CONFLUENTES MATHEMATICI

Denis SERRE

The role of the Hilbert metric in a class of singular elliptic boundary value problems in convex domains

Tome 9, n^{o} 1 (2017), p. 105-117.

<http://cml.cedram.org/item?id=CML_2017__9_1_105_0>

Confluentes Math.
9, 1 (2017) 105-117

THE ROLE OF THE HILBERT METRIC IN A CLASS OF SINGULAR ELLIPTIC BOUNDARY VALUE PROBLEMS IN CONVEX DOMAINS

DENIS SERRE

Abstract. In a recent paper [7], we were led to consider a distance over a bounded open convex domain. It turns out to be the so-called *Thompson metric*, which is equivalent to the *Hilbert* metric. It plays a key role in the analysis of existence and uniqueness of solutions to a class of elliptic boundary-value problems that are singular at the boundary.

Introduction

Let Ω be a connected open set in $\mathbb{R}^n$. If $p \in \mathbb{R}^n$, we denote $|p|$ its usual Euclidian norm. The class of boundary value problems that we are interested in is

$$\operatorname{div}(a(|\nabla w|)\nabla w) + \frac{F(|\nabla w|)}{w} = 0 \quad \text{in } \Omega, \tag{0.1}$$

$$w > 0 \quad \text{in } \Omega, \tag{0.2}$$

$$w = 0 \quad \text{on } \partial\Omega. \tag{0.3}$$

Hereabove, a is a smooth numerical even function, which satisfies the requirements for ellipticity:

$$a(r) > 0, \qquad a(r) + ra'(r) > 0, \qquad \forall r \geqslant 0. \tag{0.4}$$

We warn the reader that we do not assume *a priori* a uniform ellipticity ; it may happen that the ratio

$$\frac{a(r)}{a(r) + ra'(r)}$$

tends either to 0 or to $+\infty$ as $r \to +\infty$. For instance, we allow the principal part to be the minimal surface operator, where $a(r) = (1 + r^2)^{-1/2}$, for which

$$\frac{a(r)}{a(r) + ra'(r)} = 1 + r^2 \to +\infty.$$

We suppose that F is a smooth, non-negative function, and that

$$F(0) > 0. \tag{0.5}$$

The lower order term in (0.1) therefore becomes singular at the boundary, where w vanishes.

Notations. In the sequel, we denote $b(r) = ra(r)$, so that $a + ra' = b'$. We define a strictly increasing function

$$G(r) = \int_0^r \frac{sb'(s)}{F(s)}\, ds.$$

The inverse G^{-1} will be denoted H.

Math. classification: 35J75, 52A99.
Keywords: Elliptic PDEs, convex domain, Hilbert metric, singular BVP.

Data. At first glance, it may look strange that neither the equation, nor the boundary contain some explicit data ; both are "homogeneous". Our data is nothing but the domain itself. The assumption about Ω meets that in other works on non-uniformly elliptic BVPs: it is a *bounded convex* domain in $\mathbb{R}^n$.

Motivations. We came to this class of problems through the analysis of the two-dimensional Riemann problem for the Euler system of a compressible flow, when the gas obeys the so-called Chaplygin equation of state. This problem can be recast as

$$\operatorname{div}\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}+\frac{2}{w\sqrt{1+|\nabla w|^2}}=0, \tag{0.6}$$

which is (0.1) with

$$n=2, \qquad a(r)=\frac{1}{\sqrt{1+r^2}}\,, \qquad F=2a.$$

We first proved the existence and uniqueness (see [6]) for this problem whenever Ω is uniformly convex, in the sense that the curvature is bounded away from zero along the boundary. Later on, we removed the uniform assumption and proved the existence for every convex bounded planar domain [7]. This improvement involves an interior Lipschitz estimate of $\log w$ in terms of a special metric over Ω, for which the boundary is a horizon. We shall show below that this distance is nothing but the Hilbert metric d_H, giving meanwhile a new and rather simple proof of the triangle inequality.

It turns out that the very same BVP also governs those graphs $x_3=w(x_1,x_2)$ that are complete minimal surfaces in the 3-dimensional hyperbolic space $\mathbb{H}_3$, the upper half-space in $\mathbb{R}^3$, equipped with the metric

$$ds^2=\frac{dx_1^2+dx_2^2+dx_3^2}{x_3^2}\,,$$

of constant negative curvature. The existence of such minimal surfaces was studied by Anderson [1] in the parametric and the non-parametric cases, the latter involving the graph of w. The non-parametric part of Anderson's paper is however incomplete, in that the author contents himself to establish L^∞-bounds (by below and above) and claims that it automatically implies regularity estimates in the interior. This claim is not true because the principal part of the PDE, the operator for minimal surfaces, is not *a priori* uniformly elliptic. Uniform ellipticity requires the knowledge of a prior Lipschitz estimate, which can not be overlooked. The same flaw occurs in Lin's paper [4].

We point out that in both of these motivations, the convexity of Ω is a necessary condition for existence (and therefore a necessary and sufficient one). In the Chaplygin Riemann problem, this convexity is guaranted by the analysis of the propagation of shock waves. If Ω is not convex, a complete minimal surface in $\mathbb{H}_3$, asymptotic to $\partial\Omega$, exists[1] as a current [1], but it is not a graph over Ω.

[1] This is the parametric part, by far the main one, of Anderson's paper, on which we have no doubt at all.

Content of this paper. We start by showing in Section 1 the equality between our (not so) new distance and the Hilbert metric in Ω.

Then we turn towards the class of BVPs (0.1, 0.2, 0.3). We show that essentially the same strategy as the one designed in [7] works out under the rather mild assumption that

$$\int^{+\infty} e^{-G(s)} \frac{b'(s)}{F(s)} \, ds < \infty. \tag{0.7}$$

Our main result is therefore

Theorem 0.1. — *Let a, F be even smooth functions, satisfying (0.4,0.5,0.7). Then, for every bounded convex domain $\Omega \subset \mathbb{R}^d$, there exists one and only one function*

$$w \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$$

solving the BVP (0.1,0.2,0.3).

In addition, $\log w$ *is Lipschitz, with constant* 1, *with respect to the Hilbert metric* d_H.

Of course, if a or F has only finite regularity, then w has only finite regularity.

1. A distance over a bounded convex domain

Let Ω be a non-void, bounded convex open domain in $\mathbb{R}^n$. Given two points $p, q \in \Omega$, $\Omega - p$ contains a ball centered at the origin and is therefore absorbing. Thus there exists some $\lambda > 0$ such that $\Omega - q \subset \lambda(\Omega - p)$. If $\mu > \lambda$, then also $\Omega - q \subset \mu(\Omega - p)$, by convexity. Likewise, the infimum $m(p,q)$ of all such numbers satisfies the same inclusion, by continuity. Hence the set of these numbers is of the form $[m(p,q), +\infty)$. Considering the volumes, we have

$$|\Omega| = |\Omega - q| \leqslant m(p,q)^n |\Omega - p| = m(p,q)^n |\Omega|,$$

which implies

$$m(p,q) \geqslant 1. \tag{1.1}$$

The equality in (1.1) stands only if

$$\Omega - q = \Omega - p,$$

that is if $p = q$.

If $r \in \Omega$ is a third point, then

$$\Omega - r \subset m(q,r)(\Omega - q) \subset m(q,r)m(p,q)(\Omega - p)$$

and therefore

$$m(p,r) \leqslant m(q,r)m(p,q).$$

All this shows that the logarithm of m is a non-negative function over $\Omega \times \Omega$, which vanishes only along the diagonal and satisfies the triangle inequality. In other words, the function

$$d_\Omega(p,q) = \log m(p,q) + \log m(q,p)$$

is a *distance* over Ω. In our paper [7], we used the equivalent metric

$$d'_\Omega(p,q) = \max\{\log m(p,q), \log m(q,p)\}.$$

We prove here that d_Ω is nothing but the Hilbert distance d_H over Ω (see [3]). Let us recall the definition of the latter. If $p, q \in \Omega$, let r, s be the intersection points of the line L passing through p and q, with the boundary $\partial\Omega$; we label the points so that r, p, q, s are in this order along L. Then d_H is the logarithm of a cross-ratio :

$$d_H(p,q) = \log \frac{\overline{rq}\cdot\overline{ps}}{\overline{rp}\cdot\overline{qs}}\,.$$

Our first result therefore reads

Proposition 1.1. — *For every non-void, bounded convex open domain $\Omega \subset \mathbb{R}^n$, one has*

$$d_\Omega \equiv d_H. \tag{1.2}$$

The equality (1.2) follows immediately from the

Lemma 1.2. — *With the notation above, there holds*

$$m(p,q) = \frac{\overline{rq}}{\overline{rp}}\,. \tag{1.3}$$

Proof. — Let $u \in \mathbb{R}$ be the barycentric coordinate of r on the line L, that is $r = up + (1-u)q$. Because r is exterior to (p,q), on the same side as p, we have $u > 1$. Let us define $\theta = \frac{1}{u} \in (0,1)$. Because r is a boundary point of Ω, and Ω is convex open, we have $(1-\theta)\Omega + \theta r \subset \Omega$. This is equivalent to writing

$$\Omega - q \subset \frac{u}{u-1}(\Omega - p),$$

which is the inequality $\leqslant$ in (1.3).

Conversely, suppose that $\Omega - q \subset \lambda(\Omega - p)$. Then we have $\overline{\Omega} - q \subset \lambda(\overline{\Omega} - p)$ and therefore $r - q \in \lambda(\overline{\Omega} - p)$. This amounts to writing

$$\left(1 + \frac{u}{\lambda}\right)p - \frac{u}{\lambda}q \in \overline{\Omega},$$

but this is equivalent to

$$v \leqslant 1 + \frac{u}{\lambda} \leqslant u,$$

where v is likewise the barycentric coordinate of s. The second inequality gives $\lambda \geqslant \frac{u}{u-1}$. This implies the inequality $\geqslant$ in (1.3). □

Remarks 1.3. — This characterization of the Hilbert metric is related to the construction of the Hilbert projective metric over the cone

$$C = \{(t, tx) \,|\, t > 0 \text{ and } x \in \Omega\},$$

see [8]. The proof above provides a much simpler proof of the triangle inequality than the original one. For the classical proof, which involves projective geometry, see the introductory article in *Image des Mathématiques* [5]. Lemma 1.2 also implies that d'_Ω is identical to the Thompson metric in Ω.

2. The strategy for existence and uniqueness to the BVP

Our first observation is that the PDE (0.1) is of the quasilinear form

$$\sum_{i,j} a_{ij}(\nabla w)\partial_i\partial_j w + N(w, \nabla w) = 0,$$

where the principal part is elliptic,

$$c(p)|\xi|^2 \leqslant \sum_{i,j} a_{ij}(p)\xi_i\xi_j \leqslant C(p)|\xi|^2, \qquad 0 < c(p) < C(p) < \infty.$$

The coefficients a_{ij} involve the gradient of w, but not w itself. Finally the lower order term N is non-increasing in w. Therefore the PDE satisfies the maximum principle (MP).

The MP allows us to compare a sub-solution and a super-solution. A locally Lipschitz function $u : \Omega \to (0, +\infty)$ is a sub-solution of (0.1) if it satisfies, in the distributional sense,

$$\operatorname{div}(a(|\nabla u|)\nabla u) + \frac{F(|\nabla u|)}{u} \geqslant 0. \tag{2.1}$$

It is a super-solution if it satisfies the opposite inequality $\leqslant$ in (2.1). If in addition u is continuous over $\overline{\Omega}$, we say that u is a super-solution of the BVP if it is a super-solution of (0.1), and it satisfies $u \geqslant 0$ on $\partial\Omega$. It is a sub-solution if it satisfies (2.1), and $u \leqslant 0$ over $\partial\Omega$ (but then this means $u \equiv 0$ on the boundary, because $u > 0$ in the interior).

If u and v are a sub-solution and a super-solution respectively, of the BVP in some domain Ω, then $u \leqslant v$ in Ω. In particular, if w is a solution in Ω, then $u \leqslant w \leqslant v$. This immediately implies the uniqueness part of Theorem 0.1.

The method for existence is based on the one hand on a continuation argument, described in Section 5, and on the other hand on *a priori* estimates. The latter must be robust enough to allow us to pass to the limit in a sequence of solutions. To ensure the boundary condition, we shall use sub- and super-solution respectively to construct barrier functions $w_\pm > 0$, with $w_\pm \equiv 0$ on the boundary. The fact that w is clamped between w_- and w_+ implies the boundary condition. It also ensures that w is positive and bounded in Ω. In order to pass to the limit in the PDE, we need a precompactness property of ∇w in $L^\infty_{loc}(\Omega)$. This will be given by a $\mathcal{C}^{1,\beta}_{loc}$-regularity estimate for some $\beta > 0$ and the Ascoli–Arzela theorem. The regularity is a well-known fact (see Gilbarg & Trudinger [2]) whenever the operator

$$L(p) = \sum_{i,j} a_{ij}(p)\partial_i\partial_j$$

is uniformly elliptic. Since we have not assumed the latter property, it must come as a consequence of the fact that $p = \nabla w$ takes its values in a compact subset of $\mathbb{R}^n$. In other words, we need an *a priori* estimate of ∇w in $L^\infty_{loc}(\Omega)$.

We summarize below the tasks we are going to address:

- Construct a finite upper bound w_+ of w, continuous up to the boundary, where it satisfies $w_+ \equiv 0$.
- Construct a lower bound $w_- > 0$ of w, continuous up to the boundary, where it satisfies $w_- \equiv 0$.
- Find a Lipschitz estimate of w in Ω. This estimate may and will deteriorate near the boundary, but it must be uniform on every compact subset of Ω. This is where the Hilbert metric is at stake.
- Make all these estimates uniform with respect to some approximation.

Of course, the only tool at our disposal is the maximum principle.

3. The barrier functions

We shall use repeatedly the fact that the PDE (0.1) is invariant under a scaling: if z is a solution in some domain ω, then the function $z^\mu(x) := \frac{1}{\mu} z(\mu x)$ is again a solution, in $\frac{1}{\mu}\,\omega$.

3.1. **The upper barrier.** We write our convex domain as the intersection of slabs

$$\Omega = \bigcap_{\nu\in S^{n-1}} \Pi_\nu, \qquad \Pi_\nu = \{x \in \mathbb{R}^n \,|\, \alpha_-(\nu) < x\cdot\nu < \alpha_+(\nu)\},$$

where we have of course $\alpha_\pm(-\nu) = -\alpha_\mp(\nu)$. Notice that $\nu \mapsto \alpha_\pm$ is continuous.

Our upper bound will be given as the infimum of super-solutions. The building block is the solution of the BVP in the interval $(0,1)$:

Lemma 3.1 (1-D case.). — *When $n = 1$ and the domain is $(0,1)$, then the BVP admits a unique solution W.*

We infer that the BVP in a slab Π_ν admits a solution, namely

$$W_\nu(x) = (\alpha_+(\nu) - \alpha_-(\nu)) W\left(\frac{x\cdot\nu - \alpha_-(\nu)}{\alpha_+(\nu) - \alpha_-(\nu)}\right).$$

Because $\Omega \subset \Pi_\nu$ and W_ν is non-negative, in particular along $\partial\Omega$, its restriction to Ω is a super-solution of the BVP in Ω. Therefore the expected solution w satisfies $w \leqslant W_\nu$. This yields to our upper bound,

$$w(x) \leqslant w_+(x) = \inf_{\nu\in S^{n-1}} W_\nu(x).$$

The continuity of $\alpha_\pm$, plus the uniform continuity of W, imply that w_+ is continuous over $\overline{\Omega}$. We point out that, because every $y \in \partial\Omega$ is a boundary point of some Π_ν, w_+ vanishes on the boundary.

Proof. — We already know uniqueness. Using the reflexion $x \leftrightarrow 1 - x$, we infer that W must be even: $W(1-x) = W(x)$. We anticipate that W is monotonous over $(0, \frac{1}{2})$ and write the PDE, now an ODE as

$$(b(W'))' + \frac{F(W')}{W} = 0, \qquad W(0) = W'(\frac{1}{2}) = 0.$$

We recall that $b' > 0$, from ellipticity.

Let us define $z = W' \circ W^{-1}$. Using $W' = z(W)$, we transform the ODE into

$$zb'(z)\frac{dz}{dW} + \frac{F(z)}{W} = 0.$$

The latter ODE amounts to writing $G(z) + \log W = \mathrm{cst}$, from which we obtain $z = H(\log \frac{c}{W})$ for some integrating factor $c \in \mathbb{R}$.

Let us make temporarily the choice that $c = 1$ and consider a maximal solution of the autonomous ODE $W' = H(-\log W)$. We have

$$\frac{dW}{H(-\log W)} = dx.$$

Because of (0.7), we have

$$\int_0 \frac{ds}{H(-\log s)} < \infty.$$

Therefore there exists a unique solution W_0 of the Cauchy problem

$$W_0' = H(-\log W_0), \qquad W_0(0) = 0.$$

This W_0 is increasing. Since the integral

$$\int_0 e^{-G(s)} \frac{b'(s)}{F(s)}\, ds$$

is converging, we have

$$\int^{e^{-G(0)}} \frac{ds}{H(-\log s)} < \infty$$

and therefore W_0 reaches the value $e^{-G(0)}$ at some finite $\bar{x} > 0$. Then $W_0'(\bar{x}) = 0$. Extending W_0 it by parity, we obtain a solution of the BVP in the interval $(0, 2\bar{x})$. Then

$$W(t) = \frac{1}{2\bar{x}} W_0(2\bar{x}t)$$

defines the solution of the BVP over $(0,1)$. □

3.2. **The lower barrier.** The construction of the lower barrier does not make use of the convexity. We begin with a building block:

Lemma 3.2. — *There exists an $\epsilon > 0$ such that the function $Z(x) = \frac{\epsilon}{2}(1 - |x|^2)$ be a sub-solution of the BVP in the unit ball $B(0;1)$.*

Proof. — Since Z is positive in the ball, it suffices to check that Z satisfies (2.1). This inequality writes

$$\frac{1}{2}(\epsilon^2 - t^2)(b(t) + (n-1)a(t)) \leqslant F(t), \qquad \forall t \in [0, \epsilon].$$

Because a and b are non-negative, it is enough to have

$$\frac{\epsilon^2}{2}(b(t) + (n-1)a(t)) \leqslant F(t), \qquad \forall t \in [0, \epsilon].$$

Let A and B be the upper bounds of a and b over $[0,1]$ respectively. If $\epsilon < 1$, it is enough to have

$$\frac{\epsilon^2}{2}(B + (n-1)A) \leqslant \sup_{t \in [0,\epsilon]} F(t),$$

which is obviously true for $\epsilon > 0$ small enough. □

By translation and scaling, we inherit a sub-solution of the BVP in any ball $B(x_0; \rho)$:

$$Z_{x_0,\rho}(x) = \rho Z\left(\frac{x - x_0}{\rho}\right).$$

If $B(x_0;\rho)$ is contained in Ω, then w is a super-solution for the BVP in this ball, and we infer $w \geqslant Z_{x_0,\rho}$. This leads us to our lower barrier function

$$w_-(x) = \sup\{Z_{x_0,\rho}(x) \,|\, B(x_0;\rho) \subset \Omega\}.$$

Because every $x \in \Omega$ is the center of some ball $B \subset \Omega$, we have $w_- > 0$ in Ω. On an other hand, w_- is continuous over $\overline{\Omega}$. Because $Z_{x_0,\rho}$ is non-positive over the boundary, the same is true for w_-, which therefore vanishes identically over $\partial\Omega$.

4. The Lipschitz estimate

The main ingredient is the

Lemma 4.1. — *The solution of the BVP (0.1,0.2,0.3) in a bounded convex open domain* Ω *satisfies, if it exists*

$$|\log w(q)-\log w(p)|\leqslant\max\{\log m(p,q),\log m(q,p)\},\qquad\forall p,q\in\Omega.\tag{4.1}$$

Consequently, $\log w$ *is Lipschitz with constant at most* 1*, with respect to the Hilbert metric.*

Because the restriction of the Hilbert metric to a compact subset $K\subset\Omega$ is equivalent to the Euclidian distance, we infer a Lipschitz estimate in the classical sense, away from the boundary. Because $\min_K w_->0$ and w_+ is bounded, this transfers into a local Lipchitz estimate of w:

Corollary 4.2. — *For every compact subset* $K\subset\Omega$*, the restriction* $w|_K$ *enjoys an* a priori *estimate in the Lipschitz semi-norm* $\sup_K|\nabla w|$.

Proof. — Given $p,q\in\Omega$, the function

$$x\mapsto m(p,q)w\left(\frac{x}{m(p,q)}+p\right)$$

is the solution of the BVP in the domain $m(p,q)(\Omega-p)$. Since the latter contains $\Omega-q$, it is also a super-solution in the domain $\Omega-q$. It is therefore larger than or equal to the solution $w(x+q)$ in the latter:

$$w(x+q)\leqslant m(p,q)w\left(\frac{x}{m(p,q)}+p\right),\qquad\forall x\in\Omega-q.$$

Setting $x=0$ in the inequality above, we derive

$$w(q)\leqslant m(p,q)w(p).$$

Exchanging the roles of p and q, we also have $w(p)\leqslant m(q,p)w(q)$, whence (4.1). □

4.1. The best Lipschitz constant.

Lemma 4.1 provides an upper bound for the Lipschitz constant of $\log w$ with respect to the Hilbert metric:

$$c_\Omega:=\sup_{x\neq y}\frac{|\log w(y)-\log w(x)|}{d_H(x,y)}\leqslant 1.$$

We may wander whether this bound is accurate or not. Remark that if O is a boundary point and L is a ray emanating from O in Ω, then the restriction of d_H to L is logarithmic, in the sense that if $x,y\in L$ tend to O, then

$$d_H(x,y)=|\log t_y+\log(T_L-t_x)-\log t_x-\log(T_L-t_y)|\sim|\log t_y-\log t_x|,$$

where t is the affine coordinate along L with origin O, and T_L is the coordinate or the other intersection point of L with $\partial\Omega$. If the solution w admits a Hölder singularity at a boundary point, of exponent $\alpha\in(0,1]$, we deduce that $c_\Omega\geqslant\alpha$.

One remarquable application of this principle is the following proposition.

Proposition 4.3. — *Let the origin be a boundary point of* $\partial\Omega$*, and denote* $\mathcal{C}$ *the tangent cone at* 0*. Suppose that the BVP in the cone* $\mathcal{C}$ *admits a super-solution of the form* $V_+(x)=|x|v_+\left(\frac{x}{|x|}\right)$*, which vanishes at the boundary. Then the BVP* *is solvable in the cone $\mathcal{C}$, with a solution $V(x)=|x|v\left(\frac{x}{|x|}\right)$, and the solution in Ω is asymptotic to V as $x\to 0$. In particular,*

$$c_\Omega = 1.$$

We warn the reader that the super-solution V_+ does not exist if $\partial\Omega$ is smooth at the boundary. The proposition produces a homogeneous solution whenever the cone $\mathcal{C}$ is contained in a circular cone of aperture $<\frac{\pi}{2}$.

The fact that V is homogeneous of degree one is a consequence of the scaling invariance of both $\mathcal{C}$ (the conical property) and the PDE, and the expected uniqueness.

Proof. — Let w be the solution of the BVP in Ω, and recall that for every $\mu>0$, the function

$$w^\mu(x) := \frac{1}{\mu}w(\mu x)$$

is the solution of the BVP in the domain $\frac{1}{\mu}\Omega$. Let us list a few properties of the sequence $(w^\mu)_{\mu>0}$:

- For $\epsilon<\eta$, one has the lower bound (maximum principle) $w^\epsilon \geqslant w^\eta$ in $\frac{1}{\eta}\Omega$. This because $\frac{1}{\eta}\Omega\subset\frac{1}{\epsilon}\Omega$.
- For $\epsilon<\eta$, one has the Lipschitz estimate

$$|\log w^\epsilon(x)-\log w^\epsilon(y)| \leqslant d_H^\epsilon(x,y)\leqslant d_H^\eta(x,y), \qquad \forall x,y\in\frac{1}{\eta}\Omega,$$

  where we have denoted d_H^η the Hilbert distance in $\frac{1}{\eta}\Omega$.
- By the maximum principle, $w^\mu\leqslant V_+$ for every $\mu>0$.

The Lipschitz estimate ensures that the PDE remains uniformly elliptic in every compact subdomain of the cone $\mathcal{C}$. Therefore the theory of elliptic regularity applies: every derivative $D^\beta w^\mu$ remains bounded as $\mu\to 0^+$, on every compact subdomain of $\mathcal{C}$. The (monotonic) limit $V=\lim w_\mu$ exists because of the bound V_+, and is again a solution of the PDE. In addition, it satisfies $V^\mu=V$, which means that it is homogeneous of degree one. Because of the upper bound $V\leqslant V+$, we know that V vanishes along the boundary, and is therefore a solution to the BVP in $\mathcal{C}$.

Let us know select two points x,y on the same ray L, close to the origin. The asymptotics above gives $|\log w(y)-\log w(x)|\sim|\log|y|-\log|x||\sim d_H(x,y)$. This implies $c_\Omega\geqslant 1$. With Lemma 4.1, we conclude that $c_\Omega=1$. □

Another interesting situation is that of the equation

$$\Delta w+\frac{1}{w}=0.$$

When $n=1$, and therefore the domain is $I=(0,\ell)$, the ODE can be integrated by hand and we find $u(x)\sim C_\ell x\sqrt{-2\log x}$. This quasi-Lipschitz behaviour at the boundary implies $c_I\geqslant 1$, and therefore $c_I=1$.

The situation is significantly better for our fundamental example :

Proposition 4.4. — *Consider the BVP for the equation (0.6). When Ω is a disk (hence $n=2$), we have $c_\Omega=\frac{1}{2}$.*

Proof. — By scaling, we may work in $D = D(0;1)$. Then

$$w(x) = \sqrt{\frac{1-|x|^2}{2}}\,,$$

a rare case where the solution is known in close form. In particular, the Hölder singularity of exponent $\frac{1}{2}$ implies $c_\Omega \geqslant \frac{1}{2}$. On the other hand d_H is given by

$$d_H(x,y) = 2\log\left(1 - x\cdot y + \sqrt{|y-x|^2 - |x\wedge y|^2}\right) - \log(1-|x|^2)(1-|y|^2).$$

Let us prove the converse inequality $c_\Omega \leqslant \frac{1}{2}$. The inequality $|\log w(x) - \log w(y)| \leqslant \frac{1}{2} d_H(x,y)$ to prove is equivalent to

$$1 - |x|^2 \leqslant 1 - x\cdot y + \sqrt{|y-x|^2 - |x\wedge y|^2}.$$

It is implied by

$$(x\cdot(y-x))^2 + |x\wedge y|^2 \leqslant |y-x|^2,$$

which is true because the left-hand side equals $(x\cdot(y-x))^2 + |x\wedge(y-x)|^2 = |x|^2|y-x|^2$, and on the other hand $|x|<1$. □

We now show that the assumption made in Proposition 4.3 is always met in our fundamental example. The cone $\mathcal{C}$ is a sector S_α of aperture $\alpha\in(0,\pi)$.

Proposition 4.5. — *The BVP for the fundamental example (0.6) is solvable in any planar sector S_α.*

Corollary 4.6. — *Let Ω be a planar open convex domain. Let us restrict to the equation (0.6). If $\partial\Omega$ has a kink (a point at which $\partial\Omega$ has more than one tangent), then*

$$c_\Omega = 1.$$

Proof. — Let us work in polar coordinates. The sector is

$$S_\alpha := \{re^{i\theta}\,|\,\theta\in(0,\alpha)\}.$$

The self-similar solution is written $w_\alpha(x) = rA(\theta)$. The boundary condition is $A(0) = A(\alpha) = 0$.

With $\nabla w_\alpha = A\vec{e}_r + A'\vec{e}_\theta$, the ODE satisfied by $\theta\mapsto A(\theta)$ is

$$\frac{A}{\sqrt{1+A^2+A'^2}} + \left(\frac{A'}{\sqrt{1+A^2+A'^2}}\right)' + \frac{2}{A\sqrt{1+A^2+A'^2}} = 0,$$

that is

$$A(1+A^2)(A''+A) + 2(1+A^2+A'^2) = 0. \tag{4.2}$$

The solutions of (4.2) may not be constant. This equation can indeed be integrated once, into

$$A^4(1+A^2+A'^2) = C(1+A^2)^2, \tag{4.3}$$

for some positive constant C. This autonomous ODE has the form $A'^2 = F_C(A)$ where F_C is positive over $(0,A^*)$ with

$$A^* = \sqrt{\frac{1}{2}\left(C+\sqrt{C^2+4C}\right)}.$$

The Cauchy problem

$$A' = \sqrt{F_C(A)}, \qquad A(0) = 0$$

admits a unique maximal solution A_C on an interval $[0,\ell]$, with

$$\ell = \int_0^{A^*} \frac{dA}{\sqrt{F_C(A)}}\,,$$

and we have $A_C'(\ell) = 0$. The maximum principle tells us that at fixed x, the map $C \mapsto A_C(x)$ is increasing. In particular, $C \mapsto \ell$ is increasing; obviously, it is also continuous.

Let us compute the limits $\ell(0)$ and $\ell(+\infty)$. We have

$$\begin{aligned}\ell &= \int_0^{A^*} \frac{A^2dA}{\sqrt{(1+A^2)(A^{*2}-A^2)(CA^{*-2}+A^2)}}\\ &= A^{*3}\int_0^1 \frac{s^2ds}{\sqrt{(1+A^{*2}s^2)(1-s^2)(C+A^{*4}s^2)}}\,.\end{aligned}$$

When $C \to 0^+$, one has $A^{*2} \sim \sqrt{C}$ and therefore

$$\ell \sim C^{1/4}\int_0^1 \frac{s^2ds}{\sqrt{1-s^4}} \to 0,$$

whence $\ell(0) = 0$. When instead $C \to +\infty$, we have $A^{*2} \sim C$ and

$$\ell = \int_0^1 \frac{s^2ds}{\sqrt{(s^2+A^{*-2})(1-s^2)(s^2+CA^{*-4})}} \to \int_0^1 \frac{ds}{\sqrt{1-s^2}} = \frac{\pi}{2}\,.$$

Extending A_C by parity, we obtain a solution A_C of (4.3) vanishing at 0 and 2ℓ, where 2ℓ ranges from 0 to π when $C \in (0,+\infty)$. Therefore, there exists a unique C for which $2\ell = \alpha$. Then $w_\alpha = rA_C(\theta)$ is the announced solution. □

5. Existence proof

So far, we have proved that if the solution w of the BVP in Ω exists, then it enjoys a finite upper bound w_+, a positive lower bound w_-, and a Lipschitz estimate over every compact subdomain $K \subset \Omega$. This ensures that the linear operator $\sum_{i,j} a_{ij}(\nabla w)\partial_i\partial_j$ is uniformly elliptic on relatively compact subdomains. From regularity theory [2], we deduce locally uniform estimates of derivatives $\partial^\beta w$ of every order.

Our existence proof deals first with a modified problem, from which the singularity at the boundary has been removed, and the *a priori* uniform ellipticity has been restored.

5.1. **Relaxation of the boundary condition.** Let $\epsilon > 0$ be given, we consider the BVP formed by the PDE (0.1), together with the boundary condition

$$w = \epsilon \qquad \text{over } \partial\Omega. \tag{5.1}$$

Because of the maximum principle, the solution w_ϵ must be unique and satisfy $w_\epsilon > \epsilon$ in Ω. We may therefore replace the singularity $\frac{1}{w}$ in (0.1) by a smooth positive, decreasing function $g_\epsilon(w)$ which coincides with $\frac{1}{w}$ over $(\epsilon,+\infty)$.

An upper barrier $w_{+,\epsilon}$ can be constructed by the following procedure. For every slab Π_ν containing Ω, we consider the function

$$z_{\epsilon,\nu}(x) = \lambda W\left(\frac{x\cdot\nu-\alpha}{\lambda}\right)$$

with the same W as provided by Lemma 3.1. The parameters are chosen so that $z\equiv\epsilon$ on the boundary of the slab:

$$\lambda=\frac{\alpha_+(\nu)-\alpha_-(\nu)}{1-2s},\qquad \alpha=\frac{1-s}{1-2s}\,\alpha_-(\nu)-\frac{s}{1-2s}\,\alpha_+(\nu),\qquad s=s(\epsilon):=W^{-1}(\epsilon).$$

Because $z_{\epsilon,\nu}$ is a solution in Π_ν, it is a super-solution in Ω. Therefore our upper barrier is

$$w_{+,\epsilon}=\inf_{\nu\in S^{n-1}} z_{\epsilon,\nu}.$$

We point out that $|\nabla z_{\epsilon,\nu}|\leqslant W'(s(\epsilon))$. This implies the same bound for $w_{+,\epsilon}$.

Let now $\bar a$ be a smooth numerical function that coincides with a over $[0,W'(s(\epsilon))]$, such that $r\mapsto r\bar a(r)$ is increasing and $\bar a$ is constant over $[1+W'(s(\epsilon)),+\infty)$. Let $\sigma\in[0,1]$ be a parameter. Then the functions $z_{\epsilon,\nu}$ is a super-solution of the modified, uniformly elliptic PDE

$$\operatorname{div}(\bar a(|\nabla w|)\nabla w)+\sigma g_\epsilon(w)F(|\nabla w|)=0. \tag{5.2}$$

We point out that $z_{\epsilon,\nu}$ is actually a solution when $\sigma=1$. The BVP (5.2,5.1) admits therefore the upper barrier function $w_{+,\epsilon}$. Because a solution satisfies $\epsilon\leqslant w\leqslant w_{+,\epsilon}$, and since $w_{+,\epsilon}=\epsilon$ on the boundary, one infers that $w|_{\partial\Omega}=\epsilon$ and the normal derivative $\partial_\nu w$ is bounded by that of $w_{+,\epsilon}$, that is by $W'(s(\epsilon))$. Then, because a PDE of the form above enjoys a maximum principle for derivatives, we find that for any solution of (5.2,5.1), one has $|\nabla w|\leqslant W'(s(\epsilon))$.

All this, together with Theorem 11.3 of [2], shows that the map T, defined by $w\mapsto z=Tw$ if

$$\operatorname{div}(\bar a(|\nabla w|)\nabla z)+\sigma g_\epsilon(w)F(|\nabla w|)=0,\qquad z|_{\partial\Omega}=\epsilon$$

admits a fixed point w_ϵ in $\mathcal{C}^{1,\beta}(\overline{\Omega})$, which is a classical solution of (5.2,5.1). It satisfies the expected bounds

$$\epsilon\leqslant w_\epsilon\leqslant w_{+,\epsilon}. \tag{5.3}$$

These bounds ensure that w_ϵ is actually a solution of (0.1,5.1). We point out that w_ϵ is unique.

5.2. **Passage to the limit.** We now prove that the w_ϵ's satisfy uniform estimates. On the one hand, the same rescaling as before can be used: if $p,q\in\Omega$, and if $\Omega-q\subset\lambda(\Omega-p)$, then

$$x\mapsto\lambda w_\epsilon\left(\frac{x}{\lambda}+p\right)$$

solves (5.2) in $\Omega-q$, and is $\geqslant\lambda\epsilon\geqslant\epsilon$ over $\partial(\Omega-q)$. By the MP, we deduce

$$w_\epsilon(x+q)\leqslant\lambda w_\epsilon\left(\frac{x}{\lambda}+p\right).$$

Setting $x=0$ in the inequality above, we obtain

$$w_\epsilon(q)\leqslant\lambda w_\epsilon(p).$$

This shows that $\log w_\epsilon$ is Lipschitz with respect to the Hilbert metric, with Lipschitz constant $\leqslant 1$. This implies a Lipschitz estimate in the usual sense, over every compact subdomain.

On the other hand, the same lower barrier w_- applies to the modified BVP, and the upper barrier $w_{+,\epsilon}$ converges uniformly towards w_+ as $\epsilon\to 0+$. By regularity theory, we therefore obtain uniform bounds for higher derivatives in every compact subdomain.

By Ascoli–Arzela and a diagonal procedure, we may extract from $(w_\epsilon)_{\epsilon>0}$ a subsequence that converges in $\mathcal{C}^{1,\beta}_{loc}(\Omega)$ for some $\beta > 0$, to some limit function w. We may pass to the limit in (0.1), so that w solves the PDE. On the other hand, passing to the limit in $w_- \leqslant w_\epsilon \leqslant w_{+,\epsilon}$ yields $w_- \leqslant w \leqslant w_+$. In particular, $w \in \mathcal{C}(\overline{\Omega})$ and w satisfies the boundary condition (0.3). This ends the proof of Theorem 0.1.

Acknowledgement

I am indebted to Ludovic Marquis for useful comments about the Hilbert metric and its variants.

References

[1] M. T. Anderson. Complete minimal varieties in hyperbolic space. *Inventiones mathematicae*, 69:477–494, 1982.

[2] D. Gilbarg, N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer-Verlag, Heidelberg, 2001.

[3] D. Hilbert. Ueber die gerade Linie als kürzeste Verbindung zweier Punkte. *Mathematische Annalen*, 46:91–96, 1895.

[4] Fang Hua Lin. On the Dirichlet problem for minimal graphs. *Inventiones mathematicae*, 96:593–612, 1989.

[5] L. Marquis. Géométrie de Hilbert. *Images des Mathématiques*, CNRS (2015). `http://images.math.cnrs.fr/Geometrie-de-Hilbert.html`.

[6] D. Serre. Multi-dimensional shock interaction for a Chaplygin gas. *Arch. Rational Mech. Anal.*, 191:539–577, 2009.

[7] D. Serre. Gradient estimate in terms of a Hilbert-like distance, for minimal surfaces and Chaplygin gas. *Comm. Partial Diff. Equ.*, 41:774–784, 2016.

[8] C. Walsh. Gauge-reversing maps on cones, and Hilbert and Thompson isometries. Preprint arXiv:1312.7871 [math.MG] (December 2013).

Manuscript received February 2, 2017,
revised May 9, 2017,
accepted May 10, 2017.

Denis SERRE
UMPA, UMR 5669, École Normale Supérieure de Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France
denis.serre@ens-lyon.fr