groups and symmetric spaces. In memory of F. I. Karpelevich*, Translations of the American Mathematical Society-Series 2, vol. 210, American Mathematical Society, 2003, p. 331-335.
- [16] ———, “A Laplace-type representation of the generalized spherical functions associated with the root systems of type A ”, *Mediterr. J. Math.* **14** (2017), no. 4, article no. 147.
- [17] B. Schapira, “Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel”, *Geom. Funct. Anal.* **18** (2008), no. 1, p. 222-250.