

# CONFLUENTES MATHEMATICI

Davide RINALDI, Giovanni SAMBIN, and Peter SCHUSTER

**The Basic Zariski Topology**

Tome 7, n° 1 (2015), p. 55-81.

[http://cml.cedram.org/item?id=CML\\_2015\\_\\_7\\_1\\_55\\_0](http://cml.cedram.org/item?id=CML_2015__7_1_55_0)

© Les auteurs et Confluentes Mathematici, 2015.

*Tous droits réservés.*

L'accès aux articles de la revue « Confluentes Mathematici » (<http://cml.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://cml.cedram.org/legal/>). Toute reproduction en tout ou partie de cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

**cedram**

Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>

## THE BASIC ZARISKI TOPOLOGY

DAVIDE RINALDI, GIOVANNI SAMBIN, AND PETER SCHUSTER

**Abstract.** We present the Zariski spectrum as an inductively generated basic topology à la Martin-Löf and Sambin. Since we can thus get by without considering powers and radicals, this simplifies the presentation as a formal topology initiated by Sigstam. Our treatment includes closed subspaces and basic opens: that is, arbitrary quotients and singleton localisations. All the effective objects under consideration are introduced by means of inductive definitions. The notions of spatiality and reducibility are characterized for the class of Zariski formal topologies, and their nonconstructive content is pointed out: while spatiality implies classical logic, reducibility corresponds to a fragment of the Axiom of Choice in the form of Russell's Multiplicative Axiom.

### 1. INTRODUCTION

The theory of locales [14, 15, 35] has shown that a large part of general topology can be described assuming the lattice of open sets, instead of the space of points, as main object of investigation. In this context, it is possible to reformulate numerous classical theorems, and to reprove them without invoking the Axiom of Choice (AC).<sup>1</sup>

A predicative and constructive approach to locale theory, known as Formal Topology, was started by Per-Martin Löf and the second author [25] in order to formalize and develop general topology in Martin-Löf's dependent type theory (dTt) [20]. Rather than the space of points or the lattice of opens, here a basis of opens (more precisely, an index set for the basis elements) is the primitive object. In the present paper, we follow this approach for analysing constructively the Zariski spectrum of a commutative ring.

We will not explicitly refer to a particular constructive foundation, but work with intuitionistic logic, and avoid using forms of the AC as well as impredicative definitions. In order to realise the latter requirement, we have to distinguish sets from collections, and restrict separation to restricted formulas (that is, formulas that do not contain quantifiers ranging over a collection). As the paradigmatic example of a collection is formed by all the subsets of a given set, we refrain from using the Axiom of Power Set (PSA).

The *prime spectrum*  $\mathfrak{Spec}(A)$  of a commutative ring  $A$  is the collection of its prime ideals  $\mathfrak{p}$ , which is usually endowed with the *Zariski topology*: the topology generated from the basis of opens  $\{D(a)\}_{a \in A}$  where

$$D(a) = \{\mathfrak{p} \in \mathfrak{Spec}(A) : a \notin \mathfrak{p}\} \quad (1.1)$$

for every  $a \in A$ . This topological space was one of the starting points for modern algebraic geometry, and its impredicative nature determines the apparent non-constructive character of large parts of the subsequent theory. Moreover, the existence of a prime ideal in general depends on Zorn's Lemma or other forms of AC.<sup>2</sup>

The Zariski spectrum lends itself naturally to a point-free description, both in terms of locales [14, 35] and formal topologies [30, 31, 33]. We develop this second approach within the so-called *basic picture* [29, 26, 27]. The basic concept is that

---

*Math. classification:* 03E25, 03F65, 13A99, 54B35.

<sup>1</sup>For instance, one can prove without AC that the locale of Dedekind reals is locally compact; for the point-wise counterpart, however, AC is required [14].

<sup>2</sup>Banaschewski [2] has proved that the existence of a prime ideal in a non-trivial ring is equivalent to the Boolean Ultrafilter Theorem. Joyal [17] has built, inside a topos, a ring without prime ideals.

of *basic topology*, a generalization of the notion of formal topology that allows a positive description of both open and closed subsets. We work directly on the index set for the basis (1.1): the ring  $A$ .

A cover relation  $\triangleleft$  between element and subsets is common to every formal topology, and corresponds to the interior operator of a topology. In addition, a basic topology has a *positivity* or reduction relation  $\times$  that describes formally the behaviour of the closure operator.<sup>3</sup> These operators are related through an appropriate *compatibility* condition.

In [21], a strategy is given to generate basic topologies by induction and coinduction. By means of this, we can equip every ring  $A$  with a basic topology, starting from the inductive generation of ideals. The novelty with respect to [30, 31, 33] is that all the topological definitions, and all the related proofs, are explicitly of inductive/coinductive sort. Therefore, regardless of foundational issues, an effective implementation of these concepts is direct. Moreover, we can get by with ideals rather than radical ideals.

To a basic topology, one often assigns an *operation* which describes formally the intersection of two basic opens [25, 3]. In our setting, the product of the ring  $A$  is a natural candidate, and the induced notion of formal point matches classically with that of (the complement of) a prime ideal of  $A$ , which is the usual notion of point of the prime spectrum. In other words, this basic topology corresponds precisely to the customary Zariski topology. The correspondence which to each ring assigns a basic topology with operation is then extended to a functor, as in the classical case.

The *formal Zariski topology* [30, 31, 33] is obtained from the *basic Zariski topology* by adding a further generation rule; and every property of the latter extends canonically to the former. One can thus return to radical ideals as occasion demands.

The last part of the article is devoted to the description of two impredicative principles associated to the formal Zariski topology, *spatiality* and *reducibility*. Assuming classical logic, each of these two principles is equivalent to Krull's Lemma, but with intuitionistic logic they must be kept apart. While spatiality corresponds to the spatiality of the locale of radical ideals and is a completeness principle, reducibility affirms the existence of a formal point—that is, an appropriate sort of model—and so is a satisfiability principle. We will show, strengthening some results from [12, 22], that these principles are constructively untenable for the class of formal Zariski topologies as a whole.

Another constructive and predicative approach to the Zariski spectrum, developed in [16, 17] and used e.g. in [9, 6, 8], is by way of *distributive lattices*<sup>4</sup>. In contrast to this, the avenue via the basic picture is closer to the customary treatment, and allows us to consider simultaneously the notions of closed and open subsets. Last but not least, the potential presence of points makes working on the formal side more intuitive. Our approach has already proved fruitful in constructive commutative algebra [19], e.g. for an elementary characterization of the height of an ideal in a commutative ring [23].

## 2. THE BASIC ZARISKI TOPOLOGY

A *basic pair* is a structure  $(X, \Vdash, S)$ , where  $X$  and  $S$  are sets and  $\Vdash: X \rightarrow S$  is a relation between them. This structure, albeit minimal, is sufficient to introduce elegantly the topological concepts of closure and interior.

---

<sup>3</sup>Such relation can be considered as a generalization of the unary positivity predicate *Pos* part of the original definition of formal topology [25].

<sup>4</sup>This is of particular relevance in view of the strong tradition started by Stone [34], especially the interplay between spectral spaces and distributive lattices.

We can in fact think of  $S$  as a set of indices for a basis  $\{B_a\}_{a \in S}$  of a topological space  $X$ , and  $\Vdash$  defined by  $x \Vdash a \equiv x \in B_a$ . In these terms, one has

$$\begin{aligned} x \in \text{cl } D &\equiv \forall a \in S(x \Vdash a \rightarrow \exists y \in X(y \Vdash a \ \& \ y \in D)), \\ x \in \text{int } D &\equiv \exists a \in S(x \Vdash a \ \& \ \forall y \in X(y \Vdash a \rightarrow y \in D)) \end{aligned}$$

for all  $x \in X$  and<sup>5</sup>  $D \subseteq X$ . Inverting formally the roles of  $X$  and  $S$ , we obtain two relations between elements  $a$  and subsets  $U$  of  $S$  as follows:

$$\begin{aligned} a \triangleleft_{\Vdash} U &\equiv \forall x \in X(x \Vdash a \rightarrow \exists b \in S(x \Vdash b \ \& \ b \in U)), \\ a \times_{\Vdash} U &\equiv \exists x \in X(x \Vdash a \ \& \ \forall b \in S(x \Vdash b \rightarrow b \in U)). \end{aligned}$$

Maintaining this topological intuition, the primitive concepts of closure and interior are reflected by a certain symmetry on the set of the basis indices, which assumes a role equal to that of space. One thus introduces the main concept of the Basic Picture:

DEFINITION 2.1. — A basic topology is a structure  $(S, \triangleleft, \times)$  where  $S$  is a set, and  $\triangleleft, \times$  are relations between elements and subsets of  $S$  satisfying:

$$\begin{array}{ll} \frac{a \in U}{a \triangleleft U} \text{ Reflexivity} & \frac{a \triangleleft U \quad U \triangleleft V}{a \triangleleft V} \text{ Transitivity} \\ \frac{a \times U}{a \in U} \text{ Coreflexivity} & \frac{a \times U \quad \forall b(b \times U \rightarrow b \in V)}{a \times V} \text{ Cotransitivity} \\ \frac{a \triangleleft U \quad a \times V}{U \times V} \text{ Compatibility} & \end{array}$$

where

$$U \triangleleft V \equiv (\forall u \in U)(u \triangleleft V) \quad \text{and} \quad U \times V \equiv (\exists u \in U)(u \times V)$$

for all  $a, b \in S$  and  $U, V \subseteq S$ . The relations  $\triangleleft$  and  $\times$  are called cover and positivity respectively.

By Coreflexivity and Cotransitivity, if  $U \subseteq V$ , then  $a \times U$  implies  $a \times V$ . Dually, by Coreflexivity and Compatibility, if  $a \triangleleft U$  and  $a \times V$ , then  $U \times V$ .

Sometimes we will denote  $a \triangleleft U$  by  $a \in U_{\triangleleft}$  and  $a \times F$  by  $a \in F_{\times}$ . In particular, from Cotransitivity follows

$$\frac{a \times U}{a \times S}$$

for all  $a \in S$  and  $U \subseteq S$ . The definition of basic topology axiomatises what happens in a basic pair on the basis side, avoiding any reference to the existence of a concrete space  $X$ , which often is not available from a constructive point of view.

Let us fix a commutative ring with unit  $(A, +, \cdot, 0, 1)$ . We define a basic topology  $\text{Zar}(A)$ , called the *basic Zariski topology*, by means of some generation axioms. For an exhaustive description of the inductive generation of basic topologies we refer to [21, 10].

We can define a cover  $\triangleleft$  on the ring  $A$  which satisfies the axioms

$$\frac{a \in U}{a \triangleleft U} \text{ Refl} \quad \frac{\top}{0 \triangleleft U} 0 \quad \frac{a \triangleleft U \quad b \triangleleft U}{a + b \triangleleft U} \Sigma \quad \frac{a \triangleleft U \quad \lambda \in A}{\lambda \cdot a \triangleleft U} \text{ II} \quad (2.1)$$

and is the least relation which satisfies this property, that is, the *induction* axiom

$$\frac{\begin{array}{c} [a \in P, b \in P] \quad [a \in P, \lambda \in A] \\ \vdots \\ a \triangleleft U \quad U \subseteq P \quad 0 \in P \quad a + b \in P \quad \lambda \cdot a \in P \end{array}}{a \in P} \triangleleft\text{-induction}$$

<sup>5</sup>A subset  $U \subseteq X$  is intended to be a proposition  $U(a)$  depending on one argument  $a$  in  $X$ . In plain terms,  $a \in U$  means  $U(a)$ . For the sake of readability, in this paper we only use  $\in$  as symbol for membership, though for predicativity's sake one could distinguish two membership symbols,  $\in$  and  $\epsilon$ , to indicate, respectively, membership to a set and membership to a subset [27].

holds for all  $U \subseteq A$  and  $a \in A$ .

We can describe the cover in this way: one has  $a \triangleleft U$  if and only if  $a = 0$  or if there exists a derivation tree which uses just the rules Refl,  $\Sigma$  and  $\Pi$ , and has leaves of the form  $c \in U$  and  $a \triangleleft U$  as root. Here is a brief analysis of the derivation trees:

- (1) Since the product is associative, if in a proof one applies the product rule twice consecutively, then one application is sufficient:

$$\frac{\frac{\frac{\pi}{\vdots} a \triangleleft U}{\lambda \cdot a \triangleleft U} \Pi}{\lambda' \cdot (\lambda \cdot a) \triangleleft U} \Pi \quad \mapsto \quad \frac{\frac{\pi}{\vdots} a \triangleleft U}{(\lambda' \cdot \lambda) \cdot a \triangleleft U} \Pi$$

- (2) Since the product distributes over the sum, if in a derivation tree the product rule follows the sum rule, we can swap the two operations. More precisely:

$$\frac{\frac{\frac{\pi_a}{\vdots} a \triangleleft U \quad \frac{\pi_b}{\vdots} b \triangleleft U}{a + b \triangleleft U} \Sigma}{\lambda \cdot (a + b) \triangleleft U} \Pi \quad \mapsto \quad \frac{\frac{\frac{\pi_a}{\vdots} a \triangleleft U}{\lambda \cdot a \triangleleft U} \Pi \quad \frac{\frac{\pi_b}{\vdots} b \triangleleft U}{\lambda \cdot b \triangleleft U} \Pi}{\lambda \cdot a + \lambda \cdot b \triangleleft U} \Sigma$$

Therefore, given a derivation tree  $\pi$ , by applying these transformations we obtain a normalized derivation tree: each leaf is followed by an invocation of Refl, then by one of  $\Pi$  and eventually by  $\Sigma$  a finite number of times.

Hence we have shown

$$a \triangleleft U \leftrightarrow a = 0 \vee (\exists n \in \mathbb{N})(\exists u_1, \dots, u_n \in U)(\exists \lambda_1, \dots, \lambda_n \in A)(a = \sum_{i=1}^n \lambda_i \cdot u_i) \quad (2.2)$$

for all  $a \in A$  and all  $U \subseteq A$ . The case  $a = 0$  can be included as combination of zero coefficients, so that  $a \triangleleft U$  if and only if  $a$  belongs to

$$I(U) = \{a \in A : (\exists n \in \mathbb{N})(\exists u_1, \dots, u_n \in U)(\exists \lambda_1, \dots, \lambda_n \in A)(a = \sum_{i=1}^n \lambda_i \cdot u_i)\},$$

the ideal generated by  $U$ .

Given  $a, b \in A$ , one has  $a \triangleleft b$  if and only if there exists  $c \in A$  such that  $a = b \cdot c$ . In particular  $b$  is invertible if and only if  $1 \triangleleft b$ .

It follows from characterization (2.2) that the cover is finitary, that is

$$a \triangleleft U \leftrightarrow \exists U_0 \in \mathcal{P}_\omega(U)(a \triangleleft U_0).$$

where  $\mathcal{P}_\omega(U)$  is the set of finite subsets<sup>6</sup> of  $U$ .

In addition to the cover  $\triangleleft$ , it is possible to generate by *coinduction* a positivity  $\times$  by means of the axioms

$$\frac{a \times F}{a \in F} \quad \frac{0 \times F}{\perp} \quad \frac{a + b \times F}{a \times F \vee b \times F} \quad \frac{a \cdot b \times F}{a \times F}$$

closed coinductively by the rule

$$\frac{a \in F \quad F \subseteq G \quad \neg(0 \in F) \quad \frac{[a + b \in F] \quad [\lambda \cdot a \in F, \lambda \in A]}{\vdots} \quad a \in F \vee b \in F \quad a \in F}{a \times G} \quad \times\text{-coinduction}$$

<sup>6</sup>We say that, following [4], a subset  $U \subseteq S$  is finite if there exist  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in S$  such that

$$x \in U \leftrightarrow x = x_1 \vee \dots \vee x = x_n.$$

One can show in dTT that the collection of finite subsets of a given set is again a set.

The general theory [21] states that  $\times$  is precisely the greatest positivity compatible with the cover  $\triangleleft$ . We denote by  $\text{Zar}(A)$  the basic topology  $(A, \triangleleft, \times)$  just defined.

A subset  $U \subseteq S$  in a basic topology  $(S, \triangleleft, \times)$  is said to be *saturated* if

$$a \triangleleft U \leftrightarrow a \in U,$$

and *reduced* if

$$a \times U \leftrightarrow a \in U.$$

As cover and positivity encode closure and interior operator, the saturated and reduced subsets correspond on the basis to open and closed subsets. Hence it is worthwhile to give an explicit characterization of these two concepts for  $\text{Zar}(A) = (A, \triangleleft, \times)$ .

The saturated subsets are exactly the  $U \subseteq A$  satisfying

$$\frac{}{0 \in U} \quad \frac{a \in U \quad b \in U}{a + b \in U} \quad \frac{a \in U \quad \lambda \in A}{\lambda \cdot a \in U}$$

for all  $a, b \in A$ , and thus are the *ideals* of  $A$ . This is easy to see: one direction is the Reflexivity axiom, the reverse one is obtained from the  $\triangleleft$ -induction axiom, setting  $P = U$ .

Symmetrically, a subset  $F \subseteq A$  is reduced if and only if

$$\frac{0 \in F}{\perp} \quad \frac{a + b \in F}{a \in F \vee b \in F} \quad \frac{a \cdot b \in F}{a \in F}$$

or, in other words,  $F$  is a *coideal* of the ring  $A$ . The coideals do not appear in the usual theory of rings, because with classical logic  $F$  is a coideal if and only if  $\neg F$  is an ideal, and the two notions are interchangeable.<sup>7</sup> The relation  $a \times F$  asserts that  $a$  belongs to the greatest coideal contained in  $F$ . Constructively, the link between ideals and coideals is richer, and it is contained in the compatibility condition:

$$I(U) \check{\delta} F \rightarrow U \check{\delta} F$$

for all  $U \subseteq A$  and  $F$  reduced. Explicitly,  $F$  is reduced if and only if

$$\frac{\lambda_1 \cdot a_1 + \dots + \lambda_n \cdot a_n \in F}{a_1 \in F \vee a_2 \in F \vee \dots \vee a_n \in F}$$

for all  $n \in \mathbb{N}$  and  $\lambda_1, \dots, \lambda_n, a_1, \dots, a_n \in A$ . In particular, we notice that a reduced subset  $F$  is inhabited if and only if it contains 1, because  $(1) = A$  and  $(1) \check{\delta} F$  implies  $\{1\} \in F$ .

The saturated subsets of a basic topology (in this case, the ideals of  $A$ ) form a collection, denoted by  $\text{Sat}(\triangleleft)$ . This  $\text{Sat}(\triangleleft)$  is endowed with a natural structure of complete lattice:

$$\bigvee_{i \in I} U_i = I\left(\bigcup_{i \in I} U_i\right) \quad \text{and} \quad \bigwedge_{i \in I} U_i = \bigcap_{i \in I} U_i.$$

We will often identify  $\text{Sat}(\triangleleft)$  with the collection  $\mathcal{PA}$  of subsets of  $A$  with the equality relation  $=_{\triangleleft}$  defined as

$$U =_{\triangleleft} V \equiv U \triangleleft V \ \& \ V \triangleleft U.$$

In the case of  $\text{Zar}(A)$ , two subsets are equal, in this sense, if they generate the same ideal. Unlike the case of the formal Zariski topology, binary meet does not distribute over set-indexed join and then the lattice of saturated subsets is not a locale [14, 30].

In the following section we will show that, in the case of the basic Zariski topology, the lattice of saturated subsets is endowed with a further operation that provides it with a structure of *commutative unital quantale*. We recall from [24, 28]

<sup>7</sup>Incidentally, the same criterion does not apply in general topology: open and closed subsets do coexist and play distinct roles, though these notions are equally interchangeable by complementation.

that this is a structure  $(Q, \vee, \cdot, 1)$  such that  $(Q, \cdot, 1)$  is a commutative monoid,  $(Q, \vee)$  is a complete join-semilattice and  $\cdot$  distributes over  $\vee$ .

### 3. THE PRODUCT AS CONVERGENCE OPERATION

In this section we will show how the basic topology  $\text{Zar}(A)$  is linked to its classical counterpart. It is worthwhile to point out its naturalness within the algebraic context, being just an accurate revision of the inductive generation process of ideals.

In order to make more precise the link to usual topology, we have to introduce an operation to describe formally the intersection of two basic opens [3]:

DEFINITION 3.1. — *A convergence operation for a basic topology  $(S, \triangleleft, \times)$  is a commutative operation  $\circ : A \times A \rightarrow \mathcal{P}A$  which satisfies*

$$\frac{a \triangleleft U \quad b \triangleleft V}{a \circ b \triangleleft U \circ V} \text{ Stability}$$

where

$$U \circ V = \bigcup_{u \in U, v \in V} u \circ v$$

for all  $a, b \in A$  and  $U, V \subseteq A$ .

Stability ensures that  $\circ$  behaves well with respect the equality  $=_{\triangleleft}$  and thus it is well-defined on the lattice of saturated subsets  $\text{Sat}(\triangleleft)$ :

$$U =_{\triangleleft} U' \ \& \ V =_{\triangleleft} V' \Rightarrow U \circ V =_{\triangleleft} U' \circ V'$$

for all  $U, U', V, V' \subseteq A$ .

In practice, if the set  $A$  comes with an operation  $\cdot$  for which  $(A, \cdot)$  is a monoid, it is convenient to consider  $\cdot$  as convergence operation. This is the case of the product  $\cdot$  of the ring  $A$ :

PROPOSITION 3.2. — *The product  $\cdot$  of the ring  $A$  is a convergence operation on  $\text{Zar}(A)$ . Moreover, the following two conditions are satisfied, for all  $a, b \in A$ :*

(Weakening):  $a \cdot b \triangleleft a$ ;

(Unitality):  $a \triangleleft a \cdot A$ .

*Proof.* — We first prove Stability. Let  $\pi_a$  and  $\pi_b$  be two derivation trees for  $a \triangleleft U$  and  $b \triangleleft V$  respectively, and let us prove  $a \cdot b \triangleleft U \cdot V$  by induction on  $\pi_a$  and  $\pi_b$ . If the two derivation trees consist of just one use of rule Refl, we proceed as follows:

$$\frac{a \in U}{a \triangleleft U} \text{ Refl}, \quad \frac{b \in V}{b \triangleleft V} \text{ Refl} \quad \mapsto \quad \frac{\frac{a \in U \quad b \in V}{a \cdot b \in U \cdot V} \text{ Refl}}{a \cdot b \triangleleft U \cdot V} \text{ Refl}$$

If the last rule used in  $\pi_a$  is  $\Sigma$  where  $a = a_1 + a_2$ , we apply the following transformation:

$$\frac{\frac{\frac{\pi_1}{\vdots} a_1 \triangleleft U \quad \frac{\pi_2}{\vdots} a_2 \triangleleft U}{a \triangleleft U} \Sigma, \quad \frac{\pi_b}{\vdots} b \triangleleft V}{a \cdot b \triangleleft U \cdot V} \mapsto \frac{\frac{\frac{\frac{\pi_1}{\vdots} a_1 \triangleleft U \quad \frac{\pi_b}{\vdots} b \triangleleft V}{a_1 \cdot b \triangleleft U \cdot V} \text{ Stab.}, \quad \frac{\frac{\pi_2}{\vdots} a_2 \triangleleft U \quad \frac{\pi_b}{\vdots} b \triangleleft V}{a_2 \cdot b \triangleleft U \cdot V} \text{ Stab.}}{a \cdot b \triangleleft U \cdot V} \Sigma$$

Suppose instead that the last rule is  $\Pi$ , where  $a = \lambda \cdot a_1$ . Hence:

$$\frac{\frac{\frac{\pi_a}{\vdots} a_1 \triangleleft U \quad \lambda \in A}{a \triangleleft U} \Pi, \quad \frac{\pi_b}{\vdots} b \triangleleft V}{a \cdot b \triangleleft U \cdot V} \mapsto \frac{\frac{\frac{\frac{\pi_1}{\vdots} a_1 \triangleleft U \quad \frac{\pi_b}{\vdots} b \triangleleft V}{a_1 \cdot b \triangleleft U \cdot V} \text{ Stab.}, \quad \lambda \in A}{a \cdot b \triangleleft U \cdot V} \Pi$$

By way of these modifications, we get the required deduction tree.

To prove Unitality, it is sufficient to notice that  $a = a \cdot 1 \in a \cdot A$  and then to use Reflexivity. Finally, the weakening property is just a special case of the rule  $\Pi$ .  $\square$

As first consequence, the product is a well-defined operation on the lattice of ideals  $\text{Sat}(\triangleleft)$ . Explicitly, for all subsets  $U, V \subseteq A$ , we have  $I(U) \cdot I(V) = I(U \cdot V)$ . It follows that this operation distributes with set-indexed joins and  $\bigvee$ :

$$V \cdot \bigvee_{i \in I} U_i \stackrel{\text{Stab.}}{=}_{\triangleleft} V \cdot \bigcup_{i \in I} U_i =_{\triangleleft} \bigcup_{i \in I} V \cdot U_i = \bigvee_{i \in I} V \cdot U_i$$

In addition, there exists an element  $A = (1)$  of  $\text{Sat}(\triangleleft)$  such that, for all subsets  $U \subseteq A$ ,

$$U \cdot A =_{\triangleleft} U.$$

In all, the lattice  $(\text{Sat}(\triangleleft), \bigvee, \cdot, A)$  is a commutative unital quantale. The weakening property implies also  $U \circ V \triangleleft U \wedge V$  for all  $U, V \subseteq A$ .

**Points.** As already stressed, the intuition behind a basic topology is that of a set of basis indices for an ideal space of points. A formal point is defined as a subset of indices that behaves as a neighbourhood filter of an imaginary point.

**DEFINITION 3.3.** — *Let  $(A, \triangleleft, \times, \circ)$  be a basic topology with operation. A subset  $\alpha \subseteq S$  is said to be a formal point if:*

- (1)  $\alpha$  is inhabited, i.e.  $\exists a(a \in \alpha)$ ;
- (2)  $\alpha$  is filtering, i.e.  $a \in \alpha \ \& \ b \in \alpha \rightarrow a \circ b \checkmark \alpha$  for all  $a, b \in A$ ;
- (3)  $\alpha$  is reduced.

In the case of  $\text{Zar}(A)$ , to say that  $\alpha$  is a formal point amounts to say that  $\alpha$  is a coideal (i.e. *splits the cover*  $\triangleleft$ ) satisfying  $1 \in \alpha$  and  $a, b \in \alpha \rightarrow a \cdot b \in \alpha$  for all  $a, b \in A$ . A subset with all these properties is called *prime coideal* or, more commonly, *prime filter*.

**Remark 3.4.** — In general, not having the *Axiom of Power Set* (PSA) at our disposal, the formal points associated with  $\text{Zar}(A)$  do not form a set. Then we say that they form a *collection*, which we denote by  $\mathfrak{Pt}(A)$ . These ideal entities do not affect the effectivity of the underlying theory, but they provide a valuable help for our intuition.

Sometimes, it will be useful to consider this collection to be a set, to describe adequately the formal points and their applications. In these cases, we will indicate the presence of PSA or of any other impredicative assumption.

We regain moreover the link with the usual notion in algebraic geometry, where the points of the prime spectrum are the prime ideals of the ring. We recall that a *prime ideal* is a subset  $\mathfrak{p} \subseteq A$  such that:

- (1)  $\neg(1 \in \mathfrak{p})$  (or, equivalently,  $\mathfrak{p} \neq A$ ),
- (2)  $a \cdot b \in \mathfrak{p} \rightarrow a \in \mathfrak{p} \vee b \in \mathfrak{p}$ , for all  $a, b \in A$ ,
- (3)  $\mathfrak{p}$  is an ideal.

With classical logic,  $\mathfrak{p}$  is a prime ideal if and only if its complement  $-\mathfrak{p}$  is a prime coideal, or formal point. We have therefore a bijective correspondence

$$\begin{array}{ccc} - : \mathfrak{Spec}(A) & \longrightarrow & \mathfrak{Pt}(A) \\ \mathfrak{p} & \longmapsto & -\mathfrak{p} \end{array}$$

between the prime spectrum and the collection of formal points of  $A$ .

In general, reasoning impredicatively, the space of formal points  $\mathfrak{Pt}(A)$  of a basic topology defines a basic pair

$$\Vdash : \mathfrak{Pt}(A) \longrightarrow A$$

where

$$\alpha \Vdash x \equiv x \in \alpha$$



for all  $x \in A$  and  $\alpha \in \mathfrak{Pt}(A)$ . We can then associate to  $\mathfrak{Pt}(A)$  a closure and an interior operator by setting

$$\begin{aligned}\alpha \in \text{cl } D &\equiv \forall a \in S(\alpha \Vdash a \rightarrow \exists \beta \in \mathfrak{Pt}(A)(\beta \Vdash a \ \& \ \beta \in D)), \\ \alpha \in \text{int } D &\equiv \exists a \in S(\alpha \Vdash a \ \& \ \forall \beta \in \mathfrak{Pt}(A)(\beta \Vdash a \rightarrow \beta \in D))\end{aligned}$$

for all  $\alpha \in \mathfrak{Pt}(A)$  and  $D$  subcollection of  $\mathfrak{Pt}(A)$ .

**PROPOSITION 3.5.** — *Let  $\text{Zar}(A)$  be the basic Zariski topology on the commutative ring  $A$ , and  $\mathfrak{Pt}(A)$  the corresponding collection of points. The operator  $\text{int}$  defined impredicatively by the basic pair*

$$\Vdash: \mathfrak{Pt}(A) \longrightarrow A$$

is a topological interior operator. This means

$$\text{int } D \cap \text{int } E = \text{int } (D \cap E)$$

for all pairs  $D, E$  of sub-collections of  $\mathfrak{Pt}(A)$ .

*Proof.* — Since  $\text{int}$  is a monotone operator,  $\subseteq$  is the only non-trivial inclusion. Suppose that  $\alpha \in \text{int } D \cap \text{int } E$ , i.e.  $\alpha \in \text{int } D$  and  $\alpha \in \text{int } E$ ; by definition there exist  $a, a' \in \alpha$  such that  $a \in \beta \rightarrow \beta \in D$  and  $a' \in \beta \rightarrow \beta \in E$  for all  $\beta \in \mathfrak{Pt}(A)$ . We must produce  $b \in \alpha$  such that  $b \in \beta \rightarrow \beta \in D \cap E$  for all  $\beta \in \mathfrak{Pt}(A)$ . Define  $b = a \cdot a' \in S$ . Then  $b \in \alpha$  because  $\alpha$  is filtering; on the other hand  $b \in \beta$  implies  $a \in \beta$  for all  $\beta \in \mathfrak{Pt}(A)$ , because  $b = a \cdot a' \triangleleft a$  (viz. Weakening) and  $\beta$  splits the cover. It follows that  $b \in \beta \rightarrow \beta \in D$ . Symmetrically, for all  $\beta \in \mathfrak{Pt}(A)$ , we have  $b \in \beta \rightarrow \beta \in E$  and thus  $b \in \beta \rightarrow \beta \in D \cap E$ .  $\square$

The proof makes only use of Weakening and thence admits a slight generalization to all basic topologies satisfying this property. On  $\mathfrak{Pt}(A)$  one obtains, even if impredicatively, a structure of topological space. The topology, in the usual sense, is precisely the one induced by the basic opens of the form  $\text{Ext}(a)$ , defined by

$$\alpha \in \text{Ext}(a) \equiv a \in \alpha$$

where  $\alpha \in \mathfrak{Pt}(A)$  and  $a \in A$ . One defines the operators:

$$\alpha \in \text{Ext}(U) \equiv \alpha \check{\Vdash} U \quad \alpha \in \text{Rest}(F) \equiv \alpha \subseteq F$$

where  $U, F \subseteq A$ . The sub-collections of the form  $\text{Ext}(U)$  or  $\text{Rest}(U)$  are, respectively, the fixed points of the operators  $\text{int}$  and  $\text{cl}$ , that is, the open and closed subsets of the new topology. Notice that these concepts are not deducible one from another through complementation. In particular

$$\text{Ext}(U) = \text{Ext}(U_{\triangleleft}) \quad \text{and} \quad \text{Rest}(F) = \text{Rest}(F_{\times}).$$

Every open subset is the image of an ideal through  $\text{Ext}$  and every closed subset is the image of a coideal through  $\text{Rest}$ .

Similarly, if we define the relation  $\Vdash: \mathfrak{Spec}(A) \rightarrow A$  as

$$\mathfrak{p} \Vdash a \equiv \neg(a \in \mathfrak{p}),$$

then we have, impredicatively, a basic pair, and a topology on  $\mathfrak{Spec}(A)$  the basis of which is precisely  $\{D(a)\}_{a \in A}$ ; in other words, the Zariski topology. The classical complementation  $(-)^c$  extends to an isomorphism of basic pairs:

$$\begin{array}{ccc} & A & \\ \Vdash \swarrow & & \searrow \Vdash \\ \mathfrak{Spec}(A) & \xleftrightarrow{(-)^c} & \mathfrak{Pt}(A) \end{array}$$

and, in particular, to an isomorphism of topological spaces  $\mathfrak{Spec}(A) \cong \mathfrak{Pt}(A)$ . In this sense, assuming classical logic, our approach is equivalent to the usual one.

To summarize, the Zariski topology has been obtained by applying the machinery of basic topologies to the inductive generation of ideals.

## 4. RING HOMOMORPHISMS AND CONTINUOUS RELATIONS

We briefly recall the notion of morphism between basic topologies and between basic topologies with operation, starting from some generalities on binary relations. For a more detailed treatment, we refer to [27]. Given a relation  $r : S \rightarrow T$  between two sets, there are four operators on subsets

$$\mathcal{P}(S) \begin{array}{c} \xrightarrow{r} \\ \xrightarrow{r^{-*}} \end{array} \mathcal{P}(T) \quad \text{and} \quad \mathcal{P}(T) \begin{array}{c} \xrightarrow{r^*} \\ \xrightarrow{r^-} \end{array} \mathcal{P}(S),$$

defined by

$$\begin{array}{ll} t \in rU & \equiv r^{-t} \checkmark U & t \in r^{-*}U & \equiv r^{-t} \subseteq U \\ s \in r^{-}V & \equiv rs \checkmark V & s \in r^*V & \equiv rs \subseteq V \end{array}$$

where  $s \in r^{-}t \equiv srt$ ,  $U \subseteq S$ ,  $V \subseteq T$ ,  $s \in S$  and  $t \in T$ . If  $r$  is a functional relation, then  $r^{-}$  and  $r^*$  coincide with the usual reverse image and  $r$  with the direct image.

If  $r : S \rightarrow T$  and  $s : T \rightarrow R$  are two relations, then the composition  $s \circ r$  is defined as follows:

$$a(s \circ r)b \equiv \exists c(arc \ \& \ csb)$$

It is straightforward to verify  $(s \circ r)^{-} = r^{-} \circ s^{-}$ ,  $(s \circ r)^* = r^* \circ s^*$  and  $(s \circ r)^{-*} = s^{-*} \circ r^{-*}$ .

DEFINITION 4.1. — *Let  $(S, \triangleleft_S, \times_S)$  and  $(T, \triangleleft_T, \times_T)$  be two basic topologies. A relation  $r : S \rightarrow T$  is continuous if*

$$\frac{b \triangleleft_T U}{r^{-}b \triangleleft_S r^{-}U} \text{Con}_{\triangleleft} \quad \text{and} \quad \frac{a \times_S s^*U}{ra \times_T U} \text{Con}_{\times} \quad (4.1)$$

for all  $a \in S, b \in T$  and  $U \subseteq T$ .

From  $\text{Con}_{\triangleleft}$  one deduces that the inverse existential image  $r^{-}U$  of a saturated subset  $U \subseteq T$  is a saturated subset, while from  $\text{Con}_{\times}$  it follows that the direct existential image  $rF$  of a reduced subset  $F \subseteq S$  is a reduced subset. It is easy to verify that basic topologies with continuous relations form a category, which we denote by BTop [27].

One verifies that the composition of two continuous relations is a continuous relation, and that so is also the identity relation [27]. If the basic topology under consideration is inductively generated, as is the case for the basic Zariski topology, then from  $\text{Con}_{\triangleleft}$  one can deduce the condition  $\text{Con}_{\times}$  [21].

DEFINITION 4.2. — *Let  $(S, \triangleleft_S, \times_S, \circ_S)$  and  $(T, \triangleleft_T, \times_T, \circ_T)$  be two basic topologies with operation. A continuous relation  $r : S \rightarrow T$  is convergent if it satisfies the conditions*

- C1.  $r^{-}(a) \circ_S r^{-}(b) \triangleleft_S r^{-}(a \circ_T b)$
- C2.  $S \triangleleft_S r^{-}T$

for all  $a, b \in A$ .

The composition of two continuous and convergent operations is again continuous and convergent [27]. We denote by BTopO the category of basic topologies with continuous and convergent relations.

Remark 4.3. — The condition C1 can be reformulated by saying that the direct existential image of a filtering reduced subset  $F \subseteq S$ , through a relation  $r : S \rightarrow T$ , is filtering. In fact, let  $a, b \in rF$ ; equivalently  $r^{-}a \checkmark F$  and  $r^{-}b \checkmark F$ , from which we get  $r^{-}a \circ_S r^{-}b \checkmark F$  because  $F$  is filtering. Since  $F$  is also reduced, from the first convergence condition, by compatibility we obtain  $r^{-}(a \circ_T b) \checkmark F$  and finally  $a \circ_T b \checkmark rF$ .

The property C2 entails that the direct image of a reduced inhabited subset  $F \subseteq S$  is inhabited. Suppose that  $F \checkmark S$ ; since  $F$  is reduced, we have  $F \checkmark r^{-}T$  and therefore  $rF \checkmark T$ .

The direct existential image of a continuous and convergent morphism  $r : S \rightarrow T$  determines an operator

$$\begin{aligned} \mathfrak{Pt}(r) : \mathfrak{Pt}(S) &\rightarrow \mathfrak{Pt}(T). \\ \alpha &\mapsto r\alpha \end{aligned}$$

This  $\mathfrak{Pt}(r)$  is continuous in the classical sense if we endow the two collections  $\mathfrak{Pt}(S)$  and  $\mathfrak{Pt}(T)$  with the topology generated by the basic opens  $\text{Ext}(a)$ . This correspondence, being nothing but the direct image, defines impredicatively a functor

$$\mathfrak{Pt} : \text{BTopO} \rightarrow \text{Top}.$$

We now analyse the Zariski case, and to this end we fix two commutative rings with unit,  $A$  and  $B$ . One of the most fundamental properties of the prime spectrum lies in the fact that the inverse image of a prime ideal  $\mathfrak{q} \in \mathfrak{Spec}(B)$  through a ring homomorphism  $f : A \rightarrow B$  is a prime ideal  $f^{-1}\mathfrak{q} \in \mathfrak{Spec}(A)$ . In other words,  $f$  induces a map  $f^{-1} : \mathfrak{Spec}(B) \rightarrow \mathfrak{Spec}(A)$ .

If the two spectra are equipped with the Zariski topology, then  $f^{-1}$  is also continuous; more generally, this correspondence extends to a contravariant functor

$$\mathfrak{Spec} : \text{CRings} \rightarrow \text{Top}$$

from the category of commutative rings to that of topological spaces. These classical observations find their constructive counterpart in the framework of the basic picture.

Let  $\text{Zar}(A) = (A, \triangleleft_A, \times_A, \cdot)$  and  $\text{Zar}(B) = (B, \triangleleft_B, \times_B, \cdot)$  be the basic Zariski topologies on  $A$  and  $B$  respectively.

PROPOSITION 4.4. — *A relation  $r : \text{Zar}(B) \rightarrow \text{Zar}(A)$  is continuous if and only if it respects the axioms, i.e.*

$$r^{-1}0_A \triangleleft_B \emptyset \quad r^{-1}(a + a') \triangleleft_B r^{-1}\{a, a'\} \quad r^{-1}(a \cdot a') \triangleleft_B r^{-1}a$$

for all  $a, a' \in A$ . In particular, if  $f : A \rightarrow B$  is a ring homomorphism, then the inverse relation  $\hat{f}^{-1} : B \rightarrow A$  defined by  $b\hat{f}^{-1}a \equiv afb$  is a continuous and convergent morphism of basic Zariski topologies.

*Proof.* — ( $\rightarrow$ ) This is an instance of continuity.

( $\leftarrow$ ) It is sufficient to show the first condition in (4.1). We will show this by induction on the deductions of  $a \triangleleft_A U$ . If the last rule is *Refl*, and therefore  $a \in U$ , then one has

$$\frac{a \in U}{r^{-1}a \subseteq r^{-1}U} \text{Refl}.$$

If  $a \triangleleft_A U$  was deduced from  $a = 0_A$ , then by  $r^{-1}a \triangleleft_B \emptyset$  we have  $r^{-1}a \triangleleft_B r^{-1}U$ . If the last rule is  $\Sigma$ , that is

$$\frac{a \triangleleft_A U \quad a' \triangleleft_A U}{a + a' \triangleleft_A U},$$

then one has  $r^{-1}a \triangleleft_B r^{-1}U$  and  $r^{-1}a' \triangleleft_B r^{-1}U$  by induction hypothesis; whence  $r^{-1}\{a, a'\} \triangleleft_B r^{-1}U$ . By transitivity and the hypotheses, we have  $r^{-1}(a + a') \triangleleft_B r^{-1}U$ . One deals with the product in the same way.

Suppose now that  $f : A \rightarrow B$  is a ring homomorphism; since  $\hat{f}^{-1} = f$  as operators on subsets, the continuity conditions read as

$$f(0_A) \triangleleft_B \emptyset \quad f(a + a') \triangleleft_B \{f(a), f(a')\} \quad f(a \cdot a') \triangleleft_B f(a),$$

which is to say that

$$0_B \triangleleft_B \emptyset \quad f(a) + f(a') \triangleleft_B \{f(a), f(a')\} \quad f(a) \cdot f(a') \triangleleft_B f(a).$$

These are clearly satisfied. Finally, the two convergence properties can be rewritten as

$$f(a) \cdot f(a') =_{\triangleleft_B} f(a \cdot a'), \quad B \triangleleft_B f(A).$$

The first one follows from the fact that  $f$  respects the product, the second one follows from  $f(1_A) = 1_B$ .  $\square$

Since  $\widehat{g \circ f} = \hat{f} \circ \hat{g}$  and  $id_{\hat{A}} = id_A$ , the correspondence that assigns to each ring homomorphism a continuous and convergent relation defines a contravariant functor

$$\text{Zar} : \text{CRings} \rightarrow \text{BTopO}$$

from the category of commutative rings to the one of basic topologies with operation.

As a further consequence, the direct existential image through  $\hat{f}$  of a formal point  $\alpha \in \mathfrak{Pt}(B)$  is again a formal point  $\hat{f}\alpha \in \mathfrak{Pt}(A)$ , and the map

$$\mathfrak{Pt}(\hat{f}) : \mathfrak{Pt}(B) \rightarrow \mathfrak{Pt}(A)$$

is continuous with respect to the induced topologies. In the light of the classical link between formal points and prime ideals, the description above matches perfectly with the usual treatment.

Finally, the relation  $\hat{f}$  induces a morphism

$$\text{Sat}(\hat{f}) : \text{Sat}(\triangleleft_A) \rightarrow \text{Sat}(\triangleleft_B)$$

of commutative unital quantales, defined by  $U \rightarrow I(f(U))$ . To  $\text{Sat}(\hat{f})$ , corresponds to the morphism  $\mathfrak{Pt}(\hat{f})^-$  between the frames of opens associated with the spectra.

More generally, we have a functor

$$\text{Sat} : \text{CRings} \rightarrow \text{CQuantU}$$

from the category of commutative rings to that of commutative unital quantales.

## 5. CLOSED SUBSPACES AND LOCALISATIONS

In this section, we will show that the class of basic Zariski topologies is closed under the construction of closed subspaces and the localisation in a submonoid. Let  $A$  be a commutative ring with unit  $A$  and let

$$\text{Zar}(A) = (A, \triangleleft, \times, \cdot)$$

be the corresponding basic Zariski topology.

**5.1. Closed subspaces.** The *closed subspace* of  $\text{Zar}(A)$  defined by a subset  $U$  of  $A$  is the basic topology  $(A, \triangleleft^U, \times^U, \cdot)$  generated by the axioms of the basic Zariski topology (see (2.1)) together with

$$\frac{u \in U}{u \triangleleft^U V} C_U^{\triangleleft} \quad \frac{u \times^U V \quad u \in U}{\perp} C_U^{\times}$$

where  $V \subseteq A$ . Since in a deduction tree for such a cover the rule  $C_U^{\triangleleft}$  can only occur at a leaf, we have

$$a \triangleleft^U V \leftrightarrow a \triangleleft V \cup U \tag{5.1}$$

for all  $a \in A$  and  $V \subseteq A$ . Hence  $\triangleleft^U$  coincides with the usual definition, given for example in [12]. Since the cover  $\triangleleft^U$  was obtained by adding a rule, we get

$$a \triangleleft V \rightarrow a \triangleleft^U V$$

for all  $a \in A$  and  $U \subseteq A$ . In other words, the identity relation  $id_A : A \rightarrow A$  is a continuous morphism from  $(A, \triangleleft^U, \times^U)$  to  $(A, \triangleleft, \times)$ .

Thanks to (5.1), one can easily verify that the product  $\cdot$  is an operation for  $(A, \triangleleft^U, \times^U)$  satisfying, as in  $\text{Zar}(A)$ , Stability and Weakening.

The points  $\alpha \in \mathfrak{Pt}_U(A)$  of the closed subspace  $(A, \triangleleft^U, \times^U, \cdot)$  are the prime coideals  $\alpha \in \mathfrak{Pt}(A)$  that split the extra axiom  $C_U$ , which is to say

$$\frac{U \not\leq \alpha}{\perp} .$$

In all,  $\mathfrak{P}t_U(A)$  can be identified with the closed subspace  $\text{Rest}(-U) \subseteq \mathfrak{P}t(A)$ .

Let us now denote by  $A/U$  the set  $A$  equipped with the equality predicate

$$x \stackrel{U}{=} y \equiv x - y \triangleleft U$$

for all  $x, y \in A$ . This is nothing but the quotient  $A/I(U)$  and therefore inherits a ring structure from  $A$ . The identity function is a well-defined ring homomorphism  $\pi_U : A \rightarrow A/U$  and therefore, as described in the previous section, we have a continuous and convergent morphism  $\pi_U^- : \text{Zar}(A/U) \rightarrow \text{Zar}(A)$  between the corresponding basic Zariski topologies. This morphism, in the light of the equivalence (5.1), restricts to an isomorphism between  $\text{Zar}(A/U)$  and the closed subspace  $\text{Zar}(A)$  defined by  $U$ .

**PROPOSITION 5.1.** — *Let  $A$  be a commutative ring,  $(A/U, \triangleleft_{/U}, \times_{/U}, \cdot)$  the basic Zariski topology on the quotient ring  $A/U$ , and  $(A, \triangleleft^U, \times^U, \cdot)$  the closed subspace of the basic Zariski topology on  $A$  defined by  $U \subseteq A$ . The relations*

$$\pi_U^- : A/U \rightarrow A \quad \text{and} \quad id_A : A \rightarrow A/U$$

form an isomorphism between the basic topologies under considerations.

In conclusion, the closed subspaces and corresponding morphisms are represented by quotient rings and projection homomorphisms.

**5.2. Localisations.** Let now  $S \subseteq A$  be any subset. The localisation of  $\text{Zar}(A)$  in  $S$  is the basic topology  $(A, \triangleleft_S, \times_S)$  generated by the usual axioms (see (2.1)) and by

$$\frac{s \in S \quad a \cdot s \triangleleft_S U}{a \triangleleft_S U} L_S^\triangleleft \quad \text{and} \quad \frac{s \in S \quad a \times_S U}{a \cdot s \times_S U} L_S^\times$$

where  $a \in A$  and  $U \subseteq A$ . The cover  $\triangleleft_S$  is generated by adding a rule to  $\triangleleft$  and therefore

$$a \triangleleft U \rightarrow a \triangleleft_S U$$

for all  $a \in A$  and  $U \subseteq A$ .

The identity function  $id_A : A \rightarrow A$  is a continuous morphism from  $(A, \triangleleft_S, \times_S)$  to  $(A, \triangleleft, \times)$ . The induction rule  $L_S$  commutes with the other rules. In particular:

(1) Any application of the rule  $L_S$  followed by  $\Pi$  can be transformed as follows:

$$\frac{\lambda \in A \quad \frac{s \in S \quad a \cdot s \triangleleft_S U}{a \triangleleft_S U} L_S \quad \frac{\pi}{\lambda \cdot a \triangleleft_S U} \Pi}{\lambda \cdot a \triangleleft_S U} \Pi \quad \mapsto \quad \frac{s \in S \quad \frac{a \cdot s \triangleleft_S U}{\lambda \cdot a \cdot s \triangleleft_S U} \Pi}{\lambda \cdot a \triangleleft_S U} L_S \quad \Pi$$

(2) We can work similarly with respect to the rule  $\Sigma$ :

$$\frac{s \in S \quad \frac{a \cdot s \triangleleft_S U}{a \triangleleft_S U} L_S \quad \frac{\pi_{a \cdot s}}{b \triangleleft_S U} \Sigma \quad \frac{\pi_b}{b \triangleleft_S U} \Pi}{a + b \triangleleft_S U} \Sigma \quad \mapsto \quad \frac{s \in S \quad \frac{a \cdot s \triangleleft_S U \quad \frac{\pi_{a \cdot s}}{b \cdot s \triangleleft_S U} \Pi}{(a + b) \cdot s \triangleleft_S U} \Sigma}{a + b \triangleleft_S U} L_S \quad \Pi$$

If  $S$  is closed under multiplication  $\cdot$ , then two consecutive applications of  $L_S$  can be rewritten as a single one:

$$\frac{s \in S \quad \frac{t \in S \quad \frac{a \cdot s \cdot t \triangleleft_S U}{a \cdot s \triangleleft_S U} L_S}{a \triangleleft_S U} L_S \quad \frac{\pi_{a \cdot s \cdot t}}{a \cdot s \cdot t \triangleleft_S U} L_S}{a \triangleleft_S U} L_S \quad \mapsto \quad \frac{s \cdot t \in S \quad \frac{a \cdot s \cdot t \triangleleft_S U}{a \triangleleft_S U} L_S}{a \triangleleft_S U} L_S \quad \frac{\pi_{a \cdot s \cdot t}}{a \cdot s \cdot t \triangleleft_S U} L_S$$

Suppose, from now on, that  $S$  is a monoid,<sup>8</sup> that is, closed under multiplication and  $1 \in S$ . As a consequence, the cover  $\triangleleft_S$  acquires, in the light of the previous observations, the following reduced form:

$$a \triangleleft_S U \leftrightarrow (\exists s \in S)(a \cdot s \triangleleft U), \quad (5.2)$$

for all  $a \in A$  and  $U \subseteq A$ .

Starting from (5.2), it is easy to verify that the product  $\cdot$  is a convergence operation for the localized basic topology, and moreover satisfies, as for  $\text{Zar}(A)$ , Stability and Weakening. We finally denote with  $\text{Zar}_S(A)$  the basic topology  $(A, \triangleleft_S, \times_S, \cdot)$  obtained in this way.

The saturated subsets of  $\text{Zar}_S(A)$  coincide with the ideals  $I$  of  $A$  which satisfy

$$\frac{s \in S \quad a \cdot s \in I}{a \in I}$$

for all  $a \in A$ . Analogously, the reduced subsets of  $A$  are the coideals  $P$  such that

$$\frac{s \in S \quad a \in P}{a \cdot s \in P}. \quad (5.3)$$

for all  $a \in A$ ; these subsets are called *S-filtering*.

In particular, a formal point of  $\text{Zar}_S(A)$  is nothing but an inhabited prime coideal  $\alpha$  such that  $S \subseteq \alpha$ . In fact, for any such  $\alpha$  the condition (5.3) is a particular instance of the filtering property:

$$\frac{\frac{s \in S}{s \in \alpha} \quad a \in \alpha}{a \cdot s \in \alpha}.$$

Vice versa, a formal point contains 1 and thence, as a particular instance of (5.3),

$$\frac{s \in S \quad 1 \in \alpha}{s \in \alpha},$$

so that  $S \subseteq \alpha$ . In particular, if  $S$  is generated by a single element  $a \in A$ , one has

$$\alpha \in \mathfrak{Pt}(\text{Zar}_S(A)) \leftrightarrow \alpha \in \text{Ext}(a) \leftrightarrow a \in \alpha.$$

We briefly recall that the localisation of a ring  $A$  in a monoid  $S$  is the ring of fractions  $(A_S, +, \cdot, 0, 1)$ . More precisely, this is the set

$$A_S = \left\{ \frac{x}{s} : x \in A, s \in S \right\}$$

of formal fractions together with the equality

$$\frac{x}{s} =_S \frac{y}{t} \equiv (\exists r \in S)(r \cdot t \cdot x = r \cdot s \cdot y),$$

and the operations and constants

$$\frac{x}{s} + \frac{y}{t} =_S \frac{x \cdot t + y \cdot s}{s \cdot t}, \quad \frac{x}{s} \cdot \frac{y}{t} =_S \frac{x \cdot y}{s \cdot t}, \quad 0 =_S \frac{0}{1}, \quad 1 =_S \frac{1}{1},$$

for all  $x, y \in A$  and  $s, t \in S$ .

An element  $\frac{x}{s}$  is invertible if and only if so is  $\frac{x}{1}$ , i.e., if there exists  $r \in A$  such that  $r \cdot x \in S$ . Moreover, the function  $\phi_S : A \rightarrow A_S$  which maps  $x$  to  $\frac{x}{1}$  is a ring homomorphism. Corresponding to  $\phi_S$  we have, as before, a continuous and convergent morphism  $\phi_S^- : A_S \rightarrow A$  from  $\text{Zar}(A_S)$  to  $\text{Zar}(A)$ . In particular  $\phi_S(A^*) \subseteq A_S^*$ , and each  $\frac{x}{s} \in A_S$  is associated to an element of  $\phi_S(A)$  because  $\frac{x}{s} = \frac{x}{1} \cdot \frac{1}{s}$ .

The following proposition establish the link between the localisation of rings and the localisation of the corresponding basic Zariski topologies:

<sup>8</sup>This is not restrictive, since otherwise we can pick the monoid generated by  $S$ .

PROPOSITION 5.2. — Let  $(A_S, \triangleleft^S, \times^S, \cdot)$  be the basic Zariski topology on the localisation  $A_S$  of the ring  $A$  in  $S$ , and  $(A, \triangleleft_S, \times_S, \cdot)$  the localisation in  $S$  of the basic Zariski topology on  $A$ . The pair of relations

$$\phi_S^- : A_S \rightarrow A \text{ and } \psi_S : A \rightarrow A_S,$$

where  $a\psi_S \frac{x}{s} \equiv a \triangleleft_S x$ , constitute an isomorphism of basic topologies with operation.

*Proof.* — We first verify that  $\psi_S$  is well defined on  $A_S$ , that is,

$$a \triangleleft_S x \ \& \ \left( \frac{x}{s} =_S \frac{y}{t} \right) \rightarrow a \triangleleft_S y$$

for all  $a, x, y \in A$  and  $s, t \in S$ . By using (5.2), we have

$$a \cdot r \triangleleft x \ \& \ (x \cdot t \cdot r' = y \cdot s \cdot r') \rightarrow (\exists r'' \in S)(a \cdot r'' \triangleleft y)$$

for some  $r, r' \in S$ . From  $a \cdot r \triangleleft x$  and  $t \cdot r' \triangleleft t \cdot r'$ , from stability follows that  $a \cdot r \cdot t \cdot r' \triangleleft x \cdot t \cdot r'$ , that is,  $a \cdot r \cdot t \cdot r' \triangleleft y \cdot s \cdot r'$ . Since  $y \cdot s \cdot r' \triangleleft y$ , it is enough to take  $r'' = r \cdot t \cdot r'$ .

Secondly, we check that  $\phi_S^-$  and  $\psi_S$  are both continuous and convergent relations. For  $\phi_S^-$ , this follows from Section 4, since  $\phi_S^-$  is the inverse relation of an homomorphism. To verify that  $\psi_S$  is continuous, we check that it respects the generating axioms:

(0): It respects the 0-axiom, namely  $a\psi_S \frac{0}{1} \rightarrow a \triangleleft_S 0$ ; in fact  $a\psi_S \frac{0}{1} \equiv a \triangleleft_S 0$ .

( $\Sigma$ ): It respects the sum axiom, that is,

$$a\psi_S \frac{x \cdot t + x' \cdot s}{s \cdot t} \rightarrow a \triangleleft_S \psi_S^- \left( \frac{x}{s} \right) \cup \psi_S^- \left( \frac{x'}{t} \right).$$

Since in general  $x \in \psi_S^- \left( \frac{x}{s} \right)$  holds and  $x \cdot t + x' \cdot s \triangleleft_S \{x, x'\}$ , from  $a \triangleleft_S x \cdot t + x' \cdot s$

we get  $a \triangleleft_S \{x, x'\}$  and finally  $a \triangleleft_S \psi_S^- \left( \frac{x}{s} \right) \cup \psi_S^- \left( \frac{x'}{t} \right)$ .

(II): It respects the product axiom, viz.

$$a\psi_S \frac{x \cdot \lambda}{s \cdot t} \rightarrow a \triangleleft_S \psi_S^- \left( \frac{x}{s} \right).$$

By hypothesis  $a \triangleleft_S x \cdot \lambda$ , and since  $x \cdot \lambda \triangleleft_S x$  and  $x \in \psi_S^- \left( \frac{x}{s} \right)$ , the conclusion follows.

The relation  $\psi_S$  is convergent: in fact, condition C2 is trivially satisfied, since  $1 \in \psi_S^- \left( \frac{1}{1} \right)$ , and therefore  $1 \triangleleft_S \psi_S^- (A_S)$ . The first convergence condition C1 can be stated explicitly as

$$a\psi_S \frac{x}{s} \ \& \ a'\psi_S \frac{x'}{t} \rightarrow a \cdot a' \triangleleft_S \psi_S^- \left( \frac{x \cdot x'}{s \cdot t} \right).$$

By hypothesis  $a \triangleleft_S x$  and  $a' \triangleleft_S x'$ , hence, by *stability*,  $a \cdot a' \triangleleft_S x \cdot x'$  so that  $x \cdot x' \in \psi_S^- \left( \frac{x \cdot x'}{s \cdot t} \right)$ .

We leave to the reader the proof that the pair  $(\phi_S^-, \psi_S)$  of convergent continuous relations is an isomorphism. This amounts to show  $a =_{\triangleleft_S} \psi_S^- \phi_S a$  and  $\frac{x}{s} =_{\triangleleft_S} \phi_S \psi_S \frac{x}{s}$  for all  $a, x \in A$  and  $s \in S$ .  $\square$

In all, the localisation in a monoid and the corresponding inclusion morphism are represented by localized rings and localisation homomorphisms.

## 6. THE TOPOLOGY INDUCED BY POINTS

Let  $A$  be a set and  $(A, \triangleleft, \times, \circ)$  a basic topology with operation. As already stressed, the relation  $\Vdash: \mathfrak{Pt}(A) \rightarrow A$  forms impredicatively a basic pair; we thus can define a new cover and a new positivity relation on  $A$ :

$$a \triangleleft_{\mathfrak{Pt}} U \equiv \forall \alpha (a \in \alpha \rightarrow \alpha \checkmark U),$$

$$a \times_{\mathfrak{Pt}} U \equiv \exists \alpha (a \in \alpha \ \& \ \alpha \subseteq U)$$

where  $a \in A$  and  $u \in U$ . In general, the implications

$$a \triangleleft U \rightarrow a \triangleleft_{\mathfrak{P}t} U \quad \text{and} \quad a \times_{\mathfrak{P}t} U \rightarrow a \times U$$

hold. The reverse implications are instead non-trivial properties: if  $\triangleleft$  coincides with  $\triangleleft_{\mathfrak{P}t}$ , then we say that the topology  $(A, \triangleleft, \times, \circ)$  is *spatial*; if  $\times$  coincides with  $\times_{\mathfrak{P}t}$  we say that the topology is *reducible*.

Admitting classical logic (CL), a basic topology is spatial if and only if it is reducible. In this case, saturated and reduced subsets are complements of each other, and therefore  $a \triangleleft U \leftrightarrow \neg(a \times -U)$  holds [26, 27]. In particular, in the case of the basic Zariski topology, we have  $a \times -U$  if and only if there is a coideal  $F$  such that  $a \in F$  and  $F \subseteq -U$ . Therefore, since ideals are, with classical logic, precisely the complements of the coideals, we have  $\neg(a \times -U)$  if and only if  $a \in I$  holds for every ideal  $I$  such that  $U \subseteq I$ . That is to say,  $a \triangleleft U$ . One therefore gets the following equivalences:

$$\begin{aligned} a \times -U \rightarrow \exists \alpha (a \in \alpha \ \& \ \alpha \subseteq -U) & \text{ iff } \neg \exists \alpha (a \in \alpha \ \& \ \alpha \subseteq -U) \rightarrow \neg a \times -U \text{ iff} \\ & \text{ iff } \forall \alpha (a \in \alpha \rightarrow \neg \alpha \subseteq -U) \rightarrow a \triangleleft U \text{ iff} \\ & \text{ iff } \forall \alpha (a \in \alpha \rightarrow \alpha \not\subseteq U) \rightarrow a \triangleleft U. \end{aligned}$$

We recall that *Restricted Excluded Middle*, or shortly REM, asserts that  $\phi \vee \neg \phi$  for every restricted<sup>9</sup> formula  $\phi$ . In other terms, REM holds if and only if every subset of a set  $S$  is complemented. In the presence of PSA and REM, the equivalence shown above between spatiality and reducibility hold.

*Remark 6.1.* — If the ring  $A$  is discrete, then the subset  $A \setminus \{0\}$  is a coideal, and is inhabited by the unit if  $1 \neq 0$ . In fact

$$a + b \neq 0 \rightarrow a \neq 0 \vee b \neq 0$$

for all  $a, b \in A$ . Since the ring is discrete,  $a \neq 0 \vee a = 0$ ; in the first case, we are done, in the second case,  $a + b = 0 + b = b$  and by hypothesis  $b \neq 0$ . For the product one argues in the same way.

**PROPOSITION 6.2.** — *If the commutative ring  $A$  has a non-zero nilpotent element and is discrete, then the basic Zariski topology  $\text{Zar}(A) = (A, \triangleleft, \times, \cdot)$  is neither spatial nor reducible.*

*Proof.* — Let  $a \in A$  be such that  $a \neq 0$  but  $a^n = 0$ . Suppose  $\text{Zar}(A)$  to be spatial. Let  $\alpha \in \mathfrak{P}t(A)$  be such that  $a \in \alpha$ ; since  $\alpha$  is filtering,  $a^n \in \alpha$  follows. Hence  $\forall \alpha (a \in \alpha \rightarrow \alpha \not\subseteq \{0\})$ , which is to say that  $a \triangleleft_{\mathfrak{P}t} 0$ . By spatiality  $a \triangleleft 0$ , a contradiction.

Analogously, suppose now  $\text{Zar}(A)$  reducible and consider the subset  $F = A \setminus \{0\}$ , for which  $a \times F$ . Since  $A$  is discrete, by Remark 6.1  $F$  is a coideal. By reducibility, we get  $a \times_{\mathfrak{P}t} F$ , that is, there exists a formal point  $\alpha$  such that  $a \in \alpha$  and  $\alpha \subseteq F$ . So  $0 = a^n \in \alpha \subseteq F$ , again a contradiction.  $\square$

Note that the first part of this proof does not require the ring  $A$  to be discrete.

**DEFINITION 6.3.** — *A formal topology with operation is a basic topology with operation such that the cover  $\triangleleft$  satisfies Stability, Weakening and  $a \triangleleft a^2$  (Contraction) for all  $a \in A$ .*

In these structures, we have  $U \wedge V =_{\triangleleft} U \circ V$ , i.e. the binary meet in  $\text{Sat}(\triangleleft)$  is represented by the operation  $\circ$ . Since  $\circ$  commutes with set-indexed joins, the lattice  $\text{Sat}(\triangleleft)$  has a locale structure.

**PROPOSITION 6.4.** — *Let  $\text{Zar}(A)$  be the basic Zariski topology with operation on  $A$ . Impredicatively,  $\text{Zar}_{\mathfrak{P}t}(A) = (A, \triangleleft_{\mathfrak{P}t}, \times_{\mathfrak{P}t}, \cdot)$  is a formal topology with operation.*

<sup>9</sup>A formula  $\varphi$  is said to be *restricted* if every quantifier that appears in  $\varphi$  is restricted, i.e. of the form  $\forall x \in Y$  or  $\exists x \in Y$  for a set  $Y$ .



*Proof.* — We have to check that  $\cdot$  satisfies Stability for  $\text{Zar}_{\mathfrak{pt}}(A)$ . Suppose that  $a \triangleleft_{\mathfrak{pt}} U$  and  $b \triangleleft_{\mathfrak{pt}} V$ , that is,  $\forall \alpha (a \in \alpha \rightarrow U \not\lesssim \alpha)$  and  $\forall \alpha (b \in \alpha \rightarrow V \not\lesssim \alpha)$ . Since  $a \cdot b \triangleleft a$  and  $a \circ b \triangleleft b$  (Weakening), once we have fixed a generic formal point  $\alpha$ , from  $a \cdot b \not\lesssim \alpha$  we get  $a \in \alpha$  and  $b \in \alpha$  by Compatibility; by hypothesis one gets  $U \not\lesssim \alpha$  and  $V \not\lesssim \alpha$  and finally  $U \cdot V \not\lesssim \alpha$  since  $\alpha$  is convergent. Inasmuch  $\alpha$  is generic, we get  $a \cdot b \triangleleft_{\mathfrak{pt}} U \cdot V$ . Analogously, one proves Weakening and Contraction.  $\square$

This proof uses just Stability and Weakening and works therefore for every basic topology with these properties. Among the consequences, we rediscover that the basic pair induces on the side of points a topology in the usual sense, for which the lattice of opens is a locale.

**The formal Zariski topology.** The counterexample in Proposition 6.2 relies on the fact that the basic Zariski topology on the ring  $\mathbb{Z}/4\mathbb{Z}$  does not satisfy the axiom of contraction. We can overcome this issue by generating inductively the least basic topology  $\triangleleft_c \supseteq \triangleleft$ , over the basic Zariski topology, satisfying contraction too: it is enough to add to the rules 0,  $\Sigma$  and  $\Pi$  the following generation rule:

$$\frac{a^2 \triangleleft_c U}{a \triangleleft_c U} \text{ Rad}$$

In this way, we can generate  $\text{Zar}_c(A) = (A, \triangleleft_c, \times_c, \cdot)$ , the *formal Zariski topology*. We will show, in a few lines, that  $\text{Zar}_c(A)$  actually is a formal topology with operation.

In the presence of the rules  $S$  and  $P$ , the rule Rad is equivalent to the rule

$$\frac{a^n \triangleleft_c U \quad n \in \mathbb{N}}{a \triangleleft_c U} N$$

One direction is trivial, for  $n = 2$ . Vice versa, if  $n$  is an even number, then the property Rad allows to divide  $n$  by 2; if  $n$  is an odd number, by the rule  $P$  we can get from  $a^n \triangleleft_c U$  to  $a^{n+1} \triangleleft_c U$  where  $n+1$  is even. In this way, after a finite number of steps, we obtain  $a \triangleleft_c U$  starting from  $a^n \triangleleft_c U$ .

One verifies directly that the rule  $N$  commutes with  $\Sigma$  and  $\Pi$ , and that two applications of the same rule can be collected into one. In other words, one obtains

$$a \triangleleft_c U \leftrightarrow \exists n (a^n \triangleleft_c U)$$

for all  $a \in A$  and  $U \subseteq A$ . Recalling that  $a \triangleleft U \leftrightarrow a \in I(U)$ , one thus has

$$a \triangleleft_c U \leftrightarrow a \in R(U) \tag{6.1}$$

where

$$R(U) = \{a \in A : (\exists m \in \mathbb{N})(\exists u_1, \dots, u_n \in U)(\exists \lambda_1, \dots, \lambda_n \in A)(a^m = \sum_{i=1}^n \lambda_i \cdot u_i)\}$$

is the radical ideal generated by  $U$ .

Besides the cover  $\triangleleft_c$ , we generate by conduction a compatible positivity  $\times_c$  starting from the axioms of  $\times$  together with

$$\frac{a \times_c U}{a^2 \times_c U}$$

for all  $a \in A$  and  $U \subseteq A$ . We indicate with  $\text{Zar}_c(A) = (A, \triangleleft_c, \times_c, \cdot)$  the basic topology generated in this way, and endowed with the product operation.

For a detailed treatment, we refer to [12, 30]. Since the cover was obtained adding a rule, we have

$$a \triangleleft U \rightarrow a \triangleleft_c U$$

for all  $a \in A$  and  $U \subseteq A$ . The reverse implication does not hold in general: in  $\mathbb{Z}/4\mathbb{Z}$ , one has  $2 \triangleleft_c 0$  but not  $2 \triangleleft 0$ .

PROPOSITION 6.5. — *The basic topology  $\text{Zar}_c(A) = (A, \triangleleft_c, \times_c, \cdot)$  is a formal topology with convergence operation, for every ring  $A$ . In particular,  $\triangleleft_c$  satisfies*

$$\frac{a \triangleleft_c U \quad a \triangleleft_c V}{a \triangleleft_c U \cdot V} \text{ Convergence}$$

for all  $a \in A$  and  $U, V \subseteq A$ .

*Proof.* — The weakening property follows directly from  $\triangleleft \subseteq \triangleleft_c$ . For Stability, one acts as in the proof of Proposition 3.2, that is, one builds by induction, starting from two deduction trees for  $a \triangleleft_c U$  and  $b \triangleleft_c V$ , a deduction tree for  $a \cdot b \triangleleft_c U \cdot V$ . For the sake of completeness, we will show that stability lifts up with respect to an application of rule  $N$  in a deduction tree:

$$\frac{\frac{\frac{\pi_a}{\vdots} a^n \triangleleft_c U}{a \triangleleft_c U} N \quad \frac{\pi_b}{\vdots} b \triangleleft_c V}{a \cdot b \triangleleft_c U \cdot V} \text{ Stab.} \quad \mapsto \quad \frac{\frac{\frac{\pi_a}{\vdots} a^n \triangleleft_c U}{(a \cdot b)^n \triangleleft_c U \cdot V} N \quad \frac{\frac{\pi_b}{\vdots} b \triangleleft_c V \quad b^{n-1} \in A}{b^n \triangleleft_c V} \text{ Stab.}}{a \cdot b \triangleleft_c U \cdot V} \Pi$$

The property Contraction coincides precisely with the rule Rad. Finally, we have

$$\frac{\frac{a^2 \triangleleft_c a^2}{a \triangleleft_c a^2} \text{ Rad} \quad \frac{a \triangleleft_c^c U \quad a \triangleleft_c^c V}{a^2 \triangleleft_c^c U \cdot V} \text{ Stab.}}{a \triangleleft_c U \cdot V} \text{ Transitivity},$$

so that convergence follows from Stability.  $\square$

As already stressed, the lattice  $\text{Zar}_c(A)$  of saturated subsets can be identified with the lattice of radical ideals of  $A$ , and has the structure of a locale. The equality  $U \wedge V =_{\triangleleft} U \circ V$  can be restated explicitly as  $R(I) \cap R(J) = R(I \cdot J)$  for all ideals  $I, J \subseteq A$ .

The reduced subsets split the axioms, and therefore they can be identified with the coideals  $P \subseteq A$  satisfying the further condition

$$\frac{a \in P}{a^n \in P}$$

for all  $a \in A$ ,  $n \in \mathbb{N}$ . We call these subsets *radical coideals* [30]. The positivity relation can be characterized as follows:

$$\frac{b^m = \lambda_1 \cdot a_1 + \cdots + \lambda_n \cdot a_n \quad b \times_c F}{a_1 \times_c F \vee a_2 \times_c F \vee \cdots \vee a_n \times_c F} .$$

for all  $n, m \in \mathbb{N}$  and  $\lambda_1, \dots, \lambda_n, a_1, \dots, a_n \in A$ . Hence  $a \times_c F \rightarrow \forall n (a^n \times_c F)$  for all  $a \in A$  and  $F \subseteq A$ .

Every formal point of the basic topology  $\text{Zar}(A)$  is filtering and it is therefore a radical coideal and a formal point for  $\text{Zar}_c(A)$ . Hence the points of the formal Zariski topology coincide with the points of the basic Zariski topology.

In the realm of formal topologies, the points can be identified with the continuous and convergent morphisms to the initial object of  $\text{FTop}$ , the category of formal topologies and continuous and convergent morphisms [27].

The results in sections 4 and 5 can easily be generalized to the formal Zariski topology.

PROPOSITION 6.6. — *Let  $A$  and  $B$  be commutative rings and  $\phi : A \rightarrow B$  a ring homomorphism. The relation  $\phi^- : B \rightarrow A$  is a continuous and convergent relation between the corresponding formal Zariski topologies  $(B, \triangleleft_c, \times_c, \cdot)$  and  $(A, \triangleleft_c, \times_c, \cdot)$ . Moreover, any such correspondence is a functor from the category  $\text{CRings}$  of commutative rings to the category  $\text{FTop}$  of formal topologies.*

Also in the context of the Zariski formal topologies, we can talk about closed subspaces and localisations, and the generation strategy remains the same. Regarding localisation, we have to prove that in a generation tree the rule  $L_S$  (with the notation of section 5) commutes with the rule Rad:

$$\frac{\frac{s \in S \quad \frac{a^2 \cdot s \triangleleft_{c,S} U}{a^2 \triangleleft_{c,S} U} L_S}{a \triangleleft_{c,S} U} \text{Rad}}{\frac{s \in S \quad \frac{a^2 \cdot s \triangleleft_{c,S} U}{a \triangleleft_{c,S} U} L_S}{a \triangleleft_{c,S} U} \text{Rad}} \mapsto \frac{\frac{s \in S \quad \frac{a^2 \cdot s^2 \triangleleft_{c,S} U}{a \cdot s \triangleleft_{c,S} U} \text{Rad}}{a \triangleleft_{c,S} U} L_S}{\frac{s \in S \quad \frac{a^2 \cdot s \triangleleft_{c,S} U}{a \triangleleft_{c,S} U} L_S}{a \triangleleft_{c,S} U} \text{Rad}} \text{Rad} \quad \Pi$$

The characterizations result:

$$a \triangleleft_c^U V \leftrightarrow a \triangleleft_c U \cup V \quad \text{and} \quad a \triangleleft_{c,S} V \leftrightarrow (\exists s \in S)(a \cdot s \triangleleft_c U)$$

for all  $a \in A$ ,  $U, V \subseteq A$ ,  $S \subseteq A$  monoid (or filter).

As before, the closed subspaces and the localisations are represented in the category of rings by quotient rings and localized rings:

PROPOSITION 6.7. — *Let  $A$  be a commutative ring,  $(A/U, \triangleleft_{c,/(U)}, \times_{c,/(U)}, \cdot)$  the formal Zariski topology on the quotient ring  $A/U$ , and  $(A, \triangleleft_c^U, \times_c^U, \cdot)$  the closed subspace of the formal Zariski topology on  $A$  defined by  $U \subseteq A$ . The relations*

$$\pi_U^- : A/U \rightarrow A \quad \text{and} \quad id_A : A \rightarrow A/U$$

form an isomorphism between those formal topologies.

PROPOSITION 6.8. — *Let  $A$  be a commutative ring,  $(A_S, \triangleleft_c^S, \times_c^S, \cdot)$  the formal Zariski topology on the localized ring  $A_S$ , and  $(A, \triangleleft_{c,S}, \times_{c,S}, \cdot)$  the localisation in  $S$  of the formal Zariski topology on  $A$ . The relations*

$$\phi_S^- : A_S \rightarrow A \quad \text{and} \quad \psi_S : A \rightarrow A_S$$

where  $a\psi_S \frac{x}{s} \equiv a \triangleleft_{c,S} x$  form an isomorphism between the formal topologies under consideration.

In particular, we recall that properties such as *spatiality* and *reducibility* are preserved by continuous and convergent morphisms, as all the properties of topological nature.

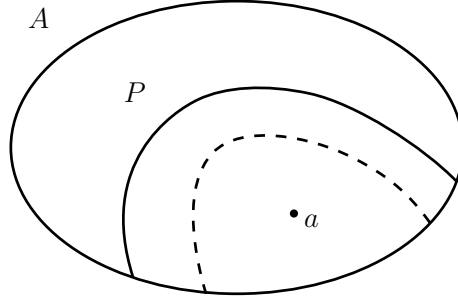
## 7. SPATIALITY AND REDUCIBILITY OF THE FORMAL ZARISKI TOPOLOGY

As shown in the previous section, the formal points of the basic Zariski topology and of the formal Zariski topology coincide. In particular, the formal topology induced impredicatively by the points of  $A$ , namely by the basic pair  $\Vdash : \mathfrak{Pt}(A) \rightarrow A$ , coincides with  $\text{Zar}_{\mathfrak{Pt}}(A)$ . We therefore have the chain of implications

$$a \triangleleft U \rightarrow a \triangleleft_c U \rightarrow a \triangleleft_{\mathfrak{Pt}} U$$

for all  $a \in A$  and  $U \subseteq A$ . In other words,  $\text{Zar}_c(A)$  is a better approximation of  $\text{Zar}_{\mathfrak{Pt}}(A)$  with respect to  $\text{Zar}(A)$ , and we are prompted to address again the issue of spatiality and reducibility for the formal topology  $\text{Zar}_c(A)$ .

The reducibility of the formal Zariski topology asserts that for every radical coideal  $P \subseteq A$  and every  $a \in P$ , there exists a prime filter  $\alpha$  containing  $a$  and contained in  $P$ .



Assuming AC (and therefore REM [11, 1, 13]) together with PSA, every finitary formal topology is reducible and hence spatial:

**PROPOSITION 7.1.** — *Let  $(S, \triangleleft, \times, \circ)$  be a finitary formal topology. Assuming Zorn's Lemma and classical logic,  $S$  is reducible (and therefore spatial).*

*Proof.* — Suppose  $a \times F$ . We have to show that there exists a formal point  $\alpha$  such that  $a \in \alpha$  and  $\alpha \subseteq F$ . To this end, we define

$$\mathcal{S}_{a,F} = \{U \in \mathcal{P}(S) : a \times U \text{ \& } U \subseteq F\}.$$

This collection is inhabited by  $F$ , and it is naturally ordered by inverse inclusion  $\supseteq$ . We show by coinduction that, if  $V_1 \supseteq V_2 \supseteq \dots$  is a chain in  $\mathcal{S}_{a,F}$ , then  $V = \bigcap_{n \in \mathbb{N}} V_n \in \mathcal{S}_{a,F}$ . The subset  $V$  is clearly a lower bound for the chain. We define  $b \in Q \equiv \forall n (b \times V_n)$ ; then we have:

- (1)  $a \in Q$  since  $V_n \in \mathcal{S}_{a,F}$  for all  $n \in \mathbb{N}$ .
- (2) If  $b \in Q$  then  $b \times V_n$ , and therefore  $b \in V_n$  for all  $n$ . In other terms,  $Q \subseteq V$ .
- (3) If  $b = b_1 + b_2$  and  $b \in Q$ , namely  $\forall n (b \times V_n)$ , then  $\forall n (b_1 \times V_n \vee b_2 \times V_n)$ .

With classical logic, since the sequence  $\{V_i\}_{i \in \mathbb{N}}$  is descending, we have  $\forall n (b_1 \times V_n) \vee \forall n (b_2 \times V_n)$ , i.e.  $b_1 \in Q \vee b_2 \in Q$ . This shows that  $Q$  splits the axiom  $\Sigma_c$ . Acting similarly for the other axioms, one shows that  $Q$  is a reduced subset.

These remarks imply that  $a \times V$  and therefore  $V \in \mathcal{S}_{a,F}$ . By means of Zorn's Lemma, we find a minimal element  $\alpha$  in  $\mathcal{S}_{a,F}$ . This minimal element is a coideal, inhabited by  $a$ , and it is filtering: suppose  $x \in \alpha$  and  $y \in \alpha$ ; then neither  $\alpha \setminus \{x\} \in \mathcal{S}_{a,F}$  nor  $\alpha \setminus \{y\} \in \mathcal{S}_{a,F}$  and hence  $\neg(a \times \alpha \setminus \{x\})$  and  $\neg(a \times \alpha \setminus \{y\})$ . Since the topology is generated and classical logic is at hand, this is equivalent to  $a \triangleleft \mathfrak{p} \cup \{x\}$  and  $a \triangleleft \mathfrak{p} \cup \{y\}$ , where  $\mathfrak{p}$  is the complement of  $\alpha$ . The axioms of formal topology allow to conclude  $a \triangleleft \mathfrak{p} \cup x \circ y$  which means  $\neg(a \times \alpha \setminus x \circ y)$ ; this is equivalent to  $x \circ y \checkmark \alpha$ .  $\square$

In particular, Proposition 7.1 holds for the formal Zariski topology: in this case the formal points coincide with the complements of the prime ideals and the statement of spatiality/reducibility has the familiar form

$$a \in \bigcap_{U \subseteq \mathfrak{p}} \mathfrak{p} \rightarrow a \in R(U) \tag{7.1}$$

where  $a \in A$  and  $U \subseteq A$ . One has in fact

$$\begin{aligned} \forall \alpha (a \in \alpha \rightarrow U \checkmark \alpha) \rightarrow a \triangleleft_c U & \text{ iff } \forall \alpha (\neg(U \checkmark \alpha) \rightarrow \neg(a \in \alpha)) \rightarrow a \triangleleft_c U \text{ iff} \\ & \text{ iff } \forall \alpha (U \subseteq -\alpha \rightarrow a \in -\alpha) \rightarrow a \triangleleft_c U \text{ iff} \\ & \text{ iff } \forall \mathfrak{p} \in \mathfrak{Spec}(A) (U \subseteq \mathfrak{p} \rightarrow a \in \mathfrak{p}) \rightarrow a \in R(U). \end{aligned}$$

The implication (7.1), or its contrapositive, is usually called ‘‘Krull's Lemma’’ and is a crucial statement in commutative algebra. Starting from the collection of points, it allows to deduce a concrete information on the ring side, that is, an equational witness for  $a \in R(U)$ . Nevertheless this existence is a consequence of AC, and a priori has no effective content [32].

From the spatiality of the formal Zariski topology on every discrete ring  $A$ , one deduces REM. The proof makes use of the following general lemma, already present in [22] and [12]. The only difference lies in the reformulation by means of  $\ltimes$ :

LEMMA 7.2. — *Let  $(S, \triangleleft, \ltimes)$  be a finitary and spatial formal topology. Then for all  $a \in S$ :*

$$a \triangleleft \emptyset \vee a \ltimes S.$$

*Proof.* — We consider the subset  $U_a$  defined by

$$x \in U_a \equiv x = a \ \& \ a \ltimes S,$$

and show  $\forall \alpha (a \in \alpha \rightarrow \alpha \checkmark U_a)$ ; by spatiality,  $a \triangleleft U_a$  follows. If  $a \in \alpha$ , then  $a \ltimes \alpha$  because  $\alpha$  is reduced, and therefore  $a \ltimes S$ ; we then have a witness for  $\alpha \checkmark U_a$ . Hence  $a \triangleleft U_a$  and, since the cover is finitary, there exists a finite  $U_0 \subseteq U_a$  such that  $a \triangleleft U_0$ . It is decidable whether  $U_0$  is empty or inhabited. If  $U_0$  is empty, then  $a \triangleleft \emptyset$ ; if  $U_0$  is inhabited, then so is  $U_a$ , and thus  $a \ltimes S$ .  $\square$

In particular, Lemma 7.2 applies to the formal Zariski topology.

Remark 7.3. — Since the class of Zariski formal topologies is stable under forming closed subspaces, from the hypothesis that the class of Zariski formal topologies is spatial follows, in the light of Lemma 7.2, that

$$a \triangleleft_c^U \emptyset \vee a \ltimes_c^U A,$$

that is,

$$a \triangleleft_c U \vee a \ltimes_c \neg U$$

for every ring  $A$ , every  $a \in A$  and every  $U \subseteq A$ . Moreover, by compatibility, we have  $a \ltimes_c \neg U \rightarrow \neg(a \triangleleft_c U)$  and we obtain

$$a \triangleleft_c U \vee \neg(a \triangleleft_c U)$$

or, in other words, every radical coideal is complemented for every ring  $A$ .

PROPOSITION 7.4. — *If spatiality holds for the class of Zariski formal topologies, then every subset  $U$  of a discrete set  $S$  is complemented.*

*Proof.* — Take the ring  $A = \mathbb{Z}[S]$  freely generated by  $S$  and consider the formal Zariski topology  $(A, \triangleleft_c, \ltimes_c)$ . We regard  $S$  and  $U$  as subsets of  $A$ . We are going to prove

$$a \triangleleft_c U \leftrightarrow a \in U$$

for any  $a \in S$ . Notice that  $A$  is the free  $\mathbb{Z}$ -module with a basis given by the monic monomials of  $A$ . Therefore every element of  $A$  can be written in a unique way as  $\mathbb{Z}$ -linear combination of such monomials. If  $a \triangleleft_c U$  then there is  $k \in \mathbb{N}$  such that

$$a^k = \sum_{i=0}^n b_i \cdot u_i \quad \text{where} \quad b_i \in A, u_i \in U;$$

more explicitly, we have  $b_i = \sum_{m \in M_i} \lambda_{i,m} \cdot m$  with  $M_i$  a finite set of monic monomials in  $A$  and  $\lambda_{i,m} \in \mathbb{Z}$ . One therefore has

$$a^k = \sum_{i=0}^n \sum_{m \in M_i} \lambda_{i,m} \cdot (m \cdot u_i).$$

Since  $a^k$  and every  $m \cdot u_i$  are monomials and therefore basis elements, this equation can be realized if and only if  $a^k = m \cdot u_i$  for some  $i = 1, \dots, n$  and  $m \in M_i$ ; this amounts to  $m = a^{k-1}$  and  $u_i = a$  and  $a \in U$ . By remark 7.3 we get

$$a \in U \vee \neg(a \in U)$$

for all  $a \in S$ .  $\square$

Not even reducibility can be accepted constructively for every ring  $A$ . In fact, assuming it, we can prove a version of Russell's *Multiplicative Axiom* in the following form  $\text{ACF}^*$ :

$\text{ACF}^*$ . — Let  $S$  be a discrete set and  $\{U_i\}_{i \in I}$  a partition of  $S$  in finite and inhabited subsets, with  $I$  discrete. Then, there exists  $\alpha \subseteq S$  such that

$$\forall i(U_i \not\subseteq \alpha) \quad \text{and} \quad \forall t, t' \forall i(t, t' \in U_i \cap \alpha \rightarrow t = t'). \quad (7.2)$$

We now fix a set  $S$  equipped with a partition  $\{U_i\}_{i \in I}$  in finite and inhabited subsets, with  $I$  and  $S$  discrete. We will define a formal topology on  $S$  such that the formal points coincide precisely with the subsets  $\alpha$  that satisfy (7.2). Consider on  $S$  the following generated basic topology:<sup>10</sup>

$$\begin{aligned} a \triangleleft_P U &\equiv a \in U \vee (\exists i \in I)(U_i \subseteq U) \\ a \times_P F &\equiv a \in F \ \& \ (\forall i \in I)(U_i \not\subseteq F). \end{aligned}$$

It follows that a subset  $U \subseteq S$  is saturated if and only if

$$\exists i(U_i \subseteq U) \rightarrow S = U$$

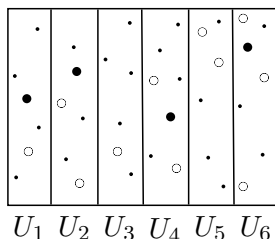
and is reduced if and only if

$$S \not\subseteq F \rightarrow \forall i(U_i \not\subseteq F)$$

for all  $i \in I$ . We define on the topology the following operation

$$a \in t \circ t' \equiv \exists i, j \in I(i \neq j \ \& \ t \in U_i \ \& \ t' \in U_j) \vee t = t' = a$$

where  $t, t' \in S$ . With this operation, the filtering subsets (viz. the  $U \subseteq S$  such that  $t, t' \in U \rightarrow t \circ t' \in U$ ), are the ones which have at most one element in each subset  $U_i$  of the partition.



In the picture above, we give as example  $S = U_1 \dot{\cup} \dots \dot{\cup} U_6$ : the fat black dots form a filtering subset, the white ones a reduced subset.

The formal points for this topology are, by direct observation, exactly the inhabited subsets  $\alpha \subseteq A$  which satisfy the conditions (7.2).

*Remark 7.5.* — The existence of a formal point for this topology is equivalent to  $\text{ACF}^*$ .

*Remark 7.6.* — Let  $A$  be a discrete ring. By Remark 6.1 the subset  $A \setminus \{0\}$  is a coideal, and is inhabited by the unit if  $1 \neq 0$ . If moreover the ring  $A$  is reduced, that is,  $a^n = 0 \rightarrow a = 0$ , then  $A \setminus \{0\}$  is a radical coideal, that is to say  $a \neq 0 \rightarrow a^n \neq 0$  for all  $n \in \mathbb{N}$ .

**LEMMA 7.7.** — Let  $S$  be a discrete set and  $\{U_i\}_{i \in I}$  a partition of  $S$  in finite and inhabited subsets, with  $I$  a discrete set; let  $(S, \triangleleft_P, \times_P, \circ)$  be the basic topology with operation assigned to  $S$  as above. Then there exists a non-trivial, reduced and discrete ring  $A$ , and a continuous and convergent morphism from  $\text{Zar}_c(A)$  to  $(S, \triangleleft_P, \times_P, \circ)$ .

<sup>10</sup>Following [21], the topology is generated by setting  $\{U_i\}_{i \in I}$  as axiom-set for all  $a \in S$ .

*Proof.* — Consider the ring  $\mathbb{Z}[S]$  freely generated by the elements of  $S$ . We apply successively the following transformations: first, we quotient  $\mathbb{Z}[S]$  by the ideal  $I(H)$  generated by

$$H = \{t \cdot t' : (\exists i \in I)(t, t' \in U_i) \ \& \ t \neq t'\};$$

secondly, we localize it in the monoid  $M(K)$  generated by

$$K = \{\sigma_i : i \in I\} \quad (\sigma_i = \sum_{u \in U_i} u)$$

for all<sup>11</sup>  $i \in I$ . Let  $A$  be the resulting ring. First of all, one can prove that  $A$  is non-trivial, that is,  $M(K) \not\wr I(H)$  leads to a contradiction. This follows from the structure of the elements of  $M(K)$  and  $I(H)$ .

Secondly, the equality on  $A$  is decidable. To see this, let  $x = \frac{a}{k} \in A$ ; we can suppose that  $a$  is a monomial, namely  $a = s_1^{m_1} \cdots s_n^{m_n} \in \mathbb{Z}[S]$ , and  $k \in M(K)$ . We want to show  $x = 0 \vee x \neq 0$ ; if two variables  $s_i$  belong to the same element of the partition  $U_j$  for some  $j \in I$ , then  $a \in I(H)$  and therefore  $\frac{a}{k} = 0$ . Suppose instead that all the  $s_i$  lie in distinct subsets  $U_j$  of the partition; we show  $k' \cdot a \notin I(H)$  for all  $k' = \sigma_{i_1}^{l_1} \cdots \sigma_{i_h}^{l_h} \in M(K)$ . In details, we have:

$$k' \cdot a = \sigma_{i_1}^{l_1} \cdots \sigma_{i_h}^{l_h} \cdot s_1^{m_1} \cdots s_n^{m_n} = \sum_{(s'_1, \dots, s'_h) \in U_{i_1} \times \dots \times U_{i_h}} s'_1, \dots, s'_h \cdot s_1^{m_1} \cdots s_n^{m_n}.$$

At least one element of this sum does not lie in  $I(H)$ : it is enough to choose  $s'_j = s_l$  if  $s_l \in U_{i_j}$ . This is sufficient to assert  $k' \cdot a \notin I(H)$  and therefore  $x \neq 0$ . Finally, the ring  $A$  is reduced; a proof can be obtained similarly to the previous point, by making explicit  $\frac{a^n}{k^n} = 0$ , where  $n \in \mathbb{N}$ ,  $a \in \mathbb{Z}[S]$  is a monomial and  $k \in M(K)$ .

We therefore have a chain of morphisms

$$\begin{array}{ccccccc} S & \xrightarrow{i} & \mathbb{Z}[S] & \xrightarrow{\pi_H} & \mathbb{Z}[S]/I(H) & \xrightarrow{\phi_K} & A \\ & & & & \searrow r & & \\ & & & & & & \end{array}$$

where  $i$  is the canonical inclusion,  $\pi_H$  is the projection on the quotients and  $\phi_K$  is the localisation homomorphism in  $M(K)$ . Finally, let  $r$  be the composition of the three morphisms, considered as relation in the opposite direction; we will prove that  $r$  is a continuous and convergent morphism if  $S$  is endowed with the topology  $(S, \triangleleft_P, \times_P, \circ)$ . For the sake of convenience, we identify  $S$  with the corresponding subset of  $A$  and the relation  $i$  with the identity.

To prove continuity, we have to check that  $r$  respects the axiom sets  $\{U_i\}_{i \in I}$  for every  $a \in A$ . This is equivalent to showing  $a \triangleleft_c U_i$  for all  $i \in I$  and  $s \in S$ , which is obvious because  $\sigma_i \in I(U_i)$  is invertible in the localized ring. As for the convergence of  $r$ , since  $A \triangleleft_c S$  as a consequence of the proof of continuity, we only have to prove

$$t \cdot t' \triangleleft_c t \circ t'$$

for all  $t, t' \in S$ . If  $t \in U_i$  and  $t' \in U_j$  with  $i \neq j$ , then  $t \circ t' = S$ . If instead  $t, t' \in U_i$  for the same  $i \in I$ , then either  $t \neq t'$  and  $t \cdot t' = \emptyset$ , or  $t = t'$  and  $t \circ t' = \{t\} = \{t'\}$ . In the first case,  $t \cdot t' \in I(H)$  and therefore  $t \cdot t' \triangleleft_c \emptyset$ ; in the second case, the convergence condition becomes  $t \cdot t \triangleleft_c t$ , which is always true by *weakening*. Hence  $r$  is a continuous and convergent morphism.  $\square$

**PROPOSITION 7.8.** — *If every formal Zariski topology is reducible, then  $\text{ACF}^*$  holds.*

*Proof.* — Proving  $\text{ACF}^*$  is equivalent to proving the existence of a formal point for the formal topology  $(S, \triangleleft_P, \times_P, \circ)$ . By Lemma 7.7, there exists a discrete, reduced and non-trivial ring  $A$  and a continuous and convergent morphism  $r$  from  $\text{Zar}_c(A)$  to  $(S, \triangleleft_P, \times_P, \circ)$ .

<sup>11</sup>Since every  $U_i$  is finite, this sum is well-defined.

By Remark 7.6, the subset  $A \setminus \{0\}$  is a radical coideal and  $1 \in A \setminus \{0\}$ . In particular  $1 \times A \setminus \{0\}$ , so, by reducibility, there exists a formal point  $\alpha$  of the formal Zariski topology such that  $\alpha \subseteq A \setminus \{0\}$ . The direct image of  $\alpha$  through  $r$  is a formal point for  $(S, \triangleleft_P, \times_P, \circ)$ .  $\square$

It remains to be seen whether Proposition 7.4 and 7.8 have any kind of converse. To conclude this section, we list some principles constructively equivalent to spatiality for the class of Zariski formal topologies. Before, we make the following observations:

- (1) For all the monoids  $S \subseteq A$  and for all ideals  $I \subseteq A$  one has

$$S \check{\text{Y}} R(I) \leftrightarrow S \check{\text{Y}} I.$$

In particular, for  $S = \{1\}$ , one gets  $A = R(I) \leftrightarrow A = I$ .

- (2) For every radical ideal  $I$  and monoid  $S$  of  $A$ , if we set

$$a \in L_S(I) \equiv (\exists s \in S)(a \cdot s \in I);$$

then  $L_S(I)$  is the saturation of  $I$  with respect to the cover  $\triangleleft_{c,S}$ , and moreover every saturated subset is obtained in this way. Also,  $S \check{\text{Y}} L_S(I)$  if and only if  $S \check{\text{Y}} I$ . By definition

$$\exists s(s \in S \ \& \ s \in L_S(I)),$$

that is, there exists  $s' \in S$  such that  $s \in S$  and  $s \cdot s' \in I$ ; then  $s \cdot s'$  is a witness of  $S \check{\text{Y}} I$ .

- (3) Putting together the previous remarks, for every monoid  $S$  and every ideal  $I$ , we get

$$S \check{\text{Y}} L_S(R(I)) \leftrightarrow S \check{\text{Y}} R(I) \leftrightarrow S \check{\text{Y}} I.$$

We can now prove the following equivalences:

PROPOSITION 7.9. — *Asserting the spatiality of  $\text{Zar}_c(A)$  for every ring  $A$  is equivalent to each of the following:*

- (1) For every ring  $A$ , for every monoid  $S \subseteq A$  and every ideal  $I \subseteq A$ ,

$$(\forall \alpha \in \mathfrak{P}\mathfrak{t}(A))(S \subseteq \alpha \rightarrow \alpha \check{\text{Y}} I) \rightarrow S \check{\text{Y}} I.$$

In particular, this holds for every filter  $S$ .

- (2) (Sufficiency) For every ring  $A$  and for every  $a \in A$ ,

$$(\forall \alpha \in \mathfrak{P}\mathfrak{t}(A))(\neg(a \in \alpha)) \rightarrow a \triangleleft_c \emptyset.$$

In terms of prime ideals, this property can be rewritten with classical logic as

$$(\forall \alpha \in \mathfrak{P}\mathfrak{t}(A))\forall \mathfrak{p}(a \in \mathfrak{p}) \rightarrow a \in \sqrt{(0)}.$$

- (3) For every ring  $A$  and for every ideal  $I \subseteq A$ ,

$$(\forall \alpha \in \mathfrak{P}\mathfrak{t}(A))(\alpha \check{\text{Y}} I) \rightarrow A = I.$$

Assuming classical logic, this statement corresponds to

$$(\forall \mathfrak{p} \in \mathfrak{S}\mathfrak{p}\mathfrak{e}\mathfrak{c}(A))(I \not\subseteq \mathfrak{p}) \rightarrow I = A.$$

*Proof.* — (1) ( $\leftarrow$ ) Let  $A$  be a commutative ring which satisfies 1. Given  $a \in A$  and  $U \subseteq A$ , let  $S(a)$  be the monoid generated by  $a$  and  $I = R(U)$ . Then

$$S(a) \subseteq \alpha \leftrightarrow a \in \alpha, \quad \alpha \check{\text{Y}} I \leftrightarrow \alpha \check{\text{Y}} U, \quad S(a) \check{\text{Y}} I \leftrightarrow a \triangleleft_c U.$$

By substituting these equivalents we get

$$\forall \alpha(a \in \alpha \rightarrow \alpha \check{\text{Y}} U) \rightarrow a \triangleleft_c U,$$

which is spatiality. ( $\rightarrow$ ) Suppose that the Zariski formal topologies are spatial, and fix a ring  $A$ , a monoid  $S$  and a radical ideal  $I$  in  $A$ . Let  $A_S$  be the localisation in  $S$ , equipped with the formal topology  $\text{Zar}_c(A_S)$ . Thanks to the isomorphism of Proposition 6.8, this formal topology is spatial if and only if the formal topology



$\text{Zar}_{c,S}(A)$  is spatial. Remembering that the formal points in  $\text{Zar}_{c,S}(A)$  are precisely the formal points of  $\text{Zar}_c(A)$  containing  $S$ , the spatiality in  $(1, L_S(R(I)))$  becomes

$$\forall \alpha \in \mathfrak{Pt}(A)(S \subseteq \alpha \rightarrow \alpha \checkmark L_S(R(I))) \rightarrow 1 \in L_S(R(I)).$$

Since  $\alpha \checkmark L_S(R(I)) \leftrightarrow \alpha \checkmark I$  and  $1 \in L_S(R(I)) \rightarrow S \checkmark L_S(R(I)) \leftrightarrow S \checkmark I$ , we obtain

$$\forall \alpha \in \mathfrak{Pt}(A)(S \subseteq \alpha \rightarrow \alpha \checkmark I) \rightarrow S \checkmark I,$$

for every ideal  $I$ .

(2) ( $\rightarrow$ ) Let  $A$  be a commutative ring satisfying spatiality. For every  $a \in A$ , we have in particular

$$\forall \alpha(a \in \alpha \rightarrow \alpha \checkmark \emptyset) \rightarrow a \triangleleft_c \emptyset,$$

that is,  $\forall \alpha(\neg(a \in \alpha)) \rightarrow a \triangleleft_c \emptyset$ , or *Sufficiency*.

( $\leftarrow$ ) Let  $a \in A$  and  $U \subseteq A$ . By hypothesis,  $A/U$  satisfies *Sufficiency*, so that

$$\forall \alpha \in \mathfrak{Pt}(A/U)(\neg(a \in \alpha)) \rightarrow a \triangleleft_c^U \emptyset.$$

Since  $\alpha \in \mathfrak{Pt}(A/U)$  if and only if  $\alpha \in \mathfrak{Pt}(A)$  and  $\neg(\alpha \checkmark U)$ , and  $a \triangleleft_c^U \emptyset$  if and only if  $a \triangleleft_c U$ , we have

$$\forall \alpha \in \mathfrak{Pt}(A)(a \in \alpha \rightarrow \alpha \checkmark U) \rightarrow a \triangleleft_c U,$$

which is spatiality.

(3) ( $\rightarrow$ ) We give a sketch, following the previous points. Let  $A$  be a commutative ring satisfying spatiality. For every ideal  $I \subseteq A$ , spatiality in  $(1, R(I))$  is expressed by

$$\forall \alpha(1 \in \alpha \rightarrow \alpha \checkmark R(I)) \rightarrow 1 \in R(I)$$

which is to say that  $\forall \alpha(\alpha \checkmark I) \rightarrow A = I$ .

( $\leftarrow$ ) Let  $a \in A$  and  $U \subseteq A$ . Applying the hypothesis to  $A_{S(a)}$ , the localisation of  $A$  in  $S(a)$ , and using the isomorphism of Proposition 6.8, we get

$$\forall \alpha \in \mathfrak{Pt}(A)(S(a) \subseteq \alpha \rightarrow \alpha \checkmark U) \rightarrow 1 \triangleleft_{c,S(a)} U$$

for all  $U \subseteq A$ . Since  $S(a) \subseteq \alpha$  if and only if  $a \in \alpha$ , and  $1 \triangleleft_{c,S(a)} U$  if and only if  $a \triangleleft_c U$ , we finally have

$$\forall \alpha \in \mathfrak{Pt}(A)(a \in \alpha \rightarrow \alpha \checkmark U) \rightarrow a \triangleleft_c U,$$

that is, spatiality for  $A$ . □

It is worthwhile to stress that the proof of ( $\leftarrow$ ) in point 1 and that of ( $\rightarrow$ ) in points 2 and 3 hold instance by instance of the ring  $A$ .

Apart from Proposition 7.8, it remains to be seen to what extent reducibility for the Zariski formal topology is related to any non-constructive principle (e.g., fragments of the law of excluded middle and/or the axiom of choice), as is the case in other contexts [5].

## 8. THE ZARISKI LATTICE AND THE FORMAL HILBERT NULLSTELLENSATZ

We present a different and well-known point-free interpretation of the Zariski spectrum (due to Joyal [17]) and briefly recall the link to the formal Zariski topology. In particular, we give an alternative proof of the so-called *formal Nullstellensatz*.

We recall that the Zariski topology on a ring  $A$  has as basis the subsets of the form

$$D(a) = \{\mathfrak{p} \in \mathfrak{Spec}(A) : a \notin \mathfrak{p}\}$$

with  $a \in A$ . This  $\mathfrak{Spec}(A)$  is a spectral space: that is, it is sober, i.e. every non-empty irreducible closed subset is the closure of a unique point, and the compact opens form a basis for the topology. It is possible to describe this basis in an formal way, as the collection of opens of the form  $D(a_1) \cup \dots \cup D(a_n)$  with  $a_1, \dots, a_n \in A$ ; these opens form a distributive lattice satisfying

$$D(0) = \emptyset, \quad D(1) = \mathfrak{Spec}(A), \quad D(ab) = D(a) \cap D(b), \quad D(a+b) \subseteq D(a) \cup D(b)$$

for all  $a, b \in A$ . At this point, one can avoid reference to the collection  $\mathfrak{Spec}(A)$  of the prime ideals and formally describe the distributive lattice freely generated by the expressions  $\{D(a)\}_{a \in A}$  and subject to the relations

$$D(0) = 0, \quad D(1) = 1, \quad D(ab) = D(a) \wedge D(b), \quad D(a + b) \leq D(a, b) \quad (8.1)$$

for all  $a, b, b' \in A$ , where

$$D(a_1, \dots, a_n) \equiv D(a_1) \vee \dots \vee D(a_n).$$

This lattice is called the *Zariski lattice* of  $A$  [7, 18]. We notice that every element of the lattice can be written in the form  $D(a_1, \dots, a_n)$ . Hence the Zariski lattice can be identified with the set  $\mathcal{Z} = \mathcal{P}_\omega(A)$  of finite subsets of  $A$ , equipped with the minimal partial order relation  $\leq$  satisfying

$$\{0\} \leq \emptyset \quad \{a \cdot b\} \leq \{a\} \quad \frac{\{c\} \leq \{a\} \quad \{c\} \leq \{b\}}{\{c\} \leq \{a \cdot b\}} \quad \{a + b\} \leq \{a, b\}. \quad (8.2)$$

and

$$U_0 \leq V_0 \leftrightarrow (\forall u \in U_0)(\{u\} \leq V_0)$$

for all  $a, b \in A$  and  $U_0, V_0 \in \mathcal{Z}$ . The condition  $\{a\} \leq \{1\}$  is already entailed by  $\{a \cdot b\} \leq \{a\}$ .

In particular, the distributive lattice  $(\mathcal{Z}, \leq)$  satisfies

$$\{0\} \leq \emptyset \quad \{a \cdot b\} \leq \{a\} \quad \{a + b\} \leq \{a, b\} \quad \{a\} \leq \{a^2\}.$$

for all  $a, b \in A$ , and therefore  $U_0 \triangleleft_c V_0 \rightarrow U_0 \leq V_0$  by the induction rule associated to  $\triangleleft_c$ . Vice versa,  $\triangleleft_c$  satisfies Convergence (Proposition 6.5) and thus the conditions (8.2) from which we derive  $\triangleleft_c = \leq$ . In other terms,  $(\mathcal{Z}, \triangleleft_c)$  is exactly the lattice generated by the conditions (8.1). The characterization (6.1) of the formal cover can then be rewritten as follows:

PROPOSITION 8.1. —  $D(a) \leq D(b_1, \dots, b_n)$  if and only if  $a \in R(\{b_1, \dots, b_n\})$ .

The statement above is also called *formal Nullstellensatz* and establishes the link between the lattice and the given structure of the ring, and is the core theorem for the application of formal methods [6, 7, 8, 18]. In our treatment, Proposition 8.1 corresponds to the explicit characterization (6.1) of the formal cover.

**Acknowledgements.** The authors are grateful to the referee for the accurate comments and the useful suggestions. The idea to put the formal Zariski topology as a basic topology is due to Per Martin-Löf. Rinaldi collaborated while he was holding an ESR position at the Mathematisches Institut of the Universität München, funded from the European Community's Seventh Framework Programme under grant agreement no. 238381 (MALOA). Schuster was holding a Feodor-Lynen-Forschungsstipendium, granted by the Alexander-von-Humboldt-Stiftung and funded by the German Bundesministerium für Bildung und Wissenschaft, at the Dipartimento di Matematica of the Università di Padova and at the Dipartimento di Filosofia of the Università di Firenze, upon invitation by Giovanni Sambin and Andrea Cantini. Final amendments to this paper were carried out within the project "Abstract Mathematics for Actual Computation: Hilbert's Program in the 21st Century" funded by the John Templeton Foundation. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

#### REFERENCES

- [1] Peter Aczel and Michael Rathjen. *Notes on Constructive Set Theory*. Available from Aczel's webpage, 2008.
- [2] Bernhard Banaschewski. The power of the ultrafilter theorem. *Journal of the London Mathematical Society*, 27:193–202, 1983.
- [3] Francesco Ciraulo, Maria Emilia Maietti, and Giovanni Sambin. Convergence in formal topology: a unifying notion. *Journal of Logic and Analysis*, 5(2):1–45, 2013.

- [4] Francesco Ciraulo and Giovanni Sambin. Finitary formal topologies and Stone's representation theorem. *Theoretical Computer Science*, 405(1-2):11–23, 2008.
- [5] Francesco Ciraulo and Giovanni Sambin. Reducibility, a constructive dual of spatiality. *Preprint, Università di Padova*, 2015.
- [6] Thierry Coquand. Space of valuations. *Annals of Pure and Applied Logic*, 157:97–109, 2009.
- [7] Thierry Coquand and Henri Lombardi. A logical approach to abstract algebra. *Mathematical Structures in Computer Science*, 16:885–900, 2006.
- [8] Thierry Coquand, Henri Lombardi, and Peter Schuster. The projective spectrum as a distributive lattice. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 48:220–228, 2007.
- [9] Thierry Coquand and Henrik Persson. Valuations and Dedekind's Prague theorem. *Journal of Pure and Applied Algebra*, 155:121–129, 2001.
- [10] Thierry Coquand, Giovanni Sambin, J. Smith, and Silvio Valentini. Inductively generated formal topologies. *Annals of Pure and Applied Logic*, 124:71–106, 2003.
- [11] Radu Diaconescu. Axiom of choice and complementation. *Proceedings of the American Mathematical Society*, 51:176–178, 1975.
- [12] Nicola Gambino and Peter Schuster. Spatiality for formal topologies. *Mathematical Structures in Computer Science*, 17(1):65–80, 2007.
- [13] N. Goodman and J. Myhill. Choice implies excluded middle. *Mathematical Logic Quarterly*, 24(25-30):461–461, 1978.
- [14] Peter T. Johnstone. *Stone Spaces*. Number 3 in Cambridge Studies in Advanced Mathematics. Cambridge etc.: Cambridge University Press, 1982.
- [15] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Number 43-44 in Oxford Logic Guides. Oxford University Press, 2002.
- [16] André Joyal. Spectral spaces and distributive lattices. *Notices of the American Mathematical Society*, 18:393, 1971.
- [17] André Joyal. Les théorèmes de Chevalley-Tarski et remarques sur l'algèbre constructive. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 16:256–258, 1976.
- [18] Henri Lombardi. Algèbre dynamique, espaces topologiques sans points et programme de Hilbert. *Annals of Pure and Applied Logic*, 137:256–290, 2006.
- [19] Henri Lombardi and Claude Quitté. *Algèbre Commutative, Méthodes constructives, Modules projectifs de type fini*. Calvage & Mounet, Paris, 2012.
- [20] Per Martin-Löf. *Intuitionistic Type Theory*. Notes by Giovanni Sambin of a series of lectures given in Padua, Bibliopolis, Napoli, 1984.
- [21] Per Martin-Löf and Giovanni Sambin. Generating positivity by coinduction. In *The Basic Picture and Positive Topology. New structures for constructive mathematics*, Oxford Logic Guides, Oxford, forthcoming. Clarendon Press.
- [22] Sara Negri. Continuous domains as formal spaces. *Mathematical Structures in Computer Science*, 12:19–52, 2002.
- [23] Davide Rinaldi. A constructive notion of codimension. *Journal of Algebra*, 383:178–196, 2013.
- [24] Kimmo I. Rosenthal. *Quantales and their applications*, volume 234 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1990.
- [25] G. Sambin. Intuitionistic formal spaces - a first communication. In D. Skordev, editor, *Mathematical Logic and its Applications*, pages 187–204. Plenum, 1987.
- [26] Giovanni Sambin. Some points in formal topology. *Theoretical Computer Science*, 305(1-3):347–408, 2003.
- [27] Giovanni Sambin. *The Basic Picture and Positive Topology. New structures for constructive mathematics*. Oxford Logic Guides. Clarendon Press, Oxford, forthcoming.
- [28] Giovanni Sambin and Giulia Battilotti. Pretopologies and a uniform presentation of sup-lattices, quantales and frames. *Annals of Pure and Applied Logic*, 137(1-3):30–61, 2006.
- [29] Giovanni Sambin and Silvia Gebellato. A preview of the basic picture: a new perspective on formal topology. In *Selected papers from the International Workshop on Types for Proofs and Programs, TYPES '98*, pages 194–207, London, UK, 1999. Springer-Verlag.
- [30] Peter Schuster. Formal Zariski topology: positivity and points. *Annals of Pure and Applied Logic*, 137(1-3):317–359, 2006.
- [31] Peter Schuster. The Zariski spectrum as a formal geometry. *Theoretical Computer Science*, 405:101–115, 2008.
- [32] Peter Schuster. Induction in algebra: a first case study. *Logical Methods in Computer Science*, 9(3):20, 2013.
- [33] Inger Sigstam. Formal spaces and their effective presentations. *Archive for Mathematical Logic*, 34(4):211–246, 1995.
- [34] Marshall Harvey Stone. Topological representations of distributive lattices and Brouwerian logics. *Časopis pro pěstování matematiky a fyziky*, 67(1):1–25, 1938.

- [35] Steven Vickers. *Topology via logic*, volume 5 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 1989.

Manuscript received December 5, 2014,  
revised July 16, 2015,  
accepted July 16, 2015.

Davide RINALDI  
Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England  
d.rinaldi@leeds.ac.uk

Giovanni SAMBIN  
Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy  
sambin@math.unipd.it

Peter SCHUSTER  
Dipartimento di Informatica, Università di Verona, Strada le Grazie 15, 37134 Verona, Italy  
peter.schuster@univr.it (Corresponding author)