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Finite groups with Quaternion Sylow subgroup

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Abstract. In this paper we show that a finite group $G$ with Quaternion Sylow 2-subgroup is 2-nilpotent if, either $3 \nmid |G|$ or $G$ is solvable and the order of its Sylow 2-subgroup is strictly greater than 16.


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1. Introduction

A 2-group of order $2^n$, with a cyclic maximal subgroup is isomorphic to one of the following group:

(i) Cyclic group $Z_{2^n}$ and abelian group $Z_{2^{n-1}} \times Z_2$.

(ii) Dihedral group $D_{2^n}$.

(iii) Semi-dihedral group $SD_{2^n}$.

(iv) Modular group $M(2^n)$.

(v) Quaternion group $Q_{2^n}$.

Let $G$ be a group with a Sylow 2-subgroup $S$ of type (i). Then $G$ is 2-nilpotent, since Aut$(S)$ is a 2-group and so $\mathcal{N}_G(S) = \mathcal{G}_G(S)$. In [5], D. Gorenstein and J. Walter gave the characterization of finite groups with dihedral Sylow 2-subgroups. They proved that, a finite group $G$ with a dihedral Sylow 2-subgroup is 2-nilpotent if $G$ contains a normal subgroup of index 4 (see [5, Lemma 2.1(iii)]). In [8, 9], W. Wong obtained the structure of finite groups whose Sylow 2-subgroups are semi-dihedral or modular 2-group. In the latter case $G$ has a normal 2-complement (see [8, Theorem 1]), so a modular 2-group can not be a Sylow 2-subgroup of a non-solvable group. In the semi-dihedral case, $G$ is 2-nilpotent if foc$(S) = S'$, where $S$ denotes a Sylow 2-subgroup of $G$ and foc$(S)$, the focal subgroup of $S$ [8, Theorem 2 (I)].

In [1], R. Brauer and M. Suzuki proved that any group with a Quaternion Sylow 2-subgroup is not simple.
Now this question seems to be natural: when a group $G$ with a Quaternion Sylow 2-subgroup is 2-nilpotent?

In this paper we obtain sufficient conditions for a group $G$ with a Quaternion (ordinary or generalized) Sylow 2-subgroup to be 2-nilpotent.

Our notations are standard and can be found in [6].

**Main Theorem.** Let $G$ be a finite group with a Quaternion (ordinary or generalized) Sylow 2-subgroup. Then $G$ is 2-nilpotent if:

(i) either $3 \nmid |G|$;

(ii) or $G$ is solvable with a Sylow 2-subgroup of order strictly greater than 16.

**Proof.** Assume that $Q \cong Q_{2^n}$ is a Sylow 2-subgroup of $G$. If $Q \leq G$ then $G = QN$, where $N$ is a complement of $Q$. Now $N$ acts trivially on $Q$, since Aut($Q_{2^n}$) is a 2-group unless for $n = 3$, in the latter case Aut($Q_8$) $\cong S_4$ and $3 \nmid |G|$, again $N$ acts trivially on $Q$. Hence $N \leq G$ and $G \cong Q \times N$. Therefore we can assume that $Q \ncong G$.

(i). By [1], $G$ is not simple. Assume that $G$ is a minimal counterexample and $M$ is a maximal normal subgroup of $G$. If $|M|$ is odd, then $G/M$ is a simple group with a Quaternion Sylow subgroup which is a contradiction, so $2 \mid |M|$. By assumption $M \neq Q$. Suppose that $M \leq Q$, as $G/M$ is a non-abelian simple group and $3 \nmid |G/M|$, as the Suzuki groups are the only non-abelian simple groups which 3 does not divides its order [4], then $G/M$ is a Suzuki group with $Q/M$ as its Sylow 2-subgroup, which is a contradiction for $Q/M$ is either cyclic or dihedral but a Sylow 2-subgroup of a Suzuki group is of exponent 4 and order greater than or equal 64, (see [7, Lemma 1.6(4)] and [2, Lemma 1 & Proposition 3]).

Therefore either $Q \leq M$, then $M$ has a normal 2-complement, for $|M| < |G|$, or $M \cap Q$ is a proper subgroup of $Q$ which is cyclic or $Q_{2^n-1}$, again $M$ has a normal 2-complement. Thus in either case $M$ has a normal 2-complement $M_1$ which is normal in $G$. Now $G/M_1$ has a Quaternion Sylow 2-subgroup. By the choice of $G$, $G \neq QM_1$ so $G/M_1$ has a normal 2-complement $N/M_1$. Hence $G = NQ$ which is a contradiction. Therefore $G$ is 2-nilpotent.

(ii). Proof by induction on $|G|$. Assume that $G$ is solvable and $n \geq 5$. Obviously $Q \neq Q'$. If $Q \subseteq Q'$, as $Q \nsubseteq G'$, then by induction $G'$ is 2-nilpotent with a normal 2-complement $M$ which is normal in $G$. Now $G/M$ has a normal Sylow 2-subgroup so is 2-nilpotent with a normal 2-complement $N/M$. Obviously $N$ is normal a 2-complement of $G$. So assume that $G' \cap Q$ is a proper subgroup of $Q$, then either $G' \cap Q$ is cyclic which again implies that $G'$ is 2-nilpotent and we are done, or $G' \cap Q \cong Q_{2^n-1}$, and in this case $G'$ is 2-nilpotent unless $n = 5$, $3 \nmid |G'|$ and $G' \cap Q$ is non-cyclic and a non-normal subgroup of $G'$. In the latter case, $Q_1 = G' \cap Q \cong Q_{16}$ and $G^{(\ell)} \neq Q_1$, for all $\ell \geq 2$. By solvability for some $\ell$, $G^{(\ell)} \cap Q_1$ is a proper subgroup of $Q_1$. We can assume that $\ell$ is the smallest number such that $G^{(\ell)} \cap Q_1 \neq Q_1$, hence $G^{(\ell)} \cap Q_1$ is cyclic, since $G^{(\ell)} \cap Q_1 \leq Q$. If $G^{(\ell)} \leq Q_1$, then $G^{(\ell-1)}/G^{(\ell)}$ is abelian with a normal 2-complement $M/G^{(\ell)}$, thus $MG^{(\ell)} \leq G^{(\ell-1)}$ and so $M \leq G^{(\ell-1)}$, for $G^{(\ell-1)} = MQ_1$. This implies that $M \leq G$ and so $G/M$ is 2-nilpotent by induction. Therefore $G$ is 2-nilpotent. Otherwise $G^{(\ell)}$ is 2-nilpotent with a normal 2-complement $M$ which is normal in $G$, again $G/M$ is 2-nilpotent by induction and we are done. \square

**Corollary 1.** Let $G$ be a finite group with Quaternion (ordinary or generalized) Sylow 2-subgroup. If a Sylow 3-subgroup of $G$ is normal, then $G$ is 2-nilpotent.

**Proof.** Assume that $P$ is the Sylow 3-subgroup of $G$. As $3 \nmid |G/P|$, $G/P$ is 2-nilpotent by the main Theorem. So $G$ is 2-nilpotent. \square

**Corollary 2.** Let $G$ be a finite group with Quaternion (ordinary or generalized) Sylow 2-subgroup such that $3 \nmid |G|$, then $G$ is solvable.
Remark 3. In the above lemma for $n = 3$ and 4 where $3 \mid |G|$, there exist many examples for which $G$ is not 2-nilpotent. The smallest order of such groups is a group of order 48. Assume that $G := \text{SmallGroup}(48, 28)$ the small group of GAP library [3], in this group $G' \cong \text{SL}(2, 3)$ and a Sylow 2-subgroup of $G$ is non-normal, obviously $G/G' \cong Z_2$. For another example we consider $G \cong Z_5 \times \text{SL}(2, 3)$. In this case $G' \cong Q_8$ and $G \cong (Z_7 \times Z_7) \times \text{SL}(2, 3)$, where $Q_8$ acts irreducibly on $Z_7 \times Z_7$, in this case $G' \cong (Z_7 \times Z_7) \times Q_8$. For non-solvable case let $G := \text{SmallGroup}(672, 1045)$, in this group $G' \cong \text{SL}(2, 7)$ and a Sylow 2-subgroup of $G$ is a Quaternion group of order 32.

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References