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ON PROBABILISTIC GENERALIZATIONS OF THE NYMAN-BEURLING CRITERION FOR THE ZETA FUNCTION

SÉBASTIEN DARSES AND ERWAN HILLION

To Luis Báez-Duarte,
In Memoriam.

Abstract. The Nyman-Beurling criterion is an approximation problem in the space of square integrable functions on $(0,\infty)$, which is equivalent to the Riemann hypothesis. This involves dilations of the fractional part function by factors $\theta_k \in (0,1)$, $k \geqslant 1$. We develop probabilistic extensions of the Nyman-Beurling criterion by considering these θ_k as random: this yields new structures and criteria, one of them having a significant overlap with the general strong Báez-Duarte criterion.

The main goal of the present paper is the study of the interplay between these probabilistic Nyman-Beurling criteria and the Riemann hypothesis. We are able to obtain equivalences in two main classes of examples: dilated structures as exponential $\mathcal{E}(k)$ distributions, and random variables $Z_{k,n}$, $1 \leqslant k \leqslant n$, concentrated around $1/k$ as n is growing. By means of our probabilistic point of view, we bring an answer to a question raised by Báez-Duarte in 2005: the price to pay to consider non compactly supported kernels is a controlled condition on the coefficients of the involved approximations.

1. Introduction

Open problem since Riemann's memoir in 1859, the Riemann hypothesis (RH) enjoys numerous equivalent reformulations from many areas of mathematics. We refer to two expository papers [14] and [8] for discussions about various approaches. One of these stems from functional analysis, which goes back to the works of Nyman [18] and Beurling [11], strengthened by Báez-Duarte [3].

The Nyman-Beurling criterion is an approximation problem in the space of square integrable functions on $(0,\infty)$, which involves dilations of the fractional part function by factors $\theta_k \in (0,1)$, $k \geqslant 1$. We develop in the current paper a new approach based on considering these dilation factors as random and possibly in the whole range $(0,\infty)$. This probabilistic point of view provides new structures and yields an answer to a question raised by Báez-Duarte in [4]: It is possible to obtain a sufficient condition (implying RH) while considering analytic kernels in the general strong Báez-Duarte criterion introduced in [4].

In this introduction, we first start with basic notations. Second, we recall the known deterministic criteria. We then introduce what we call the probabilistic and the general Nyman-Beurling criteria. We finally describe the main results of our paper.

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1.1. **Basic notations.** We adopt the following conventions and notations for *Functions*:

- The indicator function of a set A is defined as $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if not. In particular, we set $\chi = \mathbf{1}_{(0,1]}$. The fractional part (resp. the integral part) of a real number $x \geqslant 0$ reads $\{x\}$ (resp. $\lfloor x \rfloor$), and then $\{x\} = x - \lfloor x \rfloor$. For $\theta > 0$, we set
$$\rho_\theta(t) = \left\{ \frac{\theta}{t} \right\}, \quad t > 0.$$
- The Riemann zeta function ζ is defined for $\sigma > 1$ as
$$\zeta(s) = \sum_{k \geqslant 1} \frac{1}{k^s}, \quad s = \sigma + i\tau.$$
- The Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined as $\mu(1) = 1$, $\mu(n) = (-1)^r$ if $n = p_1 \dots p_r$ where $p_1, \dots, p_r$ are distinct primes, and $\mu(n) = 0$ if not (i.e. $\mu(n) = 0$ if $p^2 | n$).
- We use either Landau's notation $f = O(g)$ or Vinogradov's $f \ll_\alpha g$ to mean that $|f| \leqslant C|g|$ for some constant $C > 0$ that may depend on a parameter α.

Hilbert spaces

- The Hilbert space $H = L^2(0, \infty)$ of real valued square integrable functions for the Lebesgue measure is endowed with its scalar product (and associated norm $\|f\|_H$):
$$\langle f, g \rangle_H = \int_0^\infty f(t) g(t) dt.$$
- Let $(f_\alpha)_{\alpha \in A}$ be a family in a Hilbert space F. We define $\operatorname{span}_F\{f_\alpha, \alpha \in A\}$ as the closure in F of the vector space spanned by $(f_\alpha)_{\alpha \in A}$.

Probability

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. We set $\mathcal{H} = L^2(\Omega, H)$.
- The space of non negative random variables (r.v.) having p-moment is denoted by $L_+^p(\Omega)$, $p \geqslant 1$. The expectation (resp. the variance) of $X \in L_+^2(\Omega)$ reads $\mathbb{E}[X]$, or simply $\mathbb{E}X$ (resp. $\operatorname{Var}(X)$). We also set $\|X\|_p = (\mathbb{E}X^p)^{1/p}$ when $X \in L_+^p(\Omega)$.
- We write $X \sim \Gamma(\beta, \lambda)$ to mean that the r.v. X is Gamma distributed with parameters (β, λ). In that case, $\mathbb{E}X = \frac{\beta}{\lambda}$ and $\operatorname{Var}(X) = \frac{\beta}{\lambda^2}$. The particular case of the exponential law $\mathcal{E}(\lambda) = \Gamma(1, \lambda)$ of parameter λ will be one basic example throughout the paper. Recall that if $X \sim \mathcal{E}(1)$ and $\lambda > 0$, then $X/\lambda \sim \mathcal{E}(\lambda)$.

1.2. **The deterministic criteria.** Let us recall the fundamental identity (see e.g. [20, (2.1.5)])
$$\int_0^\infty \left\{ \frac{1}{t} \right\} t^{s-1} dt = -\frac{\zeta(s)}{s}, \quad 0 < \sigma < 1,$$
which gives, by means of a change of variable, the following relationship between ζ and the Mellin transform of ρ_θ:
$$\widehat{\rho_\theta}(s) = \int_0^\infty \rho_\theta(t) t^{s-1} dt = -\theta^s \frac{\zeta(s)}{s}, \quad 0 < \sigma < 1. \tag{1.1}$$

RH states that the non-trivial zeros of ζ belong to the critical line $\sigma = \frac{1}{2}$. See [20] and [19] for basic and advanced theory on ζ. Equation (1.1) allows for different equivalent restatements of RH, which has been first done in [18], [11]. The Nyman-Beurling criterion (NB) can be stated as follows:

Theorem 1.1 ([5]). — *RH holds if and only if*

$$\chi \in \mathrm{span}_H\{\rho_\theta,\ 0 < \theta \leqslant 1\}. \tag{1.2}$$

A proof of the if part of Theorem 1.1 will be given at the beginning of Section 3.

Remark 1.2. — Theorem 1.1 is stated in a slightly different form than in the original papers [18], [11], in which the Hilbert space considered by the authors is $L^2(0,1)$. See [5] for the extension to the case of H.

Hence, RH holds if, given $\varepsilon > 0$, there exist $n \geqslant 1$, coefficients $c_1, \dots, c_n \in \mathbb{R}$, and $\theta_1, \dots, \theta_n \in (0,1]$ such that

$$\int_0^\infty \Big(\chi(t) - \sum_{k=1}^n c_k \Big\{\frac{\theta_k}{t}\Big\}\Big)^2 dt < \varepsilon. \tag{1.3}$$

Equation (1.3) is reminiscent to the following convergence result:

$$\lim_{n\to\infty} \sum_{k=1}^n \mu(k)\Big\{\frac{1}{kt}\Big\} = -\chi(t), \quad t > 0. \tag{1.4}$$

This convergence holds point-wise and does not hold in H, see [2, p.5-6] for details, but this identity lead Báez-Duarte towards a stronger form of the Nyman-Beurling criterion, namely,

Theorem 1.3 ([3]). — *RH holds if and only if*

$$\chi \in \mathrm{span}_H\{\rho_{1/n}, n \geqslant 1\}. \tag{1.5}$$

The Báez-Duarte criterion (BD) can be restated as follows. For $n \geqslant 1$, let χ_n be the orthogonal projection of χ onto the linear subspace $H_n \subset H$ spanned by the family $(\rho_k)_{1\leqslant k \leqslant n}$. The quantity

$$d_n = \|\chi - \chi_n\|_H$$

is the distance between χ and H_n. Then RH holds if and only if $\lim_{n\to\infty} d_n = 0$ (It is furthermore equivalent to a particular asymptotic behaviour of the coefficients of χ_n, see [22]). A stronger statement is actually conjectured, namely

$$d_n^2 \sim \frac{C}{\log(n)},$$

where $C = 2 + \gamma - \log(4\pi)$, see [10]. Burnol proved the inequality $d_n^2 \geqslant \frac{C+o(1)}{\log(n)}$ for the same constant C, see [12]. The inequality $d_n < \varepsilon$ provides zero-free regions for ζ, see [17] for details when considering NB, and [16] for more general results on Dirichlet series.

A more general criterion has been stated in 2005 by Báez-Duarte, see [4]. It is based on the Müntz transform

$$Pf(t) = \sum_{k\geqslant 1} f(tk) - \frac{1}{t}\int_0^{+\infty} f(x)dx, \quad f \in L^1(0,\infty), \tag{1.6}$$

which is related to ζ via the Müntz formula, see [13, Theorem 3.1], and [13] for a general study. Báez-Duarte considers in [4] good kernels f, see [4, Definition 1.1] (this notion will be studied in Section 3.3). The general strong Báez–Duarte criterion (gBD) is stated as follows:

Theorem 1.4. — [4, Theorem 1.2] *Let $f : (0, \infty) \to \mathbb{R}$ be a good kernel. If RH holds then*

$$f \in \operatorname{span}_H \{t \mapsto Pf(nt), n \geqslant 1\}. \tag{1.7}$$

Conversely, if (1.7) *is satisfied for a good kernel f that is* compactly supported *and whose Mellin transform has no zeros in the critical strip $\{1/2 < \sigma < 1\}$, then RH holds.*

The extension of the sufficient criterion for kernels f that are not compactly supported is left open in [4]. The compactness assumption prevents from using analytical kernels, which are crucial for regularization and explicit calculations of the Müntz transform. One contribution of our paper is the extension of Theorem 1.4 to a class of non-compactly supported kernels.

1.3. **The probabilistic and general criteria.** The basic idea of our work is to randomize the variables θ_k in NB, i.e. by replacing them by random variables. More precisely:

Definition 1.5. — Given a family $(Z_{k,n})_{n \geqslant 1, 1 \leqslant k \leqslant n}$ of random variables and a family of coefficients $(c_{k,n})_{n \geqslant 1, 1 \leqslant k \leqslant n}$, we consider the distances

$$\mathcal{D}_n^2 = \mathbb{E} \int_0^\infty \Big(\chi(t) - \sum_{k=1}^n c_{k,n} \Big\{ \frac{Z_{k,n}}{t} \Big\} \Big)^2 dt \qquad \text{(pNB)},$$

$$D_n^2 = \int_0^\infty \Big(\chi(t) - \sum_{k=1}^n c_{k,n} \mathbb{E} \Big\{ \frac{Z_{k,n}}{t} \Big\} \Big)^2 dt \qquad \text{(gNB)}.$$

We say that the family $(Z_{k,n})$ satisfies the probabilistic (resp. general) Nyman–Beurling criterion pNB$(Z_{k,n})$ (resp. gNB$(Z_{k,n})$), if one can find coefficients $(c_{k,n})$ such that $\mathcal{D}_n \to 0$ (resp. $D_n \to 0$).

Let us notice that $D_n^2 \leqslant \mathcal{D}_n^2$, due to $(\mathbb{E}X)^2 \leqslant \mathbb{E}X^2$, which means that pNB implies gNB.

1.4. **Main results and Outline.** In Section 2, we show the implication pNB $\Longrightarrow$ RH under an assumption $(\mathcal{P})$ that is suited for random variables on $(0, 1)$, but mild enough to be satisfied for some families supported on $(0, \infty)$. The proof of this implication is based on Erdös' probabilistic method. In Section 2.3, we study families of random variables that are more and more concentrated around $1/k$, $k \geqslant 1$. For such families, we prove the equivalence pNB $\Longleftrightarrow$ RH. The intuitive underlying idea is that the random perturbation around $1/k$ is sufficiently small to be able to use a quantitative version of the Báez-Duarte criterion.

In Section 3, we prove the implication gNB $\Longrightarrow$ RH under a moment assumption on the sequence $(Z_{k,n})$. These r.v. may have a non-compact support, but the price to pay is a condition (C) on the growth of the coefficient $c_{k,n}$. We finally prove the implication RH $\Longrightarrow$ gNB + (C) for dilated r.v. using [4, Theorem 1.2] and a probabilistic interpretation of the Müntz operator.

In Section 4, we illustrate our various criteria with exponential r.v. $\mathcal{E}(k)$ (prototype for dilated r.v.) and Gamma distribution $\Gamma(k,n)$ (prototype for concentrated r.v.).

2. The pNB criterion

2.1. Probabilistic framework and preliminaries. The Hilbert space $\mathcal{H} = L^2(\Omega, H) \simeq L^2(\Omega \times (0,\infty))$ is endowed with the scalar product

$$\langle Z, Z'\rangle_{\mathcal{H}} = \mathbb{E}\langle Z(\omega,\cdot), Z'(\omega,\cdot)\rangle_H = \mathbb{E}\int_0^\infty Z(\omega,t)Z'(\omega,t)dt.$$

To any random variable $X : \Omega \to \mathbb{R}$, we associate the random Beurling function $\rho_X(t) = \left\{\frac{X}{t}\right\}$, which belongs to $\mathcal{H}$ when $X \in L^1_+(\Omega)$, see Lemma 2.1 below. We also introduce the "indicator random variable" $\chi(\omega,t) = \mathbf{1}_{[0,1]}(t)$, which is constant as an element of $L^2(\Omega, H)$.

A natural generalization of the deterministic Nyman-Beurling criterion to a probabilistic framework is the pNB criterion, defined in Definition 1.5.

As in the deterministic case, see [6], the interesting point of such a criterion relies on the formula expressing the squared distance in $\mathcal{H}$ between χ and any subspace $\mathrm{span}_{\mathcal{H}}\{\rho_{Z_{k,n}}, 1 \leqslant k \leqslant n\}$ as a quotient of Gram determinants. One goal is then to figure out laws that reveal remarkable structures in the scalar products, leading to calculable determinants.

We recall that if $A \in \mathcal{F}$ is an event, then $\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A)$, and that if $X \in L^1(\Omega)$, $X \geqslant 0$, Fubini theorem yields the identity

$$\mathbb{E}X = \mathbb{E}\int_0^\infty \mathbf{1}_{t\leqslant X}dt = \int_0^\infty \mathbb{P}(X \geqslant t)dt.$$

Proposition 2.1. — *Let $X \in L^1_+(\Omega)$. Then $\rho_X \in \mathcal{H}$ and*

$$\|\rho_X\|^2_{\mathcal{H}} = \mathbb{E}\int_0^\infty \left\{\frac{X}{t}\right\}^2 dt = (\log(2\pi) - \gamma)\ \|X\|_{L^1_+(\Omega)},$$

where γ is the Euler constant.

Proof. — If $X(\omega) = 0$ then

$$\int_0^\infty \left\{\frac{X(\omega)}{t}\right\}^2 dt = 0.$$

By the change of variable $u = t/X(\omega)$ when $X(\omega) \neq 0$, we obtain

$$\mathbb{E}\int_0^\infty \left\{\frac{X}{t}\right\}^2 dt = \mathbb{E}\int_0^\infty \left\{\frac{1}{u}\right\}^2 X du = \mathbb{E}[X]\int_0^\infty \left\{\frac{1}{u}\right\}^2 du.$$

The last integral can be bounded by 2 but is actually computed in [7, Prop.87 p.38]:

$$\int_0^\infty \left\{\frac{1}{u}\right\}^2 du = \int_0^\infty \left\{t\right\}^2 \frac{dt}{t^2} = \log(2\pi) - \gamma. \quad \square$$

Proposition 2.2. — *Let $Z \in L^2_+(\Omega)$ and $\alpha \in (0,1)$. Then, for any $M > 0$,*

$$M^\alpha \int_M^\infty \mathbb{E}\left\{\frac{Z}{t}\right\}^2 dt \leqslant \|Z\|_2^{1+\alpha}\int_0^\infty u^\alpha\left\{\frac{1}{u}\right\}^2 du < \infty.$$

Proof. — By the change of variable $t = Zu$ and Fubini

$$\int_M^\infty \mathbb{E}\Big\{\frac{Z}{t}\Big\}^2 dt = \mathbb{E}\int_0^\infty \mathbf{1}_{t\geqslant M}\Big\{\frac{Z}{t}\Big\}^2 dt$$
$$= \mathbb{E}\int_0^\infty \mathbf{1}_{uZ\geqslant M}\Big\{\frac{1}{u}\Big\}^2 Zdu.$$

But, by the Cauchy-Schwarz and Markov's inequalities,

$$\mathbb{E}[\mathbf{1}_{uZ\geqslant M}Z] \leqslant \sqrt{\mathbb{P}(uZ \geqslant M)}\|Z\|_2 \leqslant \frac{u^\alpha}{M^\alpha}\sqrt{\mathbb{E}Z^{2\alpha}}\|Z\|_2.$$

Noting that $\sqrt{\mathbb{E}Z^{2\alpha}} = \|Z\|_{2\alpha}^\alpha \leqslant \|Z\|_2^\alpha$ we obtained the desired result. □

2.2. pNB implies RH under a mild condition.

Definition 2.3. — A family $(Z_{k,n})_{1\leqslant k\leqslant n, n\geqslant 1}$ in $L^1_+(\Omega)$ is said to satisfy Assumption $(\mathcal{P})$ if

$$\exists \nu > 0\ ,\ \forall n \geqslant 1\ ,\ \mathbb{P}(B_n) > \nu, \qquad (\mathcal{P})$$

where

$$B_n = \bigcap_{1\leqslant k\leqslant n}\{0 < Z_{k,n} \leqslant 1\}.$$

As an example, let $(X_k)_{k\geqslant 1}$ be a sequence of independent r.v. such that $X_k \sim \mathcal{E}(k)$. Then

$$\mathbb{P}\Big(B_n\Big) = \prod_{k=1}^n (1 - e^{-k}) \geqslant \prod_{k=1}^\infty (1 - e^{-k}) > 0,$$

the later product being convergent since $\sum_{k\geqslant 1} e^{-k} < \infty$. Thus, $(X_k)_{k\geqslant 1}$ verifies Assumption $(\mathcal{P})$.

Theorem 2.4. — *Let $(Z_{k,n})_{1\leqslant k\leqslant n, n\geqslant 1}$ be a collection of r.v. in $L^1_+(\Omega)$ satisfying Assumption $(\mathcal{P})$ and the pNB criterion. Then RH holds.*

The underlying idea of the proof consists in showing that the classical Nyman-Beurling criterion holds via Erdös' probabilistic method: in order to prove that an object exists, it suffices to show that it belongs to a set of positive measure, as explained in Chapter 1 of [1].

Proof. — Fix $\varepsilon > 0$. Since $(Z_{k,n})_{1\leqslant k\leqslant n, n\geqslant 1}$ satisfies pNB and Assumption $(\mathcal{P})$, there exist $\nu > 0$, $n \geqslant 1$, and $c_{1,n}, \cdots, c_{n,n} \in \mathbb{R}$ such that $\mathbb{P}\Big(B_n\Big) > \nu$ and

$$\mathcal{D}_n^2 = \mathbb{E}\Big\|\chi(t) - \sum_{k=1}^n c_{k,n}\Big\{\frac{Z_{k,n}}{t}\Big\}\Big\|_H^2 < \varepsilon^2\nu.$$

Let us consider the event

$$A_n = \Big\{\Big\|\chi(t) - \sum_{k=1}^n c_{k,n}\Big\{\frac{Z_{k,n}}{t}\Big\}\Big\|_H \leqslant \varepsilon\Big\}.$$

By Markov's inequality, $\mathbb{P}({}^c A_n) \leqslant \mathcal{D}_n^2/\varepsilon^2$, and then

$$\mathbb{P}(A_n) \geqslant 1 - \frac{\mathcal{D}_n^2}{\varepsilon^2} > 1 - \nu.$$

Hence $\mathbb{P}(A_n) + \mathbb{P}(B_n) > 1$ and so

$$\mathbb{P}(A_n \cap B_n) = \mathbb{P}(A_n) + \mathbb{P}(B_n) - \mathbb{P}(A_n \cup B_n) > 0.$$

Thus, there exists $\omega \in \Omega$ such that, writing $\theta_k = Z_{k,n}(\omega)$,

$$\Big\|\chi(t) - \sum_{k=1}^{n} c_k \Big\{\frac{\theta_k}{t}\Big\}\Big\|_H \leqslant \varepsilon \ , \quad 0 < \theta_k \leqslant 1, \quad 1 \leqslant k \leqslant n.$$

Therefore, from the classical Nyman-Beurling criterion (Theorem 1.1), RH holds. □

2.2.1. *A lower bound.* Let $(Z_{k,n})$ be a family of r.v. in $L^2_+(\Omega)$. Set

$$\begin{aligned} m_n &= \min_{1\leqslant k\leqslant n} Z_{k,n}, \\ M_n &= \max_{1\leqslant k\leqslant n} Z_{k,n}, \\ \mathcal{D}_n &= \inf_{c_{1,n},\cdots,c_{n,n}} \Big\|\chi(t) - \sum_{k=1}^{n} c_{k,n} \Big\{\frac{Z_{k,n}}{t}\Big\}\Big\|_{\mathcal{H}}. \end{aligned}$$

Lemma 2.5. — *Let $(Z_{k,n})$ satisfy Assumption $(\mathcal{P})$. Then the following lower bound holds:*

$$\mathcal{D}_n^2 \gg \frac{1}{\log 2 + \mathbb{E}\big|\log m_n\big|}.$$

Proof. — Let us define

$$B_\lambda = \Big\{\sum_{k=1}^{n} c_{k,n}\rho_{\theta_k}\,,\ n \geqslant 1, c_{k,n} \in \mathbb{R}, 0 < \theta_k \leqslant 1, \min_{1\leqslant k\leqslant n} \theta_k \geqslant \lambda\Big\},$$

and $d(\lambda)$ the distance in H between χ and B_λ. We recall a fundamental inequality obtained in [5, p.131]:

$$d(\lambda)^2 \gg \frac{1}{\log(2/\lambda)}.$$

We then deduce that for all $n \geqslant 1$, $c_{k,n} \in \mathbb{R}$ and almost surely,

$$\Big\|\chi(t) - \sum_{k=1}^{n} c_{k,n} \Big\{\frac{Z_{k,n}}{t}\Big\}\Big\|_H \mathbf{1}_{M_n\leqslant 1} \gg \frac{\mathbf{1}_{M_n\leqslant 1}}{\sqrt{\log(2/m_n)}}.$$

Since $(Z_{k,n})_{1\leqslant k\leqslant n, n\geqslant 1}$ satisfies Assumption $(\mathcal{P})$, $\mathbb{E}\mathbf{1}_{M_n\leqslant 1} \gg 1$, and then by Cauchy-Schwarz inequality,

$$1 \ll \sqrt{\mathbb{E}\Big|\log\Big(\frac{2}{m_n}\Big)\Big|\mathbf{1}_{M_n\leqslant 1}}\sqrt{\mathbb{E}\Big\|\chi(t) - \sum_{k=1}^{n} c_{k,n}\Big\{\frac{Z_{k,n}}{t}\Big\}\Big\|_H^2}.$$

In particular,

$$\mathcal{D}_n\sqrt{\mathbb{E}\Big|\log\Big(\frac{2}{m_n}\Big)\Big|} \gg 1,$$

which yields the conclusion by the triangle inequality. □

2.3. **RH implies pNB under concentration.** The goal of this section is to prove the following:

Theorem 2.6. — *For any $n \geqslant 1$, let $(X_{k,n})_{1 \leqslant k \leqslant n}$ be r.v. in $L^1_+(\Omega)$ such that, setting $Y_{k,n} = \sqrt{X_{k,n}}$,*

$$\mathbb{E}\, Y_{k,n} = \frac{1}{\sqrt{k}}, \quad 1 \leqslant k \leqslant n, \tag{2.1}$$

$$\sup_{1 \leqslant k \leqslant n} \operatorname{Var} Y_{k,n} \ll n^{-3-\vartheta}, \tag{2.2}$$

$$\mathbb{P}(Y_{1,n} \geqslant 1) \leqslant 1 - \nu, \tag{2.3}$$

for some $\vartheta > 0$ and $\nu \in (0,1)$. Therefore, RH holds if and only if $(X_{k,n})_{n \geqslant 1, k \in [\![1,n]\!]}$ satisfies pNB.

One can check for instance that the r.v. $Y_{k,n} \sim \Gamma\big(\frac{n^{3+\vartheta}}{k}, \frac{n^{3+\vartheta}}{\sqrt{k}}\big)$, $\vartheta > 0$, satisfy Conditions (2.1), (2.2) and (2.3), see Section 4 for discussions about examples.

In order to prove Theorem 2.6, RH will be used at two different places and in two different ways, first via an explicit version of Báez-Duarte criterion; combining Proposition 1, 2 and 3 in [9], one obtains

Theorem 2.7 ([9]). — *For $\varepsilon > 0$ and $n \geqslant 1$, set*

$$\nu_{n,\varepsilon} = \Big\| \chi(t) + \sum_{k \leqslant n} \mu(k) k^{-\varepsilon} \Big\{ \frac{1}{kt} \Big\} \Big\|_H^2. \tag{2.4}$$

Under RH, the following limit holds

$$\limsup_{n \to \infty} \nu_{n,\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0.$$

In order to prove Theorem 2.6, we will use some information about the coefficients in these linear combinations, namely that $|\mu(n)| \leqslant 1$. RH will be also used via the Lindelöf hypothesis about the rate of growth of the ζ function on the critical line (see [20, p. 336-337]):

Theorem 2.8 ([20]). — *Under RH, the Lindelöf hypothesis holds:*

$$\big|\zeta(1/2 + it)\big| \ll_\eta t^\eta, \quad \eta > 0. \tag{2.5}$$

We now turn to the proof of Theorem 2.6.

Proof. — To prove that pNB implies RH, by virtue of Theorem 2.4, it suffices to show that the family $(X_{k,n})_{1 \leqslant k \leqslant n}$ satisfies Assumption $(\mathcal{P})$. First, by union bound and Chebyshev's inequality, we have

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{k=2}^n \{Y_{k,n} \geqslant 1\}\Big) &\leqslant \sum_{k=2}^n \mathbb{P}(Y_{k,n} \geqslant 1) \leqslant \sum_{k=2}^n \frac{\operatorname{Var} Y_{k,n}}{(1 - 1/\sqrt{k})^2} \\ &\leqslant \frac{1}{(1 - 1/\sqrt{2})^2} \frac{n}{n^{3+\vartheta}} \leqslant \frac{12}{n^{2+\vartheta}} \xrightarrow[n \to \infty]{} 0.\end{aligned}$$

Moreover, we have $\mathbb{P}(Y_{1,n} \geqslant 1) \leqslant 1 - \nu$ for all $n \geqslant 1$. Therefore, for all n sufficiently large,

$$\mathbb{P}\Big(\bigcup_{k=1}^n \{Y_{k,n} \geqslant 1\}\Big) \leqslant 1 - \nu/2,$$

and then, taking the complement, $(\mathcal{P})$ holds.

Let us now prove that pNB$((X_{k,n}))$ holds under RH. We have

$$\begin{aligned}\inf_{a_1,\cdots,a_n} \mathbb{E}\Big\|\chi - \sum_{k=1}^n a_k \rho_{X_{k,n}}\Big\|_H^2 &\leqslant \mathbb{E}\Big\|\chi + \sum_{k=1}^n \mu(k)k^{-\varepsilon}\rho_{X_{k,n}}\Big\|_H^2 \\ &\leqslant \mathbb{E}\Big\|\chi + \sum_{k=1}^n \mu(k)k^{-\varepsilon}\rho_{1/k} + \sum_{k=1}^n \mu(k)k^{-\varepsilon}(\rho_{X_{k,n}} - \rho_{1/k})\Big\|_H^2 \\ &\ll \nu_{n,\varepsilon} + \mathbb{E}\Big\|\sum_{k=1}^n \mu(k)k^{-\varepsilon}(\rho_{X_{k,n}} - \rho_{1/k})\Big\|_H^2 = \nu_{n,\varepsilon} + \mathcal{R}_{n,\varepsilon}.\end{aligned}$$

It thus remains to study $\mathcal{R}_{n,\varepsilon}$. Using Plancherel's formula, see [9, Prop.1], we obtain

$$\mathcal{R}_{n,\varepsilon} = \mathbb{E}\Big\|\sum_{k=1}^n \mu(k)k^{-\varepsilon}\Big(\frac{1}{k^s} - X_{k,n}^s\Big)\frac{\zeta(s)}{s}\Big\|_{L^2}^2 = \mathbb{E}\int_{-T_n}^{T_n} V_n(t)dt + \mathbb{E}\int_{|t|\geqslant T_n} V_n(t)dt, \tag{2.6}$$

where

$$V_n(t) = \Big|\sum_{k=1}^n \mu(k)k^{-\varepsilon}\Big(\frac{1}{k^s} - X_{k,n}^s\Big)\frac{\zeta(s)}{s}\Big|^2, \quad s = \frac{1}{2} + it,$$

and where the parameter T_n is to be chosen later. The quantity $V_n(t)$ depends on ε, but we do not mention this dependence as a subscript since we will bound it independently of ε just below.

Let us recall the following useful inequality for $a, b \in \mathbb{R}$, $\Re(s) = 1/2$,

$$|e^{as} - e^{bs}| = \Big|\int_a^b s e^{us} du\Big| \leqslant |s|\Big|\int_a^b e^{u/2} du\Big| = 2|s|\Big|e^{a/2} - e^{b/2}\Big|. \tag{2.7}$$

Using the Cauchy-Schwarz inequality, $|\mu(k)k^{-\varepsilon}| \leqslant 1$ and (2.7), we obtain

$$V_n(t) \leqslant n\sum_{k=1}^n \Big|\frac{1}{k^s} - X_{k,n}^s\Big|^2 \frac{|\zeta(s)|^2}{|s|^2} \tag{2.8}$$

$$\leqslant 4n\sum_{k=1}^n \Big|\frac{1}{\sqrt{k}} - Y_{k,n}\Big|^2 |\zeta(s)|^2. \tag{2.9}$$

Let us consider the term $\mathbb{E}\int_{-T_n}^{T_n}$ in (2.6). From (2.9), the Lindelöf hypothesis (cf. Theorem 2.8) written as $|\zeta(s)| \ll t^{\eta/2}$, and (2.2), we can write for any $\eta > 0$,

$$\begin{aligned}\mathbb{E}\int_{-T_n}^{T_n} V_n(t)dt &\leqslant 4n\sum_{k=1}^n \mathrm{Var}\,(Y_{k,n}) \int_{-T_n}^{T_n} |\zeta(s)|^2 dt \\ &\ll n^2 \sup_{1\leqslant k\leqslant n} \mathrm{Var}\,(Y_{k,n})\; T_n^{1+\eta} \\ &\ll n^{-1-\vartheta}\; T_n^{1+\eta}.\end{aligned} \tag{2.10}$$

We now study the term $\mathbb{E}\int_{|t|\geqslant T_n}$. From (2.8), we obtain

$$\mathbb{E}V_n(t) \leqslant 2n\sum_{k=1}^n \Big(\frac{1}{k} + \mathbb{E}X_{k,n}\Big)\frac{|\zeta(s)|^2}{|s|^2}.$$

But $\mathbb{E}X_{k,n} = (\mathbb{E}Y_{k,n})^2 + \mathrm{Var}\ (Y_{k,n}) \ll 1/k$, therefore, for any $\eta \in (0,1)$,

$$\mathbb{E}\int_{|t|\geqslant T_n} V_n(t)dt \ll n\log n\ T_n^{\eta-1}. \tag{2.11}$$

We finally need to tune η and T_n accordingly. Recall that $\vartheta > 0$ is given. Choose $\eta > 0$ such that $\frac{1+\eta}{1-\eta} < 1+\vartheta/2$, and $\alpha > 1$ so that

$$0 < \frac{1}{1-\eta} < \alpha < \frac{1+\vartheta/2}{1+\eta}.$$

Set $T_n = n^\alpha$. Hence, from (2.10) and (2.11)

$$\mathbb{E}\int_{|t|\geqslant T_n} V_n(t)dt \ll n^{1-\alpha(1-\eta)}\log n \xrightarrow[n\to\infty]{} 0,$$
$$\mathbb{E}\int_{-T_n}^{T_n} V_n(t)dt \ll n^{-\vartheta/2} \xrightarrow[n\to\infty]{} 0.$$

Finally, we conclude with Theorem 2.7. □

3. The gNB criterion

3.1. **Proof of the sufficient part in the deterministic criterion.** It is first important to write a short proof of the sufficient implication in the NB criterion, stated in Theorem 1.1 since the proof of Theorem 3.1 will follow a similar structure. Let us prove that RH holds if

$$\chi \in \mathrm{span}_H\{\rho_\theta,\ 0 < \theta \leqslant 1\}, \tag{3.1}$$

by adapting the original proof, see e.g. [11], with [5].

Proof. — We recall the proof of this sufficient condition by Nyman and completed by an argument in [5], see [5, Lemme 1 & Prop. 1 p.133]. Assume that (3.1) is satisfied, i.e. that there exist coefficients $c_k = c_{k,n}$ and $\theta_k = \theta_{k,n}$ such that

$$d_n^2 = \int_0^\infty \Big(\chi(t) - \sum_{k=1}^n c_k\Big\{\frac{\theta_k}{t}\Big\}\Big)^2 dt \xrightarrow[n\to\infty]{} 0. \tag{3.2}$$

Let $s \in \mathbb{C}$ be such that $1/2 < \sigma < 1$ and assume for contradiction that $\zeta(s) = 0$. We have:

$$\int_0^\infty \Big(\chi(t) - \sum_{k=1}^n c_k\Big\{\frac{\theta_k}{t}\Big\}\Big)t^{s-1}dt = \frac{1}{s} + \frac{\zeta(s)}{s}\sum_{k=1}^n c_k\theta_k^s = \frac{1}{s}. \tag{3.3}$$

By the Cauchy-Schwarz inequality,

$$\Big|\int_0^1 \Big(\chi(t) - \sum_{k=1}^n c_k\Big\{\frac{\theta_k}{t}\Big\}\Big)t^{s-1}dt\Big|^2 \leqslant \int_0^1 \Big(\chi(t) - \sum_{k=1}^n c_k\Big\{\frac{\theta_k}{t}\Big\}\Big)^2 dt \int_0^1 t^{2\sigma-2}dt$$
$$\leqslant \frac{d_n^2}{2\sigma-1}.$$

Moreover

$$\Big|\int_1^\infty \Big(\chi(t) - \sum_{k=1}^n c_k\Big\{\frac{\theta_k}{t}\Big\}\Big)t^{s-1}dt\Big|^2 = \Big|\int_1^\infty \sum_{k=1}^n c_k\theta_k t^{s-2}dt\Big|^2 \leqslant \frac{1}{(1-\sigma)^2}\Big|\sum_{k=1}^n c_k\theta_k\Big|^2.$$

But

$$\Big|\sum_{k=1}^{n} c_k\theta_k\Big|^2 = \int_1^\infty \Big(\chi(t) - \sum_{k=1}^{n} c_k\Big\{\frac{\theta_k}{t}\Big\}\Big)^2 dt \leqslant d_n^2.$$

Hence,

$$\int_0^\infty \Big(\chi(t) - \sum_{k=1}^{n} c_k\Big\{\frac{\theta_k}{t}\Big\}\Big) t^{s-1} dt \xrightarrow[n\to\infty]{} 0,$$

which contradicts Eq. (3.3). □

3.2. **gNB implies RH under a controlled approximation.** We can replace χ in Definition 1.5 by a more general function ϕ. We say that $\phi : (0,\infty) \to \mathbb{R}$ is an admissible target function if

(T1) $\widehat{\phi}(s) = \int_0^\infty \phi(t)t^{s-1}dt$ exists and does not vanish in the strip $\frac{1}{2} < \sigma < 1$,
(T2) $\sup_{M>0}\Big(M\int_M^\infty \phi(t)^2 dt\Big) < \infty$.

Theorem 3.1. — *Let $(Z_{k,n})_{1\leqslant k\leqslant n, n\geqslant 1}$ be r.v. in $L^2_+(\Omega)$ satisfying, for any $\epsilon > 0$,*

$$\sum_{k=1}^{n} \|Z_{k,n}\|_2^{1+\epsilon} \ll_\epsilon 1. \tag{3.4}$$

Let $\phi : (0,\infty) \to \mathbb{R}$ be an admissible target function. We suppose that there exist coefficients $(c_{k,n})_{1\leqslant k\leqslant n, n\geqslant 1}$ such that

(gNB) $D_n^2 = \displaystyle\int_0^\infty \Big|\phi(t) - \sum_{k=1}^{n} c_{k,n}\mathbb{E}\Big\{\frac{Z_{k,n}}{t}\Big\}\Big|^2 dt \xrightarrow[n\to\infty]{} 0$ *;*
(C) *For any $M_n \to \infty$,*

$$\sum_{k=1}^{n} |c_{k,n}|^2 \mathbb{P}(Z_{k,n} \geqslant M_n) \xrightarrow[n\to\infty]{} 0. \tag{3.5}$$

Then RH holds.

Proof. — We first compute the following Mellin transform:

$$\int_0^\infty \Big(\phi(t) - \sum_{k=1}^{n} c_{k,n}\mathbb{E}\Big\{\frac{Z_{k,n}}{t}\Big\}\Big) t^{s-1}dt = \widehat{\phi}(s) + \frac{\zeta(s)}{s}\sum_{k=1}^{n} c_{k,n}\mathbb{E}Z_{k,n}^s.$$

Suppose for contradiction that $\zeta(s) = 0$ for some fixed s with $\frac{1}{2} < \sigma < 1$. We thus have from (T1),

$$0 < \Big|\widehat{\phi}(s)\Big|^2 = \Big|\int_0^\infty \Big(\phi(t) - \sum_{k=1}^{n} c_{k,n}\mathbb{E}\Big\{\frac{Z_{k,n}}{t}\Big\}\Big)t^{s-1}dt\Big|^2 = |I_n|^2. \tag{3.6}$$

We will prove that the right-hand side of (3.6) goes to 0 as $n \to \infty$, which contradicts $\widehat{\phi}(s) \neq 0$.

We split I_n and use the inequality

$$\begin{aligned}|I_n|^2 &\ll |I_{1,n}|^2 + |I_{2,n}|^2 \\ &= \Big| \int_0^{M_n} \Big(\phi(t) - \sum_{k=1}^n c_{k,n} \mathbb{E}\Big\{\frac{Z_{k,n}}{t}\Big\}\Big) t^{s-1} dt \Big|^2 \\ &\quad + \Big| \int_{M_n}^\infty \Big(\phi(t) - \sum_{k=1}^n c_{k,n} \mathbb{E}\Big\{\frac{Z_{k,n}}{t}\Big\}\Big) t^{s-1} dt \Big|^2,\end{aligned}$$

where the moving threshold $M_n \geqslant 1$ is chosen so that

- $M_n \xrightarrow[n\to\infty]{} \infty$,
- $M_n^{2\sigma-1} D_n^2 \xrightarrow[n\to\infty]{} 0$ (this is possible since we assume $D_n \to 0$).

The first integral is bounded with the Cauchy-Schwarz inequality:

$$\Big| I_{1,n} \Big|^2 \leqslant D_n^2 \int_0^{M_n} t^{2\sigma-2} dt = D_n^2 \frac{M_n^{2\sigma-1}}{2\sigma-1} \xrightarrow[n\to\infty]{} 0.$$

In order to bound $|I_{2,n}|$, we write

$$I_{2,n} = \int_{M_n}^\infty \phi(t) t^{s-1} dt - \sum_{k=1}^n c_{k,n} \mathbb{E} \int_{M_n}^\infty \Big\{\frac{Z_{k,n}}{t}\Big\} t^{s-1} dt.$$

As $\widehat{\phi}(s)$ is well-defined, we have $\int_{M_n}^\infty \phi(t) t^{s-1} dt \xrightarrow[n\to\infty]{} 0$. We split the other integrals:

$$\begin{aligned}&\mathbb{E} \int_{M_n}^\infty \Big\{\frac{Z_{k,n}}{t}\Big\} t^{s-1} dt \\ &\qquad = \mathbb{E} \int_{M_n}^\infty \mathbf{1}_{Z_{k,n} \leqslant M_n} \Big\{\frac{Z_{k,n}}{t}\Big\} t^{s-1} dt + \mathbb{E} \int_{M_n}^\infty \mathbf{1}_{Z_{k,n} \geqslant M_n} \Big\{\frac{Z_{k,n}}{t}\Big\} t^{s-1} dt \\ &\qquad = A_{k,n} + B_{k,n}.\end{aligned}$$

(1) We first bound the term $\sum_{k=1}^n c_{k,n} B_{k,n}$, by splitting again each integral:

$$B_{k,n} = \mathbb{E} \int_{M_n}^{Z_{k,n}} \mathbf{1}_{Z_{k,n} \geqslant M_n} \Big\{\frac{Z_{k,n}}{t}\Big\} t^{s-1} dt + \mathbb{E} \int_{Z_{k,n}}^\infty \mathbf{1}_{Z_{k,n} \geqslant M_n} \Big\{\frac{Z_{k,n}}{t}\Big\} t^{s-1} dt. \quad (3.7)$$

For the first integral, we use the bound $\Big|\Big\{\frac{Z_{k,n}}{t}\Big\}\Big| \leqslant 1$. For the second integral, we notice that $\Big\{\frac{Z_{k,n}}{t}\Big\} = \frac{Z_{k,n}}{t}$ when $t \geqslant Z_{k,n}$. The triangle inequality then gives:

$$\begin{aligned}|B_{k,n}| &\leqslant \mathbb{E}\mathbf{1}_{Z_{k,n} \geqslant M_n} \int_{M_n}^{Z_{k,n}} t^{\sigma-1} dt + \mathbb{E}\mathbf{1}_{Z_{k,n} \geqslant M_n} Z_{k,n} \int_{Z_{k,n}}^\infty t^{\sigma-2} dt \quad (3.8) \\ &\ll \mathbb{E}\mathbf{1}_{Z_{k,n} \geqslant M_n} Z_{k,n}^\sigma. \quad (3.9)\end{aligned}$$

Thus, by the Cauchy-Schwarz inequality,

$$|B_{k,n}| \leqslant \sqrt{\mathbb{P}(Z_{k,n} \geqslant M_n)} \sqrt{\mathbb{E} Z_{k,n}^{2\sigma}}. \quad (3.10)$$

Thus, by the triangle and Cauchy-Schwarz inequalities again,

$$\Big| \sum_{k=1}^n c_{k,n} B_{k,n} \Big|^2 \leqslant \sum_{k=1}^n |c_{k,n}|^2 \mathbb{P}(Z_{k,n} \geqslant M_n) \sum_{k=1}^n \mathbb{E} Z_{k,n}^{2\sigma}.$$

Let us notice that $\mathbb{E}Z_{k,n}^{2\sigma} = \|Z_{k,n}\|_{2\sigma}^{2\sigma} \leqslant \|Z_{k,n}\|_2^{2\sigma}$ since $2\sigma \leqslant 2$. Since $2\sigma > 1$, we can deduce from (3.4) and (3.5) that

$$\sum_{k=1}^{n} c_{k,n} B_{k,n} \xrightarrow[n\to\infty]{} 0.$$

(2) It remains to bound the term $\sum_{k=1}^{n} c_{k,n} A_{k,n}$. We notice that, for $t \geqslant M_n$,

$$\mathbf{1}_{Z_{k,n}\leqslant M_n} \Big\{ \frac{Z_{k,n}}{t} \Big\} = \mathbf{1}_{Z_{k,n}\leqslant M_n} \frac{Z_{k,n}}{t},$$

so

$$\Big| \sum_{k=1}^{n} c_{k,n} A_{k,n} \Big|^2 = \Big| \sum_{k=1}^{n} c_{k,n} \mathbb{E}\mathbf{1}_{Z_{k,n}\leqslant M_n} Z_{k,n} \int_{M_n}^{\infty} t^{s-2} dt \Big|^2 \tag{3.11}$$

$$\ll \Big| \sum_{k=1}^{n} c_{k,n} \mathbb{E}\mathbf{1}_{Z_{k,n}\leqslant M_n} Z_{k,n} \Big|^2 M_n^{2\sigma-2}. \tag{3.12}$$

But

$$M_n^{2\sigma-2} \Big| \sum_{k=1}^{n} c_{k,n} \mathbb{E}\mathbf{1}_{Z_{k,n}\leqslant M_n} Z_{k,n} \Big|^2 = M_n^{2\sigma-1} \int_{M_n}^{\infty} \Big(\sum_{k=1}^{n} c_{k,n} \mathbb{E}\mathbf{1}_{Z_{k,n}\leqslant M_n} \frac{Z_{k,n}}{t} \Big)^2 dt$$

$$= M_n^{2\sigma-1} \int_{M_n}^{\infty} \Big(\sum_{k=1}^{n} c_{k,n} \mathbb{E}\mathbf{1}_{Z_{k,n}\leqslant M_n} \Big\{ \frac{Z_{k,n}}{t} \Big\} \Big)^2 dt,$$

and

$$\int_{M_n}^{\infty} \Big(\sum_{k=1}^{n} c_{k,n} \mathbb{E}\mathbf{1}_{Z_{k,n}\leqslant M_n} \Big\{ \frac{Z_{k,n}}{t} \Big\} \Big)^2 dt$$

$$\ll D_n^2 + \int_{M_n}^{\infty} \phi(t)^2 dt + \int_{M_n}^{\infty} \Big(\sum_{k=1}^{n} c_{k,n} \mathbb{E}\mathbf{1}_{Z_{k,n}\geqslant M_n} \Big\{ \frac{Z_{k,n}}{t} \Big\} \Big)^2 dt.$$

The sequence $(M_n)_{n\geqslant 1}$ has been chosen such that $M_n^{2\sigma-1} D_n^2 \xrightarrow[n\to\infty]{} 0$. Due to assumption (T2) we have

$$M_n^{2\sigma-1} \int_{M_n}^{\infty} \phi(t)^2 dt \xrightarrow[n\to\infty]{} 0$$

since $2\sigma - 1 < 1$. Let us bound the third term:

$$\mathbb{E}\mathbf{1}_{Z_{k,n}\geqslant M_n} \Big\{ \frac{Z_{k,n}}{t} \Big\} \leqslant \sqrt{\mathbb{P}(Z_{k,n} \geqslant M_n)} \sqrt{\mathbb{E}\Big\{ \frac{Z_{k,n}}{t} \Big\}^2}.$$

Therefore

$$\Big(\sum_{k=1}^{n} c_{k,n} \mathbb{E}\mathbf{1}_{Z_{k,n}\geqslant M_n} \Big\{ \frac{Z_{k,n}}{t} \Big\} \Big)^2 dt \leqslant \sum_{k=1}^{n} |c_{k,n}|^2 \mathbb{P}(Z_{k,n} \geqslant M_n) \sum_{k=1}^{n} \mathbb{E}\Big\{ \frac{Z_{k,n}}{t} \Big\}^2 dt,$$

and

$$\int_{M_n}^{\infty} \Big(\sum_{k=1}^{n} c_{k,n} \mathbb{E}\mathbf{1}_{Z_{k,n}\geqslant M_n} \Big\{ \frac{Z_{k,n}}{t} \Big\} \Big)^2 dt$$

$$\leqslant \sum_{k=1}^{n} |c_{k,n}|^2 \mathbb{P}(Z_{k,n} \geqslant M_n) \sum_{k=1}^{n} \int_{M_n}^{\infty} \mathbb{E}\Big\{ \frac{Z_{k,n}}{t} \Big\}^2 dt.$$

Due to Proposition 2.2 (take $\alpha = 2\sigma - 1$), we have

$$M_n^{2\sigma-1}\mathbb{E}\int_{M_n}^{\infty}\Big\{\frac{Z_{k,n}}{t}\Big\}^2 dt \ll_\sigma \|Z_{k,n}\|_2^{2\sigma}.$$

Since $2\sigma > 1$ we can use (3.4), together with condition (3.5), to obtain

$$M_n^{2\sigma-1}\int_{M_n}^{\infty}\Big(\sum_{k=1}^{n} c_{k,n}\mathbb{E}\mathbf{1}_{Z_{k,n}\geqslant M_n}\Big\{\frac{Z_{k,n}}{t}\Big\}\Big)^2 dt \xrightarrow[n\to\infty]{} 0.$$

(3) Hence $|I_{2,n}| \xrightarrow[n\to\infty]{} 0$, which concludes the proof. □

3.3. **RH implies gNB for dilated random variables.** The proof of the implication RH $\Longrightarrow$ gNB for dilated r.v. $Z_k = X/k$ is based on the Necessary part of Báez-Duarte's theorem [4, Theorem 1.2] regarding Müntz transform P. We first give a probabilistic interpretation of P.

Lemma 3.2. — *Let $X \in L^1_+(\Omega)$ and set $f(x) = \mathbb{P}(X \geqslant x)$, $x \geqslant 0$. Then Pf is well defined and*

$$\mathbb{E}[\{X/t\}] = -Pf(t), \quad t > 0. \tag{3.13}$$

Proof. — First, notice that $0 \leqslant f(k+1) \leqslant \int_0^1 f(k+x)dx$, $k \geqslant 0$, so that the following quantities are well defined:

$$\sum_{k\geqslant 0} f(k+1) \leqslant \int_0^{\infty} f(x)dx = \mathbb{E}X < \infty.$$

Since $0 \leqslant \{X\} \leqslant 1$, we can write

$$\begin{aligned}\mathbb{E}[\{X\}] &= \int_0^1 \mathbb{P}(\{X\} \geqslant x)dx \\ &= \int_0^1 \sum_{k\geqslant 0}\mathbb{P}(k+x \leqslant X < k+1)dx = \int_0^1 \sum_{k\geqslant 0}(f(k+x) - f(k+1))dx \\ &= \sum_{k\geqslant 0}\int_0^1 f(k+x)dx - \sum_{k\geqslant 0} f(k+1) = \int_0^{+\infty} f(x)dx - \sum_{k\geqslant 1} f(k).\end{aligned}$$

Set $t > 0$. Then $\mathbb{P}(X/t \geqslant x) = f(tx)$ and so

$$\mathbb{E}[\{X/t\}] = \int_0^{+\infty} f(tx)dx - \sum_{k\geqslant 1} f(tk) = \frac{1}{t}\int_0^{+\infty} f(x)dx - \sum_{k\geqslant 1} f(tk), \tag{3.14}$$

as desired. □

Báez-Duarte introduced in [4, Definition 1.1] the definition of a good kernel f, i.e. f is a continuously differentiable function on $(0,\infty)$ with $\int_0^\infty |f(t)|dt < \infty$ and $\int_0^\infty t|f'(t)|dt < \infty$. Let us notice that if X is a positive integrable r.v. with a continuous density ϕ then $f(t) = \mathbb{P}(X \geqslant t)$ is a good kernel, and $f' = -\phi$. We also obtain the probabilistic interpretation of the formula (2.4) in [4]:

$$\begin{aligned}Pf(t) &= \sum_{k\geqslant 1} f(kt) - \int_0^{\infty} f(ut)du = t\int_0^{\infty} f'(ut)\{u\}du \\ &= \int_0^{\infty}\{x/t\}f'(x)dx = -\mathbb{E}[\{X/t\}].\end{aligned}$$

Theorem 3.3. — *Let $X \in L^q_+(\Omega)$, $q > 1$, be a r.v. with a continuous density. If RH holds, then there exist coefficients $c_{k,n}$ such that*

(gNB) $$D_n^2 = \int_0^\infty \Big|\mathbb{P}(X \geqslant t) - \sum_{k=1}^n c_{k,n}\mathbb{E}\Big\{\frac{X}{kt}\Big\}\Big|^2 dt \xrightarrow[n\to\infty]{} 0\ ;$$

(C) *For any $M_n \to \infty$,*

$$\sum_{k=1}^n |c_{k,n}|^2 \mathbb{P}(X/k \geqslant M_n) \xrightarrow[n\to\infty]{} 0.$$

Proof. — Since RH holds, there exists coefficients $c_{k,n}$ bounded in k and n (see [9]), such that

$$\int_0^\infty \Big(\chi(t) - \sum_{k=1}^n c_{k,n}\{1/kt\}\Big)^2 dt \xrightarrow[n\to\infty]{} 0.$$

Then, Báez-Duarte deduces in [4, Section 3.1] that, for a good kernel f,

$$\int_0^\infty \Big(f(t) - \sum_{k=1}^n c_{k,n}Pf(kt)\Big)^2 dt \xrightarrow[n\to\infty]{} 0.$$

Hence using our Lemma 3.2, we deduce that (gNB) holds for the r.v. X/k and the target function $f : t \mapsto \mathbb{P}(X \geqslant t)$, which is a good kernel (see above).

Condition (C) then follows from the boundedness of the coefficients $c_{k,n}$ and the inequality

$$\mathbb{P}(X/k \geqslant M_n) \leqslant \frac{\mathbb{E}X^q}{k^q M_n^q},$$

since $q > 1$. □

4. Examples

To illustrate our theorems, we give two typical examples:

(1) *Dilation*: Let $X_k = X/k$ where $X \sim \mathcal{E}(1)$.
We have $\mathbb{E}X^q < \infty$, $q > 1$, and $\|X_k\|_2 = \frac{\sqrt{2}}{k}$, so we can apply Theorem 3.3 and Theorem 3.1.

(2) *Concentration*: Let $Z_{k,n} = Y_{k,n}^2$ where $Y_{k,n} \sim \Gamma\Big(\frac{n^4}{k}, \frac{n^4}{\sqrt{k}}\Big)$, $1 \leqslant k \leqslant n$.
We have $\mathbb{E}Y_{k,n} = 1/\sqrt{k}$ and $\mathrm{Var}(Y_{k,n}) = n^{-4}$. Since $Y_{1,n}$ is distributed as $\frac{E_1+\cdots+E_{n^4}}{n^4}$ where the E_k's are i.i.d. $\mathcal{E}(1)$, the Central Limit Theorem gives $\mathbb{P}(Y_{1,n} \geqslant 1) \to 1/2 < 1$. So we can apply Theorem 2.6. Let us notice that

$$\mathbb{E}Y_{k,n}^4 = \left(\frac{\sqrt{k}}{n^4}\right)^4 \left(\frac{n^4}{k}+3\right)\left(\frac{n^4}{k}+2\right)\left(\frac{n^4}{k}+1\right)\frac{n^4}{k}, \quad 1 \leqslant k \leqslant n.$$

When assuming RH the coefficients $c_{k,n}$ of the approximation are bounded in k and n, see proof of Theorem 2.6. Thus, using

$$\mathbb{P}(Z_{k,n} \geqslant M_n) \leqslant \mathbb{E}Z_{k,n}^2/M_n^2 \ll 1/(k^2M_n^2),$$

Condition (C) in Theorem 3.1 is verified. Finally, one can check Assumption (3.4) due to $\|Z_{k,n}\|_2 = \sqrt{\mathbb{E}Y_{k,n}^4} \ll 1/k$.

We summarize below the relationships between the various criteria :

$$\begin{array}{ccccc}
 & & & & \mathrm{gNB}\Big(\Gamma\Big(\frac{n^4}{k},\frac{n^4}{\sqrt{k}}\Big)^2_{1\leqslant k\leqslant n}\Big)+(C) \\
 & & & \swarrow\!\!\!\swarrow & \Uparrow \\
\mathrm{gNB}\big(\mathcal{E}(k)_{k\geqslant 1}\big)+(C) & \Longleftrightarrow & \mathrm{RH} & \Longrightarrow & \mathrm{pNB}\Big(\Gamma\Big(\frac{n^4}{k},\frac{n^4}{\sqrt{k}}\Big)^2_{1\leqslant k\leqslant n}\Big)+(C) \\
 & & \Updownarrow & & \\
 & & \mathrm{NB} & &
\end{array}$$

Notice that condition (C) is not necessary for the implication

$$\mathrm{pNB}\Big(\Gamma\Big(\frac{n^4}{k},\frac{n^4}{\sqrt{k}}\Big)^2_{1\leqslant k\leqslant n}\Big)\Longrightarrow RH,$$

see Theorem 2.6. This is one of the interests of pNB. The correlation structure of these r.v., which is not explored here, might also be of some importance in pNB.

The computation of the scalar products in Example (1) is studied in [15]: the main formula shows a striking simplification compared to Vasyunin's formula [21].

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