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Comptes Rendus
Mathématique

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Volume 359, issue 5 (2021), p. 523-531

Published online: 13 July 2021

https://doi.org/10.5802/crmath.182

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Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
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Complex analysis and geometry / *Analyse et géométrie complexes*

A note on pseudo-effective vector bundles with vanishing first Chern number over non-Kähler manifolds

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Abstract. In this note, We show that over a compact Hermitian manifold (X,ω) whose metric satisfies $\partial\bar{\partial}\omega^{n-1} = \partial\bar{\partial}\omega^{n-2} = 0$, every pseudo-effective vector bundle with vanishing first Chern number is in fact a numerically flat vector bundle.

Funding. The author are partially supported by NSF in China No.11625106, 11801535. The research was partially supported by the project "Analysis and Geometry on Bundles" of Ministry of Science and Technology of the People's Republic of China.

Manuscript received 25th October 2020, revised 19th December 2020, accepted 20th January 2021.

1. Introduction

In algebraic geometry and complex geometry, it is a very important topic to study the various kinds of notions for geometric positivity. For instance, let L be a smooth line bundle over a compact Hermitian manifold (X, ω). We say that L is a positive (resp. numerically effective) line bundle if there exists a smooth Hermitian metric h (resp. h_ϵ) on L such that the curvature $i\Theta(L,h) > 0$ (resp. $i\Theta(L,h_\epsilon) > -\epsilon\omega$, where ϵ is an arbitrary small positive constant). Later, Demailly [2] first introduced the concept of a singular metric h on a line bundle L and gave the definition of pseudo-effectivity: L is pseudo-effective if and only if there exists a singular metric h on L such that the curvature $i\Theta(L,h) \geq 0$ in the sense of currents.

There are also many other concepts of positivity on vector bundles, such as Griffiths positivity and Nakano positivity. The readers refer to [3, Chapter VII] for more details. In [10], Wu proved that a pseudo-effective vector bundle E over a compact Kähler manifold (X,ω) with $c_1(E) = 0$ is a numerically effective vector bundle, hence numerically flat since $c_1(E) = 0$. His proof relies heavily on the construction of Segre currents and the Demaily's regularization theorem [2]. We want to give an overview here. First, he proved that there exists a $(1,1)$-positive current in the class $\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \epsilon\pi^*[\omega])^r$ under the Kähler condition; then he proved that the Lelong numbers of the weight functions of the singular metrics h_ϵ can be controlled by the Lelong numbers of the corresponding currents $T_\epsilon = \pi_*(i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\epsilon))^r$. The limit of the Lelong numbers of the currents T_ϵ is 0 under the condition $c_1(E) = 0$. Thus the Lelong numbers of the weight functions uniformly tend to zero as $\epsilon \to 0$. Then by the Demaily's regularization theorem in [2] we know E is numerically effective.

In this note, we will generalize the [10, main Theorem] to certain non-Kähler manifolds as depicted in [8, Theorem 1.3]. Let E be a holomorphic vector bundle over a compact Hermitian manifold (X,ω) whose Hermitian metric ω satisfies $\partial\bar\partial\omega^{n-1} = \partial\bar\partial\omega^{n-2} = 0$. In [8], Li, Nie and Zhang proved that E is numerically flat if and only if E is semistable with vanishing first Chern number and second Chern number. In order to show that the vanishing of the second Chern number, we first prove that it is nonpositive by constructing a sequence of currents with bounded potential functions on $\mathbb{P}(E)$. And then by the Bogomolov's inequality, we know that the second Chern number is nonnegative, hence the desired result follows.

Theorem 1. *Let E be a pseudo-effective vector bundle over a compact Hermitian manifold (X,ω) whose metric ω satisfies Gauduchon property and Astheno–Kähler property. Assume that $c_1(E)\cdot[\omega^{n-1}] = 0$. Then E is a numerically flat vector bundle.*

2. Preliminaries

In this section, we recall some definitions about nef (short for numerically effective) vector bundles and psef (short for pseudo-effective) vector bundles. Let E be a holomorphic vector bundle over a compact complex manifold X. We denote by $\mathbb{P}(E)$ the projectivized bundle of hyperplanes of E and by $\mathcal{O}_{\mathbb{P}(E)}(1)$ the associated tautological line bundle.

Definition 2 ([4]). *We say that a holomorphic vector bundle E is nef over X if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef over $\mathbb{P}(E)$. Furthermore, we say that E is numerically flat if both E and the dual bundle E^* is nef.*

Definition 3 ([2]). *A singular Hermitian metric on a line bundle L is a Hermitian metric h which is given in any trivialization by a weight function $e^{-\varphi}$ such that φ is locally integrable. L is called pseudo-effective if there exists a singular metric h on L such that the curvature $i\Theta(L,h)$ is a closed positive $(1,1)$-current.*

Definition 4 ([10]). *Let (X,ω) be a compact complex manifold and let E be a holomorphic vector bundle over X. Then E is said to be pseudo-effective if for every $\epsilon > 0$ there exists a singular metric h_ϵ with analytic singularities on $\mathcal{O}_{\mathbb{P}(E)}(1)$, such that the curvature current $i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\epsilon) \geqslant -\epsilon\pi^*\omega$, and the projection $\pi(\mathrm{Sing}(h_\epsilon))$ of the singular set of h_ϵ is not equal to X.*

Remark 5. Recall that a plurisubharmonic function u is said to have analytic singularities if u can be written locally as
$$u = \frac{\alpha}{2}\log\left(\left|f_1\right|^2 \dots + \left|f_1\right|^N\right) + v,$$
where v is a smooth function, f_i are holomorphic functions and α is a positive constant. In [1], the definition of psef vector bundles is expressed in terms of the non-nef locus. Wu [10] gave the above definition and showed that these two definitions are equivalent to each other. In this note, we will use the above definition.

Let (X,ω) be a n-dimensional compact Hermitian manifold and ω be the Hermitian metric over X. The Hermitian metric ω is called Gauduchon if it satisfies $\partial\bar\partial(\omega^{n-1}) = 0$. If $\partial\bar\partial(\omega^{n-2}) = 0$, then the Hermitian metric ω is said to be Astheno–Kähler which was introduced by Jost and Yau in [7].

Let $(E,\, h)$ be a Hermitian vector bundle over (X,ω), where ω is assumed be Gauduchon. The ω-degree of E is defined by
$$\deg_\omega(E) := \int_X c_1(E,h) \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

Since $\partial\bar{\partial}(\omega^{n-1}) = 0$, $\deg_\omega(E)$ is well-defined and independent of the choices of the metric h. Assume that $\mathcal{F}$ is a coherent analytic sheaf of rank p, and we consider the determinant line bundle $\det(\mathcal{F}) = (wedge^p\mathcal{F})^{**}$. Define the ω-degree of $\mathcal{F}$ by

$$\deg_\omega(\mathcal{F}) := \deg_\omega(\det\mathcal{F}).$$

The slope of $\mathcal{F}$ is given by

$$\mu_\omega(\mathcal{F}) = \frac{\deg_\omega(\mathcal{F})}{\operatorname{rank}(\mathcal{F})}.$$

Definition 6. *Let $(E,\bar{\partial}_E)$ be a holomorphic vector bundle over X. We say that E is ω-stable (resp. ω-semistable) in the sense of Mumford–Takemoto if every proper coherent sub-sheaf $\mathcal{F} \hookrightarrow E$ satisfies the inequality*

$$\mu_\omega(\mathcal{F}) < \mu_\omega(E) \ \left(\text{resp. } \mu_\omega(\mathcal{F}) \le \mu_\omega(E)\right).$$

Remark 7. This definition of semistability is equivalent to the following condition: for any torsion free quotient sheaf $\mathcal{Q}$ of E

$$\mu_\omega(\mathcal{Q}) \ge \mu_\omega(E)$$

Since $\partial\bar{\partial}\omega^{n-1} = \partial\bar{\partial}\omega^{n-2} = 0$,

$$c_1(E)\cdot\left[\omega^{n-1}\right] = \int_X c_1(E,H)\wedge\omega^{n-1}$$

and

$$c_2(E)\cdot\left[\omega^{n-2}\right] = \int_X c_2(E,H)\wedge\omega^{n-2},$$
$$c_1(E)^2\cdot\left[\omega^{n-2}\right] = \int_X c_1(E)^2\wedge\omega^{n-2}$$

are well-defined and independent of the Hermitian metrics on E [8], where

$$\left[\omega^{n-1}\right] \in H_A^{n-1,n-1}(X), \quad \left[\omega^{n-2}\right] \in H_A^{n-2,n-2}(X)$$

and

$$c_1(E) \in H_{BC}^{1,1}(X),\ c_2(E),\ c_1(E)^2 \in H_{BC}^{2,2}(X).$$

Here $H_A^{\bullet,\bullet}(X)$ is the Aeppli cohomology and $H_{BC}^{\bullet,\bullet}(X)$ is the Bott–Chern cohomology.

3. Proof of Theorem 1

In this section, we will prove that a psef vector bundle with vanishing first Chern number is semistable and its second Chern number vanishes.

The theorem below gives an equivalent description of psef vector bundles motivated by a similar theorem about nef vector bundles [4, Theorem 1.12].

Theorem 8 ([10, Propostion 1]). *E is a psef vector bundle over a Hermitian manifold if and only if there exists a sequence of smooth metrics h_m on S^mE^* (which may degenerate at some points), such that the singularity locus projects into a proper Zariski set Z_m and*

$$i\Theta\left(S^mE^*, h_m\right) \le m\epsilon_m\omega\otimes Id$$

on $X\setminus Z_m$ in the sense of Griffiths, as $\epsilon_m \to 0$.

The following Stokes formula is well-known.

Lemma 9. *Let S be a current of degree* $n-1$ *over a compact complex manifold X. Then*

$$\int_X dS = 0.$$

Proof. Let η be a smooth cut-off funtion in X such that $\eta \equiv 1$ in a neighborhood of the support of S. Then

$$\int_X dS = \int_X \eta dS = \int_X S \wedge d\eta = 0.$$

□

Lemma 10. *Let E be a psef vector bundle over a compact Hermition manifold* (X,ω) *with* $c_1(E)\cdot[\omega^{n-1}] = 0$ *and* $\partial\bar{\partial}(\omega^{n-1}) = 0$. *Then E is semistable.*

Proof. Let $\mathcal{F}$ be a proper subsheaf of $\mathcal{E} = \mathcal{O}(E)$ and let

$$\det \mathcal{F} = \left(\bigwedge^p \mathcal{F}^{**}\right)$$

be its determinant bundle. So we have an injection of sheaves

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{E}/\mathcal{F} \to 0.$$

For simplicity, we denote $\mathcal{Q} = \mathcal{E}/\mathcal{F}$ and rank $\mathcal{Q} = r$. Then we obtain

$$0 \to \mathcal{Q}^* \to \mathcal{E}^*$$

and

$$0 \to \det\left(\mathcal{Q}^*\right) \to \bigwedge^r \mathcal{E}^*$$

Since E is psef, $\wedge^r E$ is also psef (cf. [10, Corollary 3]). By Theorem 8, there exists a sequence of Hermitian metrics h_m on $S^m \wedge^r E^*$, such that the singularity locus of each h_m projects into a proper Zariski closed set Z_m and

$$i\Theta\left(S^m \bigwedge^r E^*, h_m\right) \le m\epsilon_m \omega \otimes Id$$

on $X \setminus Z_m$ in the sense of Griffiths, as $\epsilon_m \to 0$.

The morphism between sheaves induces a morphism between bundles, which moreover is injective outside an analytic set W_m. So we have

$$0 \to \det\left(\mathcal{Q}^*\right)^m \to S^m \bigwedge^r E^*$$

outside $Z_m \cup W_m$. The Hermitian metric h_m on $S^m \wedge^r E^*$ induces a smooth bounded metric $\widetilde{h_m}$ on $\det(\mathcal{Q}^*)^m$ inside $X \setminus (Z_m \cup W_m)$. By the Gauss–Codazzi equation, we have

$$i\Theta\left(\left(\det\left(\mathcal{Q}^*\right)\right)^m, \widetilde{h_m}\right) \le i\Theta\left(S^m \bigwedge^r E^*, h_m\right)\Big|_{\det\left(\mathcal{Q}^*\right)^m} \le m\epsilon_m \omega$$

on $X \setminus Z_m \cup W_m$. The induced metics $\widetilde{h_m}^{\frac{1}{m}}$ on $\det(\mathcal{Q}^*)$ are bounded from above. In other words, the local weight funtions φ_m of the dual metrics on $\det(\mathcal{Q})$ are locally bounded from above.

For any point $x \in Z_m \cup W_m$, there exists a coordinate chart U, which is biholomorphic to an open ball $B(x,r) \subseteq \mathbb{C}^n$ of x with radius $r > 0$ sufficiently small such that

$$\omega \le \omega_j \le 2\omega$$

for some $(1,1)$ form ω_j with constant coefficients on $B(x,r)$ satisfying $w_j(x) = w(x)$. Hence, ω_j is closed in $B(x,r)$ and we have

$$i\partial\bar{\partial}\varphi_m \ge -\epsilon_m\omega \ge -\epsilon_m\omega_j \ge -2\epsilon_m\omega.$$

By [3, Theorem 5.23] we know that φ_m can be uniquely extended through $Z_m \cup W_m$ such that the curvature inequalities hold in the sense of currents throughout X

$$i\Theta\left(\det(\mathcal{Q}), \widetilde{h_m}^{-\frac{1}{m}}\right) = i\partial\bar{\partial}\varphi_m \geq -2\epsilon_m\omega.$$

Also there exists a smooth form α and a real-valued function $\varphi \in L^1(X)$ such that

$$i\Theta\left(\det(\mathcal{Q}), \widetilde{h_m}^{-\frac{1}{m}}\right) = \alpha + dd^c\varphi.$$

Setting $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$, by Lemma 1.1, we deduce

$$\begin{aligned}
&\int_X dd^c\varphi \wedge \omega^{n-1} \\
&= \int_X \varphi dd^c\omega^{n-1} + \int_X d\left(\omega^{n-1} \wedge d^c\varphi - \varphi \wedge d^c\omega^{n-1}\right) \\
&= \int_X \varphi dd^c\omega^{n-1} = 0.
\end{aligned}$$

So the degree of coherent sheaf $\mathcal{Q}$

$$\deg\mathcal{Q} = \int_X c_1(\mathcal{Q}) \wedge \omega^{n-1} = \int_X \frac{i}{2\pi}\Theta\left(\det(\mathcal{Q}), \widetilde{h_m}^{-\frac{1}{m}}\right) \wedge \omega^{n-1} \geq -\frac{\epsilon_m}{\pi}\int_X \omega^n.$$

Let $\epsilon_m \to 0$, it follows that $\deg\mathcal{Q} \geq 0$. By Definition 6, we conclude that E is semistable. □

Now, we wish to introduce the pushforward formula of Segre forms, which was proved by Guler [6] for projective manifolds and by Diverio [5] for general compact complex manifolds. Let E be a holomorphic vector bundle of rank r on a complex manifold X, and

$$c_\bullet(E) = 1 + c_1(E) + \cdots + c_r(E) \in H^\bullet(X, \mathbb{Z})$$

be the total Chern class of E. The inverse of $c_\bullet(E)$ is defined by the total Segre class

$$s_\bullet(E) = 1 + s_1(E) + \cdots + s_r(E) \in H^\bullet(X, \mathbb{Z}).$$

Choose a Hermitian metric H on E, then these Segre forms $s_k(E)$ can be defined by the following relation:

$$s_k(E,H) + c_1(E,H)s_{k-1}(E,H) + \cdots + c_k(E,H) = 0,\ 0 \leq k \leq \min\{r,n\}$$

For example,

$$s_1(E,H) = -c_1(E,H)$$

and

$$s_2(E,H) = c_1(E,H)^2 - c_2(E,H).$$

Let M, N be two complex manifolds of complex dimensions m, n respectively and $F : M \to N$ be a proper submersion. Set $s = m - n$, then we can define the direct image of a current T of dimension q, denoted by $F_*(T)$, by the following relation

$$\langle F_* T, u\rangle = \left\langle T, F^* u\right\rangle$$

where u is any smooth q-form. It is easy to check that

$$d(F_* T) = F_*(dT)$$

and

$$F_*\left(T \wedge F^* g\right) = F_*(T) \wedge g$$

for each smooth form g. In particular, given a Hermitian metric H on E, denote by h the induced metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ and let $\alpha = \frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h)$. Consider the map $\pi : \mathbb{P}(E) \to X$. Then

$$\left\langle \pi_* \left(\alpha^{r-1+k}\right), \omega^{n-k} \right\rangle = \int_X \pi_* \left(\alpha^{r-1+k}\right) \wedge \omega^{n-k}$$
$$\left\langle \alpha^{r-1+k}, \pi^* \omega^{n-k} \right\rangle = \int_X \alpha^{r-1+k} \wedge \pi^* \omega^{n-k}$$

We have the following formula of Segre forms:

Lemma 11 ([6]). *For each $0 \le k \le n$, we have the equality*

$$\pi_* \left(\alpha^{r-1+k}\right) = s_k(E, H).$$

Lemma 12. *Let E be a psef vector bundle over a compact Hermitian manifold (X, ω) with $c_1(E) \cdot [\omega^{n-1}] = 0$ and $\partial\bar{\partial}\omega^{n-1} = \partial\bar{\partial}\omega^{n-2} = 0$. Then*

$$ch_2(E) \cdot \left[\omega^{n-2}\right] = 0.$$

Proof. By the definition of pseudo-effectivity, for every $\epsilon > 0$, there exists a singular metric h_ϵ with analytic singularity on $\mathcal{O}_{\mathbb{P}(E)}(1)$, such that the curvature current

$$\frac{i}{2\pi}\Theta\left(\mathcal{O}_{\mathbb{P}(E)}(1), h_\epsilon\right) \geqslant -\epsilon\pi^*\omega$$

and $\pi(\mathrm{Sing}(h_\epsilon))$ is not equal to X. Choose a smooth Hermitian metric H on E, denoted by h the induced metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$, let $\alpha = c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h)$ and consider the singular metric $h_\epsilon = he^{-\varphi_\epsilon}$, where φ_ϵ is a function in $\mathbb{P}(E)$. Then we have

$$\alpha + i\partial\bar{\partial}\varphi_\epsilon + \epsilon\pi^*\omega \ge 0.$$

When the metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ is singular, we can not use Lemma 11 directly, because the Monge–Ampère operator is not always well-defined for an arbitrary positive current. Nevertheless, we overcome this difficulty by constructing new currents $S_{\delta,\epsilon}$ with bounded potential functions. Namely,

$$S_{\delta,\epsilon} = \alpha + \epsilon\pi^*\omega + i\partial\bar{\partial}\log\left(e^{\varphi_\epsilon} + \delta\right),$$

where δ is a positive constant. Next we will show that the $S_{\delta,\epsilon}$ have a positive lower bound which does not depend on δ.

Since $\mathcal{O}_{\mathbb{P}(E)}(1)$ is relatively π-ample, there exists a sufficiently large constant $K > 0$ such that

$$\alpha + K\pi^*\omega > 0. \tag{1}$$

A straightforward computation yields that

$$\begin{aligned} S_{\delta,\epsilon} &= \alpha + \epsilon\pi^*\omega + \frac{e^{\varphi_\epsilon}}{e^{\varphi_\epsilon} + \delta} i\partial\bar{\partial}\varphi_\epsilon + \frac{\delta e^{\varphi_\epsilon}}{e^{\varphi_\epsilon} + \delta} i\partial\varphi_\epsilon \wedge \bar{\partial}\varphi_\epsilon \\ &\ge \alpha + \epsilon\pi^*\omega + \frac{e^{\varphi_\epsilon}}{e^{\varphi_\epsilon} + \delta} i\partial\bar{\partial}\varphi_\epsilon \\ &= \frac{e^{\varphi_\epsilon}}{e^{\varphi_\epsilon} + \delta}\left(\alpha + i\partial\bar{\partial}\varphi_\epsilon + \epsilon\pi^*\omega\right) + \frac{\delta}{e^{\varphi_\epsilon} + \delta}\left(\alpha + K\pi^*\omega + \epsilon\pi^*\omega\right) \\ &\quad - \frac{\delta}{e^{\varphi_\epsilon} + \delta} K\pi^*\omega. \end{aligned} \tag{2}$$

By (1) and (2) we have

$$S_{\delta,\epsilon} + K\pi^*\omega \ge 0.$$

By the definition of the Monge–Ampère operator [3], we know that

$$\left(S_{\delta,\epsilon} + K\pi^*\omega\right)^{r+1} \ge 0.$$

Therefore,

$$\begin{aligned} 0 &\le \int_{\mathbb{P}(E)} \pi^*\eta_\epsilon \left(S_{\delta,\epsilon} + K\pi^*\omega\right)^{r+1} \wedge \pi^*\omega^{n-2} \\ &= \int_{\mathbb{P}(E)} \pi^*\eta_\epsilon S_{\delta,\epsilon}^{r+1} \wedge \pi^*\omega^{n-2} + (r+1)K \int_{\mathbb{P}(E)} \pi^*\eta_\epsilon S_{\delta,\epsilon}^{r} \wedge \pi^*\omega^{n-1} \\ &\quad + r(r+1)K^2 \int_X \eta_\epsilon \omega^n. \end{aligned} \tag{3}$$

For each $\epsilon > 0$, we can choose a smooth function $0 \le \eta_\epsilon \le 1$, which is equal to 1 in a neighborhood of $\pi(Z_\epsilon)$ (Z_ϵ is the set of singularities of φ_ϵ) such that

$$r(r+1)K^2 \int_X \eta_\epsilon \omega^n < \epsilon.$$

Since the support of $1 - \pi^*\eta_\epsilon$ is included in $\mathbb{P}(E) \setminus Z_\epsilon$, we have

$$\int_{\mathbb{P}(E)} \left(1 - \pi^*\eta_\epsilon\right) S_{\delta,\epsilon}^{r+1} \wedge \pi^*\omega^{n-2} = \int_{\mathbb{P}(E)\setminus Z_\epsilon} \left(1 - \pi^*\eta_\epsilon\right) S_{\delta,\epsilon}^{r+1} \wedge \pi^*\omega^{n-2}$$

and

$$\int_{\mathbb{P}(E)} \left(1 - \pi^*\eta_\epsilon\right) S_{\delta,\epsilon}^{r} \wedge \pi^*\omega^{n-1} = \int_{\mathbb{P}(E)\setminus Z_\epsilon} \left(1 - \pi^*\eta_\epsilon\right) S_{\delta,\epsilon}^{r} \wedge \pi^*\omega^{n-1}.$$

By the continuity of Monge–Ampère operators along the bounded decreasing sequences (cf. [3, Corollary 3.6]), we know that

$$\begin{aligned} &\int_{\mathbb{P}(E)\setminus Z_\epsilon} \left(1 - \pi^*\eta_\epsilon\right) S_{\delta,\epsilon}^{r+1} \wedge \pi^*\omega^{n-2} \\ &\xrightarrow{\delta\to 0} \int_{\mathbb{P}(E)\setminus Z_\epsilon} \left(1 - \pi^*\eta_\epsilon\right) \left(\alpha + i\partial\bar\partial\varphi_\epsilon + \epsilon\pi^*\omega\right)^{r+1} \wedge \pi^*\omega^{n-2} \ge 0, \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\int_{\mathbb{P}(E)\setminus Z_\epsilon} \left(1 - \pi^*\eta_\epsilon\right) S_{\delta,\epsilon}^{r} \wedge \pi^*\omega^{n-1} \\ &\xrightarrow{\delta\to 0} \int_{\mathbb{P}(E)\setminus Z_\epsilon} \left(1 - \pi^*\eta_\epsilon\right) \left(\alpha + i\partial\bar\partial\varphi_\epsilon + \epsilon\pi^*\omega\right)^{r} \wedge \pi^*\omega^{n-1} \ge 0. \end{aligned} \tag{5}$$

We can choose $\delta > 0$ sufficiently small such that

$$\int_{\mathbb{P}(E)} \left(1 - \pi^*\eta_\epsilon\right) S_{\delta,\epsilon}^{r} \wedge \pi^*\omega^{n-2} > -\frac{\epsilon}{3} \tag{6}$$

and that

$$\int_{\mathbb{P}(E)} \left(1 - \pi^*\eta_\epsilon\right) S_{\delta,\epsilon}^{r} \wedge \pi^*\omega^{n-1} > -\frac{\epsilon}{3K(r+1)}. \tag{7}$$

Because the first Chern number is zero, by the pushforward formula of Segre forms (Lemma 11), we have

$$\begin{aligned} \int_{\mathbb{P}(E)} S_{\delta,\epsilon}^{r} \wedge \pi^*\omega^{n-1} &= \int_{\mathbb{P}(E)} \left(\alpha + i\partial\bar\partial \log\left(e^{\varphi_\epsilon} + \delta\right)\right)^r \wedge \pi^*\omega^{n-1} + r\epsilon \int_X \omega^n \\ &= \int_X c_1(E,H) \wedge \omega^{n-1} + r\epsilon \int_X \omega^n = r\epsilon \int_X \omega^n \end{aligned} \tag{8}$$

and

$$\int_{\mathbb{P}(E)} \left(1 - \pi^*\eta_\epsilon\right) S_{\delta,\epsilon}^{r} \wedge \pi^*\omega^{n-1} = \int_{\mathbb{P}(E)} -\pi^*\eta_\epsilon S_{\delta,\epsilon}^{r} \wedge \pi^*\omega^{n-1} + \epsilon \int_X \omega^n. \tag{9}$$

Combining with (3), (6), (7), (8), we get

$$\begin{aligned}0\le \text{ R.H.S. of (3)} &= \int_{\mathbb{P}(E)} \left(\pi^*\eta_\epsilon - 1\right) S_{\delta,\epsilon}^{r+1}\wedge \pi^*\omega^{n-2} + \int_{\mathbb{P}(E)} S_{\delta,\epsilon}^{r+1}\wedge\pi^*\omega^{n-2} \\ &\quad + \int_{\mathbb{P}(E)} (r+1)K\pi^*\eta_\epsilon S_{\delta,\epsilon}^{r}\wedge\pi^*\omega^{n-1} + r(r+1)K^2\int_X \eta_\epsilon\omega^n \\ &< \epsilon + \epsilon\int_X \omega^n + \int_{\mathbb{P}(E)} S_{\delta,\epsilon}^{r+1}\wedge\pi^*\omega^{n-2}\end{aligned} \tag{10}$$

and

$$\begin{aligned}\int_M s_2(E,H)\wedge\omega^{n-2} &= \int_{\mathbb{P}(E)} \left(\alpha + i\partial\bar\partial\log\left(e^{\varphi_\epsilon}+\delta\right)\right)^{r+1}\wedge\pi^*\omega^{n-2} \\ &= \int_{\mathbb{P}(E)} S_{\delta,\epsilon}^{r+1}\wedge\pi^*\omega^{n-2} - (r+1)\epsilon\int_{\mathbb{P}(E)} \left(\alpha + i\partial\bar\partial\log\left(e^{\varphi_\epsilon}+\delta\right)\right)^{r}\wedge\pi^*\omega^{n-1} \\ &\quad + \frac{r(r+1)}{2}\epsilon^2\int_X\omega^n.\end{aligned} \tag{11}$$

Moreover,

$$\begin{aligned}\int_X c_2(E,H)\wedge\omega^{n-2} &= -\int_X s_2(E,H)\wedge\omega^{n-2} \\ &= -\int_{\mathbb{P}(E)} S_{\delta,\epsilon}^{r+1}\wedge\pi^*\omega^{n-2} - \frac{r(r+1)}{2}\epsilon^2\int_X\omega^n \\ &< \epsilon + \epsilon\int_X\omega^n - \frac{r(r+1)}{2}\epsilon^2\int_X\omega^n\end{aligned} \tag{12}$$

for each $\epsilon > 0$. Consequently, one has

$$\int_M c_2(E,H)\wedge\omega^{n-2}\le 0.$$

On the other hand, by [10, Corollary 1], we know that $\det(E)$ is a psedo-effective line bundle. In fact, we can conclude that $\det(E)$ is Hermitian flat under the condition $c_1(E)\cdot[\omega^{n-1}] = 0$. This implies

$$c_1(E)^2\cdot\left[\omega^{n-2}\right] = 0.$$

Since E is semistable, by the Bogomolov inequality (cf. [8, Proposition 2.6]), we have

$$c_2(E)\cdot\left[\omega^{n-2}\right] \ge \frac{r-1}{2r}c_1(E)^2\cdot\left[\omega^{n-2}\right] = 0.$$

Hence we deduce that

$$ch_2(E)\cdot\left[\omega^{n-2}\right] = \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right)\cdot\left[\omega^{n-2}\right] = 0.$$ □

Remark 13. If the first Chern class of E is not zero, this lemma is incorrect. A counterexample is given in [10]. Let X be the blow up of $\mathbb{P}^2$ at some point and D the exceptional divisor. Consider the vector bundle $E = \mathcal{O}(D)^{\oplus 2}$. Then E is a psef vector bundle, but $s(E) = s(\mathcal{O}(D))^{\oplus 2}$ with $s_2(E) = c_1(\mathcal{O}(D))^2 = -1$.

Proof of Theorem 1. We prove the theorem following the argument of [8]. By Lemma 1.1 and Lemma 1.2, we know that E is semistable with vanishing first and second Chern numbers. Then E has an approximately Hemitian flat structure [8], and there exists a filtration $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_k = E$ by subbundles whose quotients are Hermitian flat [9]. Therefore E is numerically flat. □

Acknowledgement

I thank my supervisor, Professor Xi Zhang, for leading me to the research of this problem and offering many helpful discussions. I would like to thank Doctor Chao Li for his help and useful suggestions.

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