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Ergodic Dilation of a Quantum Dynamical System


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ERGODIC DILATION OF A QUANTUM DYNAMICAL SYSTEM

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Abstract. Using the Nagy dilation of linear contractions on Hilbert space and the Stinespring’s theorem for completely positive maps, we prove that any quantum dynamical system admits a dilation in the sense of Muhly and Solel which satisfies the same ergodic properties of the original quantum dynamical system.

1. Introduction

A quantum dynamical system is a pair \((\mathcal{M}, \Phi)\) consisting of a von Neumann algebra \(\mathcal{M}\) and a normal, i.e. \(\sigma\)-weakly continuous, unital completely positive map \(\Phi : \mathcal{M} \to \mathcal{M}\).

In this work we will prove that it is possible to dilate any quantum dynamical system to a quantum dynamical system where the dynamics \(\Phi\) is a \(*\)-homomorphism of a larger von Neumann algebra.

The existence of a dilation for a quantum dynamical system has been proven by Muhly and Solel [8, Prop. 2.24] using the minimal isometric dilation of completely contractive covariant representations of particular \(W^*\)-correspondences over von Neumann algebras. In contrast, we prove the existence of a dilation for a quantum dynamical system using the Nagy dilations for linear contractions on Hilbert spaces (see [9]) and a particular representation obtained by the Stinespring theorem for completely positive maps (see [13]).

Throughout this paper we will use the abbreviation ucp-map for unital completely positive maps, and we denote by \(\mathcal{B}(\mathcal{H})\) the \(C^*\)-algebra of all bounded linear operators on a Hilbert space \(\mathcal{H}\).

In the present paper by a dilation of a quantum dynamical system \((\mathcal{M}, \Phi)\), with \(\mathcal{M}\) defined on a Hilbert space \(\mathcal{H}\) we mean a quadruple \((\mathcal{R}, \Theta, K, Z)\) where \(\mathcal{R}\) defined on Hilbert space \(K\) and \(\Theta\) is a \(*\)-homomorphism of \(\mathcal{R}\); and \(Z : \mathcal{H} \to K\) is an isometry satisfying the following properties (see [8]):

- \(Z\mathcal{M}Z^* \subset \mathcal{R}\);
- \(Z^*\mathcal{R}Z \subset \mathcal{M}\);
- \(\Phi^n(A) = Z^*\Theta^n(ZAZ^*)Z\) for \(A \in \mathcal{M}\) and \(n \in \mathbb{N}\);
- \(Z^*\Theta^n(X)Z = \Phi^n(Z^*XZ)\) for \(X \in \mathcal{R}\) and \(n \in \mathbb{N}\).

Hence, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\Phi^n} & \mathcal{M} \\
Z \cdot Z^* & \uparrow & \downarrow \quad Z^* \cdot Z \\
\mathcal{M} & \xrightarrow{\Theta^n} & \mathcal{M}
\end{array}
\]

Notice that in the literature of dynamical systems the dilation problem has taken meanings different from that used here, see e.g. [2, 3, 4, 12].

By a representation of a quantum dynamical system \((\mathcal{M}, \Phi)\) we mean a triple \((\pi, \mathcal{H}, V)\), where \(\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H})\) is a normal faithful representation on the Hilbert space \(\mathcal{H}\) and \(V\) is an isometry on \(\mathcal{H}\) such that

\[\pi(\Phi(A)) = V^*\pi(A)V \quad \text{for} \quad A \in \mathcal{M}.
\]

Keywords: Quantum Markov process, completely positive maps, Nagy dilation, ergodic state.
Since \( \pi \) is faithful and normal, we identify the quantum dynamical system \((\mathcal{M}, \Phi)\) with \((\pi(\mathcal{M}), \Phi)\) where \( \Phi \) is the ucp-map \( \Phi(A) = V^*\pi(A)V \), for any \( A \in \mathcal{M} \). This this leads us to the study of invariant algebras under the action of isometries.

In fact, in Section 3, we consider a concrete \( C^* \)-algebra \( \mathfrak{A} \) with unit, fundamental for the proof of the main result of this paper.

Before proving the existence of a dilation of a quantum dynamical system, it is necessary to recall the fundamental Nagy dilation theorem. This is the subject of Section 4. We prove a Stinespring-type theorem for ucp-maps between \( C^* \)-algebras (see [1, 5, 7, 10] for the relation between reversible processes, modular operators and \( \varphi \)-adjointness). If \( (\mathcal{M}, \Theta) \) is our dilation of the quantum dynamical system \((\mathcal{M}, \Phi)\), we shall prove that if the dynamics \( \Phi \) admits a \( \varphi \)-adjoint (see [6]) if there is a normal ucp-map \( \Phi : \mathcal{M} \to \mathcal{M} \) such that for each \( A, B \in \mathcal{M} \)

\[
\varphi(\Phi(A)B) = \varphi(A\Phi(B)),
\]

(see [1, 5, 7, 10] for the relation between reversible processes, modular operators and \( \varphi \)-adjointness). If \( (\mathcal{M}, \Theta) \) is our dilation of the quantum dynamical system \((\mathcal{M}, \Phi)\), we shall prove that if the dynamics \( \Phi \) admits a \( \varphi \)-adjoint and

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |\varphi(\Phi^k(B)) - \varphi(A)\varphi(B)| = 0 \quad \text{for} \quad A, B \in \mathcal{M},
\]

then

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |\varphi(Z^*X\Theta^k(Y)Z) - \varphi(Z^*XZ)\varphi(Z^*YZ)| = 0 \quad \text{for} \quad X, Y \in \mathcal{M}.
\]

Before proving the existence of a dilation of a quantum dynamical system, it is necessary to recall the fundamental Nagy dilation theorem. This is the subject of the next section.

### 2. Nagy dilation theorem

If \( V \) is an isometry on a Hilbert space \( \mathcal{H} \), there is a triple \((\hat{V}, \hat{\mathcal{H}}, Z)\) where \( \hat{\mathcal{H}} \) is a Hilbert space, \( Z : \mathcal{H} \to \hat{\mathcal{H}} \) is an isometry and \( \hat{V} \) is a unitary operator on \( \hat{\mathcal{H}} \) with

\[
\hat{V}Z = ZV
\]

satisfying the following minimal property:

\[
\hat{\mathcal{H}} = \bigvee_{k \in \mathbb{Z}} \hat{V}^kZ\mathcal{H},
\]

see [9]. However, for our purposes it is still useful to recall here the structure of the unitary minimal dilation of an isometry.
For a Hilbert space $\mathcal{K}$ recall that $l^2(\mathcal{K})$ denotes the Hilbert space $\{\xi : \mathbb{N} \rightarrow \mathcal{K} : \sum_{n \geq 0} |\xi(n)|^2 < \infty\}$. Consider the Hilbert space

$$\hat{\mathcal{H}} = \mathcal{H} \oplus l^2(\mathcal{F} \mathcal{H}) \quad (2.3)$$

and the unitary operator on $\hat{\mathcal{H}}$ defined as

$$\hat{V} = \begin{vmatrix} V & F \Pi_0 \\ 0 & W \end{vmatrix}, \quad (2.4)$$

where $F = I - VV^*$ and $\Pi_j : l^2(\mathcal{F} \mathcal{H}) \rightarrow \mathcal{H}$ is the canonical projection

$$\Pi_j(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = \xi_j \quad \text{for} \quad j \in \mathbb{N},$$

while $W : l^2(\mathcal{F} \mathcal{H}) \rightarrow l^2(\mathcal{F} \mathcal{H})$ is the operator

$$W(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = (0, 0, \ldots, 0, \xi_0, \xi_1, \ldots).$$

If $Z : \mathcal{H} \rightarrow \hat{\mathcal{H}}$ is the isometry defined by $Zh = h \oplus 0$ for all $h \in \mathcal{H}$, it is simple to prove that the relations (2.1) and (2.2) are verified.

We observe that for each $n \in \mathbb{N}$ we have

$$\hat{V}^n = \begin{vmatrix} V^n & C(n) \\ 0 & W^n \end{vmatrix}, \quad (2.5)$$

where $C(n) : l^2(\mathcal{F} \mathcal{H}) \rightarrow \mathcal{H}$ are the following operators:

$$C(n) := \sum_{j=1}^{m} V^n - j F \Pi_{j-1} \quad \text{for} \quad n \geq 1.$$ 

Furthermore, for each $n, m \in \mathbb{N}$ we obtain:

$$\Pi_n W^m = \Pi_{n+m} \quad \text{and} \quad \Pi_n W^m = \begin{cases} \Pi_{n-m} & \text{if} \quad n \geq m \\ 0 & \text{if} \quad n < m \end{cases}, \quad (2.6)$$

since

$$W^m(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = (0, 0, \ldots, 0, \xi_0, \xi_1, \ldots),$$

while for each $k, p \in \mathbb{N}$ we obtain:

$$\Pi_p C(k)^* = \begin{cases} FV^{(k-p-1)} & \text{if} \quad k > p \\ 0 & \text{elsewhere} \end{cases} \quad (2.7)$$

since for each $h \in \mathcal{H}$ we have:

$$C(k)^*h = (FV^{(k-1)}h \ldots FV^*h, Fh, 0, 0 \ldots). \quad (2.8)$$

3. Isometric dilation and invariant algebras

In this section we consider a concrete unital $C^*$-algebra $\mathfrak{A}$ of $\mathfrak{B}(\mathcal{H})$ and an isometry $V$ on the Hilbert space $\mathcal{H}$ such that

$$V^* \mathfrak{A} V \subseteq \mathfrak{A}.$$ 

If $(\hat{V}, \hat{\mathcal{H}}, Z)$ denotes the minimal unitary dilation of the isometry $V$, we will prove the following proposition:

PROPOSITION 3.1. — There exists a unital $C^*$-algebra $\hat{\mathfrak{A}} \subseteq \mathfrak{B}(\hat{\mathcal{H}})$ such that:

(a) $Z \mathfrak{A} Z^* \subseteq \hat{\mathfrak{A}}$;
(b) $Z^* \hat{\mathfrak{A}} Z \subseteq \mathfrak{A}$;
(c) $\hat{V}^* \hat{\mathfrak{A}} \hat{V} \subseteq \hat{\mathfrak{A}}$;
(d) $Z^* \hat{V}^* X \hat{V} Z = V^* Z^* X Z V$ for $X \in \hat{\mathfrak{A}}$;
(e) $Z^* \hat{V}^*(Z \mathfrak{A} Z^*) \hat{V} Z = V^* A V$ for $A \in \mathfrak{A}$.
The statements (d) and (e) are straightforward consequences of (a) and (b) and of the relationship $\tilde{V}Z = ZV$. In order to prove the other statements, we must study two classes of operators on the Hilbert space $\mathcal{H}$, associated to the pair $(\mathfrak{A}, V)$ defined above, which we shall call the gamma and the napla operators.

3.1. Gamma operators. We consider the sequences

$$\alpha := (n_1, n_2 \ldots n_r, A_1, A_2 \ldots A_r),$$

with $n_j \in \mathbb{N}$ and $A_j \in \mathfrak{A}$ for $j = 1, 2, \ldots, r$. These elements $\alpha$ are called strings of $\mathfrak{A}$ of length $l(\alpha) := r$ and weight $\dot{\alpha} := \sum_{i=1}^{r} n_i$. 

To any string $\alpha$ of $\mathfrak{A}$ correspond two operators of $\mathfrak{B}(\mathcal{H})$ defined by

$$|\alpha\rangle := A_1 V^{n_1} A_2 V^{n_2} \ldots A_r V^{n_r} \quad \text{and} \quad |\alpha| := V^{n_r} A_r V^{n_{r-1}} A_{r-1} \ldots V^{n_1} A_1. $$

Furthermore for each natural number $n$ we define the sets

$$[n] := \{\alpha \in \mathfrak{B}(\mathcal{H}) : \dot{\alpha} = n\},$$

and

$$[n] \mathfrak{A} = \{[\alpha] A \in \mathfrak{B}(\mathcal{H}) : A \in \mathfrak{A} \text{ and } \alpha\text{-string of } \mathfrak{A} \text{ with } \dot{\alpha} = n\}.$$  

The symbols $[n]$ and $\mathfrak{A}(n)$ have analogous meanings.

**Proposition 3.2.** Let $\alpha$ and $\beta$ be strings of $\mathfrak{A}$. For each $R \in \mathfrak{A}$ we have:

$$\langle \alpha | R | \beta \rangle \in \begin{cases} \mathfrak{A}(\dot{\alpha} - \dot{\beta}) & \text{if } \dot{\alpha} \geq \dot{\beta} \\ (\dot{\beta} - \dot{\alpha}) \mathfrak{A} & \text{if } \dot{\alpha} < \dot{\beta} \end{cases} \quad (3.1)$$

and

$$|\alpha| R |\beta\rangle \in [\dot{\alpha} + \dot{\beta}] \quad (3.2)$$

**Proof.** For each $m, n \in \mathbb{N}$ and $R \in \mathfrak{A}$ we have:

$$V^{m_r} R V^n \in \begin{cases} V^{(m-n)r} \mathfrak{A} & \text{if } m \geq n \\ \mathfrak{A} V^{(n-m)} & \text{if } m < n \end{cases} \quad (3.3)$$

Given $\alpha = (m_1, m_2 \ldots m_r, A_1, A_2 \ldots A_r)$ and $\beta = (n_1, n_2 \ldots n_s, B_1, B_2 \ldots B_s)$ we have that

$$\langle \alpha | R | \beta \rangle = V^{m_r} A_r \ldots V^{m_1} A_1 R B_1 V^{n_1} \ldots B_s V^{n_s} = (\tilde{\alpha}|I|\tilde{\beta}),$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are strings of $\mathfrak{A}$ with $l(\tilde{\alpha}) + l(\tilde{\beta}) = l(\alpha) + l(\beta) - 1$. Moreover if $\dot{\alpha} \geq \dot{\beta}$ then $\dot{\tilde{\alpha}} \geq \dot{\tilde{\beta}}$, while if $\dot{\alpha} < \dot{\beta}$ then $\dot{\tilde{\alpha}} < \dot{\tilde{\beta}}$. In fact if $m_1 \geq n_1$ we obtain:

$$\langle \alpha | R | \beta \rangle = V^{m_r} A_r \ldots V^{m_1} A_1 V^{(m_1-n_1)} R_1 B_2 V^{n_2} \ldots B_s V^{n_s} = (\tilde{\alpha}|I|\tilde{\beta}),$$

where

$$R_1 = V^{n_1} A_1 R B_1 V^{n_1},$$

$$\tilde{\alpha} = (m_1 - n_1, m_2 \ldots m_r, R_1, A_2 \ldots A_r) \quad \text{and} \quad$$

$$\tilde{\beta} = (n_2 \ldots n_s, B_2 \ldots B_s).$$

If $m_1 < n_1$ then we can write:

$$\langle \alpha | R | \beta \rangle = V^{m_r} A_r \ldots V^{m_2} A_2 R_1 V^{(n_1-m_1)} B_2 \ldots B_s V^{n_s} = (\tilde{\alpha}|I|\tilde{\beta}),$$

where

$$R_1 = V^{m_1} A_1 R B_1 V^{m_1},$$

$$\tilde{\alpha} = (m_2 \ldots m_r, A_2 \ldots A_r) \quad \text{and} \quad$$

$$\tilde{\beta} = (n_1 - m_1, n_2 \ldots n_s, R_1, B_2 \ldots B_s).$$

The proof of (3.1) follows by induction on the number $\nu = l(\alpha) + l(\beta)$. The equation (3.2) follows by a direct calculation. $\square$
Now, given the orthogonal projection $F = I - VV^*$ (see Section 2), for each string $\alpha$ of $\mathfrak{A}$ with $\alpha \geq 1$ we define

$$\Gamma(\alpha) := (\alpha|F\Pi_A - 1,$$

which we call the gamma operator associated to $G(\mathfrak{A}, V)$. The linear space generated by all gamma operators $\Gamma(\alpha)$ for $\alpha \geq 1$ will be denoted by $G(\mathfrak{A}, V)$.

**Proposition 3.3.** For any strings $\alpha$ and $\beta$ of $\mathfrak{A}$ with $\alpha, \beta \geq 1$, we have

$$\Gamma(\alpha)\Gamma(\beta)^* \in \mathfrak{A}.$$

**Proof.** Note that

$$\Gamma(\alpha)\Gamma(\beta)^* = (\alpha|F\Pi_{\alpha - 1}\Pi_{\beta - 1}^*F|\beta) = \begin{cases} (\alpha|F|\beta) & \text{if } \hat{\alpha} = \hat{\beta} \\ 0 & \text{if } \hat{\alpha} \neq \hat{\beta} \end{cases}.$$

In fact if $\hat{\alpha} = \hat{\beta}$ we have that

$$(\alpha|F|\beta) = (\alpha|(I - VV^*)|\alpha) = (\alpha|I|\alpha) - (\alpha|VV^*|\alpha) \in \mathfrak{A},$$

since $(\alpha|V| \in (\alpha - 1)$ and $V^*|\alpha) \in |\alpha - 1)$, and $(\alpha - 1|I|\alpha - 1) \in \mathfrak{A}$ by relationship (3.1). \hfill \Box

The gamma operators associated to $G(\mathfrak{A}, V)$ define an operator system $\Sigma$ of the form $\mathfrak{B}(l^2(FH))$ by

$$\Sigma := \{ T \in \mathfrak{B}(l^2(FH)) : \Gamma_1T\Gamma_2^* \in \mathfrak{A} \quad \text{for all } \Gamma_1, \Gamma_2 \in G(\mathfrak{A}, V) \}.$$ (3.4)

We observe that the unit $I$ belongs to $\Sigma$ and that

$$\Gamma_1^*A\Gamma_2 \in \Sigma \quad \text{for } A \in \mathfrak{A},$$

for any pair of gamma operators $\Gamma_1, \Gamma_2$. Furthermore, it is easy to prove that $\Sigma$ is norm closed, and it is weakly closed if $\mathfrak{A}$ is a $W^*$-algebra.

### 3.2. Napla operators

For strings $\alpha$ and $\beta$ of $\mathfrak{A}$, any $A \in \mathfrak{A}$ and $k \in \mathbb{N}$ we define

$$\Delta_k(A, \alpha, \beta) := \Pi_{\alpha+k}^*F|\alpha)A(\beta|F\Pi_{\beta+k}^*.$$

We call these operators of $\mathfrak{B}(l^2(FH))$ the napla operators associated to the pair $(\mathfrak{A}, V)$.

In the next lines we show that the linear space generated by the napla operators form a $\ast$-algebra. To this end, it is easily seen that $\Delta_k(A, \alpha, \beta)^* = \Delta_k(A^{*\prime}, \beta^{\prime}, \alpha)$ for any $h, k \geq 0$. Moreover we have the following two relationships: if $k + \beta \neq h + \gamma$, then

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = 0,$$ (3.5)

while if $k + \beta = h + \gamma$, then there is $\vartheta$ and $R \in \mathfrak{A}$ with

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = \begin{cases} \Delta_k(R, \alpha, \vartheta) & \text{if } h - k \geq 0, \text{ where } \hat{\vartheta} = \hat{\delta} + h - k \\ \Delta_h(R, \vartheta, \delta) & \text{if } h - k < 0, \text{ where } \hat{\vartheta} = \hat{\delta} + k - h. \end{cases}$$ (3.6)

In fact, notice that

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = \Pi_{\alpha+k}^*F|\alpha)A(\beta|F\Pi_{\beta+k}^*\Pi_{\gamma+h}^*F|\gamma)B(\delta|F\Pi_{\delta+h}^*.$$ If $k + \beta \neq h + \gamma$ it follows that $\Pi_{\beta+k}\Pi_{\gamma+h}^* = 0$, and this shows (3.5). If $k + \beta = h + \gamma$, without loss of generality we can assume that $h \geq k$. So $\beta = \gamma + h - k \geq \gamma$ and, by relationship (3.1), we have that $|\beta|F|\gamma) \in \mathfrak{A}(\beta - \gamma|$. Consequently, $A(\beta|F|\gamma)B(\delta| \in \mathfrak{A}(\delta + \beta - \gamma|$, and there exists a $\vartheta$ string of $\mathfrak{A}$ and an element $R \in \mathfrak{A}$ such that $\hat{\vartheta} = \hat{\delta} + \beta - \gamma$ and $A(\beta|F|\gamma)B(\delta| = R(\vartheta)$. Now, since $\hat{\vartheta} = \hat{\delta} + h - k$ we have:

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = \Pi_{\alpha+k}^*F|\alpha)R(\vartheta|F\Pi_{\delta+k}^*$$

$$= \Pi_{\alpha+k}^*F|\alpha)R(\vartheta|F\Pi_{\delta+k} = \Delta_k(R, \alpha, \vartheta),$$

showing relationship (3.6).
Proposition 3.4. — The linear space $\mathcal{X}_\alpha$ generated by the napla operators is a *-subalgebra of $\mathfrak{B}(l^2(FH))$ included in the operator systems $\Sigma$ defined in (3.4).

Proof. — From relationships (3.5),(3.6) the linear space $\mathcal{X}_\alpha$ is a *-algebra. Furthermore for each pair $\Gamma(\alpha)$, $\Gamma(\beta)$ of gamma operators we obtain:

$$\Gamma(\alpha)\Delta_k(A,\gamma,\delta)\Gamma(\beta)^* = (\alpha|F\Pi_{\alpha-1}^{\gamma}F_\gamma|\gamma)A(\delta|F\Pi_{\beta-1}^{\gamma}F_\beta|\beta) \in \mathfrak{A},$$

since by the relationships (3.1) and (3.2) we have

$$(\alpha|F\Pi_{\alpha-1}^{\gamma}F_\gamma|\gamma)A(\delta|F\Pi_{\beta-1}^{\gamma}F_\beta|\beta) \in \left\{ \begin{array}{ll}(k+1)|\mathfrak{A}|k+1 & \text{if } \hat{\alpha} - 1 = \hat{\gamma} + k, \\
0 & \text{elsewhere} \end{array} \right.$$ 

In fact, if $\hat{\alpha} = \hat{\gamma} + k + 1$ we can write

$$(\alpha|F\Pi_{\alpha-1}^{\gamma}F_\gamma|\gamma) = (\alpha|F|\gamma) - (\alpha|VV^*|\gamma) \in \mathfrak{A}(k+1),$$

since $(\alpha|I|\gamma) \in \mathfrak{A}(k+1)$ and $(\alpha|VV^*|\gamma) \in \mathfrak{A}(k+1)$. If $\hat{\beta} = \hat{\delta} + k + 1$ we have

$$(\delta|F\Pi_{\beta-1}^{\gamma}F_\beta|\beta) \in (k+1)|\mathfrak{A}|, \text{ completing the proof.} \quad \square$$

The next result is concerned with $W$-invariance.

Proposition 3.5. — The *-algebra $\mathcal{X}_\alpha$ and the operator system $\Sigma$ are $W$-invariants:

$$W^*\mathcal{X}_\alpha W \subset \mathcal{X}_\alpha \quad \text{and} \quad W^*\Sigma W \subset \Sigma.$$ 

Proof. — The first inclusion follows by (2.6). Concerning the second one, let $T \in \Sigma$. For each pair $\Gamma(\alpha)$, $\Gamma(\beta)$ of gamma operators

$$\Gamma(\alpha)(W^*TW)\Gamma(\beta)^* = (\alpha|F\Pi_{\alpha-1}^{\gamma}F_\gamma|\gamma)A(\beta|F\Pi_{\beta-1}^{\gamma}F_\beta|\beta) \in (\alpha|W^*\Gamma_1(o_\alpha)TT_2(\beta_\alpha)V\mathfrak{A}|,$$

where $\alpha_\alpha$ and $\beta_\beta$ are strings of $\mathfrak{A}$ with $\hat{\alpha}_\alpha = \hat{\alpha} - 1$ and $\hat{\beta}_\beta = \hat{\beta} - 1$. In fact if $\alpha = (m_1, m_2, \ldots, m_r, A_1, A_2, \ldots, A_r)$, then, by definition of the gamma operator, there is $i \leq r$ with $m_i \geq 1$ such that

$$(\alpha|F\Pi_{\alpha-1}F_\gamma|\gamma) = A_1 \cdots A_{r-1}V^*(\alpha_\alpha|F\Pi_{\alpha-2}F_\gamma|\gamma) = A_1 \cdots A_rV^*(\alpha_\alpha),$$

where

$$\alpha_\alpha = (0, \ldots, 0, m_i - 1, m_{i+1} \ldots, m_r, A_1, A_2, \ldots, A_r)$$

with $\hat{\alpha}_\alpha = \hat{\alpha} - 1$. Consequently

$$\Gamma(\alpha)(W^*TW)\Gamma(\beta)^* \subset V^*\mathfrak{A}W \subset \mathfrak{A},$$

completing the proof. \quad \square

3.3. The algebra generated by the napla and gamma operators. Let $\mathcal{X}$ be the closure in norm of the *-algebra $\mathcal{X}_\alpha$ of the apla operators previously defined. Since the operator system $\Sigma$ defined in (3.4) is a norm closed set, we have $\mathcal{X} \subset \Sigma$. Notice that in case $\mathfrak{A}$ is a von Neumann algebra of $\mathfrak{B}(H)$, the operator system $\Sigma$ is weakly closed and $\mathcal{X}'' \subset \Sigma$.

Proposition 3.6. — The set

$$\mathcal{S} = \left\{ \begin{array}{c} \begin{array}{c} A \\ \Gamma_1^* \\ \Gamma_2^* \\ T \end{array} \\ : A \in \mathfrak{A}, T \in \mathcal{X} \text{ and } \Gamma_1, \Gamma_2 \in \mathfrak{G}(\mathfrak{A}, \mathfrak{V}) \end{array} \right\} \quad (3.7)$$

is an operator system of $\mathfrak{B}(\tilde{H})$ such that:

$$\tilde{V}^*\mathcal{S}\tilde{V} \subset \mathcal{S}.$$ 

Furthermore

$$\tilde{V}^*\mathcal{A}^*(\mathcal{S})\tilde{V} \subset \mathcal{A}^*(\mathcal{S}),$$

where $\mathcal{A}^*(\mathcal{S})$ is the *-algebra generated by the set $\mathcal{S}$. 
Proof. — From relationship (2.4) we obtain:
\[ \hat{V}^* \hat{S} \hat{V} = \begin{vmatrix} V^*AV & V^*AC(1) + V^*\Gamma_1W \\ C(1)^*AV + W^*\Gamma_2V & C(1)^*AC(1) + W^*\Gamma_2C(1) + C(1)^*\Gamma_1W + W^*TW \end{vmatrix} \]

We observe that \( V^*\Gamma_1W \) and \( V^*AC(1) \) are gamma operators associated to the pair \((\mathfrak{A}, V)\), while \( C(1)^*AC(1), C(1)^*\Gamma(1)W \) and \( W^*TW \) are operators belonging to \( \mathfrak{X} \). In fact we have \( V^*AC(1) = V^*AF\Pi_0 = \Gamma(\vartheta) \) with \( \vartheta = (1, A) \); while if \( \alpha = (m_1, m_2 \ldots m_r, A_1, A_2 \ldots A_r) \), then \( V^*\Gamma_1W = V^*\alpha|F\Pi_{a-1}W = \Gamma(\vartheta) \), with \( \vartheta = (m_1 + 1, m_2 \ldots m_r, A_1, A_2 \ldots A_r) \) since \( \Pi_{a-1}W = \Pi_a \). Furthermore
\[ C(1)^*AC(1) = \Pi_0^*FAF\Pi_0 = \Delta_0(A, \alpha, \beta), \]
with \( \alpha = \beta = (0, I) \); while

\[ C(1)^*\Gamma_1W = \Pi_0^*F(\alpha|F\Pi_{a-1}W = \Pi_0^*F|\gamma)(\alpha|F\Pi_{a+0} = \Delta_0(I, \gamma, \alpha) \]

with \( \gamma = (0, I) \), where the last statement follows from the fact that \( \hat{V} \) is unitary. \( \square \)

We observe that \( \mathcal{A}^*(\mathcal{S}) \), the *-algebra generated by the operator system \( \mathcal{S} \) defined in (3.7), is the linear space generated by the following elements of \( \mathfrak{B}(\hat{\mathcal{H}}) \):
\[ \begin{vmatrix} A_1 \\ T_2 \Gamma_2 A_3 \\ T_3 \end{vmatrix} \]
with \( A_i \in \mathfrak{A}, \Gamma_j \in \mathfrak{G}(\mathfrak{A}, V) \) and \( T_k \in \mathfrak{X} \) for all \( i, k = 1, 2, 3 \) and \( j = 1, 2 \). We list here some easy properties of the *-algebra \( \mathcal{A}^*(\mathcal{S}) \):

(a) \( \mathfrak{Z}\mathfrak{A}\mathfrak{Z}^* \subset \mathcal{A}^*(\mathcal{S}) \);
(b) \( \mathfrak{Z}^*\mathcal{A}^*(\mathcal{S}) \mathfrak{Z} \subset \mathfrak{A} \);
(c) \( \hat{V}^*\mathcal{A}^*(\mathcal{S})\hat{V} \subset \mathcal{A}^*(\mathcal{S}) \).

Furthermore, since \( \hat{V}Z = ZV \) we have:

(d) \( \mathfrak{Z}^*\hat{V}^*X\hat{V}Z = \mathfrak{V}^*Z^*XZV \);
(e) \( \mathfrak{Z}^*\hat{V}^*(Z\mathfrak{A}\mathfrak{Z}^*)\hat{V}Z = \mathfrak{V}^*AV \).

Using these results we prove the Proposition 3.1.

Proof of Proposition 3.1. — Let \( \hat{\mathfrak{A}} \) be the C*-subalgebra of \( \mathfrak{B}(\hat{\mathcal{H}}) \) generated by
\[ \bigcup_{k=0}^{\infty} \hat{V}^k \mathfrak{Z} \mathfrak{A} \mathfrak{Z}^* \hat{V}^k \quad \text{for} \quad A \in \mathfrak{A}. \] (3.8)

For each natural number \( k \) we have that \( \hat{V}^{k+} \mathfrak{Z} \mathfrak{A} \mathfrak{Z}^* \hat{V}^k \subset \hat{V}^{k+} \mathcal{S} \hat{V}^k \subset \mathcal{S} \), since \( \mathfrak{Z} \mathfrak{A} \mathfrak{Z}^* \subset \mathcal{S} \); so \( \hat{\mathfrak{A}} \subset \mathcal{A}^*(\mathcal{S}) \), the norm closure of the *-algebra \( \mathcal{A}^*(\mathcal{S}) \). It is easily seen that \( \hat{\mathfrak{A}} \) satisfies the conditions of Proposition 3.1, completing the proof. \( \square \)

Remark 3.7. — It is straightforward to show that if \( \mathfrak{A} \) is a von Neumann algebra of \( \mathfrak{B}(\mathcal{H}) \), then the Proposition 3.1 still holds true, with \( \hat{\mathfrak{A}} \) the von Neumann algebra of \( \mathfrak{B}(\hat{\mathcal{H}}) \) generated by the elements (3.8).

4. STINESPREG REPRESENTATION AND QUANTUM DYNAMICAL SYSTEMS

We consider a concrete C*-algebra \( \mathfrak{A} \) of \( \mathfrak{B}(\mathcal{H}) \) with unit and a ucp-map \( \Phi : \mathfrak{A} \to \mathfrak{A} \).

On the algebraic tensor product \( \mathfrak{A} \otimes \mathcal{H} \) we can define a semi-inner product by
\[ \langle A_1 \otimes h_1, A_2 \otimes h_2 \rangle_\Phi := (h_1, \Phi(A_1^*A_2)h_2)_\mathcal{H}, \]
for all \( A_1, A_2 \in \mathfrak{A} \) and \( h_1, h_2 \in \mathcal{H} \). We denote by \( \overline{\mathfrak{A} \otimes \Phi \mathcal{H}} \) the Hilbert space completion of the quotient space of \( \mathfrak{A} \otimes \mathcal{H} \) by the linear subspace \( \{ T \in \mathfrak{A} \otimes \mathcal{H} : \langle T, T \rangle_\Phi = 0 \} \),
So, iterating this procedure we obtain, for each natural number \( n \) satisfying the equation

\[ \langle A_1 \overline{\Phi} h_1, A_2 \overline{\Phi} h_2 \rangle_{\mathfrak{A} \overline{\Phi} \mathcal{H}} = \langle h_1, \Phi(A_1^* A_2) h_2 \rangle_{\mathcal{H}} \]

for all \( A_1, A_2 \in \mathfrak{A} \) and \( h_1, h_2 \in \mathcal{H} \).

Moreover, we define a representation \( \sigma_\Phi : \mathfrak{A} \to \mathcal{B}(\mathfrak{A} \overline{\Phi} \mathcal{H}) \) by

\[ \sigma_\Phi(A)(X \overline{\Phi} h) := AX \overline{\Phi} h \quad \text{for} \quad A \in \mathfrak{A} \text{ and } X \overline{\Phi} h \in \mathfrak{A} \overline{\Phi} \mathcal{H}, \]

and a linear isometry \( V_\Phi : \mathcal{H} \to \mathfrak{A} \overline{\Phi} \mathcal{H} \) by

\[ V_\Phi h := \overline{\Phi} h \quad \text{for} \quad h \in \mathcal{H}, \]

satisfying the equation

\[ \Phi(A) = V_\Phi^* \sigma_\Phi(A) V_\Phi \quad \text{for} \quad A \in \mathfrak{A}. \tag{4.1} \]

The triple \((V_\Phi, \sigma_\Phi, \mathfrak{A} \overline{\Phi} \mathcal{H})\) is the Stinespring representation of the ucp-map \( \Phi \) (see [13]).

Our aim is to analyze the behaviour of the isometry \( V_\Phi \) and of its adjoint \( V_\Phi^* \) on the multiplicative domain of the ucp-map \( \Phi \). To this end note that the adjoint \( V_\Phi^* \) verifies \( V_\Phi^* A \overline{\Phi} h = \Phi(A) h \) for any \( A \in \mathfrak{A} \) and \( h \in \mathcal{H} \). Furthermore, recall that the multiplicative domain of the ucp-map \( \Phi : \mathfrak{A} \to \mathfrak{A} \) is the C*-subalgebra with unit of \( \mathfrak{A} \) defined as

\[ D_\Phi = \{ A \in \mathfrak{A} : \Phi(A^*) \Phi(A) = \Phi(A^* A) \text{ and } \Phi(A) \Phi(A^*) = \Phi(A A^*) \}, \]

see [11]. The multiplicative domain is characterized by the following relationship

\[ A \in D_\Phi \iff \sigma_\Phi(A) V_\Phi V_\Phi^* = V_\Phi V_\Phi^* \sigma_\Phi(A). \tag{4.2} \]

In fact, we first note that

\[ A \overline{\Phi} h = 1 \overline{\Phi} \Phi(A) h \quad \text{for all} \quad h \in \mathcal{H} \iff \Phi(A^* A) = \Phi(A^*) \Phi(A), \]

since

\[ |A \overline{\Phi} h - 1 \overline{\Phi} \Phi(A) h|^2 = \langle h, \Phi(A^* A) h \rangle - \langle h, \Phi(A^*) \Phi(A) h \rangle. \]

Consequently, for any \( A \in D_\Phi \) and \( B \overline{\Phi} h \in \mathfrak{A} \overline{\Phi} \mathcal{H} \) we have

\[ \sigma_\Phi(A) V_\Phi V_\Phi^* B \overline{\Phi} h = A \overline{\Phi} \Phi(B) h = 1 \overline{\Phi} \Phi(A) \Phi(B) h = 1 \overline{\Phi} \Phi(AB) h = V_\Phi V_\Phi^* \sigma_\Phi(A) B \overline{\Phi} h, \]

where we have used the property of the multiplicative domain \( \Phi(A) \Phi(B) = \Phi(AB) \) (see [13]). Conversely, if \( \sigma_\Phi(A) V_\Phi V_\Phi^* = V_\Phi V_\Phi^* \sigma_\Phi(A) \) then

\[ \Phi(A^* A) = V_\Phi^* \sigma_\Phi(A^* A) V_\Phi = V_\Phi^* \sigma_\Phi(A^*) \sigma_\Phi(A) V_\Phi V_\Phi^* V_\Phi = V_\Phi^* \sigma_\Phi(A^*) V_\Phi^* \sigma_\Phi(A) V_\Phi = \Phi(A^*) \Phi(A), \]

and this completes the proof of (4.2).

It is easily seen from (4.2) that \( \Phi \) is a *-homomorphism if, and only if, \( V_\Phi \) is a unitary operator.

The next steps provides some simple applications of the Stinespring representation of ucp-maps.

Let \( \mathfrak{A} \) be a concrete C*-subalgebra with unit of \( \mathcal{B}(\mathcal{H}) \) and \( \Phi : \mathfrak{A} \to \mathfrak{A} \) a ucp-map. By the Stinespring's theorem we obtain a triple \( (V_0, \sigma_1, \mathcal{H}_1) \), with \( \mathcal{H}_1 = \mathfrak{A} \overline{\Phi} \mathcal{H} \) such that \( \Phi(A) = V_0^* \sigma_1(A) V_0 \) for all \( A \in \mathfrak{A} \). Moreover the application \( \Phi_1 : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_1) \) defined by \( \Phi_1(A) := \sigma_1(\Phi(A)) \), for \( A \in \mathfrak{A} \), is a ucp-map because it is a composition of ucp-maps. By applying the Stinespring's theorem to \( \Phi_1 \), we have a new triple \( (V_1, \sigma_2, \mathcal{H}_2) \), with \( \mathcal{H}_2 = \mathfrak{A} \overline{\Phi} \mathcal{H}_1 \) such that \( \Phi_1(A) = V_1^* \sigma_2(A) V_1 \) for all \( A \in \mathfrak{A} \). So, iterating this procedure we obtain, for each natural number \( n \geq 1 \), a ucp-map \( \Phi_n : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_n) \) such that

\[ \Phi_n(A) = \sigma_n(\Phi(A)) \quad \text{for} \quad A \in \mathfrak{A}, \tag{4.3} \]
and a new triple \((V_n, \sigma_{n+1}, H_{n+1})\), where \(H_{n+1} = \mathcal{A} \otimes \phi_n H_n\), and an isometry \(V_n : H_n \rightarrow H_{n+1}\) such that \(\Phi_n(A) = V_n^* \sigma_{n+1}(A)V_n\) for all \(A \in \mathcal{A}\).

Now we prove the following Stinespring-type theorem (see [14]):

**Proposition 4.1.** — Let \(\mathcal{A}\) be a concrete C*-algebra with unit of \(\mathcal{B}(H)\) and \(\Phi : \mathcal{A} \rightarrow \mathcal{A}\) a ucp-map. There exists an injective representation \((\pi_\infty, H_\infty)\) of \(\mathcal{A}\) and a linear isometry \(V_\infty\) on the Hilbert Space \(H_\infty\) such that

\[
\pi_\infty(\Phi(A)) = V_\infty^* \pi_\infty(A) V_\infty \quad \text{for} \quad A \in \mathcal{A}.
\]

Furthermore, \(A \in \mathcal{D}_\Phi\) if, and only if, \(V_\infty V_\infty^* \pi_\infty(A) = \pi_\infty(A) V_\infty V_\infty^*\).

**Proof.** We consider for each natural number \(n\) the ucp-map \(\Phi_n : \mathcal{A} \rightarrow \mathcal{B}(H_0)\) defined in (4.3) and its Stinespring representation \((V_n, \sigma_{n+1}, H_{n+1})\) with \(H_0 = H\) and \(\sigma_0 = \text{id}\). Then, we obtain a faithful representation \(\pi_\infty : \mathcal{A} \rightarrow \mathcal{B}(H_\infty)\) on the Hilbert space \(H_\infty = \bigoplus_{n \geq 0} H_n\) by defining

\[
\pi_\infty(A) := \bigoplus_{n \geq 0} \sigma_n(A) \quad \text{for} \quad A \in \mathcal{A}.
\]

Now, let \(V_\infty : H_\infty \rightarrow H_\infty\) be the isometry defined by

\[
V_\infty(h_0, h_1, \ldots, h_n) := (0, V_0 h_0, V_1 h_1 \ldots V_n h_n),
\]

for all \(h_n \in H_n\) and \(n \in \mathbb{N}\). Note that the adjoint of \(V_\infty\) is

\[
V_\infty^*(h_0, h_1, \ldots, h_n) = (V_0^* h_1, V_1^* h_2 \ldots V_n^* h_n)
\]

for all \(h_n \in H_n\) and \(n \in \mathbb{N}\). Hence, for any \(n\) and \(h_n \in H_n\) we have

\[
V_\infty^* \pi_\infty(A) V_\infty \bigoplus_{n \geq 0} h_n = \bigoplus_{n \geq 0} \Phi_n(A) h_n = \bigoplus_{n \geq 0} \sigma_n(\Phi(A)) h_n = \pi_\infty(\Phi(A)) \bigoplus_{n \geq 0} h_n.
\]

Finally, the last statement easily follows by 4.2.

In fact if \(A \in \mathcal{D}_\Phi\) then \(A \in \mathcal{D}_{\Phi_n}\) for all natural number \(n\), where \(\mathcal{D}_{\Phi_n}\) is the multiplicative domain of the ucp-map (4.3), then

\[
V_\infty V_\infty^* \in \pi_\infty(\bigcap_{n \geq 0} \mathcal{D}_{\Phi_n})' \subset \pi_\infty(\mathcal{D}_\Phi)'.
\]

\[\square\]

We have the following remark on the existence of a representation of a quantum dynamical system:

**Remark 4.2.** — Let \((\mathcal{M}, \Phi)\) be a quantum dynamical system. The injective representation \(\pi_\infty(A) : \mathcal{M} \rightarrow \mathcal{B}(H_\infty)\) defined in proposition 4.1 is normal, since the Stinespring representation \(\sigma_\Phi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{L}_\Phi)\) is a normal map. Then \((\pi_\infty, H_\infty, V_\infty)\) is a representation of the quantum dynamical system \((\mathcal{M}, \Phi)\).

4.1. Dilation of a quantum dynamical system. We use the results of the previous section to analyze the problem of dilation of quantum dynamical systems.

Consider a ucp-map \(\Phi : \mathcal{A} \rightarrow \mathcal{A}\) with \(\mathcal{A}\) a concrete C*-algebra with unit of \(\mathcal{B}(H)\). If \((H_\infty, \pi_\infty, V_\infty)\) is the Stinespring representation of Proposition 4.1, then

\[
V_\infty^* \pi_\infty(\mathcal{A}) V_\infty \subset \pi_\infty(\Phi(\mathcal{A})) \subset \pi_\infty(\mathcal{A}).
\]

Hence, we can define a normal ucp-map \(\Phi_\infty : \pi_\infty(\mathcal{A})' \rightarrow \pi_\infty(\mathcal{A})'\) as

\[
\Phi_\infty(B) := V_\infty^* B V_\infty \quad \text{for} \quad B \in \pi_\infty(\mathcal{A})'.
\]

Clearly we have that \(\Phi_\infty(\pi_\infty(A)) = \pi_\infty(\Phi(A))\) for all \(A \in \mathcal{A}\).

Now, if \((\hat{V}, \hat{H}, \hat{Z})\) is minimal unitary dilation of the isometry \(V_\infty : H_\infty \rightarrow H_\infty\), then by Proposition 3.1 there is a C*-algebra with unit \(\hat{\mathcal{A}}\) of \(\mathcal{B}(\hat{H})\) such that:

(a) \(Z \pi_\infty(\hat{\mathcal{A}}) Z^* \subset \hat{\mathcal{A}}\),
(b) \(Z^* \hat{\mathcal{A}} Z = \pi_\infty(\mathcal{A})\),
(c) \(\hat{V}^* \hat{\mathcal{A}} \hat{V} \subset \hat{\mathcal{A}}\).
Furthermore, we have a *-homomorphism \( \hat{\Phi} : \hat{A} \to \hat{A} \) defined by
\[
\hat{\Phi}(X) = \hat{V}^*X\hat{V} \quad \text{for} \quad X \in \hat{A},
\]
(4.6)
such that for any \( A \in \mathfrak{A} \), \( X \in \hat{A} \) and any natural number \( n \) we have:
\[
\pi_{\infty}(\Phi^n(A)) = Z^*\hat{\Phi}^n(ZAZ^*)Z,
\]
and
\[
\pi_{\infty}(\Phi^n(Z^*XZ)) = Z^*\hat{\Phi}^n(X)Z.
\]
In conclusion, it is straightforward to prove that \((\hat{A}'', \Theta, \hat{H}, Z)\), with \( \Theta : \hat{A}''' \to \hat{A}''' \) the normal *-homomorphism
\[
\Theta(X) := \hat{V}^*X\hat{V} \quad \text{for} \quad X \in \hat{A}'',
\]
is a dilation of the quantum dynamical system \((\pi_{\infty}(\mathfrak{M}), \Phi_{\infty})\) above defined.

**4.2. The deterministic part of a quantum dynamical system and its dilations.** In this section we study which relationships there are between the dilations and the deterministic part of a quantum dynamical system.

Let \( \Phi : \mathfrak{A} \to \mathfrak{A} \) be a ucp-map as described in previous section and \( C^\ast(S) \) the C*-algebra generated by the operator systems \( S \) defined in (3.7).

We recall that \( S \subset A^\ast(S) \subset C^\ast(S) \subset \mathfrak{B}(\hat{H}) \) where \( \hat{H} = H_\infty \oplus l^2(FH_\infty) \) with \( F = I - V_\infty V_\infty^* \). By relationships (a), (b) and (c) of Section 3.3, we can define a *-homomorphism \( \Lambda : C^\ast(S) \to C^\ast(S) \) as follows:
\[
\Lambda(X) = \hat{V}^*X\hat{V} \quad \text{for} \quad X \in C^\ast(S).
\]
(4.7)
Furthermore, we have a ucp-map \( \mathcal{E} : C^\ast(S) \to \mathfrak{A} \) such that
\[
\pi_{\infty}(\mathcal{E}(X)) = Z^*XZ \quad \text{for} \quad X \in C^\ast(S)
\]
and for any natural number \( n \in \mathbb{N} \)
\[
\mathcal{E} \circ \Lambda^n = \Phi^n \circ \mathcal{E}.
\]
Hence, we have the following diagram:
\[
\begin{array}{ccc}
C^\ast(S) & \xrightarrow{\Lambda^n} & C^\ast(S) \\
\mathcal{E} \downarrow & & \downarrow \mathcal{E} \\
\mathfrak{A} & \xrightarrow{\Phi^n} & \mathfrak{A}
\end{array}
\]
where \( \mathcal{E}(ZAZ^*) = A \) for all \( A \in \mathfrak{A} \).

We consider now the C*-algebra \( \mathcal{D} := \bigcap_{n \geq 0} D_{\Phi^n} \) where the set \( D_{\Phi^n} \) is the multiplicative domain of the ucp-map \( \Phi^n : \mathfrak{A} \to \mathfrak{A} \) for all natural numbers \( n \). The restriction of \( \Phi \) to \( \mathcal{D} \) is a *-homomorphism \( \Phi_{\infty} : \mathcal{D} \to \mathcal{D} \) of C*-algebras. It is said to be the **deterministic part** of the ucp-map \( \Phi : \mathfrak{A} \to \mathfrak{A} \).

The *-homomorphism \( \Lambda \) defined above is related to the deterministic part of \( \Phi \) in the following way:

**PROPOSITION 4.3.** — There is an injective *-homomorphism \( i : \mathcal{D} \to C^\ast(S) \) such that for each natural number \( n \) and \( D \in \mathcal{D} \) we have:
\[
\mathcal{E}(\Lambda^n(i(D))) = \Phi^n(D)
\]
and
\[
\Lambda^n(i(D)) = i(\Phi^n(D)).
\]
Proof. — Since \( F \in \pi_\infty(D \Phi) \subset \pi_\infty(D)' \) by Proposition 4.1, the map \( \Xi : \mathcal{D} \rightarrow \mathcal{B}(l^2(FH_\infty)) \) defined by

\[
\Xi(D) = \sum_{k \geq 0} \Pi_k^* F \pi_\infty(\Phi_r^{- (k + 1)}(D)) F \Pi_k \quad D \in \mathcal{D}
\]

is a representation. Furthermore for any \( D \in \mathcal{D} \) we have that \( \Xi(D) \) belongs to \( \mathcal{X}_0 \), the linear space generated by the napla operators defined in Proposition 3.4, since \( \Pi_k^* F \pi_\infty(\Phi_r^{- (k + 1)}(D)) F \Pi_k \) is the napla operator \( \Delta_k(\pi_\infty(\Phi_r^{- (k + 1)}(D)), \alpha, \beta) \) with the strings \( \alpha = \beta = (0, I) \).

We define a *-homomorphism \( i : \mathcal{D} \rightarrow C^*(\mathcal{S}) \) as follows

\[
i(D) = \pi_\infty(D) \oplus \Xi(D) \quad \text{for} \quad D \in \mathcal{D},
\]

and by relationship (2.5) we obtain that

\[
\Lambda^n(i(D)) = \begin{vmatrix} V^n \pi_\infty(D)V^n, & V^n \pi_\infty(D)C_n \\ C_n^* \pi_\infty(D)V^n, & C_n^* \pi_\infty(D)C_n + W^n \Xi(D)W^n \end{vmatrix}.
\]

It is straightforward to prove that

\[
C_n^* \pi_\infty(D)C_n + W^n \Xi(D)W^n = \Xi(\Phi^n(D))
\]

and \( C_n^* \pi_\infty(D)V^n = 0 \), since by relationship (2.8) we have

\[
FV^{(n-k)^*} \pi_\infty(D)V^n = \pi_\infty(\Phi^{(n-k)}(D))FV^k = 0
\]

for all \( 1 \leq k \leq n \), completing the proof.

Finally, we observe that there is the following relationship between dilations and the deterministic part of a quantum dynamical system:

If \( (\mathcal{M}, \Theta, \mathcal{K}, \mathcal{Z}) \) is any dilation of quantum dynamical system \( (\mathcal{M}, \Phi) \), then for any natural number \( n \) and \( D \in \mathcal{D} \) we have:

\[
\Theta^n(\mathcal{Z}D\mathcal{Z}^*)\mathcal{Z} = Z\Phi^n(D),
\]

since if \( Y = \Theta^n(\mathcal{Z}D\mathcal{Z}^*)\mathcal{Z} - Z\Phi^n(D) \), then \( Y^*Y = 0 \).

5. Ergodic properties

Let \( \mathfrak{A} \) be a concrete C*-algebra of \( \mathcal{B}(\mathcal{H}) \) with unit, \( \Phi : \mathfrak{A} \rightarrow \mathfrak{A} \) a ucp-map and \( \varphi \) a state on \( \mathfrak{A} \) such that \( \varphi \circ \Phi = \varphi \). We recall that \( \varphi \) is an ergodic state, relative to the ucp-map \( \Phi \) (see [10]), if for each \( A, B \in \mathfrak{A} \)

\[
\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n (\varphi(A\Phi^k(B)) - \varphi(A)\varphi(B)) = 0,
\]

and that \( \varphi \) is weakly mixing if for each \( A, B \in \mathfrak{A} \)

\[
\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n |\varphi(A\Phi^k(B)) - \varphi(A)\varphi(B)| = 0.
\]

By Proposition 4.1 we can assume that \( \mathfrak{A} \) is a concrete C*-algebra of \( \mathcal{B}(\mathcal{H}) \), and that there is an isometry \( V \) on \( \mathcal{H} \) such that:

\[
\Phi(A) = V^* AV \quad \text{for} \quad A \in \mathfrak{A}.
\]

Let \( (\hat{V}, \hat{\mathcal{H}}, \mathcal{Z}) \) be the minimal unitary dilation of \( (V, \mathcal{H}) \) defined in (2.4), let \( \hat{\mathfrak{A}} \) be the C*-algebra included in \( \mathcal{B}(\hat{\mathcal{H}}) \) defined in Proposition 3.1, and let \( \hat{\Phi} : \hat{\mathfrak{A}} \rightarrow \hat{\mathfrak{A}} \) be the ucp-map defined in (4.6).

Proposition 5.1. — If the ucp-map \( \Phi \) admits a \( \varphi \)-adjoint and \( \varphi \) is an ergodic state, then:

\[
\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N |\varphi(Z^*X\Phi^k(Y)Z) - \varphi(Z^*XZ\varphi(Z^*YZ))| = 0
\]
for all $X, Y \in \mathfrak{K}$, while if $\varphi$ is weakly mixing, then:

$$
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} |\varphi(Z^k X \hat{\Phi}^k(Y)Z) - \varphi(Z^k XZ)\varphi(Z^k YZ)| = 0
$$

for all $X, Y \in \mathfrak{K}$.

The proof of this proposition is a straightforward consequence of the next lemma.

To this purpose, we make a preliminary observation. Recall that $\mathcal{H} = \mathcal{H} \oplus l^2(F\mathcal{H})$ and that, writing an element $X$ of $B(\mathcal{H})$ in matrix representation

$$
X = \begin{bmatrix}
X_{1,1} & X_{1,2} \\
X_{2,1} & X_{2,2}
\end{bmatrix},
$$

the following relationship holds:

$$
\varphi(Z^k X \hat{\Phi}^k(Y)Z) = \varphi(X_{1,1} \Phi^k(Y_{1,1})) + \varphi(X_{1,2} C(k)^* Y_{1,1} V^k) + \varphi(X_{1,2} W^k Y_{2,1} V^k).
$$

**Lemma 5.2.** — Let $X \in \mathcal{A}^*(\mathcal{S})$, the $*$-algebra generated by the operator system $\mathcal{S}$ defined in (3.7) and $Y \in \mathfrak{K}$. The following relations hold:

(a) If $\varphi$ is an ergodic state then we have:

$$
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(X_{1,2} C(k)^* Y_{1,1} V^k + X_{1,2} W^k Y_{2,1} V^k) = 0, \quad (5.1)
$$

(b) If $\varphi$ is weakly mixing then we have:

$$
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} |\varphi(X_{1,2} C(k)^* Y_{1,1} V^k + X_{1,2} W^k Y_{2,1} V^k)| = 0. \quad (5.2)
$$

**Proof.** — Since $X \in \mathcal{A}^*(\mathcal{S})$, we can assume without loss of generality that $X_{1,2} = A\Gamma(\gamma) \Delta_m(B, \alpha, \beta)$ with $A, B \in \mathfrak{A}$ and $\alpha, \beta, \gamma$ strings of $\mathfrak{A}$. Then we can write

$$
X_{1,2} = \begin{cases}
A(\gamma|F|\alpha)B(\beta|F\Pi_{\beta+m}^\gamma) & \text{if } \gamma - 1 = \hat{\alpha} + m \\
0 & \text{elsewhere}
\end{cases} \quad (5.3)
$$

since

$$
X_{1,2} = A(\gamma|F\Pi_{\gamma-1} \Pi_{\hat{\alpha}+m}^\gamma F|\alpha)B(\beta|F\Pi_{\beta+m}^\gamma).
$$

Observe that we can find a natural number $k_0$ such that the relation

$$
X_{1,2} W^k Y_{2,1} V^k = 0 \quad (5.4)
$$

holds for each $k > k_0$. In fact

$$
W^k (\xi_0, \xi_1, \ldots, \xi_n, \ldots) = (0, 0, \ldots, 0, \xi_0, \xi_1, \ldots),
$$

for all vectors $(\xi_0, \xi_1, \ldots, \xi_n, \ldots) \in l^2(F\mathcal{H})$; so $\Pi_{\beta+m} W^k = 0$ for all $k > \hat{\beta} + m$. Then by equation (5.4) it follows that

$$
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(X_{1,2} C(k)^* Y_{1,1} V^k + X_{1,2} W^k Y_{2,1} V^k) = \lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(X_{1,2} C(k)^* Y_{1,1} V^k),
$$

Hence we have to compute only $\varphi(X_{1,2} C(k)^* Y_{1,1} V^k)$. Notice that

$$
X_{1,2} C(k)^* Y_{1,1} V^k = A(\gamma|F|\alpha)B(\beta|F\Pi_{\beta+m} C(k)^* Y_{1,1} V^k
$$

by relationship (5.3), and that

$$
\Pi_{\beta+m} C(k)^* = F V^{(k-\hat{\beta}-m-1)^*} \quad \text{for } k > \hat{\beta} + m,
$$
by relationship (2.7). It follows that

\[ X_{1,2}C(k)^* Y_{1,1} V^k = A(\gamma | F(\alpha)) B(\beta | F V^{(k-\bar{\beta}+m+1)^*} Y_{1,1} V^k \]

\[ = A(\gamma | F(\alpha)) B(\beta | F \Phi^{(k-\bar{\beta}+1)}(Y_{1,1}) V^{\bar{\beta}+m+1} ). \]

Since \( \bar{\gamma} = \alpha + m + 1 \), we have \( A(\gamma | F(\alpha)) B(\beta \in \mathfrak{A}(\bar{\beta} + m + 1) \) by relationship (3.1). Hence there is a string \( \vartheta \) of \( \mathfrak{A} \) with \( \vartheta = \bar{\beta} + m + 1 \) and an operator \( R \in \mathfrak{A} \), such that \( A(\gamma | F(\alpha)) B(\beta) = R(\vartheta) \). So we can write

\[ X_{1,2}C(k)^* Y_{1,1} V^k = R(\vartheta | F \Phi^{(k-\bar{\beta}+1)}(Y_{1,1}) V^{\bar{\beta}+m+1} ). \]

If we set \( \vartheta = (n_1, n_2, \ldots, n_r, A_1, A_2, \ldots, A_r) \) then we have \( n_1 + n_2 + \ldots + n_r = \bar{\beta} + m + 1 \) and

\[ R_k = \Phi^{n_1}(A_1 \Phi^{(k-\bar{\beta}+1)}(Y_{1,1})) - \Phi^{n_1-1}(\Phi(A_1) \Phi^{(k-\beta)}(Y_{1,1})) \in \mathfrak{A}. \]

Using the \( \varphi \)-adjoint, we have

\[ \varphi(X_{1,2}C(k)^* Y_{1,1} V^k) = \varphi(\Phi^{n_2}(\Phi^{n_3}(\ldots \Phi^{n_r} \Phi^{(R)(A_1) \ldots} \Phi(A_2) R_k)}. \]

In fact,

\[ \varphi(X_{1,2}C(k)^* Y_{1,1} V^k) = \varphi(R \Phi^{n_r}(A_r \Phi^{n_{r-1}}(A_{r-1} \ldots \Phi^{n_2} \Phi^{(R)(A_1) \ldots} \Phi(A_2) R_k))) \]

\[ = \varphi(\Phi^{n_r}(R) A_r \Phi^{n_{r-1}}(A_{r-1}(\ldots \Phi^{n_2} \Phi^{(R)(A_1) \ldots} \Phi(A_2) R_k))) \]

\[ = \varphi(\Phi^{n_r}(\Phi^{n_3}(\ldots \Phi^{n_r} \Phi^{(R)(A_1) \ldots} \Phi(A_2) R_k))) \]

\[ = \varphi(\Phi^{n_2}(\Phi^{n_3}(\ldots \Phi^{n_r} \Phi^{(R)(A_1) \ldots} \Phi(A_2) R_k))). \]

and replacing \( R_k \) we obtain that

\[ \Phi^{n_2}(\Phi^{n_3}(\ldots \Phi^{n_r} \Phi^{(R)(A_1) \ldots} \Phi(A_2) R_k) = \Phi^{n_2}(\Phi^{n_3}(\ldots \Phi^{n_r} \Phi^{(R)(A_1) \ldots} \Phi(A_2) R_k) \ldots \Phi^{n_1}(A_1 \Phi^{(k-\beta+1)}(Y_{1,1}))). \]

Therefore

\[ \varphi(X_{1,2}C(k)^* Y_{1,1} V^k) = \varphi(\Phi^{n_1}(\Phi^{n_2}(\ldots \Phi^{n_{r-1}}(\Phi^{n_r}(R) A_r) \ldots) \Phi(A_1) \Phi^{(k-\beta+1)}(Y_{1,1}))) \]

\[ \varphi(\Phi^{n_1}(\Phi^{n_2}(\ldots \Phi^{n_{r-1}}(\Phi^{n_r}(R) A_r) \ldots) \Phi(A_1) \Phi^{(k-\beta)}(Y_{1,1}))). \]

Now, assume that \( \varphi \) is ergodic. Then we have that

\[ \lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \varphi(\Phi^{n_1}(\Phi^{n_2}(\ldots \Phi^{n_{r-1}}(\Phi^{n_r}(R) A_r) \ldots) \Phi(A_1) \Phi^{(k-\beta+1)}(Y_{1,1}))) \]

\[ \varphi(\Phi^{n_1}(\Phi^{n_2}(\ldots \Phi^{n_{r-1}}(\Phi^{n_r}(R) A_r) \ldots) \Phi(A_1) \Phi^{(k-\beta)(Y_{1,1}))), \]

and that

\[ \lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \varphi(\Phi^{n_1}(\Phi^{n_2}(\ldots \Phi^{n_{r-1}}(\Phi^{n_r}(R) A_r) \ldots) \Phi(A_1) \Phi^{(k-\beta)(Y_{1,1}))) \]

\[ \varphi(\Phi^{n_1}(\Phi^{n_2}(\ldots \Phi^{n_{r-1}}(\Phi^{n_r}(R) A_r) \ldots) \Phi(A_1) \Phi^{(k-\beta)}(Y_{1,1}))), \]

\[ \varphi(\Phi^{n_1}(\Phi^{n_2}(\ldots \Phi^{n_{r-1}}(\Phi^{n_r}(R) A_r) \ldots) \Phi(A_1) \Phi^{(k-\beta)}(Y_{1,1}))). \]
Thus
\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(X_{1,2}C(k)^*Y_{1,1}V^k) = 0,
\]
completing the proof of item (a).

In the weakly mixing case, using relationship (5.5) we obtain:
\[
|\varphi(X_{1,2}C_k^*Y_{1,1}V^k)| = |\varphi(T\Phi^{n_1}(A_1)\Phi^{(k-\beta-1)}(Y_{1,1})) - \varphi(T\Phi^{n_1-1}(\Phi(A_1)\Phi^{k-\beta})(Y_{1,1}))|,
\]
where \(T = \Phi^{n_2}R\cdots\Phi^{n_r}(R)\cdots\Phi^{n_2}A_2\).

Adding and subtracting the element \(\varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})\) we can write:
\[
|\varphi(X_{1,2}C_k^*Y_{1,1}V^k)| \leq |\varphi(T\Phi^{n_1}(A_1)\Phi^{(k-\beta-1)}(Y_{1,1})) - \varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})|
+ |\varphi(T\Phi^{n_1-1}(\Phi(A_1)\Phi^{(k-\beta)})(Y_{1,1})) - \varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})|.
\]
Moreover
\[
|\varphi(T\Phi^{n_1-1}(\Phi(A_1)\Phi^{(k-\beta)})(Y_{1,1})) - \varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})| = |\varphi(\Phi^{n_1-1}(T)\Phi(A_1)\Phi^{(k-\beta)})(Y_{1,1})) - \varphi(\Phi^{n_1-1}(T)\Phi(A_1))\varphi(Y_{1,1})|,
\]
and by the weakly mixing properties we obtain:
\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} |\varphi(T\Phi^{n_1}(A_1)\Phi^{(k-\beta-1)}(Y_{1,1})) - \varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})| = 0,
\]
and
\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} |\varphi(\Phi^{n_1-1}(T)\Phi(A_1)\Phi^{(k-\beta)})(Y_{1,1})) - \varphi(\Phi^{n_1-1}(T)\Phi(A_1))\varphi(Y_{1,1})| = 0
\]
completing the proof of item (b).

Finally, the proof of proposition Proposition 5.1 is a simple consequence of this lemma since the C*-algebra \(\mathfrak{A}\) is included in \(C^*(S)\), the norm closure of \(*\)-algebra \(\mathcal{A}(S)\).

It is clear that Proposition 5.1 can be extended to a quantum dynamical system \((\mathfrak{M}, \Phi)\) with \(\varphi\) a normal faithful state on \(\mathfrak{M}\).

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**References**


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