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Abstract

Let $(\Omega, \Sigma, \mathrm{P})$ be a nonatomic probability space. If $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ is a filtration of Ω and if $f = (f_n)_{n\in\mathbb{Z}_+}$ is a stochastic process on Ω such that f_n is integrable for all $n \in \mathbb{Z}_+$, the optional projection ${}^{O(\mathcal{F})}f = ({}^{O(\mathcal{F})}f_n)_{n\in\mathbb{Z}_+}$ of $f = (f_n)_{n\in\mathbb{Z}_+}$ is defined by ${}^{O(\mathcal{F})}f_n = \mathrm{E}[\,f_n \,|\, \mathcal{F}_n\,]$. Given a Banach function space X over Ω and $r \in [1, \infty)$, let $X[\ell_r]$ denote the Banach space consisting of all processes $f = (f_n)_{n\in\mathbb{Z}_+}$ such that $\left(\sum_{n=0}^{\infty}|f_n|^r\right)^{1/r} \in X$, and let $\|f\|_{X[\ell_r]} = \left\|\left(\sum_{n=0}^{\infty}|f_n|^r\right)^{1/r}\right\|_X$ for $f = (f_n)_{\in\mathbb{Z}_+} \in X[\ell_r]$. One of the main results gives a necessary and sufficient condition on X for the inequality $\|{}^{O(\mathcal{F})}f\|_{X[\ell_r]} \le C\|f\|_{X[\ell_r]}$ to be valid for all $f = (f_n)_{n\in\mathbb{Z}_+} \in X[\ell_r]$.

Sur quelques inégalités pour la projection optionnelle et la projection prévisible d'un processus de paramètre discret

Résumé

Soit $(\Omega, \Sigma, \mathrm{P})$ un espace de probabilité non atomique. Si $\mathcal{F} = (\mathcal{F}_n)$ est une filtration de Ω et si $f = (f_n)_{n\in Z}$ est un processus stochastique sur Ω tel que f_n est intégrable pour tout $n \in \mathbb{Z}_+$, la projection optionnelle ${}^{O(\mathcal{F})}f = ({}^{O(\mathcal{F})}f_n)_{n\in\mathbb{Z}_+}$ de $f = (f_n)_{n\in\mathbb{Z}_+}$ est définie par ${}^{O(\mathcal{F})}f_n = \mathrm{E}[\,f_n \,|\, \mathcal{F}_n\,]$. Étant donné un espace de fonction de Banach X sur Ω et $r \in [1, \infty)$, on laisse $X[\ell_r]$ désigner l'espace de Banach constitué de tous les processus $f = (f_n)_{n\in\mathbb{Z}_+}$ tels que $\left(\sum_{n=0}^{\infty}|f_n|^r\right)^{1/r} \in X$, et on laisse $\|f\|_{X[\ell_r]} = \left\|\left(\sum_{n=0}^{\infty}|f_n|^r\right)^{1/r}\right\|_X$ pour $f = (f_n)_{\in\mathbb{Z}_+} \in X[\ell_r]$. L'un des principaux résultats donne une condition nécessaire et suffisante sur X pour que l'inégalité $\|{}^{O(\mathcal{F})}f\|_{X[\ell_r]} \le C\|f\|_{X[\ell_r]}$ soit valable pour tout $f = (f_n)_{n\in\mathbb{Z}_+} \in X[\ell_r]$.

1. Introduction

It is well known that each of the optional projection and the predictable projection of a stochastic process plays an essential role in the theory of continuous parameter martingales. On the other hand, it is rare to use the terms "optional projection" and "predictable projection" in the theory of discrete parameter martingales. In fact, the definitions of these projections of a discrete parameter process are so simple and natural that they are often used without being given specific names. However, these projections are certainly important. Burkholder, Davis, and Gundy [5] used an inequality involving the

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predictable projection of a discrete parameter process in order to prove the boundedness of some operators on spaces of martingales, and Stein [28] established an inequality involving the optional projection of a discrete parameter process in connection with the general Littlewood–Paley theory. Moreover, Delbaen and Schachermayer [8] established an inequality for the predictable projection of a discrete parameter adapted process with an application in the field of mathematical finance in mind (cf. [7]).

In this paper, we study some inequalities for the optional projection and the predictable projection of a discrete parameter process.

Let $(\Omega, \Sigma, \mathrm{P})$ be a nonatomic probability space. By a *filtration* of Ω, we mean a sequence $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ of sub-σ-algebras of Σ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{Z}_+$. We denote by $\mathbb{F}$ the collection of all filtrations of Ω, and adopt the convention that $\mathcal{F}_{-1} = \mathcal{F}_0$ for every $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$.

By a *process*, we mean a sequence of random variables on Ω indexed by the set $\mathbb{Z}_+$ of nonnegative integers, and we adopt the convention that $f_{-1} \equiv 0$ for every process $f = (f_n)_{n\in\mathbb{Z}_+}$.

Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$. A process $f = (f_n)_{n\in\mathbb{Z}_+}$ is said to be $\mathcal{F}$*-adapted* if f_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{Z}_+$, and is said to be $\mathcal{F}$*-predictable* if f_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \mathbb{Z}_+$.

We denote by $\mathbb{P}$ the collection of all processes $f = (f_n)_{n\in\mathbb{Z}_+}$ such that $f_n \in L_1(\Omega)$ for all $n \in \mathbb{Z}_+$. Given $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, we denote by $\mathbb{P}(\mathcal{F})$ the collection of all $\mathcal{F}$-adapted processes in $\mathbb{P}$.

Let $f = (f_n) \in \mathbb{P}$ and $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$. The *optional projection* ${}^{O(\mathcal{F})}f = ({}^{O(\mathcal{F})}f_n)$ and the *predictable projection* ${}^{P(\mathcal{F})}f = ({}^{P(\mathcal{F})}f_n)$ of $f = (f_n)$ are defined by

$$ {}^{O(\mathcal{F})}f_n = \mathrm{E}[\, f_n \,|\, \mathcal{F}_n \,], \qquad n \in \mathbb{Z}_+, $$

and

$$ {}^{P(\mathcal{F})}f_n = \mathrm{E}[\, f_n \,|\, \mathcal{F}_{n-1} \,], \qquad n \in \mathbb{Z}_+, $$

respectively (cf. [9, p. 115]).

Let $a \in [1, \infty]$ and $r \in [1, \infty]$. For each process $f = (f_n)$, we let

$$ \|f\|_{L_a[\ell_r]} = \begin{cases} \left\| \left(\sum_{n=0}^{\infty} |f_n|^r \right)^{1/r} \right\|_{L_a} & \text{if } r \in [1, \infty), \\ \left\| \sup_{0\le n<\infty} |f_n| \right\|_{L_a} & \text{if } r = \infty, \end{cases} $$

with the convention that $\|x\|_{L_a} = \infty$ for a random variable x which is not in L_a. We let $L_a[\ell_r]$ denote the set of all processes $f = (f_n)$ for which $\|f\|_{L_a[\ell_r]} < \infty$. It is then easily seen that $L_a[\ell_r]$ is a Banach space. In [28, Chapter IV, Section 3], Stein showed that if $a \in (1, \infty)$, the inequality

$$ \|{}^{O(\mathcal{F})}f\|_{L_a[\ell_2]} \le \kappa_a \|f\|_{L_a[\ell_2]}, $$

holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$, where κ_a is a positive constant depending only on a. His method also applies to show that if $a \in (1, \infty)$ and $r \in [1, \infty]$, the inequality

$$\|{}^{O(\mathcal{F})}f\|_{L_a[\ell_r]} \le K_{a,r}\|f\|_{L_a[\ell_r]} \tag{1.1}$$

holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$, where $K_{a,r}$ is a positive constant depending only on a and r. Furthermore, it is easy to see that (1.1) is also valid when $a = r = 1$ or $a = r = \infty$. An explicit value for the constant $K_{a,r}$ was given by Dilworth [11]. He showed that (1.1) holds with

$$K_{a,r} = \begin{cases} \left(\frac{a}{r}\right)^{1/r} & \text{if } 1 \le r \le a < \infty, \\ \left(\frac{a'}{r'}\right)^{1/r'} & \text{if } 1 < a \le r \le \infty, \end{cases}$$

where a' and r' denote the conjugate exponents of a and r, respectively. It is easily checked that if $a \in (1, \infty)$, then

$$\max_{1 \le r \le a} K_{a,r} = K_{a,1} = a.$$

Furthermore, if $1 < a \le r \le \infty$, then $K_{a,r} = K_{a',r'}$, and hence

$$\max_{a \le r \le \infty} K_{a,r} = \max_{1 \le r \le a'} K_{a',r} = a'.$$

Therefore (1.1) holds with $K_{a,r}$ replaced by $K_a := a \vee a' = \max\{a, a'\}$.

As for the predictable projections, Lépingle [24] showed that the inequality

$$\|{}^{P(\mathcal{F})}f\|_{L_1[\ell_2]} \le 2\|f\|_{L_1[\ell_2]}$$

holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and then Delbaen and Schachermayer [8] showed that if $1 \le a \le r \le \infty$, the inequality

$$\|{}^{P(\mathcal{F})}f\|_{L_a[\ell_r]} \le 2\|f\|_{L_a[\ell_r]} \tag{1.2}$$

holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}(\mathcal{F})$. They also showed that in the case where $a = 1$ and $r = \infty$, the constant 2 in (1.2) is sharp. Moreover, Osękowski [26] showed that the inequality

$$\|{}^{P(\mathcal{F})}f\|_{L_1[\ell_r]} \le 2^{(r-1)/r}\|f\|_{L_1[\ell_r]}$$

holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and that the constant $2^{(r-1)/r}$ is sharp.

Johnson et al. [13] considered an analogous inequality in a more general setting. Let X be a Banach function space over Ω (see Definition 2.1), and let $1 \le r \le \infty$. For each process $f = (f_n)$, we let

$$\|f\|_{X[\ell_r]} = \begin{cases} \left\| \left(\sum_{n=0}^{\infty} |f_n|^r \right)^{1/r} \right\|_X & \text{if } r \in [1, \infty), \\ \left\| \sup_{0 \le n < \infty} |f_n| \right\|_X & \text{if } r = \infty, \end{cases} \tag{1.3}$$

with the convention that $\|x\|_X = \infty$ if x is a random variable which is not in X; and we define $X[\ell_r]$ to be the set of all processes $f = (f_n)$ for which $\|f\|_{X[\ell_r]} < \infty$. It is easily seen that $X[\ell_r]$ is a Banach space.

In the case where X is a rearrangement-invariant Banach function space over $[0, 1]$ (see Definition 2.3), Johnson et al. [13] showed that if $0 < \alpha_X \le \beta_X < 1$, the inequality

$$\|{}^{O(\mathcal{F})}f\|_{X[\ell_2]} \le C\|f\|_{X[\ell_2]}$$

holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$, where C is a positive constant depending only on X and where α_X and β_X denote the Boyd indices of X (see Subsection 2.5).

In this paper, we give not only a sufficient condition but also a necessary condition for the inequalities

$$\|{}^{O(\mathcal{F})}f\|_{X[\ell_r]} \le C\|f\|_{X[\ell_r]} \tag{1.4}$$

and

$$\|{}^{P(\mathcal{F})}f\|_{X[\ell_r]} \le C\|f\|_{X[\ell_r]}. \tag{1.5}$$

to be valid.

In Section 3, we consider inequality (1.4) for $f = (f_n) \in \mathbb{P}$. In Theorem 3.1, we give a necessary and sufficient condition for (1.4) to be valid for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$ *when* $r = 1$; in Theorem 3.2, we give a necessary and sufficient condition for (1.4) to be valid for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$ *when* $r = \infty$; and in Theorem 3.3, we give a necessary and sufficient condition for (1.4) to be valid for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, all $f = (f_n) \in \mathbb{P}$, and all $r \in (1, \infty)$.

In Section 4, we consider inequality (1.5) for $f = (f_n) \in \mathbb{P}(\mathcal{F})$. In Theorem 4.1, we give a necessary and sufficient condition for (1.5) to be valid for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}(\mathcal{F})$ *when* $r = 1$. The condition given there is also necessary and sufficient for (1.5) to be valid for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, all $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and all $r \in (1, \infty]$.

In Section 5, we consider analogous inequalities in a quasi-Banach function space w-X associated with X. Given a Banach function space X, let w-X denote the set of all random variables x for which $\|x\|_{\text{w-}X} := \sup_{\lambda>0} \lambda\|\mathbb{1}_{\{|x|>\lambda\}}\|_X < \infty$, and let w-$X[\ell_r]$ denote the set of all processes $f = (f_n)$ for which $\|f\|_{\text{w-}X[\ell_r]} < \infty$, where

$$\|f\|_{\text{w-}X[\ell_r]} = \begin{cases} \left\|\left(\sum_{n=0}^{\infty}|f_n|^r\right)^{1/r}\right\|_{\text{w-}X} & \text{if } 1 \le r < \infty, \\ \left\|\sup_{0\le n<\infty}|f_n|\right\|_{\text{w-}X} & \text{if } r = \infty. \end{cases} \tag{1.6}$$

Then w-X is a quasi-Banach function space and w-$X[\ell_r]$ is a quasi-Banach space of processes. We consider the inequalities

$$\|{}^{O(\mathcal{F})}f\|_{\text{w-}X[\ell_r]} \le C\|f\|_{\text{w-}X[\ell_r]} \tag{1.7}$$

and

$$\|{}^{P(\mathcal{F})}f\|_{\text{w-}X[\ell_r]} \le C\|f\|_{\text{w-}X[\ell_r]}. \tag{1.8}$$

In Theorem 5.1, we give a necessary and sufficient condition on X for (1.7) to be valid for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$ *when* $r = 1$. The condition given there is also necessary and sufficient for (1.8) to be valid for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}(\mathcal{F})$ *when* $r = 1$. Moreover the condition is necessary and sufficient for (1.7) (resp. (1.8)) to be valid for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, all $f = (f_n) \in \mathbb{P}$ (resp. $f = (f_n) \in \mathbb{P}(\mathcal{F})$), and all $r \in (1, \infty]$.

From Theorems 3.3 and 4.1, it follows that if (1.4) holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, all $f = (f_n) \in \mathbb{P}$, and all $r \in (1, \infty)$, then (1.5) holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, all $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and all $r \in (1, \infty]$. However the converse is false. In contrast, it follows from Theorem 5.1 that (1.7) holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, all $f = (f_n) \in \mathbb{P}$, and all $r \in (1, \infty]$ if and only if (1.8) holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, all $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and all $r \in (1, \infty]$.

2. Preliminaries

In this slightly long section, we fix some notation and give some definitions. We also recall some results used in this paper.

2.1. General notation

Throughout the paper, we assume that the probability space $(\Omega, \Sigma, \mathrm{P})$ is *nonatomic*, i.e., that there is no P-atom in Σ. This assumption is essential and will be used explicitly or implicitly. Recall that $\mathbb{F}$ denotes the collection of all filtrations of Ω and that $\mathbb{P}$ denotes the collection of all processes $f = (f_n)_{n \in \mathbb{Z}_+}$ such that each f_n is integrable. Recall also that if $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, then $\mathbb{P}(\mathcal{F})$ denotes the collection of all $\mathcal{F}$-adapted processes in $\mathbb{P}$.

In addition to Ω, we consider the interval $I := (0, 1]$ as a probability space equipped with Lebesgue measure μ. From this point of view, a measurable function on I is also called a random variable. If $0 \le a \le b \le 1$ and if η is a μ-integrable function on the interval (a, b), we write $\int_a^b \eta(s)\, \mathrm{d}s$ instead of $\int_{(a,b)} \eta(s)\, \mu(\mathrm{d}s)$.

If A is a subset of Ω (resp. I), we denote by $\mathbb{1}_A$ the indicator function of A defined on Ω (resp. I).

We let $L_0(\Omega)$ (resp. $L_0(I)$) denote the linear space consisting of (equivalence classes of) random variables on Ω (resp. I) which are finite a.s. We assume that $L_0(\Omega)$ and $L_0(I)$ are equipped with the topology of convergence in probability (i.e., in measure). For each $p \in \{0\} \cup [1, \infty]$, we often write L_p instead of $L_p(\Omega)$ or $L_p(I)$.

Let $x \in L_0(\Omega)$ and let λ be a real number. We write $\{x > \lambda\}$ as an abbreviation for the set of all $\omega \in \Omega$ such that $x(\omega) > \lambda$; moreover we use analogous abbreviations, such as, $\{\lambda_1 \le x < \lambda_1\}$, $\{x < \infty\}$, and so on. We also use such abbreviations for $\eta \in L_0(I)$.

Let $x \in L_0(\Omega)$. The *nonincreasing rearrangement* of x is the function x^* on $I = (0, 1]$ defined by

$$x^*(t) = \inf\{\lambda > 0 \colon \mathrm{P}\{|x| > \lambda\} \le t\}, \quad t \in I.$$

The nonincreasing rearrangement x^* is a right-continuous nonincreasing function on I whose distribution (with respect to Lebesgue measure) is the same as that of $|x|$; and such a function is unique. The nonincreasing rearrangement η^* of $\eta \in L_0(I)$ is defined in the same way; i.e.,

$$\eta^*(t) = \inf\{\lambda > 0 \colon \mu\{|\eta| > \lambda\} \le t\}, \quad t \in I.$$

2.2. **Banach function spaces and quasi-Banach function spaces**

Let V be a linear space. Recall that a nonnegative real-valued function $\|\cdot\|$ on V is called a *quasi-norm* if it satisfies the following conditions:

(QN1) $\|x\| = 0$ if and only if $x = 0$.

(QN2) $\|ax\| = |a|\|x\|$ for all $x \in V$ and all scalars a.

(QN3) There is a constant $K \ge 1$ such that $\|x + y\| \le K(\|x\| + \|y\|)$ for all $x, y \in V$.

Suppose that V is a quasi-normed space, i.e., that V is equipped with a quasi-norm $\|\cdot\|$. For each $x \in V$ and each $\varepsilon > 0$, let $B(x : \varepsilon) = \{y \in V \colon \|x - y\| < \varepsilon\}$. Then one can define a topology $\mathcal{T}$ on V so that $\mathcal{N}(x) := \{B(x : \varepsilon) \colon \varepsilon > 0\}$ is the neighborhood basis for $x \in V$. Choose $p_0 \in (0, 1]$ so that $2^{(1/p_0)-1} = K$, where K is the constant in (QN3). Then there exists a quasi-norm $\|\cdot\|'$ on V and a constant C which depends only on the value of K such that:

- $\|x\|' \le \|x\| \le C\|x\|'$ for all $x \in V$.
- $\|x + y\|'^{p_0} \le \|x\|'^{p_0} + \|y\|'^{p_0}$ for all $x,\ y \in V$.

Hence one can define a metric d on V by $d(x, y) = \|x - y\|'^{p_0}$, and the topology $\mathcal{T}$ can be metrized by d (see [10, p. 20] or [22, p. 47]). Of course, a Cauchy sequence in V is defined in the same way that a Cauchy sequence in a normed space is defined. If every Cauchy sequence in V converges in V, then V is called a *quasi-Banach space*. Thus V is a quasi-Banach space if and only if the metric space (V, d) is complete.

Now let X and Y be linear topological spaces. We write $Y \hookrightarrow X$ to mean that Y is continuously embedded in X, i.e., that $Y \subset X$ and the inclusion map is continuous. When each of X and Y is a Banach space or a quasi-Banach space, $Y \hookrightarrow X$ if and only if $Y \subset X$ and there is a positive constant c such that $\|x\|_X \le c\|x\|_Y$ for all $x \in Y$.

Definition 2.1. A *Banach function space* over Ω (resp. I) is a Banach space X of (equivalence classes of) random variables on Ω (resp. I) which satisfies the following conditions:

(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$.

(B2) If $|y| \le |x|$ a.s. and $x \in X$, then $y \in X$ and $\|y\|_X \le \|x\|_X$.

(B3) If $(x_n)_{n\in\mathbb{Z}_+}$ is a nonnegative process in X such that $x_n \uparrow x_\infty$ a.s. as $n \uparrow \infty$ and if $\sup_{n\in\mathbb{Z}_+}\|x_n\|_X < \infty$, then $x_\infty \in X$ and $\|x_\infty\|_X = \sup_{n\in\mathbb{Z}_+}\|x_n\|_X$.

A *quasi-Banach function space* over Ω (resp. I) is a quasi-Banach space of (equivalence classes of) random variables on Ω (resp. I) which satisfies (B2), (B3), and the condition that

(Q1) $L_\infty \hookrightarrow X \hookrightarrow L_0$.

We adopt the convention that if X is a Banach function space or a quasi-Banach function space and if x is a random variable which is not in X, then $\|x\|_X = \infty$.

Note that the definition above of a Banach function space is the same as that in [2, Chapter 1] (since $(\Omega, \Sigma, \mathrm{P})$ is a finite measure space).

Of course, Lebesgue spaces, Orlicz spaces, and Lorentz spaces are Banach function spaces. Moreover variable exponent Lebesgue spaces are also Banach function spaces (cf. [1]).

Let X be a Banach function space or a quasi-Banach function space, and let $1 \le r \le \infty$. Recall that the space $X[\ell_r]$ is defined to be the set of all processes $f = (f_n)$ such that $\|f\|_{X[\ell_r]} < \infty$, where $\|f\|_{X[\ell_r]}$ is given by (1.3).

The following facts are immediate consequence of Definition 2.1:

- If $f = (f_n)$ is a nonnegative process such that $f_n \in X$ for all $n \in \mathbb{Z}_+$, then
$$\left\|\varliminf_{n\to\infty} f_n\right\|_X \le \varliminf_{n\to\infty} \|f_n\|_X.$$
- If $f = (f_r)_{r\in[0,\infty)}$ is a process with values in $[0, \infty]$ such that $f_r \uparrow f_s$ a.s. as $r \uparrow s$ (resp. $r \downarrow s$), then
$$\|f_r\|_X \uparrow \|f_s\|_X \quad \text{as } r \uparrow s \ \text{(resp. } r \downarrow s)$$
- If $r \in [1, \infty)$ and $f = (f_n) \in X[\ell_r]$, then $\sum_{n=1}^{\infty}|f_n|^r < \infty$ a.s.
- If $f = (f_n) \in X[\ell_\infty]$, then $\sup_{n\in\mathbb{Z}_+}|f_n| < \infty$ a.s.

Let X be a Banach function space over Ω, and let $\mathbb{B}_X$ denote the closed unit ball of X. For each $y \in L_0$, we let

$$\|y\|_{X'} = \sup_{x \in \mathbb{B}_X} \mathrm{E}\big[\,|xy|\,\big].$$

It is then easy to see that

$$\|y\|_{X'} = \sup_{x \in \mathbb{B}_X \cap L_\infty} \mathrm{E}\big[\,|xy|\,\big].$$

Definition 2.2. Let X be a Banach function space over Ω or I. The *associate space* X' of X consists of those $y \in L_0$ for which $\|y\|_{X'} < \infty$.

The associate space X' of a Banach function space X is a Banach function space (see [2, Chapter 1, Section 2]). From the definition of $\|\cdot\|_{X'}$ it is easy to see that if $x \in X$ and $y \in X'$, then $xy \in L_1$ and

$$\mathrm{E}\big[\,|xy|\,\big] \le \|x\|_X \|y\|_{X'}.$$

For every $p \in [1, \infty]$, the associate space $(L_p)'$ of L_p coincides with $L_{p'}$, where p' stands for the conjugate exponent of p. It follows that $L''_p := ((L_p)')'$ coincides with L_p. More generally, if X is a Banach function space, then $X'' := (X')'$ coincides with X and $\|x\|_{X''} = \|x\|_X$ for all $x \in X''$ (see [2, p. 10]). From the fact that $(L_\infty)' = L_1$, one sees that the associate space of a Banach function space X does not coincide with the Banach space dual X^* in general.

Definition 2.3. A Banach function space X is said to be *rearrangement-invariant* or *r.i.* if whenever two random variables x and y have the same distribution and $x \in X$, then $y \in X$ and $\|x\|_X = \|y\|_X$.

By an *r.i. space*, we mean a rearrangement-invariant Banach function space.

For example, Lebesgue spaces, Orlicz spaces, and Lorentz spaces are r.i., while variable exponent Lebesgue spaces are not r.i. in general (see [1, Theorem 1]).

Note that $x \in L_p(\Omega)$ if and only if $x^* \in L_p(I)$, and that $\|x\|_{L_p(\Omega)} = \|x^*\|_{L_p(I)}$ for all $x \in L_p(\Omega)$, where x^* denotes the nonincreasing rearrangement of x. More generally, if X is an r.i. space over Ω, then there exists a unique r.i. space $\widehat{X}$ over I which satisfies the following conditions:

- $x \in X$ if and only if $x^* \in \widehat{X}$.
- $\|x\|_X = \|x^*\|_{\widehat{X}}$ for all $x \in X$.

For the proof of this fact, see [2, pp. 62–64]. We call $\widehat{X}$ the *Luxemburg representation* of X. Thus the Luxemburg representation of $L_p(\Omega)$ is $L_p(I)$.

In order to prove our main results, we will use the following lemma.

Proposition 2.4 (cf. [17, Proposition 1]). *Let X be a Banach function space over Ω. The following are equivalent:*

(i) *There exists a positive constant C such that for every sub-σ-algebra $\mathcal{A}$ of Σ and every $x \in X$,*

$$\left\| \mathrm{E}\left[x \,|\, \mathcal{A} \right] \right\|_X \leq C \|x\|_X .$$

(ii) *There exists a norm $\|\cdot\|_X^*$ on X which is equivalent to the original norm $\|\cdot\|_X$ and such that X is r.i. with respect to $\|\cdot\|_X^*$.*

We say that X *can be renormed so as to be r.i.* if (ii) of the proposition above is satisfied. For a proof the proposition, see [14, Lemma 2] and [17, Proposition 1].

In Section 5, we will deal with the space w-X defined as follows.

Definition 2.5. Let X be a Banach function space over Ω. For each $x \in L_0(\Omega)$, we let

$$\|x\|_{\text{w-}X} = \sup_{\lambda > 0} \lambda \|\mathbb{1}_{\{|x| > \lambda\}}\|_X .$$

The space w-X consists of those random variables x for which $\|x\|_{\text{w-}X} < \infty$.

It is easy to see that $X \subset$ w-X and

$$\|x\|_{\text{w-}X} \leq \|x\|_X \quad \text{for all } x \in X,$$

and that

$$\|\mathbb{1}_A\|_{\text{w-}X} = \|\mathbb{1}_A\|_X \quad \text{for all } A \in \Sigma.$$

If $1 \leq p < \infty$ and $X = L_p$, then w-X coincides with the Lorentz space $L_{p,\infty}$ (cf. [12, p. 156]). Thus, in some cases, w-X can be renormed so as to be a Banach function space. However, in general, w-X is not a Banach space but a quasi-Banach space. In fact, the function $\|\cdot\|_{\text{w-}X}$ on w-X does not satisfy the triangle inequality, but satisfies the inequality

$$\|x + y\|_{\text{w-}X} \leq 2(\|x\|_{\text{w-}X} + \|y\|_{\text{w-}X}).$$

It is straightforward to check that w-X is a quasi-Banach function space in the sense of Definition 2.1.

Recall that the space w-$X[\ell_r]$ is defined to be the set of all processes $f = (f_n)$ such that $\|f\|_{\text{w-}X[\ell_r]} < \infty$, where $\|f\|_{\text{w-}X[\ell_r]}$ is given by (1.6)

2.3. Generalized fundamental functions

In our investigation, the generalized fundamental functions of a Banach function space, defined as follows, play an important role. For each $t \in [0,1]$, let

$$\Sigma(t) = \{A \in \Sigma \colon \mathrm{P}(A) = t\}.$$

Note that $\Sigma(t)$ is not empty, because the probability space $(\Omega, \Sigma, \mathrm{P})$ is nonatomic. We define the functions $\underline{\varphi}_X \colon [0,1] \to [0,\infty)$ and $\overline{\varphi}_X \colon [0,1] \to [0,\infty)$ by setting

$$\underline{\varphi}_X(t) = \inf\left\{\|\mathbb{1}_A\|_X \colon A \in \Sigma(t)\right\}, \quad t \in [0,1],$$

and

$$\overline{\varphi}_X(t) = \sup\left\{\|\mathbb{1}_A\|_X \colon A \in \Sigma(t)\right\}, \quad t \in [0,1].$$

Note that since $X \hookrightarrow L_1$, there is a positive constant c such that $t \leq c\underline{\varphi}_X(t)$ for all $t \in I = (0,1]$.

For a Banach function space X, we define

$$k_X = \sup_{0<t\leq 1} \frac{\overline{\varphi}_X(t)}{\underline{\varphi}_X(t)}. \tag{2.1}$$

Then clearly $1 \leq k_X \leq \infty$. It is clear that if X is an r.i. space, then $k_X = 1 < \infty$. However X is not necessarily an r.i. space even if $k_X < \infty$. In fact, one can construct a Banach function space X such that $k_X < \infty$ but is not r.i. (see [18, Lemma 3 and Example 2]).

2.4. Marcinkiewicz function spaces

Recall that a function $\varphi \colon [0,1] \to [0,\infty)$ is said to be *quasi-concave* if it satisfies the following conditions:

- $\varphi(t) = 0$ if and only if $t = 0$.
- $\varphi(t)$ is nondecreasing on $[0,1]$.
- $\varphi(t)/t$ is nonincreasing on $(0,1]$.

For example, a nondecreasing concave function $\varphi \colon [0,1] \to [0,\infty)$ which vanishes only at $t = 0$ is quasi-concave. Moreover if X is a Banach function space over Ω, then $\overline{\varphi}_X$ is quasi-concave (see [18, Lemma 1]). Note that every quasi-concave function φ is continuous on $(0,1] = I$ (see [23, p. 49]).

Let $\varphi \colon [0,1] \to [0,\infty)$ be a quasi-concave function. For each $\eta \in L_0(I)$, we let

$$\|\eta\|_{M(\varphi:I)} = \sup_{t\in I} \frac{\varphi(t)}{t} \int_0^t \eta^*(s)\,\mathrm{d}s \quad \text{and} \quad \|\eta\|_{M^*(\varphi:I)} = \sup_{t\in I}\left[\varphi(t)\eta^*(t)\right].$$

The space $M(\varphi : I)$ (resp. $M^*(\varphi : I)$) is defined to be the set of all $\eta \in L_0(I)$ such that $\|\eta\|_{M(\varphi:I)} < \infty$ (resp. $\|\eta\|_{M^*(\varphi:I)} < \infty$). One can show that $M(\varphi : I)$ is an r.i. space over I (see [2, p. 69]). On the other hand, $M^*(\varphi : I)$ is a quasi-Banach function space over I. In fact, the function $\|\cdot\|_{M^*(\varphi:I)}$ on $M^*(\varphi : I)$ does not satisfy the triangle inequality, but it satisfies the inequality

$$\|\eta + \xi\|_{M^*(\varphi:I)} \le 2\big(\|\eta\|_{M^*(\varphi:I)} + \|\xi\|_{M^*(\varphi:I)}\big)$$

(cf. [20, Eq. (2.8)]). In addition, for each $x \in L_0(\Omega)$, we let

$$\|x\|_{M(\varphi:\Omega)} = \|x^*\|_{M(\varphi:I)} \quad \text{and} \quad \|x\|_{M^*(\varphi:\Omega)} = \|x^*\|_{M^*(\varphi:I)}.$$

The space $M(\varphi : \Omega)$ (resp. $M^*(\varphi : \Omega)$) is defined to be the set of all $x \in L_0(\Omega)$ such that $\|x\|_{M(\varphi:\Omega)} < \infty$ (resp. $\|x\|_{M^*(\varphi:\Omega)} < \infty$). Of course, $M(\varphi : \Omega)$ is an r.i. space over Ω and $M^*(\varphi : \Omega)$ is a quasi-Banach function space over Ω. The spaces $M(\varphi : I)$, $M^*(\varphi : I)$, $M(\varphi : \Omega)$, and $M^*(\varphi : \Omega)$ are called the *Marcinkiewicz function spaces*.

If X is a Banach function space over Ω, then since $\overline{\varphi}_X$ is a quasi-concave function, we can associate the spaces $M(\overline{\varphi}_X : I)$, $M^*(\overline{\varphi}_X : I)$, $M(\overline{\varphi}_X : \Omega)$, and $M^*(\overline{\varphi}_X : \Omega)$ with X.

2.5. **Indices of function spaces**

In the proof of our results, we need to discuss the boundedness of some operators on a (quasi-)Banach function space over I. Suppose that Y is a Banach or quasi-Banach function space over I. We denote by $B(Y)$ the set of all linear operators T which satisfy the following conditions:

- The domain of T contains Y and the range of T is contained in $L_0(I)$.
- The restriction of T to Y is a bounded operator from Y to itself.

Given $T \in B(Y)$, we write $\|T\|_{B(Y)}$ for the (quasi-)norm of the restriction of T to Y, i.e., $\|T\|_{B(Y)} = \sup\{\|Tx\|_Y : x \in \mathbb{B}_Y\}$.

We now recall the definition of Boyd indices of an r.i. space. For each $s \in (0, \infty)$, let D_s denote the *dilation operator* on $L_0(I)$ defined for $\eta \in L_0(I)$ by

$$(D_s\eta)(t) = \begin{cases} \eta(st) & \text{if } st \in I, \\ 0 & \text{if } st \notin I, \end{cases} \qquad t \in I.$$

If X is an r.i. space over Ω, then $D_s \in B(\widehat{X})$ for all $s \in (0, \infty)$, where $\widehat{X}$ is the Luxemburg representation of X. Define a function $h_X : (0, \infty) \to (0, \infty)$ by $h_X(s) = \|D_{1/s}\|_{B(\hat{X})}$. The *Boyd indices* α_X and β_X of X are defined by

$$\alpha_X = \sup_{0<s<1} \frac{\log h_X(s)}{\log s} \quad \text{and} \quad \beta_X = \inf_{1<s<\infty} \frac{\log h_X(s)}{\log s}. \tag{2.2}$$

One can show that h_X is submultiplicative, i.e.,

$$h_X(st) \le h_X(s)h_X(t) \quad \text{whenever } 0 < s,\ t < \infty.$$

One can also show that $h_X(s) \le \max\{s, 1\}$ for all $s \in (0, \infty)$ (see [2, pp. 148, 165]). Hence we have that

$$\alpha_X = \lim_{s\to 0+} \frac{\log h_X(s)}{\log s}, \qquad \beta_X = \lim_{s\to\infty} \frac{\log h_X(s)}{\log s},$$

and $0 \le \alpha_X \le \beta_X \le 1$ (see [23, p. 53]). The Boyd indices are ones which extends the role of the index p of L_p. In fact, $\alpha_{L_p} = \beta_{L_p} = 1/p$ for all $p \in [1, \infty]$. In particular, we have $\alpha_{L_p} + \beta_{(L_p)'} = (1/p) + (1/p') = 1$. More generally, we have

$$\alpha_X + \beta_{X'} = \alpha_{X'} + \beta_X = 1 \tag{2.3}$$

for any r.i. space X (see [2, pp. 149, 165, 166]).

The following lemma is one of the key tools for the proof of our main results.

Proposition 2.6 ([16, Corollary 3.4])**.** *Let X be an r.i. space over Ω, let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$, let $\rho = (\rho_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}(\mathcal{F})$, and let z be a nonnegative random variable on Ω. Suppose that $0 \le \rho_n \le \rho_{n+1}$ a.s. for all $n \in \mathbb{Z}_+$, and let $\rho_\infty = \lim_{n\to\infty} \rho_n$ a.s.*

(a) *If $\alpha_X > 0$, $z \in L_1(\Omega)$, $\rho_\infty \in L_1(\Omega)$, and*

$$\mathrm{E}[\rho_\infty - \rho_{n-1} | \mathcal{F}_n] \le \mathrm{E}[z|\mathcal{F}_n] \quad \text{a.s.}$$

for all $n \in \mathbb{Z}_+$, then

$$\|\rho_\infty\|_X \le C_X \|z\|_X,$$

where C_X is a positive constant which depends only on X.

(b) *Let $a \in (1, \infty)$. If $0 < \alpha_X,\ \beta_X < 1/a$, $z \in L_a(\Omega)$, $\rho_\infty \in L_a(\Omega)$, and*

$$\mathrm{E}\big[(\rho_\infty - \rho_{n-1})^a \big| \mathcal{F}_n\big] \le \mathrm{E}[z^a|\mathcal{F}_n] \quad \text{a.s.}$$

for all $n \in \mathbb{Z}_+$, then

$$\|\rho_\infty\|_X \le C_{X,a} \|z\|_X,$$

where $C_{X,a}$ is a positive constant which depends only on X and a.

In addition to Boyd indices, we need the indices of a quasi-concave function. Given a quasi-concave function $\varphi\colon [0, 1] \to [0, \infty)$, we define a function $m_\varphi\colon (0, \infty) \to (0, \infty)$ by

$$m_\varphi(s) = \sup_{0<t\le(1/s)\wedge 1} \frac{\varphi(st)}{\varphi(t)} = \sup_{0<t\le s\wedge 1} \frac{\varphi(t)}{\varphi(t/s)},$$

where we write $a \wedge b$ for $\min\{a, b\}$. The indices p_φ and q_φ are defined by

$$p_\varphi = \sup_{0<s<1} \frac{\log m_\varphi(s)}{\log s} \quad \text{and} \quad q_\varphi = \inf_{1<s<\infty} \frac{\log m_\varphi(s)}{\log s}. \tag{2.4}$$

It is easily seen that m_φ is submultiplicative. Moreover, since $\varphi(t)/t$ is nonincreasing, one sees that $m_\varphi(s) \le \max\{s, 1\}$. It follows from [23, p. 53] that

$$p_\varphi = \lim_{s\to 0+} \frac{\log m_\varphi(s)}{\log s}, \qquad q_\varphi = \lim_{s\to\infty} \frac{\log m_\varphi(s)}{\log s}, \tag{2.5}$$

and $0 \le p_\varphi \le q_\varphi \le 1$.

Let X be a Banach function space over Ω. Then, as mentioned above, $\overline{\varphi}_X$ is quasi-concave, and hence we can define indices $p_{\overline{\varphi}_X}$ and $q_{\overline{\varphi}_X}$. For simplicity, we write p_X instead of $p_{\overline{\varphi}_X}$, and q_X instead of $q_{\overline{\varphi}_X}$.

Note that the indices p_X and q_X are defined for *all* Banach function spaces X, while the Boyd indices α_X and β_X are defined only for r.i. spaces X.

2.6. Linear operator on function spaces

In the proof our results, we will discuss the boundedness of linear operators which are defined as follows. For each $a \in [1, \infty)$, let $\mathrm{Dom}(\mathcal{P}_a)$ be the set of all $\eta \in L_0(I)$ such that $|\eta(s)|s^{(1/a)-1}$ is integrable over the interval $(0, t)$ for all $t \in I$. For each $\eta \in \mathrm{Dom}(\mathcal{P}_a)$, the function $\mathcal{P}_a\eta$ in $L_0(I)$ is defined by

$$(\mathcal{P}_a\eta)(t) = \frac{1}{t^{1/a}} \int_0^t \eta(s)s^{1/a}\frac{\mathrm{d}s}{s}, \quad t \in I.$$

For simplicity, we write $\mathcal{P}$ instead of $\mathcal{P}_1$.

Now let $\mathrm{Dom}(Q)$ be the set of all $\eta \in L_0(I)$ which is integrable over $(t, 1]$ for all $t \in I$. For each $\eta \in \mathrm{Dom}(Q)$, the function $Q\eta$ in $L_0(I)$ is defined by

$$(Q\eta)(t) = \int_t^1 \frac{\eta(s)}{s}\,\mathrm{d}s, \quad t \in I.$$

Let X be an r.i. space over Ω. From the proof of Boyd's theorem ([3, Theorem 1]), we see that $\mathcal{P}_a \in B(\widehat{X})$ if and only if $\beta_X < 1/a$, and that $Q \in B(\widehat{X})$ if and only if $\alpha_X > 0$ (see also [2, p. 150]). In addition to these facts, we will use the following result.

Lemma 2.7 ([16, Lemma 3.2])**.** *Let $a \in [1, \infty)$. If $\eta \in \mathrm{Dom}(\mathcal{P}_a)$ is nonnegative and nonincreasing, then $\eta^a \in \mathrm{Dom}(\mathcal{P})$ and $\big((\mathcal{P}\eta^a)(t)\big)^{1/a} \le a^{-1}(\mathcal{P}_a\eta)(t)$ for all $t \in I$.*

3. Inequalities for optional projections in a Banach function space

Recall that if a Banach function space X can be renormed so as to be r.i., then the Boyd indices α_X and β_X of X are defined by (2.2). Of course, the values of α_X and β_X do not depend on the choice of the norm for which X is r.i.

In this section, we prove the following three theorems.

Theorem 3.1. *Let X be a Banach function space over Ω. The following are equivalent:*

(i) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}$,*

$$\|{}^{O(\mathcal{F})}f\|_{X[\ell_1]} \le C_X \|f\|_{X[\ell_1]}. \tag{3.1}$$

(ii) *X can be renormed so as to be r.i. and $\alpha_X > 0$.*

Theorem 3.2. *Let X be a Banach function space over Ω. The following are equivalent:*

(i) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}$,*

$$\|{}^{O(\mathcal{F})}f\|_{X[\ell_\infty]} \le C_X \|f\|_{X[\ell_\infty]}.$$

(ii) *X can be renormed so as to be r.i. and $\beta_X < 1$.*

Theorem 3.3. *Let X be a Banach function space over Ω. The following are equivalent:*

(i) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$, all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}$, and all $r \in (1,\infty)$,*

$$\|{}^{O(\mathcal{F})}f\|_{X[\ell_r]} \le C_X \|f\|_{X[\ell_r]}. \tag{3.2}$$

(ii) *X can be renormed so as to be r.i. and $0 < \alpha_X,\ \beta_X < 1$.*

In order to prove theorems above, we need some lemmas. The following lemma is a variant of Stein's inequality (see (1.1)). Notice that if $n = 0$, then by convention $\rho^{(r)}_{n-1} = \rho^{(r)}_{-1} \equiv 0$ ($r \in [1,\infty]$) in the following lemma.

Lemma 3.4. *Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and let $f = (f_n) \in \mathbb{P}$. For each $r \in [1,\infty)$, define a random variable $z^{(r)}$ and a process $\rho^{(r)} = \big(\rho^{(r)}_n\big)$ by letting*

$$z^{(r)} = \left(\sum_{n=0}^{\infty} |f_n|^r\right)^{1/r} \quad \text{and} \quad \rho^{(r)}_n = \left(\sum_{k=0}^{n} \big|\mathrm{E}[\,f_k\,|\,\mathcal{F}_k\,]\big|^r\right)^{1/r}, \tag{3.3}$$

respectively; and define a random variable $z^{(\infty)}$ *and a process* $\rho^{(\infty)} = \big(\rho_n^{(\infty)}\big)$ *by letting*

$$z^{(\infty)} = \sup_{0 \le k < \infty} |f_k| \quad \textit{and} \quad \rho_n^{(\infty)} = \sup_{0 \le k \le n} \big|\mathrm{E}[\, f_k \,|\, \mathcal{F}_k\,]\big|,$$

respectively. Let $\rho_\infty^{(r)} = \lim_{n\to\infty} \rho_n^{(r)}$ *a.s. for each* $r \in [1, \infty]$.

(a) *If* $f = (f_n) \in L_1[\ell_1]$*, then* $z^{(1)} \in L_1(\Omega)$*,* $\rho_\infty^{(1)} \in L_1(\Omega)$*, and*

$$\mathrm{E}\big[\rho_\infty^{(1)} - \rho_{n-1}^{(1)} \big| \mathcal{F}_n\big] \le \mathrm{E}\big[z^{(1)} \big| \mathcal{F}_n\big] \quad \textit{a.s.}$$

for all $n \in \mathbb{Z}_+$.

(b) *If* $a \in (1,\infty)$*,* $r \in [1,\infty]$*, and* $f = (f_n) \in L_a[\ell_r]$*, then* $z^{(r)} \in L_a(\Omega)$*,* $\rho_\infty^{(r)} \in L_a(\Omega)$*, and*

$$\mathrm{E}\big[\big(\rho_\infty^{(r)} - \rho_{n-1}^{(r)}\big)^a \big| \mathcal{F}_n\big] \le K_a^a \mathrm{E}\big[\big(z^{(r)}\big)^a \big| \mathcal{F}_n\big] \quad \textit{a.s.}$$

for all $n \in \mathbb{Z}_+$*, where* $K_a = a \vee a'$.

Proof. (a). Let $f = (f_n) \in L_1[\ell_1]$ and $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$. It is then clear that $z^{(1)} \in L_1(\Omega)$ and $\rho_\infty^{(1)} \in L_1(\Omega)$. Moreover for all $n \in \mathbb{Z}_+$,

$$\begin{aligned}\mathrm{E}\big[\rho_\infty^{(1)} - \rho_{n-1}^{(1)} \big| \mathcal{F}_n\big] &= \sum_{k=n}^{\infty} \mathrm{E}\Big[\big|\mathrm{E}[\, f_k \,|\, \mathcal{F}_k\,]\big| \Big| \mathcal{F}_n\Big] \\ &\le \sum_{k=n}^{\infty} \mathrm{E}\Big[\mathrm{E}\big[|f_k| \big| \mathcal{F}_k\big] \Big| \mathcal{F}_n\Big] = \sum_{k=n}^{\infty} \mathrm{E}\big[|f_k| \big| \mathcal{F}_n\big] \le \mathrm{E}\big[z^{(1)} \big| \mathcal{F}_n\big] \quad \text{a.s.}\end{aligned}$$

(b). Let $a \in (1,\infty)$, $r \in [1,\infty]$, $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, and $f = (f_n) \in L_a[\ell_r]$. It is then clear that $z^{(r)} \in L_a(\Omega)$. Moreover, since (1.1) holds with $K_{a,r}$ replaced by $K_a = a \vee a'$, it follows that $\rho_\infty^{(r)} \in L_a(\Omega)$. Let $n \in \mathbb{Z}_+$ and $A \in \mathcal{F}_n$. If $r \in [1,\infty)$, then by (1.1),

$$\begin{aligned}\mathrm{E}\big[\big(\rho_\infty^{(r)} - \rho_{n-1}^{(r)}\big)^a \mathbb{1}_A\big] &\le \mathrm{E}\left[\left(\sum_{k=n}^{\infty} \big|\mathrm{E}[\, f_k \mathbb{1}_A \,|\, \mathcal{F}_k\,]\big|^r\right)^{a/r}\right] \\ &\le K_a^a \mathrm{E}\left[\left(\sum_{k=n}^{\infty} |f_k \mathbb{1}_A|^r\right)^{a/r}\right] \le K_a^a \mathrm{E}\big[\big(z^{(r)}\big)^a \mathbb{1}_A\big].\end{aligned}$$

Since $A \in \mathcal{F}_n$ is arbitrary, we have

$$\mathrm{E}\big[\big(\rho_\infty^{(r)} - \rho_{n-1}^{(r)}\big)^a \big| \mathcal{F}_n\big] \le K_a^a \mathrm{E}\big[\big(z^{(r)}\big)^a \big| \mathcal{F}_n\big] \quad \text{a.s.}$$

If $r = \infty$, then by (1.1),

$$\begin{aligned} \mathrm{E}\big[\big(\rho_\infty^{(r)} - \rho_{n-1}^{(r)}\big)^a \mathbb{1}_A\big] &\le \mathrm{E}\Bigg[\Bigg(\sup_{n\le k<\infty} \big|\mathrm{E}[f_k \mathbb{1}_A | \mathcal{F}_k]\big|\Bigg)^a\Bigg] \\ &\le K_a^a \mathrm{E}\Bigg[\Bigg(\sup_{n\le k<\infty} |f_k|\Bigg)^a \mathbb{1}_A\Bigg] \le K_a^a \mathrm{E}\big[\big(z^{(r)}\big)^a \mathbb{1}_A\big]. \end{aligned}$$

Since $A \in \mathcal{F}_n$ is arbitrary, we have $\mathrm{E}\big[\big(\rho_\infty^{(r)} - \rho_{n-1}^{(r)}\big)^a \big|\mathcal{F}_n\big] \le K_a^a \mathrm{E}\big[\big(z^{(r)}\big)^a \big|\mathcal{F}_n\big]$ a.s., as desired. □

Before stating the next lemma, note that there exists a random variable $\gamma\colon \Omega \to [0,1)$ such that

$$\gamma^*(t) = 1 - t \quad \text{for all } t \in I = (0,1].$$

This is an immediate consequence of our assumption that the probability space $(\Omega, \Sigma, \mathrm{P})$ in nonatomic (see [6, (5.6), p. 44]).

The following lemma is crucial for the proofs of our results.

Lemma 3.5. *Let $\gamma\colon \Omega \to [0,1)$ be as above, and let $\eta \in L_1(I)$ be Borel measurable. Define $x \in L_1(\Omega)$ by*

$$x(\omega) = \big|\eta(1-\gamma(\omega))\big|, \quad \omega \in \Omega,$$

and define a family of sets $\{A(t)\colon 0 < t \le 1\}$ by

$$A(t) = \{\gamma > 1 - t\}.$$

Then:

(i) *$A(s) \subset A(t)$ whenever $0 < s \le t \le 1$.*

(ii) *$x^*(t) = \eta^*(t)$ for all $t \in I$.*

(iii) *$\mathrm{P}(A(t)) = t$ for all $t \in I$.*

(iv) *$\mathrm{E}\big[x\mathbb{1}_{A(t)}\big] = \int_0^t |\eta(s)|\, ds$ for all $t \in I$.*

Proof. See, for example, [21, Section 5]. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1. (ii) ⇒ (i). Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathbb{P}$. Suppose that (ii) holds. Then we may assume that X is an r.i. space with respect to the norm $\|\cdot\|_X$. Furthermore we may assume that $f = (f_n) \in X[\ell_1]$, since otherwise (3.1) is obvious. From (B1) of Definition 2.1, it follows that $f = (f_n) \in L_1[\ell_1]$. Define a random variable

$z^{(1)}$ and a process $\rho^{(1)} = \big(\rho_n^{(1)}\big)$ by (3.3) with $r = 1$. Then by (a) of Lemma 3.4, $z^{(1)} \in L_1(\Omega)$, $\rho_\infty^{(1)} \in L_1(\Omega)$, and

$$\mathrm{E}\big[\rho_\infty^{(1)} - \rho_{n-1}^{(1)} \big| \mathcal{F}_n\big] \le \mathrm{E}\big[z^{(1)} \big| \mathcal{F}_n\big] \quad \text{a.s.}$$

for all $n \in \mathbb{Z}_+$. Since $\alpha_X > 0$, we can apply (a) of Proposition 2.6 to deduce that

$$\|{}^{O(\mathcal{F})} f\|_{X[\ell_1]} = \|\rho_\infty^{(1)}\|_X \le C_X \|z^{(1)}\|_X = C_X \|f\|_{X[\ell_1]},$$

as desired.

(i) ⇒ (ii). Suppose that (i) holds. Let $\mathcal{A}$ be a sub-σ-algebra of Σ and let $x \in X$. Define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathbb{P}$ by

$$\mathcal{F}_n = \begin{cases} \mathcal{A} & \text{if } n = 0, \\ \Sigma & \text{if } n \ge 1, \end{cases} \quad \text{and} \quad f_n = \begin{cases} x & \text{if } n = 0, \\ 0 & \text{if } n \ge 1, \end{cases}$$

respectively. Then by (3.1), we have $\big\|\mathrm{E}[x \,|\, \mathcal{A}]\big\|_X \le C_X \|x\|_X$. From Proposition 2.4 it follows that X can be renormed so as to be r.i. Hence, for the rest of the proof, we assume that X is an r.i. space with respect to the norm $\|\cdot\|_X$.

To show that $\alpha_X > 0$, it suffices to show that $Q \in B(\widehat{X})$ (cf. [2, p. 150]). Let $\eta \in \widehat{X}$ be arbitrary. Define x and $\{A(t) \colon 0 < t \le 1\}$ as in Lemma 3.5. Then $x \in X$ (because $x^* = \eta^* \in \widehat{X}$). For each $n \in \mathbb{Z}_+$ let $t_n = 2^{-n}$, and define a sequence of sets $\{A_n\}_{n \in \mathbb{Z}_+}$ by $A_n = A(t_n)$. It then follows from (i) and (iii) of Lemma 3.5 that $A_{n+1} \subset A_n$ and $\mathrm{P}(A_n) = t_n$ for all $n \in \mathbb{Z}_+$. Define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathbb{P}$ by letting

$$\mathcal{F}_n = \sigma(\{A \setminus A_n \colon A \in \Sigma\}) \quad \text{and} \quad f_n = x \mathbb{1}_{A_n \setminus A_{n+1}},$$

respectively. Then

$$\begin{aligned} \mathrm{E}[f_n \,|\, \mathcal{F}_n] &= \frac{\mathbb{1}_{A_n}}{\mathrm{P}(A_n)} \mathrm{E}\big[x \mathbb{1}_{A_n \setminus A_{n+1}}\big] \\ &= \frac{\mathbb{1}_{A_n}}{t_n} \int_{t_{n+1}}^{t_n} |\eta(s)| \,\mathrm{d}s = \frac{\mathbb{1}_{A_n}}{2} \int_{t_{n+1}}^{t_n} \frac{|\eta(s)|}{t_{n+1}} \,\mathrm{d}s \\ &\ge \frac{\mathbb{1}_{A_n}}{2} \int_{t_{n+1}}^{t_n} \frac{|\eta(s)|}{s} \,\mathrm{d}s = \frac{1}{2}\big[(Q|\eta|)(t_{n+1}) - (Q|\eta|)(t_n)\big] \mathbb{1}_{A_n} \quad \text{a.s.,} \end{aligned}$$

where the second equality follows from (iv) of Lemma 3.5. Since $\mathrm{P}(A_n) = t_n \to 0$ as $n \to \infty$, we have

$$\sum_{n=0}^{\infty} \big[(Q|\eta|)(t_{n+1}) - (Q|\eta|)(t_n)\big] \mathbb{1}_{A_n} = \sum_{n=1}^{\infty} (Q|\eta|)(t_n) \mathbb{1}_{A_{n-1} \setminus A_n} \quad \text{a.s.}$$

It follows that

$$\sum_{n=1}^{\infty}(Q|\eta|)(t_n)\mathbb{1}_{A_{n-1}\setminus A_n} \le 2\sum_{n=0}^{\infty}\mathrm{E}[f_n|\mathcal{F}_n] \quad \text{a.s.}$$

Therefore, for each $t \in I$,

$$\begin{aligned}(Q|\eta|)(t) &\le \sum_{n=1}^{\infty}(Q|\eta|)(t_n)\mathbb{1}_{[t_n,\,t_{n-1})}(t) = \Big(\sum_{n=1}^{\infty}(Q|\eta|)(t_n)\mathbb{1}_{A_{n-1}\setminus A_n}\Big)^*(t)\\ &\le 2\left(\sum_{n=0}^{\infty}\mathrm{E}[f_n|\mathcal{F}_n]\right)^*(t),\end{aligned}$$

and so

$$\big\|Q|\eta|\big\|_{\hat{X}} \le 2\left\|\left(\sum_{n=0}^{\infty}\mathrm{E}[f_n|\mathcal{F}_n]\right)^*\right\|_{\hat{X}} = 2\left\|\sum_{n=0}^{\infty}\mathrm{E}[f_n|\mathcal{F}_n]\right\|_X = 2\|{}^{O(\mathcal{F})}f\|_{X[\ell_1]}. \tag{3.4}$$

On the other hand, since

$$\sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} x\mathbb{1}_{A_n\setminus A_{n+1}} = x \quad \text{a.s.},$$

and since $x^* = \eta^*$ on I, we have that

$$\|f\|_{X[\ell_1]} = \|x\|_X = \|\eta\|_{\hat{X}}. \tag{3.5}$$

From (3.1), (3.4), and (3.5), we conclude that $\big\|Q|\eta|\big\|_{\hat{X}} \le 2C_X\|\eta\|_{\hat{X}}$. Since $|Q\eta| \le Q|\eta|$ on I, we have $\|Q\eta\|_X \le 2C_X\|\eta\|_X$, as desired. □

We now turn to the proof of Theorem 3.2. We begin with the following lemma.

Lemma 3.6. *Let X be a Banach function space over Ω. The following are equivalent:*

(i) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$,*

$$\|{}^{O(\mathcal{F})}f\|_{X[\ell_\infty]} \le C_X\|f\|_{X[\ell_\infty]}. \tag{3.6}$$

(ii) *There exists a positive constant $C_{X'}$ which depends only on X' such that for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$,*

$$\|{}^{O(\mathcal{F})}f\|_{X'[\ell_1]} \le C_{X'}\|f\|_{X'[\ell_1]}. \tag{3.7}$$

Moreover if (i) *(and hence* (ii)*) holds, then the constants C_X and $C_{X'}$ can be chosen to be the same.*

Proof. (i) $\Rightarrow$ (ii). Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, let $f = (f_n) \in \mathbb{P}$, and suppose that (i) holds. We may assume that $f = (f_n) \in X'[\ell_1]$, since otherwise (3.7) is obvious. Let $x \in \mathbb{B}_X \cap L_\infty(\Omega)$, and define $g = (g_n) \in \mathbb{P}$ by letting $g_n = |x|$ for all $n \in \mathbb{Z}_+$. Since $\|g\|_{X[\ell_\infty]} = \|x\|_X \leq 1$, it follows that

$$\begin{aligned}
\mathrm{E}\Big[|x| \sum_{n=0}^{\infty} \big|\mathrm{E}[f_n|\mathcal{F}_n]\big|\Big] &\leq \sum_{n=0}^{\infty} \mathrm{E}\Big[g_n \mathrm{E}\big[|f_n|\big|\mathcal{F}_n\big]\Big] \\
&= \sum_{n=0}^{\infty} \mathrm{E}\Big[\mathrm{E}[g_n|\mathcal{F}_n]\,|f_n|\Big] = \mathrm{E}\Big[\sum_{n=0}^{\infty} \mathrm{E}[g_n|\mathcal{F}_n]\,|f_n|\Big] \\
&\leq \Big\| \sup_{n\in\mathbb{Z}_+} \mathrm{E}[g_n|\mathcal{F}_n]\Big\|_X \Big\|\sum_{n=0}^{\infty}|f_n|\Big\|_{X'} \\
&= \|{}^{O(\mathcal{F})}g\|_{X[\ell_\infty]} \|f\|_{X'[\ell_1]} \\
&\leq C_X \|g\|_{X[\ell_\infty]} \|f\|_{X'[\ell_1]} \leq C_X \|f\|_{X'[\ell_1]}.
\end{aligned}$$

Since $x \in \mathbb{B}_X \cap L_\infty(\Omega)$ is arbitrary, we conclude that $\|{}^{O(\mathcal{F})}f\|_{X'[\ell_1]} \leq C_X\|f\|_{X'[\ell_1]}$, as desired.

(ii) $\Rightarrow$ (i). Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, let $f = (f_n) \in \mathbb{P}$, and suppose that (ii) holds. We may assume that $f = (f_n) \in X[\ell_\infty]$, since otherwise (3.6) is obvious. Let $x \in \mathbb{B}_{X'} \cap L_\infty(\Omega)$ and let $N \geq 1$ be a fixed integer. We define a random time $T\colon \Omega \to \mathbb{Z}_+$ by

$$T = \min\Big\{n \in \mathbb{Z}_+ : n \leq N,\ \big|\mathrm{E}[f_n|\mathcal{F}_n]\big| = \max_{1\leq k\leq N}\big|\mathrm{E}[f_k|\mathcal{F}_k]\big|\Big\}.$$

Define $g = (g_n) \in \mathbb{P}$ by letting

$$g_n = \begin{cases} |x|\,\mathbb{1}_{\{T=n\}} & \text{if } 0 \leq n \leq N, \\ 0 & \text{if } n > N. \end{cases}$$

Since $\sum_{n=0}^{\infty} g_n = \sum_{n=0}^{N} g_n = |x|$, we have

$$\|g\|_{X'[\ell_1]} = \|x\|_{X'} \leq 1.$$

Therefore

$$\begin{aligned}
\mathrm{E}\Big[|x| \sup_{1\leq k\leq N}\big|\mathrm{E}[f_k|\mathcal{F}_k]\big|\Big] &= \sum_{n=0}^{N} \mathrm{E}\Big[\big|\mathrm{E}[f_n|\mathcal{F}_n]\big| g_n\Big] \leq \sum_{n=0}^{\infty} \mathrm{E}\Big[|f_n|\,\mathrm{E}[g_n|\mathcal{F}_n]\Big] \\
&\leq \Big\|\sum_{n=0}^{\infty} \mathrm{E}[g_n|\mathcal{F}_n]\Big\|_{X'} \Big\|\sup_{0\leq n<\infty}|f_n|\Big\|_X
\end{aligned}$$

$$= \|{}^{O(\mathcal{F})}g\|_{X'[\ell_1]}\,\|f\|_{X[\ell_\infty]}$$
$$\le C_{X'}\|g\|_{X'[\ell_1]}\,\|f\|_{X[\ell_\infty]} \le C_{X'}\|f\|_{X[\ell_\infty]}.$$

Since $x \in \mathbb{B}_{X'} \cap L_\infty(\Omega)$ is arbitrary, it follows that

$$\left\| \sup_{1\le k\le N} \left|\mathrm{E}\,[\,f_k\,|\,\mathcal{F}_k\,]\right| \right\|_X \le C_{X'}\|f\|_{X[\ell_\infty]}.$$

Since $N \ge 1$ is an arbitrary integer and since X satisfies (B3) of Definition 2.1, we conclude that $\|{}^{O(\mathcal{F})}f\|_{X[\ell_\infty]} \le C_{X'}\|f\|_{X[\ell_\infty]}$, as desired.

The last statement is an immediate consequence of the argument above. Thus the proof is complete. □

Lemma 3.7. *Let X be a Banach function space over Ω. Suppose that the associate space X' can be renormed so as to be r.i. Then X can be renormed so as to be r.i.*

Proof. Let $\|\cdot\|^*_{X'}$ be a norm on X' which is equivalent to the original norm $\|\cdot\|_{X'}$ such that X' is r.i. with respect to $\|\cdot\|^*_{X'}$. For $x \in X$, define

$$\|x\|^*_X = \sup\left\{ \int_0^1 x^*(s)y^*(s)\,\mathrm{d}s \colon\ y \in X',\ \ \|y\|^*_{X'} \le 1 \right\}.$$

Then $\|\cdot\|^*_X$ is a norm on X such that X is r.i. with respect to it, and $(X, \|\cdot\|^*_X)$ is the associate space of $(X', \|\cdot\|^*_{X'})$ (see [2, p. 60]). Since $(X, \|\cdot\|_X)$ is the associate space of $(X', \|\cdot\|_{X'})$ and since the norms $\|\cdot\|^*_{X'}$ and $\|\cdot\|_{X'}$ are equivalent, we see that the norms $\|\cdot\|^*_X$ and $\|\cdot\|_X$ are equivalent. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. (i) ⇒ (ii). Suppose that (i) holds. Then by Lemma 3.6, we have

$$\|{}^{O(\mathcal{F})}f\|_{X'[\ell_1]} \le C_X\|f\|_{X'[\ell_1]}$$

for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$. Hence by Theorem 3.1, X' can be renormed so as to be r.i. and $\alpha_{X'} > 0$. According to Lemma 3.7, X can be renormed so as to be r.i., and we have $\beta_X < 1$ by (2.3). Thus (ii) holds.

(ii) ⇒ (i). Suppose that (ii) holds. Then $X''(= X)$ can be renormed so as to be r.i., and hence by Lemma 3.7, X' can be renormed so as to be r.i. Moreover we have $\alpha_{X'} > 0$ by (2.3). It follows from Theorem 3.1 that the inequality

$$\|{}^{O(\mathcal{F})}f\|_{X'[\ell_1]} \le C_{X'}\|f\|_{X'[\ell_1]}$$

holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$. We conclude from Lemma 3.6 that (i) holds, as desired. □

In order to prove Theorem 3.3, we need one more lemma.

Lemma 3.8. *Let X be a Banach function space over Ω, let $r \in (1, \infty)$, and let r' be the conjugate exponent of r. The following are equivalent:*

(i) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$,*

$$\|{}^{O(\mathcal{F})} f\|_{X[\ell_r]} \leq C_X \|f\|_{X[\ell_r]}. \tag{3.8}$$

(ii) *There exists a positive constant $C_{X'}$ which depends only on X' such that for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$,*

$$\|{}^{O(\mathcal{F})} f\|_{X'[\ell_{r'}]} \leq C_{X'} \|f\|_{X'[\ell_{r'}]}. \tag{3.9}$$

Moreover if (i) *(and hence* (ii)*) holds, then the constants C_X and $C_{X'}$ can be chosen to be the same.*

Proof. It suffices to show that (i) implies (ii). In fact, because $r'' = r$ and $X'' = X$, if we can show that (i) implies (ii), then it follows that (ii) implies (i).

Let $\mathcal{F} = (\mathcal{F}_n) \in \mathcal{F}$, let $f = (f_n) \in \mathbb{P}$, and suppose that (i) holds. We may assume that $f = (f_n) \in X'[\ell_{r'}]$, since otherwise (3.9) is obvious. It suffice to show that

$$\left\| \left(\sum_{n=0}^{N} \left| \mathrm{E}[f_n | \mathcal{F}_n] \right|^{r'} \right)^{1/r'} \right\|_{X'} \leq C_X \|f\|_{X'[\ell_{r'}]} \tag{3.10}$$

for all integers $N \geq 1$, because (B3) of Definition 2.1 holds with X replaced by X'. We first show that (3.10) holds under the additional assumption that

$$\sum_{n=0}^{N} \left| \mathrm{E}[f_n | \mathcal{F}_n] \right|^{r'} > 0 \quad \text{a.s.} \tag{3.11}$$

Note that since each f_n is integrable, the sum on the left-hand side of (3.11) is finite a.s. Define $g = (g_n) \in \mathbb{P}$ by

$$g_n = \left| \mathrm{E}[f_n | \mathcal{F}_n] \right|^{r'-1} \left(\sum_{n=0}^{N} \left| \mathrm{E}[f_n | \mathcal{F}_n] \right|^{r'} \right)^{-1/r}, \qquad n \in \mathbb{Z}_+.$$

Then

$$\sum_{n=0}^{N} g_n^r = 1 \quad \text{a.s.} \quad \text{and} \quad \sum_{n=0}^{N} g_n \left| \mathrm{E}[f_n | \mathcal{F}_n] \right| = \left(\sum_{n=0}^{N} \left| \mathrm{E}[f_n | \mathcal{F}_n] \right|^{r'} \right)^{1/r'} \quad \text{a.s.}$$

Let $x \in \mathbb{B}_X \cap L_\infty(\Omega)$. Then

$$\begin{aligned}\mathrm{E}\left[|x|\left(\sum_{n=0}^{N}\left|\mathrm{E}[f_n|\mathcal{F}_n]\right|^{r'}\right)^{1/r'}\right] &= \sum_{n=0}^{N}\mathrm{E}\Big[|x|g_n\left|\mathrm{E}[f_n|\mathcal{F}_n]\right|\Big] \le \sum_{n=0}^{N}\mathrm{E}\Big[\mathrm{E}\big[|x|g_n\big|\mathcal{F}_n\big]\,|f_n|\Big]\\ &\le \mathrm{E}\left[\left(\sum_{n=0}^{N}\mathrm{E}\big[|x|g_n\big|\mathcal{F}_n\big]^r\right)^{1/r}\left(\sum_{n=0}^{N}|f_n|^{r'}\right)^{1/r'}\right]\\ &\le \left\|\left(\sum_{n=0}^{N}\mathrm{E}\big[|x|g_n\big|\mathcal{F}_n\big]^r\right)^{1/r}\right\|_X\left\|\left(\sum_{n=0}^{N}|f_n|^{r'}\right)^{1/r'}\right\|_{X'}.\end{aligned}$$

Define $h = (h_n) \in \mathbb{P}$ by

$$h_n = \begin{cases}|x|g_n & \text{if } n \le N,\\ 0 & \text{otherwise.}\end{cases}$$

Applying (3.8) with f replaced by h, we have that

$$\begin{aligned}\left\|\left(\sum_{n=0}^{N}\mathrm{E}\big[|x|g_n\big|\mathcal{F}_n\big]^r\right)^{1/r}\right\|_X &= \|{}^{O(\mathcal{F})}h\|_{X[\ell_r]} \le C_X\|h\|_{X[\ell_r]}\\ &= C_X\left\|\left(\sum_{n=0}^{N}|x|^r g_n^r\right)^{1/r}\right\|_X = C_X\|x\|_X \le C_X.\end{aligned}$$

Thus

$$\mathrm{E}\left[|x|\left(\sum_{n=0}^{N}\left|\mathrm{E}[f_n|\mathcal{F}_n]\right|^{r'}\right)^{1/r'}\right] \le C_X\left\|\left(\sum_{n=0}^{N}|f_n|^{r'}\right)^{1/r'}\right\|_{X'} \le C_X\|f\|_{X'[\ell_{r'}]}.$$

Since $x \in \mathbb{B}_X \cap L_\infty(\Omega)$ is arbitrary, we see that (3.10) holds.

We now remove the additional assumption that (3.11) holds. Let $f = (f_n) \in X'[\ell_{r'}]$ be arbitrary and let $\varepsilon > 0$. Define $\widetilde{f} = (\widetilde{f}_n) \in \mathbb{P}$ by letting

$$\widetilde{f}_n = \begin{cases}\left|f_0\right| + \varepsilon & \text{if } n = 0,\\ f_n & \text{if } n \ge 1,\end{cases}$$

and let $N \ge 1$ be an integer. It is then clear that

$$\left\|\left(\sum_{n=0}^{N}\left|\mathrm{E}[f_n|\mathcal{F}_n]\right|^{r'}\right)^{1/r'}\right\|_{X'} \le \left\|\left(\sum_{n=0}^{N}\left|\mathrm{E}[\widetilde{f}_n|\mathcal{F}_n]\right|^{r'}\right)^{1/r'}\right\|_{X'} \tag{3.12}$$

and

$$\|\widetilde{f}\|_{X'[\ell_{r'}]} \le \left\|\left(\sum_{n=0}^{\infty}|f_n|^{r'}\right)^{1/r'} + \varepsilon\right\|_{X'} \le \|f\|_{X'[\ell_{r'}]} + \varepsilon\|\mathbb{1}\|_{X'}, \tag{3.13}$$

where $\mathbb{1}$ is the constant function on Ω with value 1. Since $\sum_{n=0}^{N}\bigl|\mathrm{E}[\widetilde{f}_n\,|\,\mathcal{F}_n]\bigr|^{r'} > 0$ a.s., we have

$$\left\|\left(\sum_{n=0}^{N}\bigl|\mathrm{E}[\widetilde{f}_n\,|\,\mathcal{F}_n]\bigr|^{r'}\right)^{1/r'}\right\|_{X'} \le C_X\|\widetilde{f}\|_{X'[\ell_{r'}]} \tag{3.14}$$

Combining (3.12)–(3.14) yields

$$\left\|\left(\sum_{n=0}^{N}\bigl|\mathrm{E}[f_n\,|\,\mathcal{F}_n]\bigr|^{r'}\right)^{1/r'}\right\|_{X'} \le C_X\|f\|_{X'[\ell_{r'}]} + \varepsilon C_X\|\mathbb{1}\|_{X'}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (3.10) holds without the assumption that (3.11) holds. This completes the proof. □

We now turn to the proof of Theorem 3.3. To begin with, we note that if $a \in (1,\infty)$ and if X is an r.i. space with $\beta_X < 1/a$, then $X \hookrightarrow L_a(\Omega)$. The proof of this fact can be found in [25, p. 132]; we give here another proof of this fact. Let $z \in X$ and suppose that $\beta_X < 1/a$. Since $\mathcal{P}_a \in B(\widehat{X})$, it then follows from Lemma 2.7 that

$$\bigl\|(\mathcal{P}z^{*\,a})^{1/a}\bigr\|_{\hat{X}} \le \frac{1}{a}\|\mathcal{P}_a z^*\|_{\hat{X}} \le C\|z^*\|_{\hat{X}} = C\|z\|_X,$$

where $C = a^{-1}\|\mathcal{P}_a\|_{B(\hat{X})}$. On the other hand, since

$$\|z\|_{L_a(\Omega)}^a = \|z^*\|_{L_a(I)}^a = (\mathcal{P}z^{*\,a})(1) \le (\mathcal{P}z^{*\,a})(t)$$

for all $t \in I$, we have that

$$\|z\|_{L_a(\Omega)} = \bigl\|\|z\|_{L_a(\Omega)}\mathbb{1}\bigr\|_{L_1(I)} \le \bigl\|(\mathcal{P}z^{*\,a})^{1/a}\bigr\|_{L_1(I)} \le \|\mathbb{1}\|_{\hat{X}'}\bigl\|(\mathcal{P}z^{*\,a})^{1/a}\bigr\|_{\hat{X}},$$

where $\mathbb{1}$ is the constant function on I with value 1. It follows that $\|z\|_{L_a(\Omega)} \le C\|\mathbb{1}\|_{\hat{X}'}\|z\|_X$, and thus $X \hookrightarrow L_a(\Omega)$.

Proof of Theorem 3.3. (ii) ⇒ (i). Let $r \in (1,\infty)$, $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, and $f = (f_n) \in \mathbb{P}$. Suppose that (ii) holds. Then we may assume that X is an r.i. space with respect to the norm $\|\cdot\|_X$. We may also assume that $f = (f_n) \in X[\ell_r]$, since otherwise (3.2) is obvious. Define a random variable $z^{(r)}$ and a process $\rho^{(r)} = \bigl(\rho_n^{(r)}\bigr)$ by (3.3). Choose $a > 1$ so that $\beta_X < 1/a$. Then $X[\ell_r] \hookrightarrow L_a[\ell_r]$, and therefore $f = (f_n) \in L_a[\ell_r]$. Hence by (b) of Lemma 3.4, $z^{(r)} \in L_a(\Omega)$, $\rho_\infty^{(r)} \in L_a(\Omega)$, and

$$\mathrm{E}\bigl[\bigl(\rho_\infty^{(r)} - \rho_{n-1}^{(r)}\bigr)^a\bigm|\mathcal{F}_n\bigr] \le \mathrm{E}\bigl[\bigl(K_a z^{(r)}\bigr)^a\bigm|\mathcal{F}_n\bigr] \quad \text{a.s.},$$

for all $n \in \mathbb{Z}_+$, where $K_a = a \vee a'$. Note that the constant K_a depends only on β_X. We can apply (b) of Proposition 2.6 to deduce that

$$\|{}^{O(\mathcal{F})}f\|_{X[\ell_r]} = \|\rho_\infty^{(r)}\|_X \le C_{X,a}K_a\|z^{(r)}\|_X = C_{X,a}K_a\|f\|_{X[\ell_r]}.$$

Thus (3.2) holds with the constant $C_X := C_{X,a}K_a$ which depends only on X.

(i) $\Rightarrow$ (ii). Suppose that (i) holds. We claim that (3.1) holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and all $f = (f_n) \in \mathbb{P}$. We may assume that $f = (f_n) \in X[\ell_1]$, since otherwise (3.1) is obvious. Since

$$\left(\sum_{n=0}^{\infty} |{}^{O(\mathcal{F})} f_n|^r\right)^{1/r} \uparrow \sum_{n=0}^{\infty} |{}^{O(\mathcal{F})} f_n| \quad \text{and} \quad \left(\sum_{n=0}^{\infty} |f_n|^r\right)^{1/r} \uparrow \sum_{n=0}^{\infty} |f_n| \qquad \text{as } r \downarrow 1,$$

and since the constant C_X in (3.2) does not depend on r, we have that

$$\|{}^{O(\mathcal{F})} f\|_{X[\ell_1]} = \lim_{r\downarrow 1} \|{}^{O(\mathcal{F})} f\|_{X[\ell_r]} \leq C_X \lim_{r\downarrow 1} \|f\|_{X[\ell_r]} = C_X \|f\|_{X[\ell_1]},$$

as claimed. From Theorem 3.1 it follows that X can be renormed so as to be r.i. and $\alpha_X > 0$.

It remains to show that $\beta_X < 1$. According to Lemma 3.8, the inequality

$$\|{}^{O(\mathcal{F})} f\|_{X'[\ell_{r'}]} \leq C_X \|f\|_{X'[\ell_{r'}]}$$

holds for all $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, all $f = (f_n) \in \mathbb{P}$, and all $r \in (1, \infty)$, where C_X is a positive constant depending only on X. Hence by what we have proved above, $\alpha_{X'} > 0$. This together with (2.3) implies that $\beta_X < 1$, as desired. □

4. Inequalities for predictable projections in a Banach function space

In this section, we consider predictable projections of adapted processes. Recall that by convention $\mathcal{F}_{-1} \equiv \mathcal{F}_0$ for every $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$.

Notice that (4.1) and (4.2) below are inequalities for adapted processes.

Theorem 4.1. *Let X be a Banach function space over Ω. The following are equivalent:*

(i) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$, all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}(\mathcal{F})$, and all $r \in (1, \infty]$,*

$$\|{}^{P(\mathcal{F})} f\|_{X[\ell_r]} \leq C_X \|f\|_{X[\ell_r]}. \tag{4.1}$$

(ii) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}(\mathcal{F})$,*

$$\|{}^{P(\mathcal{F})} f\|_{X[\ell_1]} \leq C_X \|f\|_{X[\ell_1]}. \tag{4.2}$$

(iii) *X can be renormed so as to be r.i. and $\alpha_X > 0$.*

Remark 4.2. From the proof of Theorem 4.1, one sees that the interval $(1, \infty]$ can be replaced by $(1, 1+\varepsilon)$ in (i) of Theorem 4.1, where $\varepsilon > 0$.

In order to prove the theorem above, we use the following lemma, which is a variant of the inequality established by Delbaen and Schachermayer (see (1.2)). Notice that if $n = 0$, then by convention $\rho_{n-1}^{(r)} = \rho_{-1}^{(r)} \equiv 0$ in the following lemma.

Lemma 4.3. *Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathbb{P}$. For each $r \in [1, \infty)$, define a random variable $z^{(r)}$ and a process $\rho^{(r)} = \big(\rho_n^{(r)}\big)$ by letting*

$$z^{(r)} = \left(\sum_{n=0}^{\infty} |f_n|^r \right)^{1/r} \quad \text{and} \quad \rho_n^{(r)} = \left(\sum_{k=0}^{n+1} \big| \mathrm{E}[\, f_k \,|\, \mathcal{F}_{k-1}] \big|^r \right)^{1/r},$$

respectively; and define a random variable $z^{(\infty)}$ and a process $\rho^{(\infty)} = \big(\rho_n^{(\infty)}\big)$ by letting

$$z^{(\infty)} = \sup_{0 \le k < \infty} |f_k| \quad \text{and} \quad \rho_n^{(\infty)} = \sup_{0 \le k \le n+1} \big| \mathrm{E}[\, f_k \,|\, \mathcal{F}_{k-1}] \big| \quad (n \in \mathbb{Z}_+),$$

respectively. Let $\rho_\infty^{(r)} = \lim_{n\to\infty} \rho_n^{(r)}$ a.s. for each $r \in [1, \infty]$.

If $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, $f = (f_n) \in L_1[\ell_r] \cap \mathbb{P}(\mathcal{F})$, and $r \in [1, \infty]$, then $z^{(r)} \in L_1(\Omega)$, $\rho_\infty^{(r)} \in L_1(\Omega)$, and

$$\mathrm{E}\big[\rho_\infty^{(r)} - \rho_{n-1}^{(r)} \big| \mathcal{F}_n\big] \le \mathrm{E}\big[2z^{(r)} \big| \mathcal{F}_n\big] \quad \text{a.s.}$$

for all $n \in \mathbb{Z}_+$.

Proof. Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, $f = (f_n) \in L_1[\ell_r] \cap \mathbb{P}(\mathcal{F})$, and $r \in [1, \infty]$. It is then clear that $z^{(r)} \in L_1(\Omega)$. Hence $\rho_\infty^{(r)} \in L_1(\Omega)$ by (1.2). Let $n \in \mathbb{Z}_+$ and $A \in \mathcal{F}_n$. We let $m_n = n + 1$ for $n \ge 1$, and $m_n = 0$ for $n = 0$. If $r \in [1, \infty)$, then by (1.2),

$$\begin{aligned} \mathrm{E}\big[\big(\rho_\infty^{(r)} - \rho_{n-1}^{(r)}\big)\mathbb{1}_A\big] &\le \mathrm{E}\left[\left(\sum_{k=m_n}^{\infty} \big| \mathrm{E}[\, f_k \mathbb{1}_A \,|\, \mathcal{F}_{k-1}] \big|^r \right)^{1/r} \right] \\ &\le 2\mathrm{E}\left[\left(\sum_{k=m_n}^{\infty} |f_k|^r \right)^{1/r} \mathbb{1}_A \right] \le \mathrm{E}\big[2z^{(r)} \mathbb{1}_A\big]. \end{aligned}$$

Since $A \in \mathcal{F}_n$ is arbitrary, we have

$$\mathrm{E}\big[\rho_\infty^{(r)} - \rho_{n-1}^{(r)} \big| \mathcal{F}_n\big] \le \mathrm{E}\big[2z^{(r)} \big| \mathcal{F}_n\big] \quad \text{a.s.}$$

If $r = \infty$. then by (1.2),

$$\begin{aligned} \mathrm{E}\big[\big(\rho_\infty^{(\infty)} - \rho_{n-1}^{(\infty)}\big)\mathbb{1}_A\big] &\le \mathrm{E}\left[\sup_{m_n \le k < \infty} \big| \mathrm{E}[\, f_k \mathbb{1}_A \,|\, \mathcal{F}_{k-1}] \big| \right] \\ &\le 2\mathrm{E}\left[\sup_{m_n \le k < \infty} |f_k| \mathbb{1}_A \right] \le \mathrm{E}\big[2z^{(\infty)} \mathbb{1}_A\big]. \end{aligned}$$

Since $A \in \mathcal{F}_n$ is arbitrary, we have

$$\mathrm{E}\big[\rho_\infty^{(\infty)} - \rho_{n-1}^{(\infty)} \big| \mathcal{F}_n\big] \le \mathrm{E}\big[2z^{(\infty)} \big| \mathcal{F}_n\big] \quad \text{a.s.}$$

Thus the lemma is proved. □

Proof of Theorem 4.1. (iii) ⇒ (i). Let $r \in (1, \infty]$, $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, and $f = (f_n) \in \mathbb{P}(\mathcal{F})$. Suppose that (iii) holds. Then we may assume that X is an r.i. space with respect to the norm $\|\cdot\|_X$. We may also assume that $f = (f_n) \in X[\ell_r]$, since otherwise (4.1) is obvious. Define a random variable $z^{(r)}$ and a process $\rho^{(r)} = \big(\rho_n^{(r)}\big)$ as in Lemma 4.3. Since $X[\ell_r] \hookrightarrow L_1[\ell_r]$, it follows that $f = (f_n) \in L_1[\ell_r]$. Hence by Lemma 4.3, $z^{(r)} \in L_1(\Omega)$, $\rho_\infty^{(r)} \in L_1(\Omega)$, and

$$\mathrm{E}\big[\rho_\infty^{(r)} - \rho_{n-1}^{(r)} \big| \mathcal{F}_n\big] \le \mathrm{E}\big[2z^{(r)} \big| \mathcal{F}_n\big] \quad \text{a.s.}$$

for all $n \in Z$. Since $\alpha_X > 0$, we can apply (a) of Proposition 2.6 to deduce that

$$\|{}^{P(\mathcal{F})}f\|_{X[\ell_r]} = \|\rho_\infty^{(r)}\|_X \le 2C_X \|z^{(r)}\|_X = 2C_X \|f\|_{X[\ell_r]},$$

where C_X is a positive constant which depends only on X. Thus (i) holds.

(i) ⇒ (ii). Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and suppose that (i) holds. We may assume that $f = (f_n) \in X[\ell_1]$, since otherwise (4.2) is obvious. Since

$$\left(\sum_{n=0}^{\infty} |{}^{P(\mathcal{F})}f_n|^r\right)^{1/r} \uparrow \sum_{n=0}^{\infty} |{}^{P(\mathcal{F})}f_n| \quad \text{and} \quad \left(\sum_{n=0}^{\infty} |f_n|^r\right)^{1/r} \uparrow \sum_{n=0}^{\infty} |f_n| \qquad \text{as } r \downarrow 1$$

and since the constant C_X in (4.1) does not depend on r, we have that

$$\|{}^{P(\mathcal{F})}f\|_{X[\ell_1]} = \lim_{r\downarrow 1} \|{}^{P(\mathcal{F})}f\|_{X[\ell_r]} \le C_X \lim_{r\downarrow 1} \|f\|_{X[\ell_r]} = C_X \|f\|_{X[\ell_1]},$$

even when $f = (f_n)$ is not in $X[\ell_1]$. Thus (ii) holds.

(ii) ⇒ (iii). Suppose that (ii) holds. Let $\mathcal{A}$ be a sub-σ-algebra of Σ and let $x \in X$. Define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathbb{P}(\mathcal{F})$ by letting

$$\mathcal{F}_n = \begin{cases} \mathcal{A} & \text{if } n = 0, \\ \Sigma & \text{if } n \ge 1, \end{cases} \quad \text{and} \quad f_n = \begin{cases} x & \text{if } n = 1, \\ 0 & \text{if } n \ne 1, \end{cases} \tag{4.3}$$

respectively. Then by (4.2), we have $\big\|\mathrm{E}[x\,|\,\mathcal{A}]\big\|_X \le C_X \|x\|_X$. From Proposition 2.4 it follows that X can be renormed so as to be r.i. Hence, for the rest of the proof, we assume that X is an r.i. space with respect to the norm $\|\cdot\|_X$.

To show that $\alpha_X > 0$, it suffices to show that $Q \in B(\widehat{X})$. Let $\eta \in \widehat{X}$, and define x and $\{A(t)\colon 0 < t \le 1\}$ as in Lemma 3.5. For each $n \in \mathbb{Z}_+$, let $t_n = 2^{-n}$ and define a sequence of sets $\{A_n\}_{n\in\mathbb{Z}_+}$ by $A_n = A(t_n)$. Define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathbb{P}(\mathcal{F})$ by letting

$$\mathcal{F}_n = \sigma(\{A \setminus A_{n+1}\colon A \in \Sigma\}) \quad \text{and} \quad f_n = \begin{cases} x\mathbb{1}_{A_n\setminus A_{n+1}} & \text{if } n \ge 1, \\ \mathrm{E}[x\mathbb{1}_{A_0\setminus A_1}] & \text{if } n = 0, \end{cases}$$

respectively. Then, as in the proof of Theorem 3.1, we have that for $n \ge 1$,

$$\frac{1}{2}\big[(Q|\eta|)(t_{n+1}) - (Q|\eta|)(t_n)\big]\mathbb{1}_{A_n} \le \mathrm{E}[f_n\,|\,\mathcal{F}_{n-1}] \quad \text{a.s.} \tag{4.4}$$

Furthermore, when $n = 0$, we have that $\mathcal{F}_{n-1} = \mathcal{F}_{-1} = \mathcal{F}_0$ and

$$\begin{aligned} \frac{1}{2}\big[(Q|\eta|)(t_{n+1}) - (Q|\eta|)(t_n)\big]\mathbb{1}_{A_0} &= \frac{1}{2}\int_{t_1}^{t_0} \frac{|\eta(s)|}{s}\,\mathrm{d}s \\ &\le \int_{t_1}^{t_0} |\eta(x)|\,\mathrm{d}s = \mathrm{E}[x\mathbb{1}_{A_0\setminus A_1}] \\ &= f_0 = \mathrm{E}[f_n\,|\,\mathcal{F}_{n-1}]. \end{aligned}$$

Thus (4.4) holds for all $n \in \mathbb{Z}_+$. Therefore, as in the proof of Theorem 3.1, we have that

$$|(Q\eta)(t)| \le (Q|\eta|)(t) \le 2\left(\sum_{n=0}^{\infty} \mathrm{E}[f_n\,|\,\mathcal{F}_{n-1}]\right)^{*}(t)$$

for all $t \in I$. Hence

$$\|Q\eta\|_{\hat{X}} \le 2\left\|\left(\sum_{n=0}^{\infty} \mathrm{E}[f_n\,|\,\mathcal{F}_{n-1}]\right)^{*}\right\|_{\hat{X}} = 2\|{}^{P(\mathcal{F})}f\|_{X[\ell_1]}.$$

Since $\|f\|_{X[\ell_1]} = \|x\|_X = \|\eta\|_{\hat{X}}$, we conclude from (4.2) that $\|Q\eta\|_{\hat{X}} \le 2C_X\|\eta\|_{\hat{X}}$. This implies $\alpha_X > 0$, as desired. □

5. Inequalities for optional and predictable projections in w-X

Recall that the indices p_φ and q_φ of a quasi-concave function $\varphi\colon [0,1] \to [0,\infty)$ are defined by (2.4), and that the indices p_X and q_X of a Banach function space X (over Ω) are given by $p_X = p_{\overline{\varphi}_X}$ and $q_X = q_{\overline{\varphi}_X}$. Recall also that the index k_X of a Banach function space X is defined by (2.1).

In this section, we prove the following theorem.

Theorem 5.1. *Let X be a Banach function space over Ω. The following are equivalent:*

(i) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$, all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}$, and all $r \in (1,\infty]$,*
$$\|^{O(\mathcal{F})} f\|_{\text{w-}X[\ell_r]} \le C_X \|f\|_{\text{w-}X[\ell_r]}. \tag{5.1}$$

(ii) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}$,*
$$\|^{O(\mathcal{F})} f\|_{\text{w-}X[\ell_1]} \le C_X \|f\|_{\text{w-}X[\ell_1]}. \tag{5.2}$$

(iii) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$, all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}(\mathcal{F})$, and all $r \in (1,\infty]$,*
$$\|^{P(\mathcal{F})} f\|_{\text{w-}X[\ell_r]} \le C_X \|f\|_{\text{w-}X[\ell_r]}. \tag{5.3}$$

(iv) *There exists a positive constant C_X which depends only on X such that for all $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{P}(\mathcal{F})$,*
$$\|^{P(\mathcal{F})} f\|_{\text{w-}X[\ell_1]} \le C_X \|f\|_{\text{w-}X[\ell_1]}. \tag{5.4}$$

(v) $0 < p_X\,,\ q_X < 1$ *and* $k_X < \infty$.

Remark 5.2. From the proof of Theorem 5.1, one sees that the interval $(1,\infty]$ can be replaced by $(1, 1+\varepsilon)$ in (i) and (iii) of Theorem 5.1, where $\varepsilon > 0$.

In order to prove the theorem above, we need some preliminary results. The following proposition is an analogue of [4, Lemma 7.1], which asserts that a distribution function inequality implies a quasi-norm inequality in Marcinkiewicz function space.

Proposition 5.3. *Let $x, y \in L_0(\Omega)$ be nonnegative, let $b \in (1,\infty)$ be a fixed number, and let $\varphi\colon [0,1] \to [0,\infty)$ be a quasi-concave function. Suppose there exists a function $\alpha\colon (0,1) \to (0,\infty)$ such that the inequality*
$$\mathrm{P}\{y > b\lambda,\ x \le \delta\lambda\} \le \alpha(\delta)\mathrm{P}\{y > \lambda\} \tag{5.5}$$
holds for all $\delta \in (0,1)$ and all $\lambda \in (0,\infty)$. If $p_\varphi > 0$ and if $\alpha(\delta) \to 0$ as $\delta \to 0+$, then there is a positive constant C, depending only on φ, b, and α, such that
$$\|y\|_{M^*(\varphi:\Omega)} \le C\|x\|_{M^*(\varphi:\Omega)}. \tag{5.6}$$

For the proof of the proposition above, we adopt the additional convention that if $y \in L_0(\Omega)$ and if $1 \le t < \infty$, then $y^*(t) = 0$.

Lemma 5.4. *Let $x, y \in L_0(\Omega)$ and $b \in (1, \infty)$ be as in Proposition 5.3, and suppose that a function $\alpha\colon (0,1) \to (0,\infty)$ satisfies* (5.5) *for all $\delta \in (0,1)$ and all $\lambda \in (0,\infty)$. Then*

$$y^*(t) \le b y^*\left(\frac{t}{2\alpha(\delta)}\right) + \frac{b}{\delta} x^*\left(\frac{t}{2}\right) \tag{5.7}$$

for all $\delta \in (0,1)$ and all $t \in I$.

Note that if $2\alpha(\delta) \le t \le 1$, then by convention, (5.7) can be written as

$$y^*(t) \le \frac{b}{\delta} x^*\left(\frac{t}{2}\right). \tag{5.8}$$

Before proving Lemma 5.4, we note that if $x \in L_0(\Omega)$ is nonnegative, then

$$\mathrm{P}\{x > x^*(t)\} \le t \quad \text{for all } t \in I.$$

This is an immediate consequence of the definition of x^*.

Proof of Lemma 5.4. Fix $\delta \in (0,1)$. Then by (5.5) we have that

$$\begin{aligned} \mathrm{P}\{y > b\lambda\} &= \mathrm{P}\{y > b\lambda,\ x \le \delta\lambda\} + \mathrm{P}\{y > b\lambda,\ x > \delta\lambda\} \\ &\le \alpha(\delta)\mathrm{P}\{y > \lambda\} + \mathrm{P}\{x > \delta\lambda\} \end{aligned} \tag{5.9}$$

for all $\lambda \in (0,\infty)$. Suppose first that $t \in I$ and $0 < t \le 2\alpha(\delta)$. If we set

$$\lambda = y^*\left(\frac{t}{2\alpha(\delta)}\right) + \frac{1}{\delta} x^*\left(\frac{t}{2}\right),$$

then by (5.9),

$$\mathrm{P}\{y > b\lambda\} \le \alpha(\delta)\mathrm{P}\left\{y > y^*\left(\frac{t}{2\alpha(\delta)}\right)\right\} + \mathrm{P}\left\{x > x^*\left(\frac{t}{2}\right)\right\} \le t.$$

This implies (5.7). Suppose now that $2\alpha(\delta) < t \le 1$. It then follows from (5.9) that

$$\mathrm{P}\{y > b\lambda\} \le \alpha(\delta) + \mathrm{P}\{x > \delta\lambda\} \le \frac{t}{2} + \mathrm{P}\{x > \delta\lambda\}. \tag{5.10}$$

If we set

$$\lambda = y^*\left(\frac{t}{2\alpha(\delta)}\right) + \frac{1}{\delta}x^*\left(\frac{t}{2}\right) = \frac{1}{\delta}x^*\left(\frac{t}{2}\right),$$

then by (5.10),

$$\mathrm{P}\{y > b\lambda\} \le \frac{t}{2} + \mathrm{P}\left\{x > x^*\left(\frac{t}{2}\right)\right\} \le t.$$

This implies (5.7). Thus Lemma 5.4 is proved. □

Proof of Proposition 5.3. Suppose that $p_\varphi > 0$ and $\alpha(\delta) \to 0$ as $\delta \to 0+$. Let $a = p_\varphi/2$. Then by (2.5) there exists $s_0 \in (0,1)$, depending only on φ, such that if $0 < s < s_0$, then $a < \log m_\varphi(s)/\log s$, or equivalently,

$$\sup_{0<u\le 1} \frac{\varphi(su)}{\varphi(u)} = m_\varphi(s) < s^a.$$

Thus $\varphi(su) \le s^a \varphi(u)$ for all $u \in I$ whenever $0 < s < s_0$. Since $\alpha(\delta) \to 0$ as $\delta \to 0+$, there exists $\delta_1 > 0$, depending only on α and s_0, such that $2\alpha(\delta) < s_0$ whenever $0 < \delta \le \delta_1$. It follows that if $0 < \delta \le \delta_1$ and $0 < t \le 2\alpha(\delta)$, then

$$\varphi(t) \le 2^a \alpha(\delta)^a \varphi\left(\frac{t}{2\alpha(\delta)}\right). \tag{5.11}$$

Furthermore, since $\varphi(t)/t$ is nonincreasing, we have

$$\varphi(t) \le 2\varphi\left(\frac{t}{2}\right) \tag{5.12}$$

for all $t \in I$. Hence if $0 < \delta \le \delta_1$ and $0 < t \le 2\alpha(\delta)$, then by (5.7), (5.11), and (5.12),

$$\begin{aligned}\varphi(t)y^*(t) &\le b\varphi(t)y^*\left(\frac{t}{2\alpha(\delta)}\right) + \frac{b}{\delta}\varphi(t)x^*\left(\frac{t}{2}\right)\\ &\le 2^a b\alpha(\delta)^a \varphi\left(\frac{t}{2\alpha(\delta)}\right)y^*\left(\frac{t}{2\alpha(\delta)}\right) + \frac{2b}{\delta}\varphi\left(\frac{t}{2}\right)x^*\left(\frac{t}{2}\right)\\ &\le 2^a b\alpha(\delta)^a \|y\|_{M^*(\varphi:\Omega)} + \frac{2b}{\delta}\|x\|_{M^*(\varphi:\Omega)}.\end{aligned}$$

On the other hand, if $2\alpha(\delta) < t \le 1$, then by (5.8) and (5.12),

$$\varphi(t)y^*(t) \le \frac{2b}{\delta}\varphi\left(\frac{t}{2}\right)x^*\left(\frac{t}{2}\right) \le \frac{2b}{\delta}\|x\|_{M^*(\varphi:\Omega)}.$$

Consequently,

$$\|y\|_{M^*(\varphi:\Omega)} \le 2^a b\alpha(\delta)^a \|y\|_{M^*(\varphi:\Omega)} + \frac{2b}{\delta}\|x\|_{M^*(\varphi:\Omega)},$$

provided $0 < \delta \le \delta_1$. Since $\alpha(\delta) \to 0$ as $\delta \to 0+$, there exists $\delta_0 \in (0, \delta_1]$, depending only on φ, b, and α, such that $2^a b\alpha(\delta_0)^a \le 1/2$. Thus if $\|y\|_{M^*(\varphi:\Omega)} < \infty$, then

$$\frac{1}{2}\|y\|_{M^*(\varphi:\Omega)} \le \left(1 - 2^a b\alpha(\delta_0)^a\right)\|y\|_{M^*(\varphi:\Omega)} \le \frac{2b}{\delta_0}\|x\|_{M^*(\varphi:\Omega)}.$$

Thus (5.6) holds with $C = 4b/\delta_0$.

To complete the proof, we must show that if the left-hand side of (5.6) is infinite, then so is the right-hand side. Note that (5.5) holds with y replaced by $y \wedge n$, where n is a positive integer. Since $\|y \wedge n\|_{M^*(\varphi:\Omega)} < \infty$, we have

$$\|y \wedge n\|_{M^*(\varphi:\Omega)} \leq C\|x\|_{M^*(\varphi:\Omega)}.$$

Letting $n \to \infty$, we see that the right-hand side of (5.6) is infinite when the left-hand side is infinite. Thus the proof of Proposition 5.3 is complete. □

The following proposition is an analogue of Proposition 2.6.

Proposition 5.5. *Let $\varphi\colon [0,1] \to [0,\infty)$ be a quasi-concave function, let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, let $\rho = (\rho_n) \in \mathbb{P}(\mathcal{F})$, and let z be a nonnegative random variable on Ω. Suppose that $0 \leq \rho_n \leq \rho_{n+1}$ a.s. for all $n \in \mathbb{Z}_+$, and let $\rho_\infty = \lim_{n\to\infty} \rho_n$ a.s.*

If $1 \leq a < \infty$, $0 < p_\varphi$, $q_\varphi < 1/a$, $\rho_\infty \in L_a(\Omega)$, $z \in L_a(\Omega)$, and

$$\mathrm{E}\big[(\rho_\infty - \rho_{n-1})^a \big| \mathcal{F}_n \big] \leq \mathrm{E}[z^a | \mathcal{F}_n] \quad a.s. \tag{5.13}$$

for all $n \in \mathbb{Z}_+$, then there exists a positive constant $C_{a,\varphi}$ which depends on a and φ such that

$$\|\rho_\infty\|_{M^*(\varphi:\Omega)} \leq C_{a,\varphi}\|z\|_{M^*(\varphi:\Omega)}. \tag{5.14}$$

To prove the proposition above, we need the following lemma.

Lemma 5.6. *Let $\varphi\colon [0,1] \to [0,\infty)$ be a quasi-concave function, and suppose that $q_\varphi < 1$. Then $M^*(\varphi:\Omega) = M(\varphi:\Omega)$ and*

$$\|x\|_{M^*(\varphi:\Omega)} \leq \|x\|_{M(\varphi:\Omega)} \leq K_\varphi\|x\|_{M^*(\varphi:\Omega)} \tag{5.15}$$

for all $x \in M^*(\varphi:\Omega) = M(\varphi:\Omega)$, where K_φ is a positive constant which depends only on φ. Moreover we may replace $M^*(\varphi:\Omega)$ and $M(\varphi:\Omega)$ above with $M^*(\varphi:I)$ and $M(\varphi:I)$, respectively.

Proof. It is clear that $M(\varphi:\Omega) \subset M^*(\varphi:\Omega)$ and the first inequality of (5.15) holds for all $x \in M(\varphi:\Omega)$.

Suppose that $q_\varphi < 1$. Then $\mathcal{P} \in B(M^*(\varphi:I))$ by [19, Proposition 3.2]. It follows that if $x \in M^*(\varphi:\Omega)$, then

$$\|x\|_{M(\varphi:\Omega)} = \|\mathcal{P}x^*\|_{M^*(\varphi:I)} \leq K_\varphi\|x^*\|_{M^*(\varphi:I)} = K_\varphi\|x\|_{M^*(\varphi:\Omega)},$$

where $K_\varphi = \|\mathcal{P}\|_{B(M^*(\varphi:I))}$. Thus $M^*(\varphi:\Omega) \hookrightarrow M(\varphi:\Omega)$.

The argument above remains valid when Ω is replaced by I. Thus the proof of lemma is complete. □

Proof of Proposition 5.5. Suppose that $1 \leq a < \infty$, $0 < p_\varphi$, $q_\varphi < 1/a$, $\rho_\infty \in L_a(\Omega)$, $z \in L_a(\Omega)$, and (5.13) holds for all $n \in \mathbb{Z}_+$. We may assume that $z \in M^*(\varphi:\Omega)$, since otherwise (5.14) is obvious. It then follows from Lemma 5.6 that $z \in M(\varphi:\Omega)$.

Let $\delta \in (0,1)$, let $\lambda \in (0,\infty)$, and let $\vartheta = (\vartheta_n)$ be the uniformly integrable martingale defined by $\vartheta_n = \mathrm{E}[\,z^a\,|\,\mathcal{F}_n\,]$ for $n \in \mathbb{Z}_+$. Define $\mathcal{F}$-stopping times σ and τ by letting

$$\sigma = \min\{n \in \mathbb{Z}_+ \colon \rho_n > \lambda\} \quad \text{and} \quad \tau = \min\{n \in \mathbb{Z}_+ \colon \vartheta_n > \delta^a \lambda^a\},$$

with the convention that $\min\emptyset = \infty$. It is then clear that $\{\rho_\infty > 2\lambda\} \subset \{\sigma < \infty\}$ and $\rho_\infty - \rho_{\sigma-1} > \lambda$ on $\{\rho_\infty > 2\lambda\}$. Let $M\vartheta = \sup_{n\in\mathbb{Z}_+}|\vartheta_n|$. Then we have

$$\{(M\vartheta)^{1/a} \le \delta\lambda\} = \{\tau = \infty\},$$

and hence

$$\big\{\rho_\infty > 2\lambda,\ (M\vartheta)^{1/a} \le \delta\lambda\big\} \subset \big\{\rho_\infty - \rho_{\sigma-1} > \lambda,\ \sigma < \tau = \infty\big\}.$$

On the other hand, by (5.13) we have

$$\mathrm{E}\big[\,(\rho_\infty - \rho_{\sigma-1})^a \big| \mathcal{F}_\sigma\,\big] \le \vartheta_\sigma \le \delta^a\lambda^a \quad \text{a.s. on } \{\sigma < \tau\}.$$

Therefore

$$\begin{aligned}
\mathrm{P}\{\rho_\infty > 2\lambda,\ (M\vartheta)^{1/a} \le \delta\lambda\} &\le \mathrm{P}\{\rho_\infty - \rho_{\tau-1} > \lambda,\ \sigma < \tau = \infty\} \\
&\le \frac{1}{\lambda^a}\,\mathrm{E}\big[\,(\rho_\infty - \rho_{\sigma-1})^a \mathbb{1}_{\{\sigma<\tau\}}\,\big] \\
&\le \frac{1}{\lambda^a}\,\mathrm{E}\big[\mathrm{E}\big[\,(\rho_\infty - \rho_{\sigma-1})^a \big| \mathcal{F}_\sigma\,\big]\mathbb{1}_{\{\sigma<\tau\}}\,\big] \\
&\le \delta^a\,\mathrm{P}\{\sigma < \tau\} \\
&\le \delta^a\,\mathrm{P}\{\rho_\infty > \lambda\}.
\end{aligned}$$

Hence, by Proposition 5.3, there exists a positive constant $C''_{a,\varphi}$ which depends on φ and a such that

$$\begin{aligned}
\|\rho_\infty\|_{M^*(\varphi:\Omega)} &\le C''_{a,\varphi}\|(M\vartheta)^{1/a}\|_{M^*(\varphi:\Omega)} \\
&\le C''_{a,\varphi}\|(M\vartheta)^{1/a}\|_{M(\varphi:\Omega)} = C''_{a,\varphi}\big\|\big((M\vartheta)^*\big)^{1/a}\big\|_{M(\varphi:I)}.
\end{aligned}$$

Note that by Doob's inequality, we have $u\mathrm{P}\{M\vartheta > u\} \le \mathrm{E}[\,z^a\mathbb{1}_{\{M\vartheta>u\}}\,]$ for all $u > 0$ (see, for example, [27, p. 150]). Hence by [15, Lemma 5],

$$(M\vartheta)^*(t) \le (\mathcal{P}z^{a\,*})(t) = (\mathcal{P}z^{*\,a})(t)$$

for all $t \in I$. It follows from Lemma 2.7 that

$$\big((M\vartheta)^*(t)\big)^{1/a} \le \big((\mathcal{P}z^{*\,a})(t)\big)^{1/a} \le a^{-1}(\mathcal{P}_a z^*)(t)$$

for all $t \in I$. According to [19, Proposition 3.1], $\beta_{M(\varphi:I)} = q_\varphi < 1/a$, and therefore $\mathcal{P}_a \in B(M(\varphi:I))$ (see [2, p. 150]). Consequently

$$\|\rho_\infty\|_{M^*(\varphi:\Omega)} \le a^{-1}C''_{a,\varphi}\|\mathcal{P}_a z^*\|_{M(\varphi:I)} \le C'_{a,\varphi}\|z^*\|_{M(\varphi:I)} = C'_{a,\varphi}\|z\|_{M(\varphi:\Omega)},$$

where $C'_{a,\varphi} = a^{-1}C''_{a,\varphi}\|\mathcal{P}_a\|_{B(M(\varphi:I))}$. Clearly $C'_{a,\varphi}$ depends only on φ and a. Since $\|z\|_{M(\varphi:\Omega)} \le K_\varphi\|z\|_{M^*(\varphi:\Omega)}$ by Lemma 5.6, we have

$$\|\rho_\infty\|_{M^*(\varphi:\Omega)} \le C_{\varphi,a}\|z\|_{M^*(\varphi:\Omega)},$$

where $C_{\varphi,a} = C'_{\varphi,a}K_\varphi$. This completes the proof. □

In addition to the lemmas above, we need the following lemma; see [18] for the proof.

Lemma 5.7 ([18, Lemma 3])**.** *Let X be a Banach function space over* Ω*. The following are equivalent:*

(i) w-$X = M^*(\overline{\varphi}_X : \Omega)$ *and there exists a positive constant c such that for all* $x \in$ w-X*,*

$$\|x\|_{\text{w-}X} \le \|x\|_{M^*(\overline{\varphi}_X:\Omega)} \le c\|x\|_{\text{w-}X}. \tag{5.16}$$

(ii) $k_X < \infty$.

Moreover if $k_X < \infty$*, then* (5.16) *holds with* $c = k_X$*.*

Combining Proposition 5.5 and Lemma 5.7, we have the following proposition, which is one of the key result for the proof of Theorem 5.1.

Proposition 5.8. *Let X be a Banach function space over* Ω*, let* $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$*, let* $\rho = (\rho_n) \in \mathbb{P}(\mathcal{F})$*, and let z be a nonnegative random variable on* Ω*. Suppose that* $0 \le \rho_n \le \rho_{n+1}$ *a.s. for all* $n \in \mathbb{Z}_+$*, and let* $\rho_\infty = \lim_{n\to\infty}\rho_n$ *a.s.*

If $1 \le a < \infty$*,* $0 < p_X,\ q_X < 1/a$*,* $k_X < \infty$*, and* (5.13) *holds for all* $n \in \mathbb{Z}_+$*, then there exists a positive constants* $C_{a,X}$ *which depends only on a and X such that*

$$\|\rho_\infty\|_{\text{w-}X} \le C_{a,X}\|z\|_{\text{w-}X}.$$

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. (i) ⇒ (ii). Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, let $f = (f_n) \in \mathbb{P}$, and suppose that (i) holds. Since

$$\left(\sum_{n=0}^{\infty}|{}^{O(\mathcal{F})}f_n|^r\right)^{1/r} \uparrow \sum_{n=0}^{\infty}|{}^{O(\mathcal{F})}f_n| \quad\text{and}\quad \left(\sum_{n=0}^{\infty}|f_n|^r\right)^{1/r} \uparrow \sum_{n=0}^{\infty}|f_n| \quad \text{as } r \downarrow 1,$$

and since the constant C_X in (5.1) does not depend on r. we have that

$$\|{}^{O(\mathcal{F})}f\|_{\text{w-}X[\ell_1]} = \lim_{r\downarrow 1}\|{}^{O(\mathcal{F})}f\|_{\text{w-}X[\ell_r]} \le C_X \lim_{r\downarrow 1}\|f\|_{\text{w-}X[\ell_r]} = C_X\|f\|_{\text{w-}X[\ell_1]},$$

even when $f = (f_n)$ is not in w-$X[\ell_1]$. Thus (5.2) holds.

(iii) ⇒ (iv). Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, let $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and suppose that (iii) holds. Replacing ${}^{O(\mathcal{F})}f = ({}^{O(\mathcal{F})}f_n)$ with ${}^{P(\mathcal{F})}f = ({}^{P(\mathcal{F})}f_n)$ in the argument above, one can show that (5.4) holds.

(i) ⇒ (iii). Let $r \in (1, \infty]$, let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, let $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and suppose that (i) holds. Define $\widetilde{\mathcal{F}} = (\widetilde{\mathcal{F}}_n) \in \mathbb{F}$ by

$$\widetilde{\mathcal{F}}_n = \begin{cases} \mathcal{F}_0 & \text{if } n = 0, \\ \mathcal{F}_{n-1} & \text{if } n \geq 1. \end{cases}$$

Then ${}^{P(\mathcal{F})}f = {}^{O(\tilde{\mathcal{F}})}f$. Applying (5.1) with ${}^{O(\mathcal{F})}f$ replaced by ${}^{O(\tilde{\mathcal{F}})}f$, we have that

$$\|{}^{P(\mathcal{F})}f\|_{\text{w-}X[\ell_r]} = \|{}^{O(\tilde{\mathcal{F}})}f\|_{\text{w-}X[\ell_r]} \leq C_X \|f\|_{\text{w-}X[\ell_r]}.$$

Thus (5.3) holds.

(ii) ⇒ (iv). Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, let $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and suppose that (ii) holds. Replacing r with 1 in the argument above, one can show that (5.4) holds.

(iv) ⇒ (v). Let $x \in X$, let $\mathcal{A}$ be a sub-σ-algebra of Σ, and suppose that (iv) holds. Define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathbb{P}(\mathcal{F})$ by (4.3). Then by (5.4),

$$\big\|\mathrm{E}[x\,|\,\mathcal{A}]\big\|_{\text{w-}X} \leq C_X \|x\|_{\text{w-}X} \leq C_X \|x\|_X.$$

From [18, Lemma 3] we see that $k_X < \infty$, and from [20, Theorem 4.1] we see that $q_X = q_{\overline{\varphi}_X} < 1$.

It only remains to show that $p_X > 0$. As shown above, $q_X < 1$ and $k_X < \infty$. Hence by Lemmas 5.6 and 5.7, we have that w-$X = M^*(\overline{\varphi}_X : \Omega) = M(\overline{\varphi}_X : \Omega)$ and

$$\|x\|_{\text{w-}X} \leq \|x\|_{M^*(\overline{\varphi}_X : \Omega)} \leq \|x\|_{M(\overline{\varphi}_X : \Omega)} \leq K_X k_X \|x\|_{\text{w-}X}$$

for all $x \in$ w-X, where K_X is the constant which depends only on X. For simplicity, write $Y = M(\overline{\varphi}_X : \Omega)$. Then (5.4) can be rewritten as

$$\|{}^{P(\mathcal{F})}f\|_{Y[\ell_1]} \leq C\|f\|_{Y[\ell_1]},$$

where $C = C_X K_X k_X$ and where C_X is the constant in (5.4). Since Y is an r.i. Banach function space, we can apply Theorem 4.1 to deduce that $\alpha_{M(\overline{\varphi}_X : \Omega)} = \alpha_Y > 0$. Since $p_X = \alpha_{M(\overline{\varphi}_X : \Omega)}$ by [19, Proposition 3.1], we conclude that $p_X > 0$, as desired.

(v) ⇒ (i). Let $\mathcal{F} = (\mathcal{F}_n) \in \mathcal{F}$, $f = (f_n) \in \mathbb{P}(\mathcal{F})$, and let $r \in (1, \infty]$. Suppose that (v) holds. Then by [19, Proposition 3.1], $0 < \alpha_{M(\overline{\varphi}_X : \Omega)}, \ \beta_{M(\overline{\varphi}_X : \Omega)} < 1$. Hence by Theorems 3.2 and 3.3, we have

$$\|{}^{O(\mathcal{F})}f\|_{M(\overline{\varphi}_X : \Omega)[\ell_r]} \leq C_X \|f\|_{M(\overline{\varphi}_X : \Omega)[\ell_r]}.$$

On the other hand, by Lemmas 5.6 and 5.7, we have w-$X = M(\overline{\varphi}_X : \Omega)$. It follows that (5.1) holds, as desired. □

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