INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Sheng Rao and Runze Zhang

L^2 extension theorem for jets with variable denominators

Volume 359, issue 2 (2021), p. 181-193

Published online: 17 March 2021

https://doi.org/10.5802/crmath.167

Les Comptes Rendus. Mathématique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2021, Vol. 359, 2, p. 181-193
https://doi.org/10.5802/crmath.167

Complex analysis / *Analyse complexe*

L^2 extension theorem for jets with variable denominators

Sheng Rao [a, b] **and Runze Zhang** [*, a]

[a] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People's Republic of China

[b] Université de Grenoble-Alpes, Institut Fourier (Mathématiques) UMR 5582 du C.N.R.S., 100 rue des Maths, 38610 Gières, France

E-mails: likeanyone@whu.edu.cn, sheng.rao@univ-grenoble-alpes.fr, runzezhang@whu.edu.cn

Abstract. By studying the variable denominators introduced by X. Zhou–L. Zhu, we generalize the results of D. Popovici for the L^2 extension theorem for jets. As a direct corollary, we also give a generalization of T. Ohsawa–K. Takegoshi's extension theorem to a jet version.

Keywords. Continuation of analytic objects in several complex variables; Sheaves and cohomology of sections of holomorphic vector bundles, general results, Kähler manifolds, Exhaustion functions.

Mathematical subject classification (2010). 32D15, 32L10, 32Q15, 32T35.

Funding. Both authors are partially supported by NSFC (Grant No. 11671305, 11771339, 11922115) and the Fundamental Research Funds for the Central Universities (Grant No. 2042020kf1065).

Manuscript received 16 October 2020, accepted 13 December 2020.

1. Introduction: main results and applications

T. Ohsawa–K. Takegoshi established a remarkable extension theorem of holomorphic functions defined on a bounded pseudoconvex domain in $\mathbb{C}^n$ with growth control in [13]. Since then, many versions and variants of the L^2 extension theorems have been studied (see [2,5,10,11,14,15], etc). These results lead to numerous applications in algebraic geometry and complex analysis.

One interesting problem is to study the L^2 extension theorem for jets. The first such result was given by D. Popovici [14], which generalized the L^2 extension theorems of Ohsawa–Takegoshi–Manivel to the case of jets of sections of a line bundle over a weakly pseudoconvex Kähler manifold. Then J.-P. Demailly [6] considered the extension from more general non reduced varieties. Following a new method of B. Berndtsson–L. Lempert [1], G. Hosono [9] proved an L^2 extension theorem for jets with optimal estimate on a bounded pseudoconvex domain in $\mathbb{C}^n$ (see also [12]).

[*] Corresponding author.

ISSN (electronic) : 1778-3569 https://comptes-rendus.academie-sciences.fr/mathematique/

The idea of considering variable denominators was first introduced by J. McNeal–D. Varolin [11]. They obtained some results on weighted L^2 extension of holomorphic top forms with values in a holomorphic line bundle, where the weights used are determined by the variable denominators. Recently, X. Zhou–L. Zhu [15] proved an L^2 extension theorem for holomorphic sections of holomorphic line bundles equipped with singular metrics on weakly pseudoconvex Kähler manifolds. Futhermore, they obtained optimal constants corresponding to variable denominators.

The method of solving undetermined functions with ODEs was first used in [8, 17]. From then on, a lot of spectacular works appear along this line, such as [7, 15, 16], etc. Several optimal L^2 extension theorems have been proved in this process.

The main goal of this paper is to apply the methods of Zhou–Zhu [15] and Demailly [5] to L^2 jet extension to slightly generalize the results obtained by Popovici [14]. As an application of our main theorem, we also generalize Ohsawa–Takegoshi's extension theorem [13] to a jet version (see Corollary 2). In near future, we will try to formulate this work to $\bar{\partial}$-closed high-degree jets on a weakly pseudoconvex Kähler case under mixed positivity conditions, such as [3].

We make precise the setting for our work. Let X be an n-dimensional weakly pseudoconvex manifold with Kähler metric ω, and E a Hermitian holomorphic vector bundle of rank $m \geq 1$ over X. Assume that $s \in H^0(X, E)$ is transverse to the zero section. Set

$$Y := \left\{x \in X : s(x) = 0\right\}.$$

Futhermore, let L be a holomorphic line bundle equipped with a smooth Hermitian metric satisfying an appropriate positivity condition.

We denote by $\Lambda^{r,s} T_X^*$ the bundle of differential forms of bidegree (r, s) on X, and $\mathscr{I}_Y$ the sheaf of germs of holomorphic functions on X which vanish on Y. For any integer $k \geq 0$, let $\mathscr{O}_X / \mathscr{I}_Y^{k+1}$ be the nonlocally free sheaf of k-jets which are "transversal" to Y. Fix a point $y \in X$ and a Stein neighborhood U in X of y. Then this gives rise to a surjective morphism

$$H^0(U, K_X \otimes L) \longrightarrow H^0(U, K_X \otimes L \otimes \mathscr{O}_X / \mathscr{I}_Y^{k+1})$$

of local section spaces, and an arbitrary local lifting $\widetilde{f} \in H^0(U, K_X \otimes L)$ of f. For any transversal k-jet $f \in H^0(U, K_X \otimes L \otimes \mathscr{O}_X / \mathscr{I}_Y^{k+1})$ and any weight function $\rho > 0$ on U, the *pointwise ρ-weighted norm associated to the section s*, was defined by [14, Definition 0.1.1]:

$$|f|^2_{s,\rho,(k)}(y) := |\widetilde{f}|^2_L(y) + \frac{|\nabla^1 \widetilde{f}|^2_L}{|\Lambda^m(\mathrm{d}s)|_E^{2\frac{1}{m}} \rho^{2(m+1)}}(y) + \cdots + \frac{|\nabla^k \widetilde{f}|^2_L}{|\Lambda^m(\mathrm{d}s)|_E^{2\frac{k}{m}} \rho^{2(m+k)}}(y),$$

and *the $L^2_{(k)}$ weighted norm* by:

$$\|f\|^2_{s,\rho,(k)} = \int_Y \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|^2_E} \mathrm{d}V_{Y,\omega}.$$

Here for $i = 0, \ldots, k$, $\nabla^i \widetilde{f}$ is constructed by induction as the projection of the $(1,0)$-part

$$\nabla^{1,0}(\nabla^k \widetilde{f}) \in C^\infty(U, K_X \otimes L \otimes S^{j-1} N^*_{Y/X} \otimes T^*_X)$$

of $\nabla(\nabla^k \widetilde{f})$ with the associated Chern connection ∇ to $C^\infty(U, K_X \otimes L \otimes S^j N^*_{Y/X})$, induced by the surjective bundle morphism $K_X \otimes L \otimes T^*_X|_Y \to K_X \otimes L \otimes N^*_{Y/X}$.

It is worthwhile to notice that the norm $|f|^2_{s,\rho,(k)}$ of the k-jet f at the point $y \in Y$ is independent of the choice of the local lifting $\widetilde{f}$. Moreover, one has the following notations [14, Notation 0.1.3]:

(a) For a transversal k-jet $f \in H^0(U, K_X \otimes L \otimes \mathscr{O}_X / \mathscr{I}_Y^{k+1})$, denote $\nabla^j f := (\nabla^j \widetilde{f})_{|U \cap Y}$, for all $j = 0, \ldots, k$ and an arbitrary lifting $\widetilde{f} \in H^0(U, K_X \otimes L)$ of f.

(b) For every integer $k \geq 0$, and every open set $U \subset X$, set

$$J_U^k : H^0(U, K_X \otimes L) \longrightarrow H^0(U, K_X \otimes L \otimes \mathscr{O}_X / \mathscr{I}_Y^{k+1})$$

as the cohomology group morphism induced by the projection $\mathscr{O}_X \to \mathscr{O}_X / \mathscr{I}_Y^{k+1}$.

We refer to [14, pp. 2–5] for more details about the notations and the construction of relevant metrics on jets.

In [15, p. 136], Zhou–Zhu defined the variable denominators. Let $\mathfrak{R}$ be the class of functions defined by

$$\left\{ R \in C^\infty(-\infty, 0] : \begin{array}{l} R > 0,\ R' \leq 0,\ \int_{-\infty}^0 \frac{1}{R(t)} \mathrm{d}t < +\infty \\ \text{and } e^t R(t) \text{ is bounded above on } (-\infty, 0] \end{array} \right\}.$$

Denote $\int_{-\infty}^0 \frac{1}{R(t)} \mathrm{d}t$ by C_R. Notice that the function $R(t)$ equals to the function $\frac{1}{c_A(-t)e^t}$ defined just before the main theorems in [7, p. 1143] when $A = 0$.

With such preparation, our main theorem is as follows.

Main Theorem 1. *Let R be a function in $\mathfrak{R}$. On a weakly pseudoconvex n-dimensional Kähler manifold (X, ω), let L be a smooth Hermitian holomorphic line bundle, E a smooth Hermitian holomorphic vector bundle of rank $m \geq 1$, and $s \in H^0(X, E)$ a section transverse to the zero section. Set*

$$Y := \left\{ x \in X : s(x) = 0 \right\}.$$

Assume also that, for an integer $k \geq 0$, the $(1,1)$-form $\sqrt{-1}\Theta_L + (m+k)\sqrt{-1}\partial\bar{\partial}\log|s|_E^2$ involving the curvature of L is semipositive on $X \setminus Y$, and that there exists a continuous function $\alpha > 0$ on X such that on $X \setminus Y$,

(i) $\sqrt{-1}\Theta_L + (m+k)\sqrt{-1}\partial\bar{\partial}\log|s|_E^2 \geq \alpha^{-1} \dfrac{\{\sqrt{-1}\Theta_E s, s\}_E}{|s|_E^2}$,

(ii) $|s|_E \leq e^{-\alpha}$.

Then for every relatively compact open subset $\Omega \subset X$ and every k-jet $f \in H^0(X, K_X \otimes \mathscr{O}_X / \mathscr{I}_Y^{k+1})$ satisfying

$$\int_Y \frac{|f|_{s,\rho,(k)}^2}{|\Lambda^m(\mathrm{d}s)|_E^2} \mathrm{d}V_{Y,\omega} < +\infty,$$

there exists $F^{(k)} \in H^0(\Omega, K_X \otimes L)$ such that $J_\Omega^k F^{(k)} = f$ and

$$\int_\Omega \frac{|F^{(k)}|_L^2}{|s|_E^{2m} R(m \log|s|_E^2)} \mathrm{d}V_{X,\omega} \leq C_{m,R}^{(k)} \int_Y \frac{|f|_{s,\rho,(k)}^2}{|\Lambda^m(\mathrm{d}s)|_E^2} \mathrm{d}V_{Y,\omega}, \tag{1}$$

where $C_{m,R}^{(k)} > 0$ is a constant depending only on m, k, E, R and $\sup_\Omega \|i\Theta(L)\|$.

Comparing the statement in [14, Main Theorem], one sees that only in the case of a compact ambient manifold X, the jet extension $F^{(k)}$ and the final L^2 estimate of it can be obtained over the whole of X; while in the general noncompact case, since the constant in his L^2 jet estimate (2) depends on $\sup_\Omega \|i\Theta(L)\|$, and then it seems difficult to obtain the jet extension and the final estimate over X by extracting weak limit as $c \to \infty$, where X_c will be introduced later in Section 3.

Our Theorem 1 is a generalization of [14, Main Theorem], where $\alpha \geq 1$. In fact, if we take $R(t) = (\frac{t}{2m})^2$ on $(-\infty, -2m]$, and then (1) implies Popovici's L^2 jet estimate [14, p. 6]

$$\int_\Omega \frac{|F^{(k)}|_L^2}{|s|_E^{2m}(-\log|s|)^2} \mathrm{d}V_{X,\omega} \leq C_m^{(k)} \int_Y \frac{|f|_{s,\rho,(k)}^2}{|\Lambda^m(\mathrm{d}s)|_E^2} \mathrm{d}V_{Y,\omega}, \tag{2}$$

where $C_m^{(k)} > 0$ is a constant depending only on m, k, E, and $\sup_\Omega \|i\Theta(L)\|$.

Take $R(t) = e^{-t}$ (then $C_R = 1$) to get the following corollary as a generalization of the main theorem in [13] to a jet version.

Corollary 2. *Let (X,ω), L, E, s, Y be the same as in Theorem 1. Then for every relatively compact open subset $\Omega \subset X$, and every k-jet $f \in H^0(X, K_X \otimes L \otimes \mathscr{O}_X / \mathscr{I}_Y^{k+1})$ satisfying*

$$\int_Y \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|_E^2} \mathrm{d}V_{Y,\omega} < +\infty,$$

there exists $F^{(k)} \in H^0(\Omega, K_X \otimes L)$ such that $J^k_\Omega F^{(k)} = f$ and

$$\int_\Omega |F^{(k)}|_L^2 \mathrm{d}V_{X,\omega} \le C_m^{(k)} \int_Y \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|_E^2} \mathrm{d}V_{Y,\omega},$$

where $C_m^{(k)} > 0$ is a constant depending only on m, k, E, and $\sup_\Omega \| i\Theta(L)\|$.

The following theorem is a special case of Theorem 1 for a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. Denote the space of all plurisubharmonic functions on Ω by $\mathrm{Psh}(\Omega)$.

Theorem 3. *Let R be a function in $\mathfrak{R}$. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex open set, and $Y \subset \Omega$ a closed nonsingular subvariety defined by some section $s \in H^0(\Omega, E)$ of a Hermitian holomorphic vector bundle E of rank $m \ge 1$ with bounded curvature form. Set $|s|_E \le 1$ on Ω. Then for any relatively compact open subset $\Omega' \subset \Omega$, any $k \ge 0$ and any $\varphi \in Psh(\Omega)$, there exists a constant $C^{(k)}_{m,R} > 0$ depending only on E, Ω, R and the modulus of continuity of φ, such that for any holomorphic section of $\mathscr{O}_\Omega / \mathscr{I}_Y^{k+1}$ satisfying*

$$\int_Y \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|_E^2} e^{-\varphi} \mathrm{d}V_Y < +\infty,$$

there exists a holomorphic function $F^{(k)}$ on Ω' such that $J^k_{\Omega'} F^{(k)} = f$ and

$$\int_{\Omega'} \frac{|F^{(k)}|^2}{|s|_E^{2m} R(m \log |s|_E^2)} e^{-\varphi} \mathrm{d}V_{\Omega'} \le C^{(k)}_{m,R} \int_Y \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|_E^2} e^{-\varphi} \mathrm{d}V_Y.$$

Even studying the special case $Y = \{z_0\}$ is quite interesting. In this case, we just take $s(z) = (\operatorname{diam}\Omega')^{-1}(z - z_0)$, viewed as a section of the trivial vector bundle $\Omega' \times \mathbb{C}^n$ with $|s| \le 1$. The jet f at z_0 is then given by $a_\alpha \in \mathbb{C}$, $|\alpha| \le k$, $\alpha = (\alpha_1, \dots, \alpha_n)$. Obviously, one has

$$\int_Y \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^n(\mathrm{d}s)|^2} e^{-\varphi} = e^{-\varphi(z_0)} \cdot \sum_{|\alpha| \le k} |a_\alpha|^2.$$

As a result, one obtains the following corollary.

Corollary 4. *Let R be a function in $\mathfrak{R}$. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex open set. Then for any relatively compact open subset $z_0 \in \Omega' \subset \Omega$, any $k \ge 0$ and any $\varphi \in Psh(\Omega)$, there exists a constant $C^{(k)}_{n,R} > 0$ depending only on R, and on the modulus of continuity of φ, such that for all complex numbers a_α, $|\alpha| \le k$, there exists a holomorphic function f on Ω' satisfying*

$$f(z_0) = a_0, \ \frac{\partial^\alpha f}{\partial z^\alpha}(z_0) = a_\alpha, \ 1 \le |\alpha| \le k,$$

$$\int_{\Omega'} \frac{|f|^2}{|z - z_0|^{2n} R\big(2n \log\big(\frac{|z - z_0|}{\operatorname{diam}\Omega'}\big)\big)} e^{-\varphi(z)} \mathrm{d}V_{\Omega'}(z) \le \frac{C^{(k)}_{n,R} e^{-\varphi(z_0)}}{(\operatorname{diam}\Omega')^{2n}} \sum_{|\alpha| \le k} |a_\alpha|^2.$$

Notation 5. Unless otherwise stated, we will always adopt the notations in Section 1 in the latter sections and in particular, we will denote $|s|_E$ and $|\Lambda^m(\mathrm{d}s)|_E$ simply by $|s|$ and $|\Lambda^m(\mathrm{d}s)|$, respectively.

2. Preliminaries for L^2 extension

In this section, we present a few results to be used in the proof of our Theorem 1.

Lemma 6. *Let X be a complex manifold, $Z \subseteq X$ a (possibly singular) subvariety, and L a holomorphic line bundle on X. Suppose that u and f are L-valued forms on X with coefficients in L^2_{loc} such that $\bar\partial u = f$ on $X \setminus Y$ (in the sense of currents). Then one has $\bar\partial u = f$ on X (also in the sense of currents).*

Proof. The proof is almost the same as [4, Lemma 6.9]. Notice that the line bundle L is locally trivial. □

Lemma 7. *Let X be a complete Kähler manifold possessing a non-necessarily complete Kähler metric ω, and Q a Hermitian vector bundle over X. Assume that μ and A are bounded smooth positive functions on X and put*

$$B := \sqrt{-1}\left[\mu\Theta_Q - \partial\bar\partial\mu - A^{-1}\partial\mu\wedge\bar\partial\mu, \Lambda\right],$$

where $\Lambda := \Lambda_\omega$ is the dual Lefschetz operator. Suppose that $\delta \geq 0$ is a number such that $B+\delta I$ is semi-positive definite everywhere on $\Lambda^{n,q}T^*_X\otimes Q$ for some $q \geq 1$. Then given a form $g \in L^2(X, \Lambda^{n,q}T^*_X\otimes Q)$ such that $\bar\partial g = 0$ and $\int_X \langle (B+\delta I)^{-1}g, g\rangle_Q \mathrm{d}V_X < +\infty$, there exists an approximate solution $u \in L^2(X, \Lambda^{n,q-1}T^*_X\otimes Q)$ and an error term $h \in L^2(X,\Lambda^{n,q}T^*_X\otimes Q)$ such that $\bar\partial u + \sqrt{\delta}h = g$ and

$$\int_X \frac{|u|^2_Q}{\mu+A}\mathrm{d}V_X + \int_X |h|^2_Q \mathrm{d}V_X \leq \int_X \langle (B+\delta I)^{-1}g, g\rangle_Q \mathrm{d}V_X.$$

Proof. Lemma 7 appeared as [15, Lemma 3.2], which is modified from [5, Remark 3.2]. Demailly presented the error term method therein aiming to overcome the lack of sufficient positivity to solve $\bar\partial$-equations. □

Lemma 8 (cf. [4, Theorem 1.5]). *Let X be a Kähler manifold and $Z \subset X$ an analytic subset. Assume that $\Omega \subset X$ is a relatively compact open subset equipped with a complete Kähler metric. Then $\Omega \setminus Z$ possesses a complete Kähler metric.*

Lemma 9 (cf. [15, Lemma 3.9]). *Let X be a complex manifold, and Q a Hermitian vector bundle over X. Let*

$$\{\cdot,\cdot\}_Q : \Lambda^{p_1,q_1}T^*_X\otimes Q \times \Lambda^{p_2,q_2}T^*_X \otimes Q \longrightarrow \Lambda^{p_1+q_2,q_1+p_2}T^*_X$$

be the sesquilinear product which combines the wedge product $(u, v) \mapsto u\wedge\bar v$ on scalar valued forms with the Hermitian inner product on Q. Then for any smooth section s of Q and any smooth section w of $T^*_X\otimes Q$ over X,

$$\sqrt{-1}\{w,s\}_Q \wedge \{s,w\}_Q \geq |s|^2_Q\sqrt{-1}\{w,w\}_Q.$$

3. Proof of Theorem 1

Without loss of generality, set $C_R = 1$. Otherwise, one replaces R with $C_R R$ in the proof. We will divide the proof into several steps.

Step 1. ***Constructing special weights and twist factors.***

Recall that a complex manifold X is said to be *weakly pseudoconvex* if there exists a smooth plurisubharmonic exhaustion function P over X. We shall focus on the relatively compact subset $X_c \setminus Y$ rather than working on X itself, where $X_c = \{P < c\}$ ($c = 1,2,\ldots,$ we choose P such that $X_1 \neq \emptyset$). Then $X_c \setminus Y$ ($c = 1,2,\ldots$) is complete Kähler thanks to Lemma 8. For an arbitrary relatively compact subset $\Omega \subset X$, select some c such that $\overline{\Omega} \subset X_c$.

From now on, the jet order k and c will be fixed during the proof.

Put $\beta := \frac{m}{m+k}$. Let $\zeta : (-\infty, 0) \to (0, +\infty)$ be a smooth strictly increasing function, and $\chi : (-\infty, 0) \to (0, +\infty)$ a smooth strictly decreasing function. Assume that $\chi(t) \geq -\frac{\beta \cdot t}{2}$ for $t \in (-\infty, 0)$. We will find more assumptions on ζ and χ in the proof, by which we will get explicit expressions of ζ and χ in the end of this section.

Let $a \in (0,1)$ and put $\sigma_\varepsilon = (m+k)\log(|s|^2 + \varepsilon^2) - a$ and $\sigma = (m+k)\log|s|^2 - a$. Then $\beta \cdot \sigma_\varepsilon = m\log(|s|^2 + \varepsilon^2) - \beta \cdot a$ and $\beta \cdot \sigma = m\log|s|^2 - \beta \cdot a$. As $|s| \leq e^{-\alpha}$ on $X \setminus Y$, there exists a positive number $\varepsilon_a \in (0,1)$ such that $\sigma_\varepsilon \leq -2(m+k)\alpha - \frac{a}{2}$ on $\overline{X_c} \setminus Y$ for $\varepsilon \in (0, \varepsilon_a)$.

The holomorphic line bundle L is equipped with a Hermitian metric h_L, which is written locally as $e^{-\varphi_L}$ for some smooth function with respect to a local holomorphic frame of L. Let $L_{a,\varepsilon}$ denote the line bundle L on $X_c \setminus Y$ equipped with the new metric $h_{a,\varepsilon} := e^{-\varphi_L - \sigma - \zeta(\sigma_\varepsilon)}$.

Set $\tau_\varepsilon = \chi(\sigma_\varepsilon)$ and let A_ε be a smooth positive function on $\overline{X_c}$, which will be determined later. Set $B_\varepsilon = [\Theta_\varepsilon, \Lambda]$ on $X_c \setminus Y$, where

$$\Theta_\varepsilon := \tau_\varepsilon \sqrt{-1}\Theta_{L_{a,\varepsilon}} - \sqrt{-1}\partial\bar{\partial}\tau_\varepsilon - \sqrt{-1}\frac{\partial\tau_\varepsilon \wedge \bar{\partial}\tau_\varepsilon}{A_\varepsilon}.$$

Set

$$\nu_\varepsilon = \frac{\{D's, s\}}{|s|^2 + \varepsilon^2}. \tag{3}$$

We want to find suitable ζ, χ and A_ε such that on $X_c \setminus Y$,

$$\Theta_\varepsilon \geq \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon \wedge \bar{\nu}_\varepsilon. \tag{4}$$

Easy computation yields

$$\begin{aligned}
\Theta_\varepsilon &= \chi(\sigma_\varepsilon)\big(\sqrt{-1}\partial\bar{\partial}\varphi_L + \sqrt{-1}\partial\bar{\partial}\sigma\big) + \big(\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon) - \chi'(\sigma_\varepsilon)\big)\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon \\
&\quad + \left(\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon) - \chi''(\sigma_\varepsilon) - \frac{(\chi'(\sigma_\varepsilon))^2}{A_\varepsilon}\right)\sqrt{-1}\partial\sigma_\varepsilon \wedge \bar{\partial}\sigma_\varepsilon \\
&= \chi(\sigma_\varepsilon)\big(\sqrt{-1}\Theta_L + (m+k)\sqrt{-1}\partial\bar{\partial}\log|s|^2\big) + \big(\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon) - \chi'(\sigma_\varepsilon)\big)\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon \\
&\quad + \left(\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon) - \chi''(\sigma_\varepsilon) - \frac{(\chi'(\sigma_\varepsilon))^2}{A_\varepsilon}\right)\sqrt{-1}\partial\sigma_\varepsilon \wedge \bar{\partial}\sigma_\varepsilon.
\end{aligned}$$

Assuming the equalities

$$\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon) - \chi'(\sigma_\varepsilon) = \beta \tag{5}$$

and

$$\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon) - \chi''(\sigma_\varepsilon) - \frac{(\chi'(\sigma_\varepsilon))^2}{A_\varepsilon} = 0 \tag{6}$$

hold, we can see that

$$\Theta_\varepsilon \geq \chi(\sigma_\varepsilon)\big(\sqrt{-1}\Theta_L + (m+k)\sqrt{-1}\partial\bar{\partial}\log|s|^2\big) + \beta\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon \tag{7}$$

on $X_c \setminus Y$. On the other hand, by (6) one can also assume that $A_\varepsilon = \eta(\sigma_\varepsilon)$ for some smooth function $\eta : (-\infty, 0) \to (0, +\infty)$ such that

$$\chi\zeta'' - \chi'' - \frac{(\chi')^2}{\eta} = 0. \tag{8}$$

As Lemma 9 gives

$$|s|^2\sqrt{-1}\{D's, D's\} \geq \sqrt{-1}\{D's, s\} \wedge \{s, D's\},$$

we obtain that on $X_c \setminus Y$,

$$\begin{aligned}
\sqrt{-1}\partial\bar\partial\sigma_\varepsilon &= \frac{(m+k)\sqrt{-1}\{D's,D's\}}{|s|^2+\varepsilon^2} - \frac{(m+k)\sqrt{-1}\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{(m+k)\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\\
&\geq \frac{(m+k)\varepsilon^2}{|s|^2}\frac{\sqrt{-1}\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{(m+k)\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\\
&= \frac{(m+k)\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar\nu_\varepsilon - \frac{(m+k)\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}.
\end{aligned}$$

Then it follows from (7) that on $X_c \setminus Y$,

$$\Theta_\varepsilon \geq \chi(\sigma_\varepsilon)\big(\sqrt{-1}\Theta_L + (m+k)\sqrt{-1}\partial\bar\partial\log|s|^2\big) - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2} + \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar\nu_\varepsilon.$$

Since $\chi(\sigma_\varepsilon) \geq -\frac{\beta\cdot\sigma_\varepsilon}{2} \geq m\alpha$ by the assumption $\chi(t) \geq -\frac{\beta\cdot t}{2}$, it follows from the curvature conditions on $X \setminus Y$ in Theorem 1 that

$$\begin{aligned}
&\chi(\sigma_\varepsilon)\big(\sqrt{-1}\Theta_L + (m+k)\sqrt{-1}\partial\bar\partial\log|s|^2\big) - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\\
&\qquad = \chi(\sigma_\varepsilon)\big(\sqrt{-1}\Theta_L + (m+k)\sqrt{-1}\partial\bar\partial\log|s|^2\big) - \frac{m\alpha|s|^2}{|s|^2+\varepsilon^2}\frac{\sqrt{-1}\{\Theta_E s,s\}}{\alpha|s|^2}\\
&\qquad \geq \frac{m\alpha|s|^2}{|s|^2+\varepsilon^2}\left(\sqrt{-1}\Theta_L + (m+k)\sqrt{-1}\partial\bar\partial\log|s|^2 - \frac{\{\sqrt{-1}\Theta_E s,s\}}{\alpha|s|^2}\right)\\
&\qquad \geq 0
\end{aligned}$$

on $X_c \setminus Y$. Hence, one gets (4) as expected.

As a result,

$$B_\varepsilon \geq \left[\frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar\nu_\varepsilon, \Lambda\right] = \frac{m\varepsilon^2}{|s|^2}T_{\bar\nu_\varepsilon}T^*_{\bar\nu_\varepsilon} \tag{9}$$

on $X_c \setminus Y$ as an operator on $(n,1)$-forms, where $T_{\bar\nu_\varepsilon}$ denotes the operator $\bar\nu_\varepsilon \wedge \cdot$ and $T^*_{\bar\nu_\varepsilon}$ is its Hilbert adjoint operator.

Step 2. ***Solving $\bar\partial$ on X_c with estimates.***

With such preparation, we now argue by induction on $k \geq 0$. The case $k=0$ is a special case of [15, Theorem 1.1]. Now, assume that the theorem has been proved for $k-1$, and we consider the short exact sequence of sheaves

$$0 \longrightarrow S^k N^*_{Y/X} \longrightarrow \mathscr{O}_X/\mathscr{I}_Y^{k+1} \longrightarrow \mathscr{O}_X/\mathscr{I}_Y^k \longrightarrow 0.$$

Let $J_X^{k-1} f \in H^0(X, K_X \otimes L \otimes \mathscr{O}_X/\mathscr{I}_Y^k)$ be the image of $f \in H^0(X, K_X \otimes L \otimes \mathscr{O}_X/\mathscr{I}_Y^{k+1})$ under the induced cohomology group morphism. By the induction hypothesis, there exists $F^{(k-1)} \in H^0(X, K_X \otimes L)$ such that

$$J_X^{k-1}F^{(k-1)} = J_X^{k-1} f, \quad \int_{X_c}\frac{|F^{(k-1)}|_L^2}{|s|^{2m}R(m\log|s|_E^2)}\mathrm{d}V_{X,\omega} \leq C_{m,R}^{(k-1)}\int_{Y_c}\frac{|f|^2_{s,\rho,(k-1)}}{|\Lambda^m(\mathrm{d}s)|^2}\mathrm{d}V_{Y,\omega}, \tag{10}$$

where $C_{m,R}^{(k-1)} > 0$ is a constant as in the statement of Theorem 1 and $Y_c := Y \cap X_c$. Thus, the image $J_X^{k-1} f - J_X^{k-1}F^{(k-1)} \in H^0(X, K_X \otimes L \otimes \mathscr{O}_X/\mathscr{I}_Y^k)$ of $f - J_X^k F^{(k-1)} \in H^0(X, K_X \otimes L \otimes \mathscr{O}_X/\mathscr{I}_Y^{k+1})$ vanishes. So we can view the jet $f - J_X^k F^{(k-1)}$ as a global holomorphic section (on Y) of the sheaf $K_X \otimes L \otimes S^k N^*_{Y/X} = K_X \otimes L \otimes S^k E^*_{|Y}$.

Using the results in [14, p. 12], one can construct an extension $\widetilde{f} \in C^\infty(X, K_X \otimes L)$ of the holomorphic k-jet $f \in H^0(X, K_X \otimes L \otimes \mathscr{O}_X/\mathscr{I}_Y^{k+1})$ by means of a partition of unity, satisfying

$$\bar\partial\widetilde{f} = 0 \quad \text{on } Y,$$

and

$$|\bar{\partial}\widetilde{f}| = O(|s|^{k+1}) \quad \text{in a neighbourhood of } Y.$$

Set

$$G_\varepsilon^{(k-1)} := \theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right)(\widetilde{f} - F^{(k-1)}) \in C^\infty(X, K_X \otimes L),$$

where $0 < \varepsilon < \varepsilon_a$, and $\theta : \mathbb{R} \to [0,1]$ is a C^∞ function such that $\theta \equiv 0$ on $\left(-\infty, \frac{1}{3}\right]$, $\theta \equiv 1$ on $\left[\frac{2}{3}, +\infty\right)$, and $|\theta'| \le 4$ on $\mathbb{R}$. Then it suffices to solve the equation

$$\bar{\partial} u_\varepsilon = \bar{\partial} G_\varepsilon^{(k-1)}, \tag{11}$$

with the extra condition $\frac{|u_\varepsilon|^2}{|s|^{2(m+k)}} \in L^1_{\text{loc}}$ in a neighbourhood of Y. This condition guarantees that u_ε, as well as all its jets of orders $\le k$, vanishes on Y.

By direct calculations, one has

$$\bar{\partial} G_\varepsilon^{(k-1)} = g_\varepsilon^{(1)} + g_\varepsilon^{(2)},$$

where

$$g_\varepsilon^{(1)} = -\frac{\varepsilon^2}{|s|^2+\varepsilon^2} \cdot \theta'\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right) \overline{\nu}_\varepsilon \wedge (\widetilde{f} - F^{(k-1)}),$$
$$g_\varepsilon^{(2)} = \theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right) \bar{\partial}(\widetilde{f} - F^{(k-1)}).$$

Recall that ν_ε is given in (3).

In this situation, $g_\varepsilon^{(2)}$ turns out to have no contribution in the limit since it converges uniformly to 0 on every compact set when ε tends to 0. Actually, $\text{Supp}(g_\varepsilon^{(2)}) \subset \{|s| < \sqrt{2}\varepsilon\}$ and $|g_\varepsilon^{(2)}| = O(|s|^{k+1})$ because of $|\bar{\partial}\widetilde{f}| = O(|s|^{k+1})$ in a neighbourhood of Y as we have previously shown.

Then

$$\int_{X_c \setminus Y} \langle B_\varepsilon^{-1} g_\varepsilon^{(2)}, g_\varepsilon^{(2)} \rangle_L |s|^{-2(m+k)} \mathrm{d}V_{X,\omega} = O(\varepsilon),$$

provided that B_ε is locally uniformly bounded below in a neighbourhood of Y. Otherwise, we shall solve the approximate equation $\bar{\partial} u + \sqrt{\delta} h = g_\varepsilon$ with $\delta > 0$ small (see Lemma 7 and [5, Remark 3.2] for more details). One can remove the extra error term $\sqrt{\delta} h$ by putting $\delta \to 0$ at the end. Since there is no essential difficulty during this procedure, for the purpose of simplicity, we will assume to have the desired lower bound for B_ε and the estimate of $g_\varepsilon^{(2)}$ as above.

Next, we turn to estimate the term involving $g_\varepsilon^{(1)}$ on $X_c \setminus Y$. By (9),

$$\langle B_\varepsilon^{-1} g_\varepsilon^{(1)}, g_\varepsilon^{(1)} \rangle_{L_{a,\varepsilon}} \le \frac{|s|^2}{m\varepsilon^2} \cdot \left| \theta'\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right) \frac{\varepsilon^2}{|s|^2+\varepsilon^2} (\widetilde{f} - F^{(k-1)}) \right|^2_{L_{a,\varepsilon}}.$$

In [14, p. 17], Popovici showed that on every compact set, $\frac{|(\tilde{f}-F^{(k-1)})(\varepsilon s, z')|_L^2}{\varepsilon^{2k}}$ converges to $|\nabla^k(f - J_X^k F^{(k-1)})(z')|_L^2$ uniformly as $\varepsilon \to 0$. Then using a partition of unity around $\overline{X_c} \setminus Y$ and the Fubini theorem, we obtain

$$\begin{aligned}\int_{X_c \setminus Y} \left\langle B_\varepsilon^{-1} g_\varepsilon^{(1)}, g_\varepsilon^{(1)} \right\rangle_{L_{a,\varepsilon}} \mathrm{d}V_X &\le \frac{16 e^a}{m} \int_{X_c \cap \{\sqrt{\frac{1}{2}}\varepsilon < |s| < \sqrt{2}\varepsilon\}} \frac{\varepsilon^2 |\widetilde{f} - F^{(k-1)}|_L^2 \mathrm{d}V_X}{\left(|s|^2+\varepsilon^2\right)^2 |s|^{2(m+k-1)}} \\ &\longrightarrow \frac{16 e^a}{m} C_{m,k} \int_{Y_c} \frac{\left|\nabla^k \left(f - J_X^k F^{(k-1)}\right)\right|_L^2}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}} \mathrm{d}V_{Y,\omega} \quad (\varepsilon \longrightarrow 0),\end{aligned}$$

where

$$C_{m,k} := \int_{z \in \mathbb{C}^m, \sqrt{\frac{1}{2}} \le |z| \le \sqrt{2}} \frac{\sqrt{-1}\Lambda^m(\mathrm{d}z) \wedge \Lambda^m(\mathrm{d}\bar{z})}{(|z|^2+1)^2 |z|^{2(m+k-1)}}$$

depends only on m and k. It may be worthwhile to note that $\left|\nabla^k\left(f-J_X^k F^{(k-1)}\right)\right|_L = |f-J_X^k F^{(k-1)}|_L$, where $f-J_X^k F^{(k-1)} \in H^0(Y, K_X\otimes L\otimes S^k N^*_{Y/X})$. Then, one has

$$\int_{X_c\setminus Y}\langle B_\varepsilon^{-1}g_\varepsilon^{(1)}, g_\varepsilon^{(1)}\rangle_{L_{a,\varepsilon}}\mathrm{d}V_X \le \frac{16e^a}{m}C_{m,k}\int_{Y_c}\frac{|\nabla^k(f-J_X^kF^{(k-1)})|_L^2}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}}\mathrm{d}V_{Y,\omega},$$

when ε is small enough. By using Lemma 7 with $\delta=0$, we can solve (11), i.e., there exists $u_{c,a,\varepsilon}\in L^2(X_c\setminus Y, K_X\otimes L_{a,\varepsilon})$ such that

$$\bar\partial u_{c,a,\varepsilon} = \bar\partial G_\varepsilon^{(k-1)} = g_\varepsilon^{(1)} + g_\varepsilon^{(2)}$$

on $X_c\setminus Y$ and

$$\int_{X_c\setminus Y}\frac{|u_{c,a,\varepsilon}|_L^2 e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon + A_\varepsilon}\mathrm{d}V_X \le \frac{16e^a}{m}C_{m,k}\int_{Y_c}\frac{|\nabla^k(f-J_X^kF^{(k-1)})|_L^2}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}}\mathrm{d}V_{Y,\omega} + O(\varepsilon). \tag{12}$$

Since $\sigma, \zeta(\sigma_\varepsilon), \tau_\varepsilon + A_\varepsilon$ are all bounded above on $\overline{X_c}$ for each fixed ε, the inequality (12) implies that $u_{c,a,\varepsilon}\in L^2(X_c, K_X\otimes L)$. As (11), (12) and G_ε^{k-1} is smooth, Lemma 6 gives that

$$\bar\partial u_{c,a,\varepsilon} = \bar\partial G_\varepsilon^{(k-1)} = g_\varepsilon^{(1)} + g_\varepsilon^{(2)} \tag{13}$$

extends across Y and

$$\int_{X_c}\frac{|u_{c,a,\varepsilon}|_L^2 e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon + A_\varepsilon}\mathrm{d}V_X \le \frac{16e^a}{m}C_{m,k}\int_{Y_c}\frac{|\nabla^k(f-J_X^kF^{(k-1)})|_L^2}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}}\mathrm{d}V_{Y,\omega} + O(\varepsilon). \tag{14}$$

The jet extension of f to X_c is then given by

$$F_{c,a,\varepsilon}^{(k)} := G_\varepsilon^{(k-1)} - u_{c,a,\varepsilon} + F^{(k-1)}.$$

Then $F_{c,a,\varepsilon}^{(k)}$ is holomorphic on X_c, thanks to (13), $F^{(k-1)}\in H^0(X, K_X\otimes L)$ as well as the ellipticity of the operator $\bar\partial$ in bidegree $(n,0)$. So $u_{c,a,\varepsilon}$ is also smooth on X_c. Locally, near an arbitrary point of Y, all partial derivatives of orders $\le k$ of $F_{c,a,\varepsilon}^{(k)}$ are prescribed by f.

By the variant of Cauchy–Schwarz inequality, we have

$$\begin{aligned}\langle\kappa_1+\kappa_2+\kappa_3, \kappa_1+\kappa_2+\kappa_3\rangle &\le 2\langle\kappa_1+\kappa_2, \kappa_1+\kappa_2\rangle + 2\langle\kappa_3,\kappa_3\rangle \\ &\le 4\langle\kappa_1,\kappa_1\rangle + 4\langle\kappa_2,\kappa_2\rangle + 2\langle\kappa_3,\kappa_3\rangle\end{aligned} \tag{15}$$

for any inner product space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$, $\kappa_1,\kappa_2,\kappa_3\in\mathscr{H}$.

Then for some sufficiently small ε, $R(\beta\cdot\sigma_\varepsilon)\le R(\beta\cdot\sigma)$, $R(m\log|s|^2)\le R(\beta\cdot\sigma)$, the induction hypothesis (10), (14) and (15) give the estimate on the relatively compact open subset X_c,

$$
\begin{aligned}
&\int_{X_c}\frac{|F^{(k)}_{c,a,\varepsilon}|^2_L}{e^{\beta\cdot\sigma}R(\beta\cdot\sigma)}\mathrm{d}V_X \qquad (16)\\
&\le 4\int_{X_c}\frac{|u_{c,a,\varepsilon}|^2_L}{e^{\beta\cdot\sigma}R(\beta\cdot\sigma)}\mathrm{d}V_X+4\int_{X_c}\frac{|F^{(k-1)}|^2_L}{e^{\beta\cdot\sigma}R(\beta\cdot\sigma)}\mathrm{d}V_X+2\int_{X_c}\frac{|G^{(k-1)}_\varepsilon|^2_L}{e^{\beta\cdot\sigma}R(\beta\cdot\sigma)}\mathrm{d}V_X\\
&\le 4\left(\sup_{X_c}\frac{(\tau_\varepsilon+A_\varepsilon)e^{\zeta}(\sigma_\varepsilon)}{R(\beta\cdot\sigma_\varepsilon)}\right)\int_{X_c}\frac{|u_{c,a,\varepsilon}|^2_L e^{-\beta\cdot\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon+A_\varepsilon}\mathrm{d}V_X\\
&\quad+4e^{\beta\cdot a}\int_{X_c}\frac{|F^{(k-1)}|^2_L}{|s|^{2m}R(m\log|s|^2)}\mathrm{d}V_X+2e^{\beta\cdot a}\int_{X_c}\frac{\theta(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})^2|\widetilde{f}-F^{(k-1)}|^2_L}{|s|^{2m}R(m\log|s|^2)}\mathrm{d}V_X\\
&\le 4e^{(\beta-1)a}\left(\sup_{X_c}\frac{(\tau_\varepsilon+A_\varepsilon)e^{\zeta}(\sigma_\varepsilon)}{R(\beta\cdot\sigma_\varepsilon)}\right)\int_{X_c}\frac{|u_{c,a,\varepsilon}|^2_L e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon+A_\varepsilon}\mathrm{d}V_X\\
&\quad+4e^{\beta\cdot a}\int_{X_c}\frac{|F^{(k-1)}|^2_L}{|s|^{2m}R(m\log|s|^2)}\mathrm{d}V_X+2C_1\,e^{\beta\cdot a}\int_{X_c}\frac{1}{|s|^{2m}R(m\log|s|^2)}\mathrm{d}V_X\\
&\le\frac{64e^{\beta\cdot a}}{m}\left(\sup_{X_c}\frac{(\tau_\varepsilon+A_\varepsilon)e^{\zeta}(\sigma_\varepsilon)}{R(\beta\cdot\sigma_\varepsilon)}\right)C_{m,k}\int_{Y_c}\frac{|\nabla^k(f-J^k_XF^{(k-1)})|^2_L}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}}\mathrm{d}V_Y\\
&\quad+4e^{\beta\cdot a}\,C^{(k-1)}_{m,R}\int_{Y_c}\frac{|f|^2_{s,\rho,(k-1)}}{|\Lambda^m(\mathrm{d}s)|^2}\mathrm{d}V_Y+C_2\int_{-\infty}^{2m\log\varepsilon+C_3}\frac{1}{R(t)}\mathrm{d}t+O(\varepsilon)\\
&\le e^{\beta\cdot a}\,C'^{(k)}_{m,R}\int_{Y_c}\frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|^2}\mathrm{d}V_Y\\
&\quad+\frac{64e^{\beta\cdot a}}{m}C_{m,k}\int_{Y_c}\frac{|\nabla^k(J^k_XF^{(k-1)})|^2_L}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}}\mathrm{d}V_Y+C_2\int_{-\infty}^{2m\log\varepsilon+C_3}\frac{1}{R(t)}\mathrm{d}t+O(\varepsilon),
\end{aligned}
$$

where $C'^{(k)}_{m,R}=\frac{64}{m}C_{m,k}+4C^{(k-1)}_{m,R}$ and C_1,C_2,C_3 are all positive numbers independent of ε. Here in (16), we also assume that

$$\frac{(\tau_\varepsilon+A_\varepsilon)e^{\zeta}(\sigma_\varepsilon)}{R(\beta\cdot\sigma_\varepsilon)}=1 \qquad (17)$$

on X_c. We will solve (17) together with (5) and (8) in Step 4.

Step 3. ***Passing to the limits to get the final jet extension on*** Ω**.**

As $\sup_{t\le0}\left(e^tR(t)\right)<\infty$, applying Montel's theorem and (16) to extract a weak limit of $\left\{F^{(k)}_{c,a,\varepsilon}\right\}_{\varepsilon>0}$ as $\varepsilon\to0$, we get a holomorphic L-valued n-form $F^{(k)}_{c,a}$ on X_c such that $J^k_{X_c}F^{(k)}_{c,a}=f$ and

$$\int_{X_c}\frac{|F^{(k)}_{c,a}|^2_L}{e^{\beta\cdot\sigma}R(\beta\cdot\sigma)}\mathrm{d}V_X\le e^{\beta\cdot a}\,C'^{(k)}_{m,R}\int_{Y_c}\frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|^2}\mathrm{d}V_Y+\frac{64e^{\beta\cdot a}}{m}C_{m,k}\int_{Y_c}\frac{|\nabla^k(J^k_XF^{(k-1)})|^2_L}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}}\mathrm{d}V_Y.$$

In other words,

$$
\begin{aligned}
&\int_{X_c}\frac{|F^{(k)}_{c,a}|^2_L}{|s|^{2m}R(m\log|s|^2-\beta\cdot a)}\mathrm{d}V_X\\
&\qquad\le C'^{(k)}_{m,R}\int_{Y_c}\frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|^2}\mathrm{d}V_Y+\frac{64}{m}C_{m,k}\int_{Y_c}\frac{|\nabla^k(J^k_XF^{(k-1)})|^2_L}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}}\mathrm{d}V_Y. \qquad (18)
\end{aligned}
$$

Since R is continuous decreasing on $(-\infty,0]$, $\sup_{t\le 0}(e^t R(t))<\infty$, similarly as before, we use Montel's theorem and extract a weak limit of $\{F^{(k)}_{c,a}\}_{a>0}$ as $a\to 0$, to obtain a holomorphic L-valued n-form $F^{(k)}_c$ on X_c from (18) such that $J^k_{X_c}F^{(k)}_c=f$ and

$$\int_{X_c}\frac{|F^{(k)}_c|^2_L}{|s|^{2m}R(m\log|s|^2)}\mathrm{d}V_X\le C'^{(k)}_{m,R}\int_{Y_c}\frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|^2}\mathrm{d}V_Y+\frac{64}{m}C_{m,k}\int_{Y_c}\frac{|\nabla^k(J^k_X F^{(k-1)})|^2_L}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}}\mathrm{d}V_Y. \tag{19}$$

As Popovici [14, Sections 0.4–0.6] has shown that the last term in the right-hand side of (19) can be controlled uniformly, a slight modification of his proof in [14, Section 0.4] in terms of the variable denominators introduced by [15, p. 136] can complete the proof of Main Theorem 1. Indeed, one just needs to modify the first and second inequalities in [14, p. 22], respectively, as

$$\begin{aligned}&\frac{\sum_{|\alpha|=k}\left|\frac{\frac{\partial^\alpha F^{(k-1)}}{\partial z'^\alpha}(0,z'')}{\alpha!}\right|^2 e^{-2\varphi(0,z'')-2A|z''|^2}}{|\Lambda^m(\mathrm{d}s)(0,z'')|^{2\frac{m+k}{m}}}\\
&\qquad\le \mathrm{Const}\cdot\frac{2(m+k)}{\rho^{2(m+k)}}e^{2(\varepsilon(\rho)+A\rho^2)}\sup_{(z',z'')\in U_j}\frac{|s(z',z'')|^{2m}R(m\log s(z',z'')^2)}{|\Lambda^m(\mathrm{d}s)(0,z'')|^{2\frac{m+k}{m}}}\\
&\qquad\qquad\times\int_{z'\in B'(0,\rho)}\frac{\|F^{(k-1)}(z',z'')\|^2}{|s(z',z'')|^{2m}R(m\log s(z',z'')^2)}\mathrm{d}\lambda(z'),\end{aligned}$$

and

$$\int_{Y_c}\frac{|\nabla^k(J^k_X F^{(k-1)})|^2}{|\Lambda^m(\mathrm{d}s)|^{2\frac{m+k}{m}}}\mathrm{d}V_Y\le D_{m,k}NM(c)\frac{1}{\rho^{2(m+k)}}e^{2(\varepsilon(\rho)+A\rho^2)}\int_{\Omega'}\frac{\|F^{(k-1)}\|^2}{|s|^{2m}R(m\log|s|^2)}\mathrm{d}V_{X,\omega},$$

where

$$M(c):=\sup_{(z',z'')\in\Omega'}\frac{|s(z',z'')|^{2m}R(m\log s(z',z'')^2)}{|\Lambda^m(\mathrm{d}s)(0,z'')|^{2\frac{m+k}{m}}}.$$

and $D_{m,k}:=\mathrm{Const}\cdot 2(m+k)$. Notice that the smoothness of the function R on $(-\infty,0]$ ensures that one can get the suprema on U_j and Ω', respectively. We refer to [14, Section 0.4] for more explanations about the above notations.

Then as a result, we get a holomorphic L-valued n-form $F^{(k)}_c$ on Ω such that $J^k_\Omega F^{(k)}_c=f$ and

$$\begin{aligned}\int_\Omega\frac{|F^{(k)}_c|^2_L}{|s|^{2m}_E R(m\log|s|^2_E)}\mathrm{d}V_{X,\omega}&\le\int_{X_c}\frac{|F^{(k)}_c|^2_L}{|s|^{2m}_E R(m\log|s|^2_E)}\mathrm{d}V_{X,\omega}\\
&\le C^{(k)}_{m,R}\int_{Y_c}\frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|^2_E}\mathrm{d}V_{Y,\omega}\\
&\le C^{(k)}_{m,R}\int_{Y}\frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|^2_E}\mathrm{d}V_{Y,\omega},\end{aligned}$$

where $C^{(k)}_{m,R}>0$ is a constant depending only on m,k,E,R and $\sup_\Omega\|i\Theta(L)\|$.

Step 4. ***Solving ordinary differential equations***

We have already proved Theorem 1, provided that there exist appropriate χ,η,ζ satisfying some assumptions. Now, we will come to use these assumptions about χ,η,ζ to get their explicit expressions.

Notice that (5), (8) and (17) are equivalent to the following system of ordinary differential equations defined on $(-\infty, 0)$:

$$\chi(t)\zeta'(t) - \chi'(t) = \beta, \tag{20}$$

$$(\chi(t) + \eta(t))e^{\zeta(t)} = R(\beta \cdot t), \tag{21}$$

$$\frac{(\chi'(t))^2}{\chi(t)\zeta''(t) - \chi''(t)} = \eta(t), \tag{22}$$

where $\beta = \frac{m}{m+k}$. It is noteworthy that when $k = 0$ (and then $\beta = 1$), our system of ODEs coincides with [15, (4.17), (4.18), (4.19)]. Moreover, we have assumed that ζ, χ and η are all smooth on $(-\infty, 0)$ and that $\zeta > 0, \chi > 0, \eta > 0, \zeta' > 0, \chi' < 0$ and $\chi(t) \geq -\frac{\beta \cdot t}{2}$ on $(-\infty, 0)$. In the proof of Theorem 1, we have assumed that $C_R = \int_{-\infty}^{0} \frac{1}{R(t)} \mathrm{d}t = 1$. Hence, we get $\int_{-\infty}^{0} \frac{1}{R(\beta \cdot t)} \mathrm{d}t = \frac{1}{\beta} \int_{-\infty}^{0} \frac{1}{R(\beta \cdot t)} \mathrm{d}(\beta \cdot t) = \frac{1}{\beta}$.

Following the argument of solving undetermined functions with ODEs introduced in [15, Section 4, pp. 151–153], we get

$$\zeta = -\log\left(1 - \beta \int_{-\infty}^{t} \frac{1}{R(\beta \cdot t_1)} \mathrm{d}t_1\right),$$

$$\chi = \frac{-\beta \cdot t - \beta^2 \int_t^0 (\int_{-\infty}^{t_2} \frac{1}{R(\beta \cdot t_1)} \mathrm{d}t_1) \mathrm{d}t_2}{1 - \beta \int_{-\infty}^{t} \frac{1}{R(\beta \cdot t_1)} \mathrm{d}t_1},$$

$$\eta = \left(1 - \beta \int_{-\infty}^{t} \frac{1}{R(\beta \cdot t_1)} \mathrm{d}t_1\right) R(\beta \cdot t) + \frac{\beta \cdot t + \beta^2 \int_t^0 (\int_{-\infty}^{t_2} \frac{1}{R(\beta \cdot t_1)} \mathrm{d}t_1) \mathrm{d}t_2}{1 - \beta \int_{-\infty}^{t} \frac{1}{R(\beta \cdot t_1)} \mathrm{d}t_1},$$

and

$$\chi' + \frac{\beta}{2} = \beta \left(\frac{-\frac{1}{2}(\lambda_1')^2 + \lambda_1 \lambda_1''}{(\lambda_1')^2} \right) \leq 0.$$

It is easy to verify all the previous assumptions about ζ, χ and η.

Acknowledgements

Both authors would like to express sincere gratitude to Professor Langfeng Zhu for a useful discussion on this work, as well as Jian Chen for many discussions on L^2 theory.

References

[1] B. Berndtsson, L. Lempert, “A proof of the Ohsawa–Takegoshi theorem with sharp estimates”, *J. Math. Soc. Japan* **68** (2016), no. 4, p. 1461-1472.

[2] Z. Błocki, “Suita conjecture and the Ohsawa–Takegoshi extension theorem”, *Invent. Math.* **193** (2013), no. 1, p. 149-158.

[3] J. Chen, S. Rao, “L^2 extension of $\bar{\partial}$-closed forms on weakly pseudoconvex Kähler manifolds”, https://arxiv.org/abs/2009.14101, 2020.

[4] J.-P. Demailly, “Estimations L^2 pour l’opérateur $\bar{\partial}$ d’un fibré vectoriel holomorphe semi-positif au dessus d’une variété kählérienne complète”, *Ann. Sci. Éc. Norm. Supér.* **15** (1982), p. 457-511.

[5] ———, “On the Ohsawa–Takegoshi–Manivel L^2 extension theorem”, in *Complex analysis and geometry (Paris, 1997)*, Progress in Mathematics, vol. 188, Birkhäuser, 2000, p. 47-82.

[6] ———, “Extension of holomorphic functions defined on non reduced analytic subvarieties”, in *The legacy of Bernhard Riemann after one hundred and fifty years. Vol. I*, Advanced Lectures in Mathematics (ALM), vol. 35, International Press, 2016, p. 191-222.

[7] Q. A. Guan, X. G. Zhou, “A solution of an L^2 extension problem with an optimal estimate and applications”, *Ann. Math.* **181** (2015), no. 3, p. 1139-1208.

[8] Q. A. Guan, X. G. Zhou, L. F. Zhu, “On the Ohsawa–Takegoshi L^2 extension theorem and the twisted Bochner–Kodaira identity”, *C. R. Math. Acad. Sci. Paris* **349** (2011), no. 13-14, p. 797-800.

[9] G. Hosono, "The optimal jet L^2 extension of Ohsawa–Takegoshi type", *Nagoya Math. J.* **239** (2020), p. 153-172.
[10] L. Manivel, "Un théorème de prolongement L^2 de sections holomorphes d'un fibré hermitien", *Math. Z.* **212** (1993), no. 1, p. 107-122.
[11] J. D. McNeal, D. Varolin, "Analytic inversion of adjunction: L^2 extension theorems with gain", *Ann. Inst. Fourier* **57** (2007), no. 3, p. 703-718.
[12] ———, "Extension of jets with L^2 estimates, and an application", https://arxiv.org/abs/1707.04483, 2017.
[13] T. Ohsawa, K. Takegoshi, "On the extension of L^2 holomorphic functions", *Math. Z.* **195** (1987), no. 2, p. 197-204.
[14] D. Popovici, "L^2 extension for jets of holomorphic sections of a hermitian line bundle", *Nagoya Math. J.* **180** (2005), p. 1-34.
[15] X. G. Zhou, L. F. Zhu, "An optimal L^2 extension theorem on weakly pseudoconvex Kähler manifolds", *J. Differ. Geom.* **110** (2018), no. 1, p. 135-186.
[16] ———, "Siu's lemma, optimal L^2 extension and applications to twisted pluricanonical sheaves", *Math. Ann.* **377** (2020), no. 1-2, p. 675-722.
[17] L. F. Zhu, Q. A. Guan, X. G. Zhou, "On the Ohsawa–Takegoshi L^2 extension theorem and the Bochner–Kodaira identity with non-smooth twist factor", *J. Math. Pures Appl.* **97** (2012), no. 6, p. 579-601.