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# On the CLT for rotations and BV functions 

Jean-Pierre Conze<br>Stéphane Le Borgne


#### Abstract

Let $x \mapsto x+\alpha \bmod 1$ be a rotation on the circle and let $\varphi$ be a step function. We denote by $\varphi_{n}(x)$ the corresponding ergodic sums $\sum_{j=0}^{n-1} \varphi(x+j \alpha)$. For a class of irrational rotations (containing the class with bounded partial quotients) and under a Diophantine condition on the discontinuity points of $\varphi$, we show that $\varphi_{n} /\left\|\varphi_{n}\right\|_{2}$ is asymptotically Gaussian for $n$ in a set of density 1 . The proof is based on decorrelation inequalities for the ergodic sums taken at times $q_{k}$, where $\left(q_{k}\right)$ is the sequence of denominators of $\alpha$. Another important point is the control of the variance $\left\|\varphi_{n}\right\|_{2}^{2}$ for $n$ belonging to a large set of integers. When $\alpha$ is a quadratic irrational, the size of this set can be precisely estimated.


## 1. Introduction

For a dynamical system $(X, \mu, T)$ and an observable $\varphi$ on $X$, a general question is the asymptotic behaviour in distribution of the ergodic sums $\sum_{0}^{L-1} \varphi \circ T^{k}$ after normalisation. For a large class of observables and chaotic systems, many results of convergence toward a Gaussian distribution are known.

When the dynamical system has zero entropy, in particular for a rotation, the situation is different. Nevertheless one can ask if, at least, there are observables satisfying a non degenerate Central Limit Theorem. In this direction there are positive answers: R. Burton and M. Denker [4] in 1987, then T. de la Rue, S. Ladouceur, G. Peskir and M. Weber [9], M. Lacey [18] proved for rotations the existence of functions whose ergodic sums satisfy a CLT after self-normalization. In general for a measure preserving aperiodic system, further results by D. Volný and P. Liardet [21], J.-P. Thouvenot and B. Weiss [23] showed that any distribution can appear as a limiting distribution of the ergodic sums of some functions after normalisation.

A different question is to ask if, for smooth systems, there is a CLT for explicit functions in a certain class of regularity. Here we consider step functions on $X=\mathbb{R} / \mathbb{Z}$ and their ergodic sums $\varphi_{n}(x):=\sum_{0}^{n-1} \varphi(x+j \alpha)$ over an irrational rotation $x \mapsto x+\alpha \bmod 1$.

By the Denjoy-Koksma inequality, if $\varphi$ is a centered function with bounded variation, the sequence $\left(\varphi_{n}\right)$ is uniformly bounded along the sub-sequence of denominators of $\alpha$. But, besides, a stochastic behaviour at a certain scale can occur along other sub-sequences $\left(n_{k}\right)$. We propose a quantitative analysis of this phenomenon.

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Let us mention the following related papers. For $\psi:=1_{\left[0, \frac{1}{2}[ \right.} 1_{\left[\frac{1}{2}, 0[ \right.}, F$. Huveneers [15] studied the existence of a sequence $\left(n_{k}\right)_{n \in \mathbb{N}}$ such that $\left(\psi_{n_{k}}\right)$ after normalization is asymptotically normally distributed. In [7] it was shown that, when $\alpha$ has unbounded partial quotients, along some subsequences the ergodic sums of $\varphi$ in a class of step functions can be approximated by a Brownian motion.

Here we will use as in [15] a method based on decorrelation inequalities which applies in particular when the sequence of partial quotients of $\alpha$ is bounded ( $\alpha$ is said to be of bounded type or bpq) or under a slightly more general Diophantine assumption. It relies on an abstract central limit theorem valid under some suitable decorrelation conditions. If $\varphi$ is a step function, we give conditions which ensure that for $n$ in a set of integers of density 1 , the distribution of $\varphi_{n} /\left\|\varphi_{n}\right\|_{2}$ is asymptotically Gaussian (Theorem 3.6). Beside the remarkable recent "temporal" limit theorems for rotations of bounded type (see $[1,2,3,10]$ ), this shows that a "spatial" asymptotic normal distribution can also be observed for $n$ in a large set of integers.

An important point is the control of the variance $\left\|\varphi_{n}\right\|_{2}^{2}$. In Section 2, we study the set of integers for which the variance $\left\|\varphi_{n}\right\|_{2}^{2}$ of the ergodic sums is big (expected to be of order $\ln n$ for $n$ belonging to a set of density 1 , in the case $\alpha \mathrm{bpq}$ ). The most precise information is obtained in the special case where $\alpha$ is a quadratic irrational in Subsection 2.4.

The central limit theorem is presented in Section 3. It is based on the decorrelation between the ergodic sums at times $q_{k}$ (the denominators of $\alpha$ ) and on an abstract central limit theorem. To apply the results to a step function, a Diophantine condition is needed on the discontinuities of $\varphi$ which holds generically.

The proofs of the CLT and the decorrelation are given in Sections 4 and 5. In Appendix A, we prove a proposition used for quadratic numbers in the study of the variance.

The results of this paper have been announced in [8]. The authors thank the referees for their very useful remarks.

## 2. Variance of the ergodic sums

Notation. The uniform measure on $\mathbb{T}^{1}$ identified with $X=[0,1[$ is denoted by $\mu$. A function $\varphi$ on $\mathbb{T}^{1}$ is viewed as a 1-periodic function of a real variable. We denote by $V(\varphi)$ the variation of the restriction of $\varphi$ to $[0,1]$ and write BV for "with bounded variation".

The class of real centered $B V$ functions on $\mathbb{T}^{1}$ is denoted by $C$. It contains the 1-periodic step functions with a finite number of discontinuities. The Fourier coefficients of a function
$\varphi$ are denoted by $\widehat{\varphi}(r)$. For $r \neq 0$, they can be written in the following form:

$$
\begin{equation*}
\widehat{\varphi}(r)=\frac{\gamma_{r}(\varphi)}{r} . \tag{2.1}
\end{equation*}
$$

If $\varphi \in C$, the $\gamma_{r}(\varphi)$ are bounded: we have

$$
\begin{equation*}
K(\varphi):=\sup _{r \neq 0}\left|\gamma_{r}(\varphi)\right| \leq \frac{V(\varphi)}{2 \pi}<+\infty . \tag{2.2}
\end{equation*}
$$

Let $\alpha=\left[0 ; a_{1}, a_{2}, \ldots\right]$ be an irrational number in $] 0,1\left[\right.$, with partial quotients $a_{n}=a_{n}(\alpha)$, numerators $p_{n}$ and denominators $q_{n}, n \geq 1$.

The ergodic sums $\sum_{j=0}^{n-1} \varphi(x+j \alpha)$ of a 1-periodic function $\varphi$ for the rotation by $\alpha$ are denoted by $\varphi_{n}(x)$. Their Fourier expansion is $\varphi_{n}(x)=\sum_{r \neq 0} \frac{\gamma_{r}(\varphi)}{r} e^{\pi i(n-1) r \alpha} \frac{\sin \pi n r \alpha}{\sin \pi r \alpha} e^{2 \pi i r x}$.

If $\varphi \in C$, then $\mathrm{V}\left(\varphi_{n}\right) \leq n V(\varphi)$ and $\left|\widehat{\varphi_{n}}(r)\right|=\frac{\left|\gamma_{r}(\varphi)\right|}{|r|} \frac{|\sin \pi n r \alpha|}{|\sin \pi r \alpha|} \leq \frac{n K(\varphi)}{|r|}, r \neq 0$.

### 2.1. Reminders on continued fractions

In this subsection, we recall some classical results on diophantine approximation. For this material we refer to [16] or [19], as well as J. Beck's book [2].

For $u \in \mathbb{R},\{u\}$ denotes its fractional part and $\|u\|:=\inf _{n \in \mathbb{Z}}|u-n|=\min (\{u\}, 1-\{u\})$ its distance to $\mathbb{Z}$. Recall that $2\|x\| \leq|\sin \pi x| \leq \pi\|x\|, \forall x \in \mathbb{R}$.

For $n \geq 1$, writing $\alpha=\frac{p_{n}}{q_{n}}+\frac{\theta_{n}}{q_{n}}$, we have

$$
\begin{gather*}
\frac{1}{a_{n+1}+2} \leq \frac{q_{n}}{q_{n+1}+q_{n}} \leq q_{n}\left\|q_{n} \alpha\right\| \leq \frac{q_{n}}{q_{n+1}}=\frac{q_{n}}{a_{n+1} q_{n}+q_{n-1}} \leq \frac{1}{a_{n+1}},  \tag{2.3}\\
\theta_{n}=(-1)^{n}\left\|q_{n} \alpha\right\|, \alpha=\frac{p_{n}}{q_{n}}+(-1)^{n} \frac{\left\|q_{n} \alpha\right\|}{q_{n}}, \frac{1}{2} q_{n+1}^{-1} \leq\left|\theta_{n}\right| \leq q_{n+1}^{-1},  \tag{2.4}\\
q_{n+1} / q_{n+k} \leq C \rho^{k}, \forall n, k \geq 1, \text { with } C=\frac{5+\sqrt{5}}{2}, \rho=\frac{\sqrt{5}-1}{2}<1 . \tag{2.5}
\end{gather*}
$$

Let us show the last inequality: for $n \geq 1$ fixed, putting $r_{0}=q_{n}, r_{1}=q_{n+1}, r_{k+1}=r_{k}+r_{k-1}$, for $k \geq 1$, we have $q_{n+k} \geq r_{k}, \forall k \geq 0$, by induction and (2.5) follows easily.

For $n \geq 1$, we denote by $m(n)$ the integer such that $n \in\left[q_{m(n)}, q_{m(n)+1}[\right.$.
If $\alpha$ has bounded partial quotients (i.e., $\sup a_{n}<\infty$ ), then $m(n)$ is of order $\ln n$.

## Ostrowski's expansion ([2, 22])

Every integer $n \geq 1$ can be represented as follows ( $\alpha$-Ostrowski's expansion):
if $n<q_{m+1}, n=\sum_{k=0}^{m} b_{k} q_{k}$, with $0 \leq b_{0} \leq a_{1}-1,0 \leq b_{k} \leq a_{k+1}$ for $1 \leq k \leq m$.

Indeed, if $n \in\left[q_{0}, q_{1}=a_{1}\left[\right.\right.$, then (2.6) is satisfied, and if $n \in\left[q_{m}, q_{m+1}[\right.$ with $m \geq 1$, we write $n=b_{m} q_{m}+r$, with $1 \leq b_{m} \leq a_{m+1}, 0 \leq r<q_{m}$. By iteration, we get either $r=0$ at some point and the algorithm stops, or $n \in\left[q_{0}, q_{1}[\right.$. In either cases we obtain (2.6).

In this way, we can code every $n<q_{m+1}$ by a word $b_{0} \ldots b_{m}$, with $b_{0} \in\{0,1, \ldots$, $\left.a_{1}-1\right\}$ and $b_{j} \in\left\{0,1, \ldots, a_{j+1}\right\}, j=1, \ldots, m$.

In this representation, $b_{m(n)} \neq 0$ and $b_{j}=0$ for $m(n)<j \leq m$ when $m>m(n)$. In the latter case, there are $m-m(n)$ zero's at the right end. For a given $m$ and $n<q_{m+1}$, this Ostrowski's expansion is "proper" (without zeros at the end) if $m=m(n)$.

For $m \geq 0$, we call admissible of length $m+1$ a finite word $b_{0} \ldots b_{m}$ such that $b_{0} \in\left\{0,1, \ldots, a_{1}-1\right\}, b_{j} \in\left\{0,1, \ldots, a_{j+1}\right\}$, for $j=1, \ldots, m$ and such that, for two consecutive letters $b_{j-1}, b_{j}$, if $b_{j}=a_{j+1}$ then $b_{j-1}=0$.

Remark that if $b_{0} \ldots b_{m}$ is admissible, $m \geq 1$, then $b_{0} \ldots b_{m-1}$ is admissible. Let us show by induction that the Ostrowski's expansion of an integer $n$ is admissible. Let $n$ be in $\left[q_{m}, q_{m+1}[\right.$. We start the construction of the expansion of $n$ as above. Now the following steps of the algorithm yield the Ostrowski's expansion of $n-b_{m} q_{m}$. Since $n-b_{m} q_{m} \in\left[0, q_{m}\left[\right.\right.$, the Ostrowski's expansion of $n-b_{m} q_{m}$ is admissible. It remains to check that, if $b_{m}=a_{m+1}$, then $b_{m-1}=0$. But if $b_{m-1} \neq 0$, we would have $n \geq a_{m+1} q_{m}+q_{m-1}=q_{m+1}$, a contradiction.

Conversely, if $b_{0} \ldots b_{m}$ is admissible, one shows by induction that $b_{0}+b_{1} q_{1}+\cdots+$ $b_{m} q_{m}<q_{m+1}$. This holds if $m=0$, since $b_{1}<q_{1}=a_{1}$. Assume that this is true for the length $m$. Let $b_{0} \ldots b_{m} b_{m+1}$ be admissible of length $m+1$.

If $b_{m+1}=a_{m+2}$, then $b_{m}=0$ and $b_{0}+b_{1} q_{1}+\cdots+b_{m} q_{m}=b_{0}+b_{1} q_{1}+\cdots+$ $b_{m-1} q_{m-1}<q_{m}$, so that $b_{0}+b_{1} q_{1}+\cdots+b_{m+1} q_{m+1}<q_{m}+a_{m+2} q_{m+1}=q_{m+2}$.

If $b_{m+1} \leq a_{m+2}-1$, then $b_{0}+b_{1} q_{1}+\cdots+b_{m+1} q_{m+1}<q_{m+1}+\left(a_{m+2}-1\right) q_{m+1}<q_{m+2}$.
Therefore, if we associate to an admissible word the integer $n=b_{0}+b_{1} q_{1}+\cdots+b_{m} q_{m}$, there is a bijection between the Ostrowski's expansions of integers $n<q_{m+1}$ and the set of admissible words of length $m+1$. The number of admissible words of length $m$ is $q_{m}-1$.

For $n$ given by (2.6), putting $n_{0}=b_{0}, n_{k}=\sum_{t=0}^{k} b_{t} q_{t}$, for $k \leq m(n)$, we have

$$
\begin{align*}
& \begin{aligned}
\varphi_{n}(x) & =\sum_{k=0}^{m(n)} \sum_{j=n_{k-1}}^{n_{k}-1} \varphi(x+j \alpha)=\sum_{k=0}^{m(n)} \sum_{j=0}^{b_{k} q_{k}-1} \varphi\left(x+n_{k-1} \alpha+j \alpha\right) \\
& =\sum_{k=0}^{m(n)} \sum_{i=0}^{b_{k}-1} \varphi_{q_{k}}\left(x+\left(n_{k-1}+i q_{k}\right) \alpha\right)=\sum_{k=0}^{m(n)} f_{k}(x),
\end{aligned} \\
& \text { with } f_{k}(x):=\sum_{i=0}^{b_{k}-1} \varphi_{q_{k}}\left(x+\left(n_{k-1}+i q_{k}\right) \alpha\right)=\varphi_{b_{k} q_{k}}\left(x+n_{k-1} \alpha\right), \tag{2.7}
\end{align*}
$$

By convention, we put $\sum_{i=0}^{b_{k}-1} \varphi_{q_{k}}\left(x+\left(n_{k-1}+i q_{k}\right) \alpha\right)=0$, if $b_{k}=0$.
If $\varphi$ is a BV centered function, then it holds (Denjoy-Koksma inequality):

$$
\begin{equation*}
\left\|\varphi_{q}\right\|_{\infty}=\sup _{x}\left|\sum_{i=0}^{q-1} \varphi(x+i \alpha)\right| \leq V(\varphi), \text { if } q \text { is a denominator of } \alpha \tag{2.9}
\end{equation*}
$$

One can also show that if $\varphi$ satisfies (2.2) then $\left\|\varphi_{q_{n}}\right\|_{2} \leq 2 \pi K(\varphi)$. By (2.9), we have for $f_{k}$ defined by (2.8): $\left\|f_{k}\right\|_{\infty} \leq b_{k} V(\varphi) \leq a_{k+1} V(\varphi)$.

### 2.2. Bounds for the variance

Let $\varphi \in C$ and $n \in\left[q_{\ell-1}, q_{\ell}[\right.$. The variance is bounded from below as follows:

$$
\begin{aligned}
\left\|\varphi_{n}\right\|_{2}^{2} & =2 \sum_{k>1}|\widehat{\varphi}(k)|^{2} \frac{(\sin \pi n k \alpha)^{2}}{(\sin \pi k \alpha)^{2}} \\
& \geq 2 \sum_{j=1}^{\ell}\left|\widehat{\varphi}\left(q_{j}\right)\right|^{2} \frac{\left(\sin \pi n q_{j} \alpha\right)^{2}}{\left(\sin \pi q_{j} \alpha\right)^{2}} \geq c_{0} \sum_{j=1}^{\ell}\left|\widehat{\varphi}\left(q_{j}\right)\right|^{2} \frac{\left\|n q_{j} \alpha\right\|^{2}}{\left\|q_{j} \alpha\right\|^{2}}
\end{aligned}
$$

with $c_{0}=\frac{8}{\pi^{2}}$. Therefore, by (2.3) we have, for $0<\delta<\frac{1}{2}$,

$$
\begin{equation*}
\left\|\varphi_{n}\right\|_{2}^{2} \geq c_{0} \sum_{j=1}^{\ell}\left|\gamma_{q_{j}}(\varphi)\right|^{2} a_{j+1}^{2}\left\|n q_{j} \alpha\right\|^{2} \geq c_{0} \delta^{2} \sum_{j=1}^{\ell}\left|\gamma_{q_{j}}(\varphi)\right|^{2} a_{j+1}^{2} 1_{\left\|n q_{j} \alpha\right\| \geq \delta} \tag{2.10}
\end{equation*}
$$

An upper bound for the variance and a lower bound for the mean of the variance are shown in [7]: there are constants $C, c>0$ such that

$$
\begin{gather*}
\left\|\varphi_{n}\right\|_{2}^{2} \leq C K(\varphi)^{2} \sum_{j=0}^{m(n)} a_{j+1}^{2},  \tag{2.11}\\
\frac{1}{n} \sum_{k=0}^{n-1}\left\|\varphi_{k}\right\|_{2}^{2} \geq c \sum_{j=0}^{m(n)-1}\left|\gamma_{q_{j}}(\varphi)\right|^{2} a_{j+1}^{2} . \tag{2.12}
\end{gather*}
$$

Inequality (2.10) gives a semi explicit lower bound for the variance. Note that by (2.9), the variance is small if $n$ is a denominator $q_{i}$ of $\alpha$. In this case, as expected, one finds that the lower bound given by (2.10) is small. Indeed, by (2.5), we have $\left\|q_{i} q_{j} \alpha\right\| \leq C_{1} \rho^{|i-j|}$, with $\rho<1$, for a constant $C_{1}$, so that, for a given $\delta>0$, the number of $j$ 's less than $\ell$ such that $\left\|q_{i} q_{j} \alpha\right\| \geq \delta$ is bounded independently from $\ell$.

Now our first goal will be to bound from below the variance $\left\|\varphi_{n}\right\|_{2}$ by a big value for $n$ in a set of large size.

## Bounds for the variance for $n$ in a large set of integers

According to (2.10), a lower bound for $\left\|\varphi_{n}\right\|_{2}$ depends on two separate conditions:
Firstly we need the following condition on the Fourier coefficients of $\varphi$ :

$$
\begin{equation*}
\exists M, \eta, \theta>0 \text { such that } \operatorname{Card}\left\{j \leq N: a_{j+1}\left|\gamma_{q_{j}}(\varphi)\right| \geq \eta\right\} \geq \theta N, \forall N \geq M . \tag{2.13}
\end{equation*}
$$

This condition clearly holds when $\varphi$ is the function $\varphi^{0}(x)=\{x\}-\frac{1}{2}$, since in this case $\left|\gamma_{q_{j}}\left(\varphi^{0}\right)\right|=\frac{1}{2 \pi}, \forall j$. Its validity, related to Diophantine conditions on the points of discontinuities, will be discussed for some step functions in Subsection 3.2.

Secondly, we need an information depending on $\alpha$, namely how often $\left\{n q_{j} \alpha\right\}$ is close to 0 or 1 for a given $j$. For $j<\ell$, we will estimate how many times $\left\{n q_{j} \alpha\right\} \in I_{\delta}:=$ $[0, \delta] \cup[1-\delta, 1]$ for $n \leq q_{\ell}$ and deduce from this estimation that $\sum_{j=1}^{\ell} 1_{I_{\delta}}\left(\left\{n q_{j} \alpha\right\}\right)=$ $\sum_{j=1}^{\ell} 1_{\left\|n q_{j} \alpha\right\| \leq \delta}$ is small for a large set of values of $n$.

Lemma 2.1. For every $\delta \in] 0, \frac{1}{2}\left[\right.$ and every interval of integers $I=\left[N_{1}, N_{1}+L[\right.$, we have

$$
\begin{equation*}
\sum_{n=N_{1}}^{N_{1}+L-1} 1_{I_{\delta}}\left(\left\{n q_{j} \alpha\right\}\right) \leq 20\left(\delta+q_{j+1}^{-1}\right) L, \quad \forall j \text { such that } q_{j+1} \leq 2 L . \tag{2.14}
\end{equation*}
$$

Proof. For a fixed $j$ and $0 \leq N_{1}<N_{1}+L$, let us describe the behaviour of the sequence $\left(\left\|n q_{j} \alpha\right\|, n=N_{1}, \ldots, N_{1}+L-1\right)$.

Recall that (modulo 1) we have $q_{j} \alpha=\theta_{j}$, with $\theta_{j}=(-1)^{j}\left\|q_{j} \alpha\right\|$ (see (2.4)). We treat the case $j$ even (hence $\theta_{j}>0$ ). The case $j$ odd is analogous.

We are going to count how many times, for $j$ even, we have $\left\{n \theta_{j}\right\}<\delta$ or $1-\delta<\left\{n \theta_{j}\right\}$.
We start with $n_{1}:=N_{1}$. Putting $w(j, 1):=\left\{n_{1} \theta_{j}\right\}$, we have $\left\{n \theta_{j}\right\}=w(j, 1)+\left(n-n_{1}\right) \theta_{j}$, for $n=n_{1}, n_{1}+1, \ldots, n_{2}-1$, where $n_{2}$ is such that $w(j, 1)+\left(n_{2}-1-n_{1}\right) \theta_{j}<1<$ $w(j, 1)+\left(n_{2}-n_{1}\right) \theta_{j}$.

Putting $w(j, 2):=\left\{n_{2} \theta_{j}\right\}$, we have $w(j, 2)=w(j, 1)+\left(n_{2}-n_{1}\right) \theta_{j}-1<\theta_{j}$. Starting now from $n_{2}$, we have $\left\{n \theta_{j}\right\}=w(j, 2)+\left(n-n_{2}\right) \theta_{j}$ for $n=n_{2}, n_{2}+1, \ldots, n_{3}-1$, where $n_{3}$ is such that $w(j, 2)+\left(n_{3}-1-n_{2}\right) \theta_{j}<1<w(j, 2)+\left(n_{3}-n_{2}\right) \theta_{j}$.

We iterate up to $R(j)$, where $n_{R(j)-1}<N_{1}+L \leq n_{R(j)}$. This construction yields a sequence $n_{1}<n_{2}<\cdots<n_{R(j)}$ such that $\left\{n \theta_{j}\right\}=w(j, i)+\left(n-n_{i}\right) \theta_{j}, \forall n \in\left[n_{i}, n_{i+1}[\right.$, and

$$
w(j, i)+\left(n_{i+1}-1-n_{i}\right) \theta_{j}<1<w(j, i)+\left(n_{i+1}-n_{i}\right) \theta_{j},
$$

with $w(j, i)$ defined recursively by $w(j, i+1)=\left\{w(j, i)+\left(n_{i+1}-n_{i}\right) \theta_{j}\right\}$ and satisfying $w(j, i)<\theta_{j}$, for $i=1, \ldots, R(j)$.

Since $\left(n_{i+1}-n_{i}+1\right) \theta_{j} \geq w(j, i)+\left(n_{i+1}-n_{i}\right) \theta_{j}>1$ for $i \neq 1$ and $i \neq R(j)$, we have $n_{i+1}-n_{i} \geq \theta_{j}^{-1}-1$, for each $i \neq 1, R(j)$. This implies $R(j) \leq \frac{L}{\theta_{j}^{-1}-1}+2$.

For each $i$, the number of integers $n \in\left[n_{i}, n_{i+1}-1\left[\right.\right.$ such that $\left\{n \theta_{j}\right\} \in[0, \delta[\cup] 1-\delta, 1[$ is bounded by $2\left(1+\delta \theta_{j}^{-1}\right)$. (This number is less than 2 if $\delta<\theta_{j}$.)

Altogether, using (2.4) and the assumption $2 L \geq q_{j+1}$, the number of integers $n \in I$ such that $\left\{n \theta_{j}\right\} \in[0, \delta[\cup] 1-\delta, 1[$ is bounded by

$$
\begin{aligned}
2 R(j)\left(1+\delta \theta_{j}^{-1}\right) & \leq\left(\frac{2 L}{\theta_{j}^{-1}-1}+4\right)\left(1+\delta \theta_{j}^{-1}\right) \leq 4\left(L+\theta_{j}^{-1}\right)\left(\delta+\theta_{j}\right) \\
& \leq 4\left(L+2 q_{j+1}\right)\left(\delta+q_{j+1}^{-1}\right) \leq 20\left(\delta+q_{j+1}^{-1}\right) L
\end{aligned}
$$

Remark 2.2. For every $\delta \in] 0, \frac{1}{2 r}$ [ and every interval [ $\left.N_{1}, N_{1}+L\right]$, we have by a slight extension of Lemma 2.1:

$$
\begin{equation*}
\operatorname{Card}\left\{n \in\left[N_{1}, N_{1}+L\right]: d\left(n q_{j} \alpha, \mathbb{Z} / r\right) \leq \delta\right\} \leq 20 r\left(\delta+q_{j+1}^{-1}\right) L, \text { if } q_{j+1} \leq 2 L \tag{2.15}
\end{equation*}
$$

Lemma 2.3. Let $I=\left[N_{1}, N_{1}+L\right]$ be an interval and $\ell$ such that $q_{\ell} \leq 2 L$.
(a) For all $\delta \in] 0, \frac{1}{2}[$ and $\zeta \in] 0,1[$, the set

$$
\begin{equation*}
A:=\left\{n \in I: \operatorname{Card}\left(j<\ell: d\left(n q_{j} \alpha, \mathbb{Z}\right) \leq \delta\right) \leq \zeta \ell\right\} \tag{2.16}
\end{equation*}
$$

satisfies

$$
\begin{equation*}
\operatorname{Card}(A) \geq\left(1-20 \zeta^{-1}\left(\delta+C \ell^{-1}\right)\right) L \tag{2.17}
\end{equation*}
$$

(b) Under Condition (2.13) on $\varphi$, there are positive constants $\eta_{0}, c$ (not depending on $\delta$ ) such that, for every $\delta \in] 0, \frac{1}{2}\left[\right.$, the subset $V(I, \delta, \ell):=\left\{n \in I:\left\|\varphi_{n}\right\|_{2} \geq\right.$ $\left.\eta_{0} \delta \sqrt{\ell}\right\}$ satisfies:

$$
\begin{equation*}
\operatorname{Card}(V(I, \delta, \ell)) \geq\left(1-c\left(\delta+\ell^{-1}\right)\right) L \tag{2.18}
\end{equation*}
$$

Proof. (a). Let $A^{c}=I \backslash A$ be the complementary of $A$. We will find an upper bound of the density $L^{-1} \operatorname{Card}\left(A^{c}\right)$ by counting the number of values of $n$ in $I$ such that $\left\|n q_{j} \alpha\right\|<\delta$ in an array indexed by $(j, n)$.

By summing (2.14) from $j=0$ to $j=\ell-1$ and using the definition of $A$, we get:

$$
\begin{aligned}
20\left(\delta \ell+\sum_{0 \leq j \leq \ell-1} q_{j+1}^{-1}\right) L & \geq \sum_{0 \leq j \leq \ell-1} \sum_{n \in I} 1_{I_{\delta}}\left(\left\{n q_{j} \alpha\right\}\right) \\
& \geq \sum_{n \in A^{c}} \sum_{0 \leq j \leq \ell-1} 1_{I_{\delta}}\left(\left\{n q_{j} \alpha\right\}\right) \geq \sum_{n \in A^{c}} \zeta \ell=\zeta \ell \operatorname{Card}\left(A^{c}\right) .
\end{aligned}
$$

With $C:=\sum_{j=0}^{\infty} q_{j}^{-1}$, we have $\operatorname{Card}\left(A^{c}\right) \leq 20 \zeta^{-1}\left(\delta+C \ell^{-1}\right) L$, so (2.17) is shown.
(b). With $\zeta=\frac{1}{2} \theta$, where $\theta$ is the constant in (2.13), in view of the definition of $A$ and (2.13), we have, for $n \in A$ :
$\operatorname{Card}\left(\left\{j \leq \ell-1:\left\|n q_{j} \alpha\right\| \geq \delta\right\} \cap\left\{j: a_{j+1}\left|\gamma_{q_{j}}(\varphi)\right| \geq \eta\right\}\right) \geq(1-(\zeta+1-\theta)) \ell=\frac{1}{2} \theta \ell$.
Putting $c:=20 \zeta^{-1} \max (1, C)$ and $\eta_{0}=\left(\frac{1}{2} c_{0} \eta^{2} \theta\right)^{\frac{1}{2}}$, this implies by (2.10) and (2.17):

$$
\begin{equation*}
\left\|\varphi_{n}\right\|_{2}^{2} \geq \frac{1}{2} c_{0} \delta^{2} \eta^{2} \theta \ell=\eta_{0}^{2} \delta^{2} \ell, \forall n \in A, \text { and } \operatorname{Card}(A) \geq 1-c\left(\delta+\ell^{-1}\right) L \tag{2.19}
\end{equation*}
$$

hence $A \subset V(I, \delta, \ell)$ and therefore $V(I, \delta, \ell)$ satisfies (2.18).
Lemma 2.3 implies the following theorem, where the constants $c$ and $\eta_{0}$ are those of the lemma:

Theorem 2.4. Under Condition (2.13) on $\varphi$, the density of the subset

$$
W:=\left\{n \in \mathbb{N}:\left\|\varphi_{n}\right\|_{2} \geq \eta_{0}\left(\frac{m(n)}{\ln m(n)}\right)^{\frac{1}{2}}\right\}
$$

satisfies for every $N \geq 1$.

$$
\frac{\operatorname{Card}(W \cap[0, N[)}{N} \geq 1-2 c(\ln m(N))^{-\frac{1}{2}} .
$$

Proof. Since $t / \ln t$ is increasing for $t \geq e$, we have, after the first terms, for $n$ in $W^{c} \cap[0, N[:$

$$
\left\|\varphi_{n}\right\|_{2}<\eta_{0}\left(\frac{m(n)}{\ln m(n)}\right)^{\frac{1}{2}} \leq \eta_{0}\left(\frac{m(N)}{\ln m(N)}\right)^{\frac{1}{2}} .
$$

Therefore, by Lemma 2.3(b) with $I=\left[0, N\left[, L=N, \delta=(\ln m(N))^{-\frac{1}{2}}\right.\right.$ and $\ell=m(N)$, it follows

$$
\frac{\operatorname{Card}\left(W^{c} \cap[0, N[)\right.}{N} \leq c(\ln m(N))^{-\frac{1}{2}}+c m(N)^{-1} \leq 2 c(\ln m(N))^{-\frac{1}{2}} .
$$

The bound from below $\left(\frac{m(n)}{\ln m(n)}\right)^{\frac{1}{2}}$, appearing in the definition of $W$ above, depends on $\alpha$. But we will see this dependance is not so strong under a condition on $\alpha$ that will be necessary later (Hypothesis 3.2). In the next sections we will show that, under a Diophantine condition on $\alpha$, for a big set of $n$, the distribution of $\varphi_{n} /\left\|\varphi_{n}\right\|_{2}$ is approximately Gaussian. For that, we will use the bound defining $W$ in Theorem 2.4. But the following counter-example shows that another condition is necessary. This will impose stronger restrictions on $\alpha$.

By (2.12), if $n_{\ell} \leq q_{\ell+1}$ is an integer such that $\left\|\varphi_{n_{\ell}}\right\|_{2}=\max _{k<q_{\ell+1}}\left\|\varphi_{k}\right\|_{2}$, then we have the lower bound $\left\|\varphi_{n_{\ell}}\right\|_{2}^{2} \geq c \sum_{j=0}^{\ell-1}\left|\gamma_{q_{j}}\right|^{2} a_{j+1}^{2}$ (the inequality which holds for the mean is true for the maximum). Remark that one can easily show that this inequality is
satisfied not only for the record indices for the variance, but also for a set of integers $n$ of positive density.

### 2.3. A counter-example

For a parameter $\gamma>0$, let the sequence $\left(a_{n}\right)_{n \geq 1}$ be defined by

$$
a_{n}= \begin{cases}\left\lfloor n^{\gamma}\right\rfloor & \text { if } n \in\left\{2^{k}: k \geq 0\right\} \\ 1 & \text { if } n \notin\left\{2^{k}: k \geq 0\right\}\end{cases}
$$

Let $\alpha$ be the number which has $\left(a_{n}\right)_{n \geq 1}$ for sequence of partial quotients. For simplicity, let us take for $\varphi$ the sawtooth function $\varphi^{0}$ defined above for which $\gamma_{k}=\frac{-1}{2 \pi i}, \forall k \neq 0$.

For $\ell>1$, let $n_{\ell}:=\max \left\{n<q_{\ell+1}:\left\|\varphi_{n}\right\|_{2}=\max _{k<q_{\ell+1}}\left\|\varphi_{k}\right\|_{2}\right\}$. As mentioned above, we have

$$
\left\|\varphi_{n_{\ell}}\right\|_{2}^{2} \geq c \sum_{j=0}^{\ell-1}\left|\gamma_{q_{j}}\right|^{2} a_{j+1}^{2} \geq c \sum_{s=0}^{\left\lfloor\log _{2}(\ell)\right\rfloor} 2^{2 \gamma s} \geq c \ell^{2 \gamma} .
$$

In the $\operatorname{sum} \varphi_{n_{\ell}}(x)=\sum_{k=0}^{\ell} \sum_{j=0}^{b_{k} q_{k}-1} \varphi\left(x+N_{k-1} \alpha+j \alpha\right)$ defined in (2.7), we can isolate the indices $k$ for which $k+1$ is a power of 2 (for the other indices $a_{k+1}=1$ ) and write $\varphi_{n_{\ell}}(x)=U_{\ell}+V_{\ell}$ with

$$
\begin{aligned}
& U_{\ell}=\sum_{p=1}^{\left\lfloor\log _{2}(\ell)\right\rfloor} \sum_{j=0}^{b_{2} p} q_{2_{2} p-1} \varphi\left(x+N_{2^{p}-1} \alpha+j \alpha\right), \\
& V_{\ell}=\sum_{k \in[0, \ell] \cap\left\{a_{k+1}=1\right\}} \sum_{j=0}^{b_{k} q_{k}-1} \varphi\left(x+N_{k-1} \alpha+j \alpha\right) .
\end{aligned}
$$

We will see in (4.5) that the variance of a sum where $b_{k}$ equals 0 or 1 is bounded as follows

$$
\left\|\sum_{k \in[0, \ell] \cap\left\{a_{k+1}=1\right\}} \sum_{j=0}^{b_{k} q_{k}-1} \varphi\left(x+N_{k-1} \alpha+j \alpha\right)\right\|_{2}^{2} \leq C \ell \log (\ell) .
$$

On the other side, we also have (because $b_{2^{p}-1} \leq a_{2^{p}}$ and Denjoy-Koksma inequality (2.9))

$$
\left|\sum_{j=0}^{b_{2} p_{-1} q_{2 p-1}-1} \varphi\left(x+N_{2^{p}-2} \alpha+j \alpha\right)\right| \leq a_{2^{p}} V(\varphi),
$$

and

$$
\left|\sum_{p=1}^{\left\lfloor\log _{2}(\ell)\right\rfloor} \sum_{j=0}^{b_{2} p_{-1} q_{2} p_{-1}-1} \varphi\left(x+N_{2^{p}-2} \alpha+j \alpha\right)\right| \leq \sum_{p=1}^{\left\lfloor\log _{2}(\ell)\right\rfloor} a_{2^{p}} V(\varphi) \leq C \sum_{p=1}^{\left\lfloor\log _{2}(\ell)\right\rfloor} 2^{\gamma p} \leq C \ell^{\gamma}
$$

The previous bounds imply $\frac{\varphi_{n_{\ell}}}{\left\|\varphi_{n_{\ell}}\right\|_{2}}=\frac{U_{\ell}}{\left\|\varphi_{n_{\ell}}\right\|_{2}}+\frac{V_{\ell}}{\left\|\varphi_{n_{\ell}}\right\|_{2}}$ with $\left\|\varphi_{n_{\ell}}\right\|_{2} \geq c \ell^{\gamma},\left\|U_{\ell}\right\|_{\infty} \leq C \ell^{\gamma}$, $\left\|V_{\ell}\right\|_{2} \leq(C \ell \log (\ell))^{1 / 2}$.

Thus, if $\gamma>1 / 2$, one has $\left\|\frac{U_{\ell}}{\left\|\varphi_{n_{\ell}}\right\|_{2}}\right\|_{\infty} \leq \frac{C}{c},\left\|\frac{V_{\ell}}{\left\|\varphi_{n_{\ell}}\right\|_{2}}\right\|_{2} \rightarrow 0$ and the limit points of the distributions of $\frac{\varphi_{n_{\ell}}}{\left\|\varphi_{n_{\ell}}\right\|_{2}}$ have all their supports included in $\left[-\frac{C}{c}, \frac{C}{c}\right]$, hence are not Gaussian.

Here we have $a_{n} \leq n^{\gamma}$, for every $n$ (with some $\gamma>1 / 2$ ), Theorem 2.4 applies, but the CLT is certainly not true for a subsequence of density one. To obtain it, we need a stricter condition on the sequence $\left(a_{n}\right)$.

### 2.4. A special case: quadratic numbers

When $\alpha$ is a quadratic number, using the ultimate periodicity of the sequence $\left(a_{n}(\alpha)\right)_{n \geq 1}$ and the good properties of the associated Ostrowski's expansion of the integers, it is possible to improve the result of Theorem 2.4 on the variance. In this subsection we show that the variance $\left\|\varphi_{n}\right\|_{2}^{2}$ of the ergodic sums of $\varphi$ under the rotation by a quadratic number $\alpha$ is of order $\ln n$ for $n$ in a big set of integers whose size is precisely estimated. For example, if we take $\varphi(x)=\varphi^{0}(x)=\{x\}-\frac{1}{2}$, Theorem 2.6 shows that there are positive constants $\eta_{1}, \eta_{2}, R$ and $\left.\xi \in\right] 0,1[$ such that,

$$
\begin{equation*}
\frac{1}{N} \operatorname{Card}\left\{n \leq N: \eta_{1} \ln n \leq\left\|\varphi_{n}\right\|_{2}^{2} \leq \eta_{2} \ln n\right\} \geq\left(1-R N^{-\xi}\right) . \tag{2.20}
\end{equation*}
$$

The main step in the proof is the following proposition showing that, in case of a quadratic number, for most of the integers $n$ (in a set whose size is precisely estimated), $\left\|n q_{j} \alpha\right\|$ is far from 0 for a big proportion of $j$ 's:

Proposition 2.5. If $\alpha$ is a quadratic number, for every $\left.\varepsilon_{0} \in\right] 0, \frac{1}{2}[$, there are $\delta \in] 0, \frac{1}{2}[$ and positive constants $C$ and $\xi$ such that for every $\ell \geq 1$ :

$$
\begin{align*}
\operatorname{Card}\left\{n<q_{\ell+1}: \operatorname{Card}\left(j<\ell: d\left(n q_{j} \alpha, \mathbb{Z}\right) \geq \delta\right) \geq(1-\right. & \left.\left.\varepsilon_{0}\right) \ell\right\} \\
& \geq\left(1-C q_{\ell+1}^{-\xi}\right) q_{\ell+1} . \tag{2.21}
\end{align*}
$$

The proof of Proposition 2.5 is given in Appendix.
Theorem 2.6. If $\alpha$ is a quadratic number and if $\varphi$ satisfies Condition (2.13), there are positive constants $\eta_{1}, \eta_{2}, R$ and $\left.\xi \in\right] 0,1[$ such that, for $N$ big enough, it holds :

$$
\begin{equation*}
\operatorname{Card}\left\{n \leq N: \eta_{1} \ln n \leq\left\|\varphi_{n}\right\|_{2}^{2} \leq \eta_{2} \ln n\right\} \geq N\left(1-R N^{-\xi}\right) . \tag{2.22}
\end{equation*}
$$

Proof. There is $\eta_{2}>0$ such that the upper bound in (2.22) holds for every $n \geq 1$ : indeed, when $\alpha$ is quadratic, as $\left(q_{k}\right)$ is equivalent to a geometric sequence, $m(n)$ is equivalent to
$\ln (n)$ up to a multiplicative constant factor. Therefore, for $n \in\left[q_{\ell}, q_{\ell+1}[\right.$ (i.e., $m(n)=\ell$ ), (2.11) implies $\left\|\varphi_{n}\right\|_{2}^{2} \leq C K(\varphi)^{2} \sum_{j=0}^{\ell} a_{j+1}^{2} \leq \eta_{2} \ln (n)$, for some positive constant $\eta_{2}$.

For the lower bound, by (2.10) we have $\left\|\varphi_{n}\right\|_{2}^{2} \geq c_{0} \delta^{2} \sum_{j=1}^{\ell}\left|\gamma_{q_{j}}(\varphi)\right|^{2} a_{j+1}^{2} 1_{\left\|n q_{j} \alpha\right\| \geq \delta}$. Let $\varphi$ in $C$ be such that (2.13) is satisfied and, for $\varepsilon_{0}=\frac{1}{2} \theta$, let $\delta=\delta\left(\varepsilon_{0}\right)$ be given by Proposition 2.5. According to (2.13) and (2.21), for $\ell$ big enough, the set of integers $n<q_{\ell+1}$ such that simultaneously $\left\|n q_{j} \alpha\right\| \geq \delta$ and $\left|\gamma_{q_{j}}(\varphi)\right| \geq \eta$, for at least $\frac{1}{2} \theta \ell$ different indices $j$, has a cardinal bigger than $q_{\ell+1}\left(1-C q_{\ell+1}^{-\xi}\right)$ for some constants $\left.C>0, \xi \in\right] 0,1[$.

Therefore we have $\left\|\varphi_{n}\right\|_{2}^{2} \geq \frac{c_{0}}{2} \eta^{2} \delta^{2} \theta \ell=\eta_{1} \ell$ for more than $q_{\ell+1}\left(1-C q_{\ell+1}^{-\xi}\right)$ values of $n$ between 1 and $q_{\ell+1}$.

This shows that, for $N \in\left[q_{\ell}, q_{\ell+1}\left[\right.\right.$, the cardinal of the set $\left\{n<N:\left\|\varphi_{n}\right\|_{2}^{2} \leq \eta_{1} \ell\right\}$ is less than $C q_{\ell+1}^{1-\xi} \leq C^{\prime} N^{1-\xi}$ (because for a quadratic number $\sup _{\ell} q_{\ell+1} / q_{\ell}<+\infty$ ). Hence, the result.

## 3. A central limit theorem and its application to rotations

### 3.1. Decorrelation and CLT

## An abstract CLT under a decorrelation property

Below $Y_{1}$ denotes a r.v. with a normal distribution $\mathcal{N}(0,1)$. Recall that, if $X, Y$ are two real random variables, their mutual (Kolmogorov) distance in distribution is defined by: $d(X, Y)=\sup _{x \in \mathbb{R}}|\mathbb{P}(X \leq x)-\mathbb{P}(Y \leq x)|$.

The notation $C$ denotes an absolute constant whose value may change from a line to the other.

Proposition 3.1. Let $N$ be a positive integer. Let $\left(q_{k}\right)_{1 \leq k \leq N}$ be an increasing sequence of positive integers such that for a constant $\rho \in] 0,1[$

$$
\begin{equation*}
q_{k} / q_{m} \leq C \rho^{m-k}, 1 \leq k<m \leq N . \tag{3.1}
\end{equation*}
$$

Let $\left(f_{k}\right)_{1 \leq k \leq N}$ be real centered BV functions such that for constants $u_{k}$

$$
\begin{equation*}
\left\|f_{k}\right\|_{\infty} \leq u_{k}, \mathrm{~V}\left(f_{k}\right) \leq C u_{k} q_{k}, 1 \leq k \leq N . \tag{3.2}
\end{equation*}
$$

Moreover assume that, for some constant $\theta$, the following decorrelation properties hold:

$$
\begin{align*}
\left|\int_{X} \psi f_{k} \mathrm{~d} \mu\right| & \leq C \mathrm{~V}(\psi) u_{k} \frac{k^{\theta}}{q_{k}}, & & 1 \leq k \leq N, \forall \psi B V,  \tag{3.3}\\
\left|\int_{X} \psi f_{k} f_{m} \mathrm{~d} \mu\right| & \leq C V(\psi) u_{k} u_{m} \frac{m^{\theta}}{q_{k}}, & & 1 \leq k \leq m \leq N, \forall \psi B V \text { centered },  \tag{3.4}\\
\left|\int_{X} \psi f_{k} f_{m} f_{t} \mathrm{~d} \mu\right| & \leq C V(\psi) u_{k} u_{m} u_{t} \frac{t^{\theta}}{q_{k}}, & & 1 \leq k \leq m \leq t \leq N, \forall \psi B V \text { centered. } . \tag{3.5}
\end{align*}
$$

Then, putting $w_{N}:=\max _{j=1}^{N} u_{j}, S_{N}:=f_{1}+\cdots+f_{N}$, there is for every $\delta>0$ a constant $C(\delta)>0$ (depending only on $\delta$ ) such that the condition

$$
\begin{equation*}
\frac{w_{N}}{\left\|S_{N}\right\|_{2}} \leq N^{p-\frac{1}{2}}, \text { with } p \in\left[0, \frac{1}{8}[\right. \tag{3.6}
\end{equation*}
$$

implies

$$
\begin{equation*}
d\left(\frac{S_{N}}{\left\|S_{N}\right\|_{2}}, Y_{1}\right) \leq C(\delta) N^{-\frac{1-8 p}{12}+\delta} \tag{3.7}
\end{equation*}
$$

The proposition is proved in Section 4. We apply it to an irrational rotation by taking for $q_{k}$ 's the denominators of $\alpha$ (they satisfy (3.1)) and for $f_{k}$ the ergodic sums $\varphi_{b_{k} q_{k}}$ of a function $\varphi$ (composed by a translation), where the $b_{k}$ 's ( $b_{k} \leq a_{k+1}$ ) are given by the Ostrowski's expansion described above.

## Decorrelation between partial ergodic sums

In order to apply the previous proposition we will prove decorrelation properties between the ergodic sums of $\varphi \in C$ at time $q_{n}$ under the following assumption on $\alpha$ :

Hypothesis 3.2. There are two constants $A \geq 1, p \geq 0$ such that

$$
\begin{equation*}
a_{n} \leq A n^{p}, \forall n \geq 1 . \tag{3.8}
\end{equation*}
$$

Remark 3.3.
(a) The case $\alpha$ of bounded type, i.e., with bounded partial quotients, corresponds to $p=0$. In this case, as we have seen, $m(n)$ is of order $\ln n$.
(b) Observe that $m(n)$ can be smaller, but at least of order $\frac{\ln n}{\ln \ln n}$ up to a bounded factor, under the more general assumption 3.2.

## Lemma 3.4.

(a) For every $p>1$, for a.e. $\alpha$, there is a finite constant $A(\alpha, p)$ such that

$$
\begin{equation*}
a_{n} \leq A(\alpha, p) n^{p}, \forall n \geq 1 . \tag{3.9}
\end{equation*}
$$

(b) If $\alpha$ satisfies (3.8), then there is $c>0$ such that

$$
\begin{equation*}
\|k \alpha\| \geq \frac{c}{|k|(\log k)^{p}}, \forall k>1 \tag{3.10}
\end{equation*}
$$

Proof. (a). We have $a_{n+1}(\alpha)=\left\lfloor 1 / \theta^{n}(\alpha)\right\rfloor$ where $\theta$ is the Gauss map. Let $\gamma>1$. Since $\alpha \rightarrow\left(a_{1}(\alpha)\right)^{\frac{1}{\gamma}}$ is integrable for the $\theta$-invariant measure $\frac{\mathrm{d} x}{1+x}$ on $\left.] 0,1\right]$, we have, for a constant $A(\gamma): \mu\left\{\alpha: a_{n}(\alpha)>n^{s}\right\} \leq A(\gamma) n^{-\frac{s}{\gamma}}$.

By the Borel-Cantelli lemma, it follows that for a.e $\alpha$ there is $C(\alpha, \gamma)$ such that, if $s>\gamma, a_{n}(\alpha) \leq C(\alpha, \gamma) n^{s}, \forall n \geq 1$.
(b). For every irrational $\alpha$, there are $C>0$ and $\lambda>1$ such that the denominators of $\alpha$ satisfy $q_{\ell} \geq C \lambda^{\ell}$, for every $\ell \geq 1$. For $k \geq 2$, let $n$ be such that $q_{n-1} \leq k<q_{n}$. Since $C \lambda^{n-1} \leq q_{n-1} \leq k$, it follows that $n \leq C^{\prime} \log k$, for some constant $C^{\prime}$. By (3.8), we have $a_{n} \leq A n^{p} \leq A\left(C^{\prime} \log k\right)^{p}$.

Since $\|k \alpha\|>\left\|q_{n-1} \alpha\right\| \geq \frac{1}{2 q_{n}} \geq \frac{1}{4 a_{n} q_{n-1}} \geq \frac{1}{4 a_{n} k}$, this implies (3.10).
As a corollary, using Theorem 2.4, it follows that for a.e. $\alpha$, under the rotation by $\alpha$, for a function $\varphi \in C$ satisfying (2.13), the growth of the variance $\left\|\varphi_{n}\right\|_{2}^{2}$ is "roughly" of order $\ln n$ on a large set of integers.

Proposition 3.5. Let $\psi$ and $\varphi$ be BV centered functions. Suppose that $\alpha$ satisfies Hypothesis 3.2. Then there are constants $C, \theta_{1}, \theta_{2}, \theta_{3}$ such that, for every $1 \leq k \leq m \leq \ell$ :

$$
\begin{align*}
\left|\int_{X} \psi \varphi_{b_{k} q_{k}} \mathrm{~d} \mu\right| & \leq C \mathrm{~V}(\psi) \mathrm{V}(\varphi) \frac{k^{\theta_{1}}}{q_{k}} b_{k},  \tag{3.11}\\
\left|\int_{X} \psi \varphi_{b_{k} q_{k}} \varphi_{b_{m} q_{m}} \mathrm{~d} \mu\right| & \leq C \mathrm{~V}(\psi) \mathrm{V}(\varphi)^{2} \frac{m^{\theta_{2}}}{q_{k}} b_{k} b_{m},  \tag{3.12}\\
\left|\int_{X} \psi \varphi_{b_{k} q_{k}} \varphi_{b_{m} q_{m}} \varphi_{b_{\ell} q_{\ell}} \mathrm{d} \mu\right| & \leq C \mathrm{~V}(\psi) \mathrm{V}(\varphi)^{3} \frac{\ell^{\theta_{3}}}{q_{k}} b_{k} b_{m} b_{\ell} . \tag{3.13}
\end{align*}
$$

The proposition is proved in Section 5. From Propositions 3.1 and 3.5 we will deduce a convergence toward a Gaussian distribution under a variance condition, by bounding the distance to the normal distribution.

Theorem 3.6. Let $\varphi$ be in $C$ satisfying (2.13).
(1) The set defined (cf. Theorem 2.4) by

$$
\begin{equation*}
W:=\left\{n \in \mathbb{N}:\left\|\varphi_{n}\right\|_{2} \geq \eta_{0}(\log m(n))^{-\frac{1}{2}} m(n)^{\frac{1}{2}}\right\} \tag{3.14}
\end{equation*}
$$

has density 1 in $\mathbb{N}$.

Suppose that $\alpha$ satisfies Hypothesis 3.2 (i.e., for constants $A \geq 1, p \geq 0$, $\left.a_{n} \leq A n^{p}, \forall n \geq 1\right)$ with $p<\frac{1}{8}$. Then, for $\left.\delta \in\right] 0, \frac{1-8 p}{12}[$, there is a constant $C(\delta)$ such that, for $n$ in $W$,

$$
\begin{equation*}
d\left(\frac{\varphi_{n}}{\left\|\varphi_{n}\right\|_{2}}, Y_{1}\right) \leq C(\delta) m(n)^{-\frac{1-8 p}{12}+\delta} \underset{n \in W, n \rightarrow \infty}{\longrightarrow} 0 . \tag{3.15}
\end{equation*}
$$

In particular when $\alpha$ has bounded partial quotients, we have $p=0$ and $m(n)$ can be replaced by $\log n$.
(2) Suppose that $\alpha$ is a quadratic irrational. With the notation of Theorem 2.6, let

$$
V:=\left\{n \geq 1: \eta_{1} \sqrt{\log n} \leq\left\|\varphi_{n}\right\|_{2} \leq \eta_{2} \sqrt{\log n}\right\} .
$$

Then, there are two constants $R, \xi>0$ such that

- the density of $V$ satisfies:

$$
\begin{equation*}
\operatorname{Card}(V \cap[1, N]) \geq N\left(1-R N^{-\xi}\right), \text { for } N \geq N_{0} \tag{3.16}
\end{equation*}
$$

- for $\delta \in] 0, \frac{1}{12}[$, there is a constant $C(\delta)$ such that, for $n \in V$ :

$$
\begin{equation*}
d\left(\frac{\varphi_{n}}{\left\|\varphi_{n}\right\|_{2}}, Y_{1}\right) \leq C(\delta)(\log n)^{-\frac{1}{12}+\delta} \underset{n \in V, n \rightarrow \infty}{\longrightarrow} 0 . \tag{3.17}
\end{equation*}
$$

Proof. (1). The result on the density of the set $W$ follows from Theorem 2.4.
For (3.15), we use Proposition 3.1 with $N=m(n)$ such that $n \in\left[q_{m(n)}, q_{m(n)+1}\left[, f_{k}\right.\right.$ defined by (2.8) and the decomposition of the ergodic sums given by (2.7), i.e.,
$\varphi_{n}(x)=\sum_{k=0}^{m(n)} f_{k}(x)$, where $f_{k}(x):=\sum_{i=0}^{b_{k}-1} \varphi_{q_{k}}\left(x+\left(n_{k-1}+i q_{k}\right) \alpha\right)=\varphi_{b_{k} q_{k}}\left(x+n_{k-1} \alpha\right)$.
The decorrelation inequalities in Proposition 3.5 are obtained for functions of the form $\varphi_{b_{k} q_{k}}$. But in the proof of the decorrelation inequalities, one sees that they remain valid for $f_{k}$, since translations on the variable do not change the modulus of the Fourier coefficients.

As $\left\|f_{k}\right\|_{\infty} \leq b_{k} V(\varphi) \leq a_{k+1} V(\varphi)$, up to a fixed factor the constant $u_{k}$ in the statement of Proposition 3.5 can be taken to be $a_{k+1} \leq k^{p}$, for some constant $p>0$, by Hypothesis 3.2.

With the notation of Proposition 3.1, we have $w_{N}:=\max _{j=1}^{N} b_{j}, \varphi_{n}=S_{N}=$ $f_{1}+\cdots+f_{N}$. For $n \in W$ and under Hypothesis 3.2, we have

$$
\frac{w_{N}}{\left\|S_{N}\right\|_{2}} \leq C N^{p-\frac{1}{2}}(\log N)^{\frac{1}{2}}
$$

The factor $(\log N)^{\frac{1}{2}}$ can be absorbed in the factor $N^{p-\frac{1}{2}}$ by taking $p$ larger and we have (3.6). By (3.7) it follows:

$$
d\left(\frac{\varphi_{n}}{\left\|\varphi_{n}\right\|_{2}}, Y_{1}\right) \leq C(\delta) m(n)^{-\frac{1-8 p}{12}+\delta} .
$$

(2). In the quadratic case, $p=0$ and the property of the set $V$ is given by Theorem 2.6.

Remark 3.7. The previous result is written with a self-normalisation. If $\alpha$ is quadratic, let us consider the normalisation by $\sqrt{\ln n}$ for the ergodic sums, i.e. $\left(\frac{\varphi_{n}}{\sqrt{\ln n}}\right)_{n \geq 1}$. Then, for $n \in V$, the accumulation points of the sequence of distributions are Gaussian non degenerated with a variance belonging to a compact interval.

### 3.2. Application to step functions, examples

If $\varphi$ belongs to the class $C$ of centered BV functions, with Fourier series $\sum_{r \neq 0} \frac{\gamma_{r}(\varphi)}{r} e^{2 \pi i r}$, to apply Theorem 3.6 we have to check Condition (2.13) on the coefficients $\gamma_{q_{k}}(\varphi)$, i.e.:

$$
\exists M, \eta, \theta>0 \text { such that } \frac{1}{N} \operatorname{Card}\left\{j \leq N: a_{j+1}\left|\gamma_{q_{j}}(\varphi)\right| \geq \eta\right\} \geq \theta, \forall N \geq M .
$$

The functions $\{x\}-\frac{1}{2}=\frac{-1}{2 \pi i} \sum_{r \neq 0} \frac{1}{r} e^{2 \pi i r x}$ and $1_{\left[0, \frac{1}{2}[ \right.}-1_{\left[\frac{1}{2}, 1[ \right.}=\sum_{r} \frac{2}{\pi i(2 r+1)} e^{2 \pi i(2 r+1)}$. are immediate examples where this condition is satisfied. In the second case, one observes that $\gamma_{q_{k}}=0$ if $q_{k}$ is even, $=\frac{2}{\pi i}$ if $q_{k}$ is odd. Clearly, (2.13) is satisfied, because two consecutive $q_{k}$ 's are relatively prime and therefore cannot be both even.

In general, for a step function, Condition (2.13) (and therefore a lower bound for the variance $\left\|\varphi_{n}\right\|_{2}^{2}$ for a large set of integers $n$ ) is related to the Diophantine properties of its discontinuities with respect to $\alpha$. We discuss now this point.

Let us consider a centered step function $\varphi$ on $[0,1$ [ taking a non null constant value $v_{j} \in \mathbb{R}$ on the interval $\left[u_{j}, u_{j+1}\left[, j=0,1, \ldots, s\right.\right.$, with $u_{0}=0<u_{1}<\cdots<u_{s}<$ $u_{s+1}=1$ :

$$
\begin{equation*}
\varphi=\sum_{j=0}^{s} v_{j} 1_{\left[u_{j}, u_{j+1}[ \right.}-c \tag{3.18}
\end{equation*}
$$

The constant $c$ above is such that $\varphi$ is centered, but it plays no role below.
Lemma 3.8. If $\varphi$ is given by (3.18), there is a continuous periodic function $H_{\varphi}\left(u_{1}, \ldots, u_{s}\right) \geq 0$ such that

$$
\begin{equation*}
\left|\gamma_{r}(\varphi)\right|^{2}=\pi^{-2} H_{\varphi}\left(r u_{1}, \ldots, r u_{s}\right) . \tag{3.19}
\end{equation*}
$$

Proof. Since $\widehat{\varphi}(r)=\sum_{j=0}^{s} \frac{v_{j}}{\pi r} e^{-\pi i r\left(u_{j}+u_{j+1}\right)} \sin \pi r\left(u_{j+1}-u_{j}\right), r \neq 0, H_{\varphi}\left(u_{1}, \ldots, u_{s}\right)$ is

$$
\left[\sum_{j=0}^{s} v_{j} \cos \pi\left(u_{j}+u_{j+1}\right) \sin \pi\left(u_{j+1}-u_{j}\right)\right]^{2}+\left[\sum_{j=0}^{s} v_{j} \sin \pi\left(u_{j}+u_{j+1}\right) \sin \pi\left(u_{j+1}-u_{j}\right)\right]^{2}
$$

Example 3.9. $\varphi=\varphi(u, \cdot)=1_{[0, u[ }-u, H_{\varphi}(u)=\sin ^{2}(\pi u)$.
Example 3.10. $\varphi=\varphi(w, u, \cdot)=1_{[0, u]}-1_{[w, u+w]}, H(\varphi)=4 \sin ^{2}(\pi u) \sin ^{2}(\pi w)$.
We show now that (2.13) is satisfied generically by the family of step functions parametrised by ( $u_{1}, \ldots, u_{s}$ ) defined by (3.18).

## Corollary 3.11.

(1) Suppose that $\varphi$ is a step function given by (3.18) for $s \geq 1$, with parameter $\left(u_{1}, \ldots, u_{s}\right)$. Then Condition (2.13) is satisfied if $\left(u_{1}, \ldots, u_{s}\right)$ is such that the sequence $\left(q_{k} u_{1}, \ldots, q_{k} u_{s}\right)_{k \geq 1}$ is uniformly distributed in $\mathbb{T}^{s}$.
(2) This latter condition holds for a.e. value of $\left(u_{1}, \ldots, u_{s}\right)$ in $\mathbb{T}^{s}$.

Proof. (1). If the sequence $\left(q_{k} u_{1}, \ldots, q_{k} u_{s}\right)_{k \geq 1}$ is uniformly distributed in $\mathbb{T}^{s}$, we have with the notation of Lemma 3.8:

$$
\begin{align*}
& \lim _{N} \frac{1}{N} \sum_{k=1}^{N}\left|\gamma_{q_{k}}(\varphi)\right|^{2}=\lim _{n} \frac{1}{n} \sum_{k=1}^{n} H_{\varphi}\left(q_{k} u_{1}, \ldots, q_{k} u_{s}\right) \\
&=\int_{\mathbb{T}^{s}} H\left(x_{1}, \ldots, x_{s}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{s}>0, \text { for a.e. }\left(u_{1}, \ldots, u_{s}\right) \in \mathbb{T}^{s} \tag{3.20}
\end{align*}
$$

Let $N_{0}$ and $\delta>0$ be such that, for $N \geq N_{0}, \frac{1}{N} \sum_{k=1}^{N}\left|\gamma_{q_{k}}(\varphi)\right|^{2} \geq \delta$. The sequence $\left(\left|\gamma_{q_{k}}(\varphi)\right|^{2}, k \geq 1\right)$ is bounded by $K:=\pi^{-2}\left\|H_{\varphi}\right\|_{\infty}$. Therefore, we have, for $N \geq N_{0}$,
$\delta \leq \frac{1}{N} \sum_{k=1}^{N}\left|\gamma_{q_{k}}(\varphi)\right|^{2} \leq \frac{K}{N} \sum_{k=1}^{N} 1_{\left|\gamma_{q_{j}}(\varphi)\right| \geq \eta}+\frac{\eta^{2}}{N} \sum_{k=1}^{N} 1_{\left|\gamma_{q_{j}}(\varphi)\right|<\eta} \leq \frac{K}{N} \sum_{k=1}^{N} 1_{\left|\gamma_{q_{j}}(\varphi)\right| \geq \eta}+\eta^{2}$.
This shows: $\frac{1}{N} \sum_{k=1}^{N} 1_{\left|\gamma_{q_{j}}(\varphi)\right| \geq \eta} \geq K^{-1}\left(\delta-\eta^{2}\right)$, for $N \geq N_{0}$.
It follows that (2.13) is satisfied with $M=N_{0}, \eta=\left(\frac{\delta}{2}\right)^{\frac{1}{2}}, \theta=K^{-1} \frac{\delta}{2}$.
(2). To prove the uniform distribution for a.e. value of $\left(u_{1}, \ldots, u_{s}\right)$ in $\mathbb{T}^{s}$, by Weyl equirepartition criterium it suffices to show, for all integers $r_{1}, \ldots, r_{s}$ not all 0 ,

$$
\begin{equation*}
\lim _{k} \frac{1}{N} \sum_{k=1}^{N} e^{2 i \pi q_{k}\left(r_{1} u_{1}+\cdots+r_{s} u_{s}\right)}=0, \text { for a.e. }\left(u_{1}, \ldots, u_{s}\right) \in \mathbb{T}^{s} \tag{3.21}
\end{equation*}
$$

Since $\left(q_{k}\right)$ is a strictly increasing sequence of integers, (3.21) follows from the law of large numbers for orthogonal bounded variables (Rajchman's theorem) which is recalled in Appendix B in a slightly more general formulation (Proposition B.1).

Besides a generic result, there are also specific values of the parameter $\left(u_{1}, \ldots, u_{s}\right)$ for which (2.13) holds. A simple example (for $s=1$ ) is:

We will give another example of special values related to the rectangular billiard model in Example 3.17 below.

Remark 3.13. For the case of Example 3.9, let us make some remarks about the degeneracy of the variance.

It is known that if $\alpha$ is bpq and if $\lim _{k}\left|\sin \left(\pi q_{k} u\right)\right|=0$, where $q_{k}$ are the denominators of $\alpha$, then $u \in \mathbb{Z} \alpha+\mathbb{Z}$ (cf. for instance [5]). But it is easily seen that there is an uncountable set of $u$ 's such that $\lim _{N} \frac{1}{N} \sum_{k=1}^{N} \sin ^{2}\left(\pi q_{k} u\right)=0$ and thus for which Condition (2.13) does not hold.

Observe also that, if $\alpha$ is not bpq, there are many $u$ 's for which the sequence $\left(q_{k} u \bmod 1\right)$ does not satisfy the equidistribution property in a strong sense and (3.20) fails.

Indeed, let $u=\sum_{n \geq 0} b_{n} q_{n} \alpha \bmod 1, b_{n} \in \mathbb{Z}, 0 \leq b_{n} \leq a_{n+1}$, be the so-called Ostrowski expansion of $u$ associated to the denominators of $\alpha$. It can be shown that, if $\lim _{n} \frac{\left|b_{n}\right|}{a_{n+1}}=0$, then $\lim _{k}\left\|q_{k} u\right\|=0([13$, Proposition 1]). There is an uncountable set of $u$ 's satisfying the condition $\lim _{n} \frac{\left|b_{n}\right|}{a_{n+1}}=0$ if $\alpha$ is not bpq. For these values of $u$, we have $\lim _{k} \gamma_{q_{k}}(\varphi(u, \cdot))=0$. Therefore Condition (2.13), which is used to get a lower estimate of the variance, fails, although, if $u$ is not in the countable set $\mathbb{Z} \alpha+\mathbb{Z}, \varphi(u, \cdot)$ is not a coboundary (and even generates an ergodic cocycle).

Remark 3.14. Another remark is about the "generic" validity of estimates of the variance.
As previously remarked, in Theorem 3.6 the CLT is written with self-normalisation (by $\left\|\varphi_{n}\right\|_{2}$ ). In Theorem 2.4 the lower bound given for the variance $\left\|\varphi_{n}\right\|_{2}^{2}$ for $n$ in the set $W$ can be smaller than the mean of the variance.

Inequalities (2.11) and (2.12) give a precise estimation of the variance in the mean when an information is available on $\gamma_{q_{i}}(\varphi)$.

For example in the case of the "saw-tooth" function, we get the estimate $\sum_{k=1}^{m(n)} a_{k}^{2}$ for the mean of the variance.

If we consider Example 3.9 or more generally $\varphi=\varphi(u, \cdot)$ given by (3.18), the same estimate is valid "generically" with respect to $u$ under a condition on $\alpha$. This is a consequence of the equidistribution argument used previously and of Proposition B.1.

Namely, using this proposition and an approximation by trigonometric polynomials, we get:

If $1 \leq a_{n} \leq n^{p}$, with $p<\frac{1}{4}$, if $H_{\varphi}\left(u_{1}, \ldots, u_{s}\right)$ is a continuous periodic function on the torus $\mathbb{T}^{s}, s \geq 1$, then:

$$
\begin{equation*}
\lim _{N} \frac{\sum_{k=1}^{N} a_{k}^{2} H\left(q_{k} u_{1}, \ldots, q_{k} u_{s}\right)}{\sum_{k=1}^{N} a_{k}^{2}}=\int_{\mathbb{T}^{s}} H\left(x_{1}, \ldots, x_{s}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{s} \text {, for a.e. } u . \tag{3.22}
\end{equation*}
$$

For instance, in Example 3.9, $\lim _{N} \frac{\sum_{k=1}^{N} a_{k}^{2} \sin ^{2}\left(\pi q_{k} u\right)}{\sum_{k=1}^{N} a_{k}^{2}}=\frac{1}{2}$, for a.e. $u$.
By (2.12), it follows that the mean of the variance, $\frac{1}{n} \sum_{k=0}^{n-1}\left\|\varphi_{k}(u, .)\right\|_{2}^{2}$, is of order $\sum_{k=1}^{m(n)} a_{k}^{2}$ generically with respect to $u$, if $\alpha$ satisfies Hypothesis 3.2, i.e., $a_{n}=O\left(n^{p}\right)$, $\forall n \geq 1$, with $p<\frac{1}{4}$.

## Vectorial case

For simplicity, we consider the case of two components. Let be given a vectorial function $\Phi=\left(\varphi^{1}, \varphi^{2}\right)$, where $\varphi^{1}, \varphi^{2}$ are two centered step functions with respectively $s_{1}, s_{2}$ discontinuities: $\varphi^{i}=\sum_{j=0}^{s_{i}} v_{j}^{i} 1_{\left[u_{j}^{i}, u_{j+1}^{i}[ \right.}-c_{i}$, for $i=1,2$.

Let the matrix $\Gamma_{n}$ be defined by $\Gamma_{n}(a, b):=(\log n)^{-1}\left\|a \varphi_{n}^{1}+b \varphi_{n}^{2}\right\|_{2}^{2}$ and denote by $I_{2}$ the 2-dimensional identity matrix.

Theorem 3.15. If $\alpha$ has bounded partial quotients and if the condition (2.13) is satisfied uniformly with respect to $(a, b)$ in the unit sphere, there are $0<r_{1}, r_{2}<+\infty$ two constants such that for a "large" set of integers $n$ as in Theorem 3.6:

- $\Gamma_{n}$ satisfies inequalities of the form $r_{1} I_{2} \leq \Gamma_{n}(a, b) \leq r_{2} I_{2}$;
- the distribution of $\Gamma_{n}^{-1} \Phi_{n}$ converges to the standard 2-dimensional normal law.

Proof. We only sketch the proof. The classical method of proof of a CLT for a vectorial function is to show a scalar CLT for all linear combinations of the components of the function. So the proof is like that of Theorem 3.1, but taking care of the bound from below of the variance for $a \varphi_{n}^{1}+b \varphi_{n}^{2}$ : (2.13) should be uniform for $(a, b)$ on the unit sphere.

Proposition 3.16. Let $\Lambda$ be a compact space and $\left(F_{\lambda}, \lambda \in \Lambda\right)$ be a family of nonnegative non identically null continuous functions on $\mathbb{T}^{d}$ depending continuously on $\lambda$. If a sequence $\left(z_{n}\right)$ is equidistributed in $\mathbb{T}^{d}$, then

$$
\exists \theta, N_{0}, \eta>0 \text { such that } \operatorname{Card}\left\{n \leq N: F_{\lambda}\left(z_{n}\right) \geq \eta\right\} \geq \theta N, \forall N \geq N_{0}, \forall \lambda \in \Lambda .
$$

Proof. For $\lambda \in \Lambda$, let $u_{\lambda} \in \mathbb{T}^{d}$ be such that $F_{\lambda}\left(u_{\lambda}\right)=\sup _{u \in \mathbb{T}^{d}} F_{\lambda}(u)$. We have $F_{\lambda}\left(u_{\lambda}\right)>0$ and there is $\eta_{\lambda}>0$ and an open neighborhood $U_{\lambda}$ of $u_{\lambda}$ such that $F_{\lambda}(u)>2 \eta_{\lambda}$ for $u \in U_{\lambda}$. Using the continuity of $F_{\lambda}$ with respect to the parameter $\lambda$, the inequality $F_{\zeta}(u)>\eta_{\lambda}$ holds for $u \in U_{\lambda}$ and $\zeta$ in an open neighborhood $V_{\lambda}$ of $\lambda$. By compactness of $\Lambda$, there is a finite set $\left(\lambda_{j}, j \in J\right)$ such that $\left(V_{\lambda_{j}}, j \in J\right)$ is an open covering of $\Lambda$. Let $\theta:=\frac{1}{2} \inf _{j \in J} \operatorname{Leb}\left(U_{\lambda_{j}}\right)$.

By equidistribution of $\left(z_{n}\right)$, there is $N_{0}$ such that $\frac{1}{N} \sum_{n=1}^{N} 1_{U_{\lambda_{j}}}\left(z_{n}\right) \geq \theta, \forall N \geq N_{0}, \forall$ $j \in J$.

Let $\eta:=\inf _{j \in J} \eta_{\lambda_{j}}$. For every $\lambda \in \Lambda$, there is $j \in J$ such that $\lambda \in V_{\lambda_{j}}$ and therefore $F_{\lambda}\left(z_{n}\right) \geq \eta_{\lambda_{j}} \geq \eta$, if $z_{n} \in U_{\lambda_{j}}$. This implies:

$$
\operatorname{Card}\left\{n \leq N: F_{\lambda}\left(z_{n}\right) \geq \eta\right\} \geq \operatorname{Card}\left\{n \leq N: z_{n} \in U_{\lambda_{j}}\right\} \geq \theta N, \forall N \geq N_{0}
$$

## A generic result

By Proposition 3.16 applied for $(a, b)$ in the unit sphere, for a.e. values of the parameter ( $u_{1}^{1}, \ldots, u_{s_{1}}^{1}, u_{1}^{2}, \ldots, u_{s_{2}}^{2}$ ), the functions $a \varphi^{1}+b \varphi^{2}$ satisfy Condition (2.13) uniformly in $(a, b)$ in the unit sphere. Hence Theorem 3.15 applies generically with respect to the discontinuities.

## Special values: an application to the rectangular billiard in the plane

Example 3.17. Now, for an application to the periodic billiard, we consider the vectorial function $\psi=\left(\varphi^{1}, \varphi^{2}\right)$ with

$$
\begin{aligned}
& \varphi^{1}=1_{\left[0, \frac{\alpha}{2}\right]}-1_{\left[\frac{1}{2}, \frac{1}{2}+\frac{\alpha}{2}\right]}=\frac{2}{\pi} \sum_{r \in \mathbb{Z}} e^{-\pi i(2 r+1) \frac{\alpha}{2}} \frac{\sin \left(\pi(2 r+1) \frac{\alpha}{2}\right)}{2 r+1} e^{2 \pi i(2 r+1)}, \\
& \varphi^{2}=1_{\left[0, \frac{1}{2}-\frac{\alpha}{2}\right]}-1_{\left[\frac{1}{2}, 1-\frac{\alpha}{2}\right]}=\frac{-2 i}{\pi} \sum_{r \in \mathbb{Z}} e^{\pi i(2 r+1) \frac{\alpha}{2}} \frac{\cos \left(\pi(2 r+1) \frac{\alpha}{2}\right)}{2 r+1} e^{2 \pi i(2 r+1)} .
\end{aligned}
$$

The Fourier coefficients of $\varphi^{1}$ and $\varphi^{2}$ of order $r$ are null for $r$ even.
Let us consider a linear combination $\varphi_{a, b}=a \varphi^{1}+b \varphi^{2}$. For $r=2 t+1$ odd, we have:

$$
c_{2 t+1}\left(a \varphi^{1}+b \varphi^{2}\right)=\frac{2}{\pi} \frac{1}{2 t+1} e^{-\pi i(2 t+1) \frac{\alpha}{2}}\left[a \sin \left(\pi(2 t+1) \frac{\alpha}{2}\right)-i b \cos \left(\pi(2 t+1) \frac{\alpha}{2}\right)\right] .
$$

If $q_{j}$ is even, $\gamma_{q_{j}}\left(\varphi_{a, b}\right)$ is null. If $q_{j}$ is odd, we have

$$
\left|\gamma_{q_{j}}\left(\varphi_{a, b}\right)\right|^{2}=4 / \pi^{2}\left|a \sin \left(\pi q_{j} \frac{\alpha}{2}\right)-i b \sin \left(\pi\left(\frac{1}{2}+q_{j} \frac{\alpha}{2}\right)\right)\right|^{2}
$$

For $q_{j}$ odd, we have by (2.3),

$$
\begin{aligned}
\left\|q_{j} \frac{\alpha}{2}\right\| & =\left\|\frac{p_{j}}{2}+\frac{\theta_{j}}{2}\right\|, \text { hence }\left|\left\|q_{j} \frac{\alpha}{2}\right\|-\left\|\frac{p_{j}}{2}\right\|\right| \leq\left|\frac{\theta_{j}}{2}\right| \leq \frac{1}{2 q_{j+1}}, \\
\left\|\frac{1}{2}+q_{j} \frac{\alpha}{2}\right\| & =\left\|\frac{1}{2}+\frac{p_{j}}{2}+\frac{\theta_{j}}{2}\right\|, \text { hence }\left\|\frac{1}{2}+q_{j} \frac{\alpha}{2}\right\|-\left\|\frac{1}{2}+\frac{p_{j}}{2}\right\|\left|\leq\left|\frac{\theta_{j}}{2}\right| \leq \frac{1}{2 q_{j+1}},\right.
\end{aligned}
$$

This implies, for $q_{j}$ odd: $\gamma_{q_{j}}\left(\varphi_{a, b}\right)=a\left(1+O\left(\frac{1}{q_{j+1}}\right)\right)$, if $p_{j}$ is odd, $=b\left(1+O\left(\frac{1}{q_{j+1}}\right)\right)$, if $p_{j}$ is even.

The computation shows that, if $\alpha$ is such that, in average, there is a positive proportion of pairs $\left(p_{j}, q_{j}\right)$ which are (even, odd) and a positive proportion of pairs $\left(p_{j}, q_{j}\right)$ which are (odd, odd), then the condition of Theorem 3.15 is fulfilled by the vectorial step function $\psi=\left(\varphi^{1}, \varphi^{2}\right)$.

For an application to the model of rectangular periodic billiard in the plane described in [6], we refer to [7].

## 4. Proof of Proposition 3.1 (CLT)

The difference $H_{X, Y}(\lambda):=\left|\mathbb{E}\left(e^{i \lambda X}\right)-\mathbb{E}\left(e^{i \lambda Y}\right)\right|$ can be used to get an upper bound of the distance $d(X, Y)$ thanks to the following inequality ([12, Chapter XVI, Inequality (3.13)]): if $X$ has a vanishing expectation, then, for every $U>0$,

$$
\begin{equation*}
d(X, Y) \leq \frac{1}{\pi} \int_{-U}^{U} H_{X, Y}(\lambda) \frac{\mathrm{d} \lambda}{\lambda}+\frac{24}{\pi} \frac{1}{\sigma \sqrt{2 \pi}} \frac{1}{U} \tag{4.1}
\end{equation*}
$$

Using (4.1), we get an upper bound of the distance between the distribution of $X$ and the normal law by bounding $\left|\mathbb{E}\left(e^{i \lambda X}\right)-e^{-\frac{1}{2} \sigma^{2} \lambda^{2}}\right|$.

We will use the following remarks:

$$
\begin{align*}
& \mathrm{V}(f g) \leq\|f\|_{\infty} \mathrm{V}(g)+\|g\|_{\infty} \mathrm{V}(f), \forall f, g \mathrm{BV},  \tag{4.2}\\
& \text { if } g \in C^{1}(\mathbb{R}, \mathbb{R}) \text { and } u \text { is } \mathrm{BV} \text {, then } \mathrm{V}(g \circ u) \leq\left\|g^{\prime}\right\|_{\infty} \mathrm{V}(u) . \tag{4.3}
\end{align*}
$$

Let $w_{k}:=\max _{j=1}^{k} u_{j}$, where $u_{j}$ is larger than $\left\|f_{j}\right\|_{\infty}$ (see Proposition 3.1).
Since $\mathrm{V}\left(f_{k}\right) \leq C u_{k} q_{k}$, (3.3) implies

$$
\begin{equation*}
\left|\int_{X} f_{k} f_{m} \mathrm{~d} \mu\right| \leq C \frac{q_{k}}{q_{m}} m^{\theta} w_{m}^{2}, \text { for } k \leq m \tag{4.4}
\end{equation*}
$$

## Bounding the moments

Lemma 4.1. Under the assumption of Proposition 3.1, there is $C_{1}$ such that

$$
\begin{align*}
& \int_{X}\left|\sum_{k=m}^{m+\ell} f_{k}\right|^{2} \leq C_{1} \ln (m+\ell) \sum_{j \in[m, m+\ell]} u_{j}^{2} \leq C_{1} \ell \ln (m+\ell) w_{m+\ell}^{2} .  \tag{4.5}\\
& \int_{X}\left|\sum_{k=m}^{m+\ell} f_{k}\right|^{3} \leq C_{1} \ell \ln ^{2}(m+\ell) w_{m+\ell}^{3}, \quad \forall m, \ell \geq 1,  \tag{4.6}\\
& \int_{X}\left|\sum_{k=m}^{m+\ell} f_{k}\right|^{4} \leq C_{1} \ell^{2} \ln ^{3}(m+\ell) w_{m+\ell}^{4}, \quad \forall m, \ell \geq 1 . \tag{4.7}
\end{align*}
$$

Proof. We show (4.6) and (4.7). The proof of (4.5) is the same.
For (4.6), it suffices to bound the sums $\sum_{m \leq s \leq t \leq u \leq m+\ell}\left|\int_{X} f_{s} f_{t} f_{u} \mathrm{~d} \mu\right|$.
Replacing $f_{k}$ by $w_{m+\ell}^{-1} f_{k}$, we will deduce the bound (4.6) from the inequalities (3.2), (3.3), (3.4) when $w_{k} \leq 1$, for $1 \leq k \leq m+\ell$. By (3.2) and (4.2), we have

$$
\left|\int_{X} f_{s} f_{t} f_{u} \mathrm{~d} \mu\right| \leq C \text { and } \mathrm{V}\left(f_{s} f_{t}\right) \leq C\left(q_{s}+q_{t}\right) \leq 2 C q_{t}
$$

From (3.3) and (3.1), then from (3.4) and (3.1), we obtain

$$
\begin{aligned}
& \left|\int_{X}\left(f_{s} f_{t}\right) \cdot f_{u} \mathrm{~d} \mu\right| \leq C \frac{V\left(f_{s} f_{t}\right)}{q_{u}} u^{\theta} \leq C \frac{q_{t}}{q_{u}} u^{\theta} \leq C \rho^{(u-t)} u^{\theta}, \\
& \left|\int_{X} f_{s} \cdot\left(f_{t} f_{u}\right) \mathrm{d} \mu\right| \leq C \frac{V\left(f_{s}\right)}{q_{t}} u^{\theta} \leq C \frac{q_{s}}{q_{t}} u^{\theta} \leq C \rho^{(t-s)} u^{\theta}
\end{aligned}
$$

Set $\kappa=\frac{\theta+3}{\ln (1 / \rho)} \ln (m+\ell)$. If $t-s$ or $u-t \geq \kappa$, the previous inequalities imply:

$$
\left|\int_{X} f_{s} f_{t} f_{u} \mathrm{~d} \mu\right| \leq C \rho^{\kappa} u^{\theta} \leq C(m+\ell)^{-\theta-3} u^{\theta} \leq C(m+\ell)^{-3} .
$$

It implies:

$$
\sum_{m \leq s \leq t \leq u \leq m+\ell: \max (t-s, u-t)>\kappa}\left|\int_{X} f_{s} f_{t} f_{u} \mathrm{~d} \mu\right| \leq C \ell^{3}(\ell+m)^{-3} \leq C .
$$

Now the result follows from:

$$
\sum_{m \leq s \leq t \leq u \leq m+\ell: \max (t-s, u-t) \leq \kappa}\left|\int_{X} f_{s} f_{t} f_{u} \mathrm{~d} \mu\right| \leq C \ell \kappa^{2}
$$

For (4.7), we bound the sums $\sum_{m \leq s \leq t \leq u \leq v \leq m+\ell}\left|\int_{X} f_{s} f_{t} f_{u} f_{v} \mathrm{~d} \mu\right|$ using (3.1) and successively (3.3), (3.4), (3.5).

We obtain (because $f_{v}$ is centered for the first inequality):

$$
\begin{aligned}
& \left|\int_{X}\left(f_{s} f_{t} f_{u}\right) \cdot f_{v} \mathrm{~d} \mu\right| \leq C \frac{V\left(f_{s} f_{t} f_{u}\right)}{q_{v}} v^{\theta} \leq C \frac{q_{u}}{q_{v}} v^{\theta} \leq C \rho^{(v-u)} v^{\theta}, \\
& \left|\int_{X}\left[f_{s} f_{t}-\mathbb{E}\left(f_{s} f_{t}\right)\right] f_{u} f_{v} \mathrm{~d} \mu\right| \leq C \frac{V\left(f_{s} f_{t}\right)}{q_{u}} v^{\theta} \leq C \frac{q_{t}}{q_{u}} v^{\theta} \leq C \rho^{(u-t)} v^{\theta}, \\
& \left|\int_{X} f_{s} f_{t} f_{u} f_{v} \mathrm{~d} \mu\right| \leq C \frac{V\left(f_{s}\right)}{q_{t}} v^{\theta} \leq C \frac{q_{s}}{q_{t}} v^{\theta} \leq C \rho^{(t-s)} v^{\theta} .
\end{aligned}
$$

Putting $\kappa=\frac{\theta+4}{\ln (1 / \rho)} \log (m+\ell)$, we get by the previous inequalities, for constants C, $C_{2}, C_{3}$ :

$$
\begin{aligned}
& \sum_{m \leq s \leq t \leq u \leq v \leq m+\ell: \max (t-s, u-t, v-u)>\kappa}\left|\int_{X} f_{s} f_{t} f_{u} f_{v} \mathrm{~d} \mu\right| \\
& \leq C \ell^{4}(\ell+m)^{-4}+\sum_{m \leq s \leq t \leq u \leq v \leq m+\ell}\left|\int_{X} f_{s} f_{t} \mathrm{~d} \mu\right|\left|\int_{X} f_{u} f_{v} \mathrm{~d} \mu\right| \\
& \leq C+\left(\sum_{m \leq s \leq t \leq m+\ell}\left|\int_{X} f_{s} f_{t} \mathrm{~d} \mu\right|\right)^{2} \leq C+C_{2} \ell^{2}(\ln (m+\ell))^{2} .
\end{aligned}
$$

The remaining terms give a bound which can be absorbed in the previous one, namely:

$$
\sum_{m \leq s \leq t \leq u \leq m+\ell: \max (t-s, u-t, v-u) \leq \kappa}\left|\int_{X} f_{s} f_{t} f_{u} f_{v} \mathrm{~d} \mu\right| \leq C \ell \kappa^{3} \leq C_{3} \ell \log (m+\ell)^{3}
$$

## Proof of Proposition 3.1

The proof is given in several steps.

## Defining blocks

We split the sum $S_{n}:=f_{1}+\cdots+f_{n}$ into small and large blocks. The small ones will be removed, providing gaps and allowing to take advantage of the decorrelation properties assumed in the statement of the proposition.

Let $\tau, \delta$ be parameters ( $\delta$ close to 0 ) such that $0<\delta<\frac{1}{2}$ and $\delta<\tau$. We set for $n \geq 1$ :

$$
\begin{aligned}
& n_{1}=n_{1}(n):=\left\lfloor n^{\tau}\right\rfloor, n_{2}=n_{2}(n):=\left\lfloor n^{\delta}\right\rfloor, \\
& v=v(n):=n_{1}+n_{2}, p(n):=\lfloor n / v(n)\rfloor+1=n^{1-\tau}+h_{n} \sim n^{1-\tau} .
\end{aligned}
$$

For $0 \leq k<p(n)$, we put (with $f_{j}=0$, if $n<j \leq n+v$ )

$$
F_{n, k}=f_{k v(n)+1}+\cdots+f_{k v(n)+n_{1}(n)}, \quad G_{n, k}=f_{k v(n)+n_{1}(n)+1}+\cdots+f_{(k+1) v(n)} .
$$

The sums $F_{n, k}, G_{n, k}$ have respectively $n_{1} \sim n^{\tau}, n_{2} \sim n^{\delta}$ terms and $S_{n}$ reads

$$
S_{n}=\sum_{k=0}^{p(n)-1}\left(F_{n, k}+G_{n, k}\right) .
$$

We put $S_{n}^{\prime}:=\sum_{k=0}^{p(n)-1} F_{n, k}, v_{k}=v_{n, k}:=\left(\int_{X} F_{n, k}^{2} \mathrm{~d} \mu\right)^{\frac{1}{2}}$.
The following inequalities are implied by (4.5):

$$
\begin{equation*}
v_{k}^{2}=v_{n, k}^{2}=\left\|F_{n, k}\right\|_{2}^{2} \leq C n^{\tau} \ln n w_{n}^{2},\left\|G_{n, k}\right\|_{2}^{2} \leq C n^{\delta} \ln n w_{n}^{2}, 0 \leq k<p(n) . \tag{4.8}
\end{equation*}
$$

Since $q_{1}+q_{2}+\cdots+q_{n} \leq C q_{n+1}, \forall n \geq 1$, by (3.1), it follows by (4.3) and hypothesis (3.2):

$$
\begin{equation*}
\mathrm{V}\left(e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)}\right) \leq C|\zeta| w_{n} q_{(k-1) v+n_{1}} \tag{4.9}
\end{equation*}
$$

## Lemma 4.2.

$$
\begin{align*}
& \left|\left\|S_{n}\right\|_{2}^{2}-\sum_{k=0}^{p(n)-1} v_{k}^{2}\right|=\left|\left\|S_{n}\right\|_{2}^{2}-\sum_{k=0}^{p(n)-1}\left\|F_{n, k}\right\|_{2}^{2}\right| \leq C n^{1-\frac{\tau-\delta}{2}} \ln n w_{n}^{2},  \tag{4.10}\\
& \left\|S_{n}-S_{n}^{\prime}\right\|_{2}^{2}=\left\|\sum_{k=0}^{p(n)-1} G_{k}\right\|^{2} \leq C n^{1-\tau+\delta} \ln n w_{n}^{2} . \tag{4.11}
\end{align*}
$$

Proof. It follows from (4.4) and (3.1), with $C_{0}=\frac{C \rho}{(1-\rho)^{2}}$,

$$
\left|\int_{X}\left(\sum_{u=a}^{b} f_{u}\right)\left(\sum_{t=c}^{d} f_{t}\right) \mathrm{d} \mu\right| \leq C_{0} \rho^{c-b} d^{\theta} w_{d}^{2}, \quad \forall a \leq b<c \leq d .
$$

Therefore, we have, with $C_{1}=C_{0} \sum_{i \geq 0} \rho^{i v}$, writing simply $F_{k}, G_{k}$ instead of $F_{n, k}, G_{n, k}$,

$$
\begin{aligned}
& \sum_{0 \leq j<k<p(n)}\left|\int F_{j} F_{k} \mathrm{~d} \mu\right| \leq C_{0} n^{\theta} w_{n}^{2} \sum_{0 \leq j<k<p(n)} \rho^{k v+1-\left(j v+n_{1}\right)} \\
& \leq C_{0} n^{\theta} w_{n}^{2} \rho^{n_{2}} \sum_{0 \leq j<k<p(n)} \rho^{(k-1) v-j v} \leq C_{0} \rho^{n_{2}} n^{\theta} w_{n}^{2} p(n) \sum_{i \geq 0} \rho^{i v} \leq C_{1} n^{1-\tau+\theta} w_{n}^{2} \rho^{n^{\delta}} .
\end{aligned}
$$

The LHS of (4.10) is less than the sum for $k=0$ to $p(n)-1$ of

$$
\begin{aligned}
& \int G_{k}^{2} \mathrm{~d} \mu+\left|\int G_{k} F_{k} \mathrm{~d} \mu\right|+\left|\int G_{k} F_{k+1} \mathrm{~d} \mu\right| \\
& \quad+2 \sum_{0 \leq j<k}\left[\left|\int F_{j}\left(F_{k}+G_{k}\right) \mathrm{d} \mu\right|+\left|\int G_{j} G_{k} \mathrm{~d} \mu\right|\right]+2 \sum_{0 \leq j<k-1}\left|\int G_{j} F_{k} \mathrm{~d} \mu\right|
\end{aligned}
$$

The first term is bounded by $C n^{\delta} \ln n w_{n}^{2}$, the second one and the third one bounded by $C n^{\frac{\delta+\tau}{2}} \ln n w_{n}^{2}$ are the biggest. The other terms are negligible as shown by the preliminary computation because of the factor $\rho^{n^{\delta}}$.

Therefore the LHS of (4.10) is less than: $C_{1} n^{1-\tau} n^{\frac{\delta+\tau}{2}} \ln n w_{n}^{2}=C_{1} n^{1-\frac{\tau-\delta}{2}} \ln n w_{n}^{2}$.
An analogous computation shows that the LHS of (4.11) behaves like $\sum_{k=0}^{p(n)-1} \int G_{k}^{2} \mathrm{~d} \mu$ which gives the bound $C n^{1-\tau+\delta} \ln n w_{n}^{2}$ of (4.11).

## Approximation of the characteristic function of the sum $S_{n}^{\prime}$ by a product

For $\zeta \in \mathbb{R}$, let $\mathrm{I}_{n,-1}(\zeta):=1, \mathrm{I}_{n, k}(\zeta):=\int_{X} e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k}\right)} \mathrm{d} \mu, 0 \leq k<p(n)$.
Lemma 4.3. For $0 \leq k<p(n)$, we have

$$
\begin{equation*}
\left|\mathrm{I}_{n, k}(\zeta)-\left(1-\frac{\zeta^{2}}{2} v_{n, k}^{2}\right) \mathrm{I}_{n, k-1}(\zeta)\right| \leq C\left(|\zeta|^{3} w_{n}^{3} n^{\tau} \ln ^{2}(n)+\zeta^{4} w_{n}^{4} n^{2 \tau} \ln ^{2}(n)\right) \tag{4.12}
\end{equation*}
$$

Proof. We use $e^{i u}=1+i u-\frac{1}{2} u^{2}-\frac{i}{6} u^{3}+u^{4} r(u)$, with $|r(u)| \leq \frac{1}{24}$, for $u \in \mathbb{R}$. Let $k \geq 1$.
We have

$$
\mathrm{I}_{n, k}(\zeta)=\int_{X} e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)}\left[1+i \zeta F_{n, k}-\frac{\zeta^{2}}{2} F_{n, k}^{2}-\frac{i}{6} \zeta^{3} F_{n, k}^{3}+\zeta^{4} F_{n, k}^{4} r\left(\zeta F_{n, k}\right)\right] \mathrm{d} \mu
$$

For the first term, using (4.9), we have:

$$
\begin{align*}
\left|\int_{X} e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)} F_{n, k} \mathrm{~d} \mu\right| & \leq \sum_{j=1}^{n_{1}}\left|\int_{X} e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)} f_{k v+j} \mathrm{~d} \mu\right| \\
& \leq C|\zeta| w_{n} \sum_{j=1}^{n_{1}} \frac{q_{(k-1) v+n_{1}}}{q_{k v+j}}(k v+j)^{\theta} w_{(k-1) v+n_{1}} \\
& \leq C|\zeta| w_{n}^{2} n^{\theta} \sum_{j=1}^{n_{1}} \rho^{v+j-n_{1}} \leq \frac{C \rho}{1-\rho}|\zeta| w_{n}^{2} n^{\theta} \rho^{n_{2}} . \tag{4.13}
\end{align*}
$$

Similarly, for the second term we apply (3.4) and (4.9) and we get:

$$
\begin{align*}
\mid \int_{X} e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)} F_{n, k}^{2} \mathrm{~d} \mu & -\mathrm{I}_{n, k-1} \int_{X} F_{n, k}^{2} \mathrm{~d} \mu \mid \\
& \leq C \mathrm{~V}\left(e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)}\right) w_{n}^{2} \sum_{j^{\prime}=1}^{n_{1}} \sum_{j=1}^{j^{\prime}} \frac{\left(k v+j^{\prime}\right)^{\theta}}{q_{k v+j}} \\
& \leq C|\zeta| w_{n}^{3} n^{\theta+\tau} \rho^{n_{2}} . \tag{4.14}
\end{align*}
$$

Likewise (3.5) and Lemma 4.1 imply:

$$
\begin{align*}
& \left|\int_{X} e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)} F_{n, k}^{3} \mathrm{~d} \mu\right| \\
& \quad \leq \mid \int_{X}\left(e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)}-\mathbb{E}\left(e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)}\right) F_{n, k}^{3} \mathrm{~d} \mu\left|+\left|\int_{X} F_{n, k}^{3} \mathrm{~d} \mu\right|\right.\right. \\
& \quad \leq C|\zeta| n^{1+2 \tau+\theta} w_{n}^{3} \rho^{n_{2}}+C n^{\tau} w_{n}^{3} \ln ^{2}(n) \tag{4.15}
\end{align*}
$$

At last, by (4.7) we have

$$
\begin{equation*}
\left|\int_{X} e^{i \zeta\left(F_{n, 0}+\cdots+F_{n, k-1}\right)} F_{n, k}^{4} r\left(\zeta F_{n, k}\right) \mathrm{d} \mu\right| \leq \int_{X} F_{n, k}^{4} \mathrm{~d} \mu \leq w_{n}^{4} n^{2 \tau} \ln (n)^{2} . \tag{4.16}
\end{equation*}
$$

From (4.13), (4.14), (4.15) and (4.16), we deduce that $\left|\mathrm{I}_{n, k}(\zeta)-\left(1-\frac{\zeta^{2}}{2} v_{n, k}^{2}\right) \mathrm{I}_{n, k-1}(\zeta)\right|$ is bounded up to a constant factor $C$ by

$$
\begin{aligned}
|\zeta|^{2} w_{n}^{2} n^{\theta} \rho^{n_{2}}+|\zeta|^{3} w_{n}^{3} n^{\theta+\tau} \rho^{n_{2}}+|\zeta|^{4} n^{1+2 \tau+\theta} & w_{n}^{4} \rho^{n_{2}} \\
& +|\zeta|^{3} w_{n}^{3} n^{\tau} \ln ^{2}(n)+|\zeta|^{4} w_{n}^{4} n^{2 \tau} \ln ^{2}(n)
\end{aligned}
$$

In the sum above, for $n$ big, we keep only the last two terms, since for $n$ big enough the first terms are smaller than the last ones.

If $X$ and $Y$ are two real square integrable random variables, then $\left|\mathbb{E}\left(e^{i X}\right)-\mathbb{E}\left(e^{i Y}\right)\right| \leq$ $\|X-Y\|_{2}$. Therefore, using (4.11), we have for $\mathrm{J}_{n}(\zeta):=\int_{X} e^{i \zeta S_{n}} \mathrm{~d} \mu$ :

$$
\begin{equation*}
\left|\mathbf{J}_{n}(\zeta)-\mathbf{I}_{n, p(n)}(\zeta)\right| \leq|\zeta|\left\|S_{n}-S_{n}^{\prime}\right\|_{2} \leq C|\zeta| w_{n} n^{\frac{1-\tau+\delta}{2}}(\ln n)^{\frac{1}{2}} \tag{4.17}
\end{equation*}
$$

then, by (4.17) and (4.12) of Lemma 4.3, we get

$$
\begin{align*}
& \left|\mathrm{J}_{n}(\zeta)-\prod_{k=1}^{p(n)}\left(1-\frac{1}{2} \zeta^{2} v_{k}^{2}\right)\right| \\
& \quad \leq\left|\mathrm{J}_{n}(\zeta)-\mathrm{I}_{n, p(n)}(\zeta)\right|+\sum_{k=0}^{p(n)-1}\left|\mathrm{I}_{n, k}(\zeta)-\left(1-\frac{\zeta^{2}}{2} v_{k}^{2}\right) \mathrm{I}_{n, k-1}(\zeta)\right| \\
& \quad \leq C\left[|\zeta| w_{n} n^{\frac{1-\tau+\delta}{2}}(\ln n)^{1 / 2}+n^{1-\tau}|\zeta|^{3} w_{n}^{3} n^{\tau} \ln ^{2}(n)+n^{1-\tau} \zeta^{4} w_{n}^{4} n^{2 \tau} \ln (n)^{2}\right. \\
& \quad \leq C\left[|\zeta| w_{n} n^{\frac{1-\tau+\delta}{2}}(\ln n)^{\frac{1}{2}}+|\zeta|^{3} w_{n}^{3} n(\ln n)^{2}+\zeta^{4} w_{n}^{4} n^{1+\tau}(\ln n)^{2}\right] . \tag{4.18}
\end{align*}
$$

## Approximation of the exponential by a product

Below, $\zeta$ will be such that $|\zeta| v_{n, k} \leq 1$. This is satisfied if

$$
\begin{equation*}
|\zeta| n^{\frac{\tau}{2}} w_{n}(\log n)^{\frac{1}{2}} \leq 1 . \tag{4.19}
\end{equation*}
$$

Lemma 4.4. If $\left(\rho_{k}\right)_{k \in J}$ is a finite family of real numbers in $[0,1[$, then

$$
\begin{equation*}
0 \leq e^{-\sum_{k \in J} \rho_{k}}-\prod_{k \in J}\left(1-\rho_{k}\right) \leq \sum_{k \in J} \rho_{k}^{2}, \quad \text { if } 0 \leq \rho_{k} \leq \frac{1}{2}, \forall k \tag{4.20}
\end{equation*}
$$

Proof. We have $\ln (1-u)=-u-u^{2} v(u)$, with $\frac{1}{2} \leq v(u) \leq 1$, for $0 \leq u \leq \frac{1}{2}$ and $1-e^{-\sum \varepsilon_{k}} \leq \sum \varepsilon_{k}$, if $\sum_{k} \varepsilon_{k} \geq 0$.

Writing $1-\rho_{k}=e^{-\rho_{k}-\varepsilon_{k}}$, with $\varepsilon_{k}=-\left(\ln \left(1-\rho_{k}\right)+\rho_{k}\right)$, the previous inequality implies $0 \leq \varepsilon_{k} \leq \rho_{k}^{2}$, if $0 \leq \rho_{k} \leq \frac{1}{2}$. Therefore, under this condition, we have:

$$
\begin{aligned}
0 & \leq e^{-\sum_{J} \rho_{k}}-\prod_{J}\left(1-\rho_{k}\right)=e^{-\sum_{J} \rho_{k}}\left(1-e^{-\sum_{J} \varepsilon_{k}}\right) \\
& \leq e^{-\sum \rho_{k}} \sum \varepsilon_{k} \leq \sum \varepsilon_{k} \leq \sum \rho_{k}^{2}
\end{aligned}
$$

We apply (4.20) with $\rho_{k}=\frac{1}{2} \zeta^{2} v_{n, k}^{2}$, under Condition (4.19). In view of (4.8) it follows:

$$
\begin{equation*}
\left|e^{\frac{1}{2} \zeta^{2} \sum v_{k}^{2}}-\prod_{k=0}^{p(n)-1}\left(1-\frac{1}{2} \zeta^{2} v_{k}^{2}\right)\right| \leq \frac{1}{4} \zeta^{4} \sum_{k=0}^{p(n)-1} v_{k}^{4} \leq C \zeta^{4} w_{n}^{4} n^{1+\tau} \ln ^{2}(n) \tag{4.21}
\end{equation*}
$$

The bound is like the last term in (4.18).

## Conclusion

From (4.18) and (4.21), it follows:

$$
\begin{aligned}
&\left|J_{n}(\zeta)-e^{-\frac{1}{2} \zeta^{2} \sum_{k=0}^{p(n)-1} v_{k}^{2}}\right| \\
& \leq C\left[|\zeta| w_{n} n^{\frac{1-\tau+\delta}{2}}(\ln n)^{\frac{1}{2}}+|\zeta|^{3} w_{n}^{3} n(\ln n)^{2}+\zeta^{4} w_{n}^{4} n^{1+\tau}(\ln n)^{2}\right]
\end{aligned}
$$

We replace $\zeta$ by $\frac{\lambda}{\left\|S_{n}\right\|_{2}}$; hence Condition (4.19) becomes

$$
\begin{equation*}
\frac{\lambda}{\left\|S_{n}\right\|_{2}} n^{\frac{\tau}{2}} w_{n}(\log n)^{\frac{1}{2}} \leq 1 \tag{4.22}
\end{equation*}
$$

We get:

$$
\begin{aligned}
& \left|\int_{X} e^{i \lambda \frac{S_{n}(x)}{\left\|S_{n}\right\|_{2}}} \mathrm{~d} \mu(x)-e^{-\frac{1}{2} \frac{\lambda^{2}}{\left\|S_{n}\right\|_{2}} \sum_{k} v_{k}^{2}}\right| \\
& \quad \leq C\left[|\lambda| \frac{w_{n}}{\left\|S_{n}\right\|_{2}} n^{1-\tau+\delta}+|\lambda|^{3} \frac{w_{n}^{3}}{\left\|S_{n}\right\|_{2}^{3}} n \ln ^{2}(n)+\lambda^{4} \frac{w_{n}^{4}}{\left\|S_{n}\right\|_{2}^{4}} n^{1+\tau} \ln ^{2}(n)\right]
\end{aligned}
$$

Since $\left|e^{-a}-e^{-b}\right| \leq|a-b|$, for any $a, b \geq 0$, we have, by (4.10):

$$
\left|e^{-\frac{1}{2} \lambda^{2}}-e^{-\frac{1}{2} \frac{\lambda^{2}}{\left\|S_{n}\right\|_{2}^{2}} \sum_{k=0}^{p(n)-1} v_{k}^{2}}\right| \leq \frac{1}{2} \frac{\lambda^{2}}{\left\|S_{n}\right\|_{2}^{2}}\left|\left\|S_{n}\right\|_{2}^{2}-\sum_{k=0}^{p(n)-1} v_{k}^{2}\right| \leq C \lambda^{2} \frac{w_{n}^{2}}{\left\|S_{n}\right\|_{2}^{2}} n^{1-\frac{\tau-\delta}{2}} \ln n .
$$

Let us call respectively $E_{1}$ the error in neglecting the sums on the small blocks, $E_{2}$ the error in the replacement of $e^{-\frac{1}{2} \lambda^{2}}$ by $\exp \left(-\frac{1}{2} \lambda^{2} \frac{\sum v_{k}^{2}}{\left\|S_{n}\right\|_{2}^{2}}\right), E_{3}$ the error of order 3 in the expansion, $E_{4}$ the approximation error of the exponential by the product.

Finally we get the bound $\left|\int_{X} e^{i \lambda \frac{S_{n}(x)}{\left\|n_{n}\right\|_{2}}} \mathrm{~d} \mu(x)-e^{-\frac{1}{2} \lambda^{2}}\right| \leq E_{1}+E_{2}+E_{3}+E_{4}$ and this sum is smaller than

$$
\begin{aligned}
& C\left[|\lambda| \frac{w_{n}}{\left\|S_{n}\right\|_{2}} n^{\frac{1-\tau+\delta}{2}} \ln ^{\frac{1}{2}}(n)+\lambda^{2} \frac{w_{n}^{2}}{\left\|S_{n}\right\|_{2}^{2}} n^{1-\frac{\tau-\delta}{2}} \ln n\right. \\
&\left.+|\lambda|^{3} \frac{w_{n}^{3}}{\left\|S_{n}\right\|_{2}^{3}} n \ln ^{2}(n)+\lambda^{4} \frac{w_{n}^{4}}{\left\|S_{n}\right\|_{2}^{4}} n^{1+\tau+\delta} \ln ^{2}(n)\right]
\end{aligned}
$$

Denote by $Y_{1}$ a r.v. with $\mathcal{N}(0,1)$-distribution. Putting $R_{n}:=\frac{w_{n}}{\left\|S_{n}\right\|_{2}}$, the bound reads:

$$
\begin{equation*}
C\left[|\lambda| R_{n} n^{\frac{1-\tau+\delta}{2}}+\lambda^{2} R_{n}^{2} n^{1-\frac{\tau-\delta}{2}} \ln n+|\lambda|^{3} R_{n}^{3} n \ln ^{2}(n)+\lambda^{4} R_{n}^{4} n^{1+\tau+\delta} \ln ^{2}(n)\right] \tag{4.23}
\end{equation*}
$$

Notice that $\delta$ can be taken arbitrary small. A change of its value modifies the generic constant $C$ in the previous inequalities. Therefore we take $\delta=0$ in the optimisation below, keeping in mind that the constant factor in the inequalities depends on $\delta$. Likewise the $\ln n$ factors can be neglected.

We have an inequality of the form $H_{\frac{S_{n}}{\left\|S_{n}\right\|_{2}}, Y_{1}}(\lambda) \leq C \sum_{i=1}^{4}|\lambda|^{\alpha_{i}} R_{n}^{\alpha_{i}} n^{\gamma_{i}}$, where the exponents are given by the previous inequality. In view of (4.1), it follows that, up to a constant factor,

$$
d\left(\frac{S_{n}}{\left\|S_{n}\right\|_{2}}, Y_{1}\right) \leq \frac{1}{U_{n}}+U_{n} R_{n} n^{\frac{1-\tau}{2}}+\frac{1}{2} U_{n}^{2} R_{n}^{2} n^{1-\frac{\tau}{2}}+\frac{1}{3} U_{n}^{3} R_{n}^{3} n+\frac{1}{4} U_{n}^{4} R_{n}^{4} n^{1+\tau} .
$$

Now, we optimize the choice of $U=U_{n}$. As $R_{n}$ is less than $n^{-\beta}$ for some $\beta>0$, if we take $U_{n}=n^{\gamma}$ with $\gamma>0$, then (4.23) gives inside the bracket the bound:

$$
n^{\frac{1-\tau}{2}-\beta+\gamma}+\frac{1}{2} n^{1-\frac{\tau}{2}-2 \beta+2 \gamma}+\frac{1}{3} n^{1-3 \beta+3 \gamma}+\frac{1}{4} n^{1+\tau-4 \beta+4 \gamma}
$$

We choose $U_{n}$ such that $1 / U_{n}$ is of the same order as the second term, i.e., we take $n^{-\gamma}=n^{1-\frac{\tau}{2}-2 \beta+2 \gamma}$, i.e., $\gamma=\frac{\frac{\tau}{2}+2 \beta-1}{3}$. If $\tau=\frac{1}{2}$ and if $\beta=\frac{1}{2}-p$ with $p>0$, then it gives:

$$
\gamma=\frac{1-8 p}{12}>0 \quad \text { if } p<\frac{1}{8}
$$

The four terms in the bound are respectively:
$(A)=-\frac{1}{6}+\frac{p}{3}$,
$(B)=-\gamma=-\frac{1}{12}+\frac{2 p}{3}$,
$(C)=-\frac{1}{4}+p, \quad(D)=-\frac{1}{6}+\frac{4 p}{3}$.

We check that $(B)$ is the biggest term: $(B)-(A)=\frac{1}{12}+\frac{p}{3}>0,(B)-(C)=\frac{1}{6}-\frac{p}{3}>0$, if $p<\frac{1}{2},(B)-(D)=\frac{1}{12}-\frac{2 p}{3}>0$, if $p<\frac{1}{8}$.

This gives the bound stated in Proposition 3.1 for the distance to the normal law:
For every $\delta>0$, for $N$ big enough, there is a constant $C(\delta)>0$ (depending only on $\delta$ ) such that, if $\frac{w_{N}}{\left\|S_{N}\right\|_{2}} \leq N^{p-\frac{1}{2}}$ with $p \in\left[0, \frac{1}{8}[\right.$, then

$$
d\left(\frac{S_{N}}{\left\|S_{N}\right\|_{2}}, Y_{1}\right) \leq C(\delta) N^{-\frac{1-8 p}{12}+\delta} \ll 1
$$

To conclude, observe that, if $w_{n} \leq n^{p}$ with $p<\frac{1}{8}$ and $\left\|S_{n}\right\|_{2} \geq C n^{\frac{1}{2}} /(\log n)^{\frac{1}{2}}$, (4.22) is satisfied for $|\lambda| \leq U_{n}=n^{\gamma}$, since

$$
n^{\gamma} n^{\frac{\tau}{2}} \frac{w_{n}(\log n)^{\frac{1}{2}}}{\left\|S_{n}\right\|_{2}} \leq C n^{\frac{1-8 p}{12}} n^{p-\frac{1}{4}}(\log n)^{\frac{1}{2}}=C n^{-\frac{1}{6}+\frac{p}{3}}(\log n)^{\frac{1}{2}} \leq 1 \text {, for } n \text { big. }
$$

## 5. Proof of Proposition 3.5 (decorrelation)

For the proof of Proposition 3.5, by homogeneity, we may assume that $\psi$ and $\varphi$ are BV centered functions with variation $\leq 1$. Moreover, we may also assume $b_{i}=1, \forall i$. Indeed, the decorrelation inequalities will follow from bounds on sums of products of quantities like $\left|\widehat{\varphi_{b_{n} q_{n}}}(j)\right| \leq b_{n}\left|\widehat{\varphi_{q_{n}}}(j)\right|$ or $\left\|\varphi_{b_{n} q_{n}}\right\|_{2} \leq b_{n}\left\|\varphi_{q_{n}}\right\|_{2}$.

First we truncate the Fourier series of the ergodic sums $\varphi_{q}$. For functions in $C$, the remainders are easily controlled and it suffices to treat the case of trigonometric polynomials.

For $\varphi \in C$, the Fourier coefficients of order $j \neq 0$ of the ergodic $\operatorname{sum} \varphi_{n}$ satisfy:

$$
\begin{equation*}
\left|\widehat{\varphi_{n}}(j)\right|=\frac{\left|\gamma_{j}(\varphi)\right|}{|j|} \frac{|\sin \pi n j \alpha|}{|\sin \pi j \alpha|} \leq \frac{\pi}{2} \frac{K(\varphi) \mid}{|j|} \frac{\|n j \alpha\|}{\|j \alpha\|} . \tag{5.1}
\end{equation*}
$$

Recall also (cf. (2.9)) that, if $q$ is a denominator of $\alpha$, then

$$
\left\|\varphi_{q}\right\|_{\infty}=\sup _{x}\left|\sum_{\ell=0}^{q-1} \varphi(x+\ell \alpha)\right| \leq V(\varphi) \quad \text { and } \quad\left\|\varphi_{q}\right\|_{2} \leq 2 \pi K(\varphi) .
$$

We will use the notations: $S_{L} f$ for the partial sum of order $L \geq 1$ of the Fourier series of $f \in L^{2}(\mathbb{T}), R_{L} f:=f-S_{L} f$ for the remainder and, $q_{n}$ denoting the denominators of $\alpha$,

$$
\begin{equation*}
a_{n}^{\prime}:=\frac{q_{n+1}}{q_{n}} \leq a_{n+1}+1, c_{n}:=\frac{q_{n+1}}{q_{n}} \ln q_{n+1} . \tag{5.2}
\end{equation*}
$$

## Preliminary inequalities and truncation

We begin by some inequalities which are valid for any irrational number $\alpha$.
Lemma 5.1. There is a constant $C$ such that, if $q$ is a denominator of $\alpha$,

$$
\begin{equation*}
\sum_{|j| \geq q} \frac{1}{j^{2}} \frac{\|L j \alpha\|^{2}}{\|j \alpha\|^{2}} \leq C \frac{L}{q}, \quad \forall L \in[1, q] . \tag{5.3}
\end{equation*}
$$

Proof. If $f$ is a non negative BV function with integral $\mu(f)$, by Denjoy-Koksma inequality applied to $f-\mu(f)$, we have

$$
\begin{array}{rl}
\sum_{j=q}^{\infty} \frac{f(j \alpha)}{j^{2}} \leq \sum_{i=1}^{\infty} \frac{1}{(i q)^{2}} \sum_{r=0}^{q-1} & f((i q+r) \alpha) \\
& \leq \frac{1}{q^{2}}\left(\sum_{i=1}^{\infty} \frac{1}{i^{2}}\right)(q \mu(f)+V(f))=\frac{\pi^{2}}{6}\left(\frac{\mu(f)}{q}+\frac{V(f)}{q^{2}}\right)
\end{array}
$$

Taking for $f(x)$ respectively $1_{\left[0, \frac{1}{L}\right]}(|x|)$ and $\frac{1}{x^{2}} 1_{\left[\frac{1}{L}, \frac{1}{2}\right.}(|x|)$, we obtain:

$$
\sum_{j:\|j \alpha\| \leq \frac{1}{L}, j \geq q} \frac{1}{j^{2}} \leq C\left(\frac{1}{q^{2}}+\frac{1}{L q}\right), \quad \sum_{j:\|j \alpha\| \geq \frac{1}{L}, j \geq q} \frac{1}{j^{2}} \frac{1}{\|j \alpha\|^{2}} \leq C\left(\frac{L^{2}}{q^{2}}+\frac{L}{q}\right) .
$$

This implies (5.3), since for $L \leq q$ :

$$
\begin{aligned}
\frac{1}{2} \sum_{|j| \geq q} \frac{1}{j^{2}} \frac{\|L j \alpha\|^{2}}{\|j \alpha\|^{2}} & \leq L^{2} \sum_{\|j \alpha\| \leq \frac{1}{L}, j \geq q} \frac{1}{j^{2}}+\sum_{\|j \alpha\|>\frac{1}{L}, j \geq q} \frac{1}{j^{2}} \frac{1}{\|j \alpha\|^{2}} \\
& \leq 2 C\left(\frac{L}{q}+\frac{L^{2}}{q^{2}}\right) \leq 4 C \frac{L}{q} .
\end{aligned}
$$

We will use the good equirepartition of the numbers $\|k \alpha\|$ when $k$ varies between 1 and $q_{n}$ through two inequalities given in the following lemma, which will be used several times.

Lemma 5.2. We have

$$
\begin{gather*}
\sum_{j=q_{t}}^{q_{t+1}-1} \frac{1}{\|j \alpha\|} \leq \sum_{j=1}^{q_{t+1}-1} \frac{1}{\|j \alpha\|} \leq C q_{t+1} \ln q_{t+1}, \quad \forall t \geq 0,  \tag{5.4}\\
\sum_{1 \leq j<q_{r+1}} \frac{1}{j\|j \alpha\|} \leq C \sum_{t=0}^{r} \frac{q_{t+1}}{q_{t}} \ln q_{t+1}=C \sum_{t=0}^{r} c_{t}, \quad \forall r \geq 0 . \tag{5.5}
\end{gather*}
$$

Proof. There is exactly one element of the set $\left\{j \alpha \bmod 1, j=1, \ldots, q_{t+1}-1\right\}$ in each interval $\left[\frac{\ell}{q_{t+1}}, \frac{\ell+1}{q_{t+1}}\left[, \ell=1, \ldots, q_{t+1}-1\right.\right.$. Moreover, for $1 \leq j<q_{t+1}$, one has $\|j \alpha\| \geq \frac{1}{2 q_{t+1}}$.

This implies: $\sum_{j=1}^{q_{t+1}-1} \frac{1}{\|j \alpha\|} \leq 2 q_{t+1}+\sum_{\ell=1}^{q_{t+1}-1} \frac{1}{\ell / q_{t+1}} \leq C q_{t+1} \ln q_{t+1}$.
From (5.4) applied for $t=1, \ldots, r$, we deduce (5.5):

$$
\sum_{1 \leq j<q_{r+1}} \frac{1}{j\|j \alpha\|}=\sum_{t=0}^{r} \sum_{q_{t} \leq j<q_{t+1}} \frac{1}{j\|j \alpha\|} \leq \sum_{t=0}^{r} \frac{1}{q_{t}} \sum_{q_{t} \leq j<q_{t+1}} \frac{1}{\|j \alpha\|} \leq C \sum_{t=0}^{r} \frac{q_{t+1}}{q_{t}} \ln q_{t+1}
$$

Lemma 5.3. For $\varphi \in C$, it holds

$$
\begin{equation*}
\left\|S_{q_{r}} \varphi_{q_{n}}\right\|_{\infty} \leq C V(\varphi) \ln \left(q_{r}\right) \tag{5.6}
\end{equation*}
$$

Proof. Using the Fejér kernel, we get

$$
\left\|S_{q_{r}} \varphi_{q_{n}}\right\|_{\infty} \leq\left\|\varphi_{q_{n}}\right\|_{\infty}+\frac{1}{q_{r}} \sum_{|j|<q_{r}}\left|j \widehat{\varphi_{q_{n}}}(j)\right| \leq\left\|\varphi_{q_{n}}\right\|_{\infty}+C K(\varphi) \frac{1}{q_{r}} \sum_{j=1}^{q_{r}-1} \frac{1}{\|j \alpha\|}
$$

(5.6) follows by (2.9) and (5.4).

## Truncation

Now we bound the truncation error for the Fourier series of the ergodic sums $\varphi_{b_{n} q_{n}}$.
Lemma 5.4. If $\psi$ is bounded and $\varphi \in C$, with $C_{1}=V(\varphi)^{2}\|\psi\|_{\infty}, C_{2}=V(\varphi)^{3}\|\psi\|_{\infty}$, up to a numerical factor, we have, for $q_{n} \leq q_{m} \leq q_{r} \leq q_{\ell}$ :

$$
\begin{gather*}
\left|\int \psi\left[\varphi_{q_{n}} \varphi_{q_{m}}-S_{q_{\ell}} \varphi_{q_{n}} S_{q_{\ell}} \varphi_{q_{m}}\right] \mathrm{d} \mu\right| \leq C_{1}\left(\frac{q_{m}}{q_{\ell}}\right)^{\frac{1}{2}},  \tag{5.7}\\
\left|\int \psi\left[\varphi_{q_{n}} \varphi_{q_{m}} \varphi_{q_{r}}-S_{q_{\ell}} \varphi_{q_{n}} S_{q_{\ell}} \varphi_{q_{m}} S_{q_{\ell}} \varphi_{q_{r}}\right] \mathrm{d} \mu\right| \leq C_{2}\left(\frac{q_{r}}{q_{\ell}}\right)^{\frac{1}{2}} \ln ^{2}\left(q_{\ell}\right) . \tag{5.8}
\end{gather*}
$$

Proof. We use the bound (5.3) which gives, for $q_{n} \leq q_{\ell}$,

$$
\left\|R_{L} \varphi_{q_{n}}\right\|_{2}^{2}=\sum_{|j| \geq q_{\ell}}\left|\widehat{\varphi_{q_{n}}}(j)\right|^{2}=\sum_{|j| \geq q_{\ell}} \frac{\left|\gamma_{j}(\varphi)\right|^{2}}{j^{2}} \frac{\left\|q_{n} j \alpha\right\|^{2}}{\|j \alpha\|^{2}} \leq C^{2} K(\varphi)^{2} \frac{q_{n}}{q_{\ell}} .
$$

For $\psi$ bounded, as $\left\|\varphi_{q_{n}}\right\|_{2} \leq C K(\varphi)$, this implies that $\left|\int \psi\left[\varphi_{q_{n}} \varphi_{q_{m}}-S_{q_{\ell}} \varphi_{q_{n}} S_{q_{\ell}} \varphi_{q_{m}}\right] \mathrm{d} \mu\right|$ is smaller than

$$
\|\psi\|_{\infty}\left[\left\|\varphi_{q_{n}}\right\|_{2}\left\|R_{q_{\ell}} \varphi_{q_{m}}\right\|_{2}+\left\|R_{q_{\ell}} \varphi_{q_{n}}\right\|_{2}\left\|\varphi_{q_{m}}\right\|_{2}\right] \leq C V(\varphi)^{2}\|\psi\|_{\infty}\left[\left(\frac{q_{n}}{q_{\ell}}\right)^{\frac{1}{2}}+\left(\frac{q_{m}}{q_{\ell}}\right)^{\frac{1}{2}}\right]
$$

This proves (5.7). For (5.8), in each term of the expansion of $\left(S_{q_{\ell}} \varphi_{q_{n}}+R_{q_{\ell}} \varphi_{q_{n}}\right)\left(S_{q_{\ell}} \varphi_{q_{m}}+\right.$ $\left.R_{q_{\ell}} \varphi_{q_{m}}\right)\left(S_{q_{\ell}}+R_{q_{\ell}}\right)-S_{q_{\ell}} \varphi_{q_{n}} S_{q_{\ell}} \varphi_{q_{m}} S_{q_{\ell}} \varphi_{q_{r}}$, we bound one factor in $L^{2}$-norm and the others in uniform norm using (5.6).

## Inequalities under Hypothesis 3.2

Recall that the decorrelation inequalities of Lemma 5.5 are based on Hypothesis 3.2 on $\alpha$.
From (3.8) in Hypothesis 3.2, one deduces: for constants $B, C$, the coefficients in Ostrowski's expansion satisfy $b_{n} \leq B n^{p}$ and, since $q_{n} \leq B^{n}(n!)^{p}$,

$$
\begin{equation*}
\ln q_{n} \leq C n \ln n, c_{n} \leq C n^{p+1} \ln n \tag{5.9}
\end{equation*}
$$

The case when $\alpha$ has bounded partial quotients corresponds to $p=0$ and we have then $\ln q_{n} \leq C n$.

Let us mention that Hardy and Littlewood in [14] considered quantities similar to that in the lemma below. One of their motivations was to study asymptotically the number of integral points contained in homothetic triangles.

Lemma 5.5. If $a_{k+1} \leq A k^{p}, \forall k \geq 1$ and $n \leq m \leq \ell$, we have for every $\Lambda \geq 1$ :

$$
\begin{gather*}
\sum_{j=1}^{\infty} \frac{\left\|q_{n} j \alpha\right\|}{j^{2}\|j \alpha\|} \leq C \frac{n^{p+2} \ln n}{q_{n+1}},  \tag{5.10}\\
\sum_{1 \leq j, k<q_{\Lambda}, j \neq k} \frac{\left\|q_{n} j \alpha\right\|\left\|q_{m} k \alpha\right\|}{|k-j| k j\|j \alpha\|\|k \alpha\|} \leq \frac{C}{q_{n+1}} \Lambda^{2 p+4}(\ln \Lambda)^{2},  \tag{5.11}\\
\sum_{-q_{\Lambda}<i, j, k<q_{\Lambda}, i+j+k \neq 0}  \tag{5.12}\\
\frac{\left\|q_{n} i \alpha\right\|\left\|q_{m} j \alpha\right\|\left\|q_{\ell} k \alpha\right\|}{|i+j+k| i j k\|i \alpha\|\|j \alpha\|\|\|k \alpha\|} \leq \frac{C}{q_{n+1}} \Lambda^{3 p+8} .
\end{gather*}
$$

Proof of (5.10). We use the inequalities: $\frac{\left\|j q_{k} \alpha\right\|}{j} \leq\left\|q_{k} \alpha\right\| \leq \frac{1}{q_{k+1}}$ for $j<q_{k+1}$, $\left\|j q_{k} \alpha\right\| \leq 1$ for $j \geq q_{k+1}$. For $\ell>n$, we write $\sum_{j=1}^{q_{\ell}-1} \frac{\left\|q_{n} j \alpha\right\|}{j^{2}\|j \alpha\|}=(A)+(B)$, with

$$
\begin{aligned}
&(A):=\sum_{j=1}^{q_{n+1}-1} \frac{1}{j} \frac{\left\|q_{n} j \alpha\right\|}{j\|j \alpha\|} \leq \frac{1}{q_{n+1}} \sum_{j=1}^{q_{n+1}-1} \frac{1}{j} \frac{1}{\|j \alpha\|} \\
& \leq \frac{1}{q_{n+1}} \sum_{k=0}^{n} \frac{1}{q_{k}} \sum_{j=q_{k}}^{q_{k+1}-1} \frac{1}{\|j \alpha\|} \leq C \frac{1}{q_{n+1}} \sum_{k=0}^{n} \frac{q_{k+1}}{q_{k}} \ln q_{k+1}, \quad \text { by (5.5); }
\end{aligned}
$$

$$
\begin{aligned}
(B):=\sum_{j=q_{n+1}}^{q_{\ell}-1} \frac{\left\|q_{n} j \alpha\right\|}{j^{2}\|j \alpha\|} & \leq \sum_{k=n+1}^{\ell-1} \sum_{j=q_{k}}^{q_{k+1}-1} \frac{1}{j^{2}\|j \alpha\|} \\
& \leq \sum_{k=n+1}^{\ell-1} \frac{1}{q_{k}^{2}} \sum_{j=q_{k}}^{q_{k+1}-1} \frac{1}{\|j \alpha\|} \leq C \sum_{k=n+1}^{\ell-1} \frac{1}{q_{k}} \frac{q_{k+1}}{q_{k}} \ln q_{k+1}, \quad \text { by (5.4). }
\end{aligned}
$$

By (2.5), we know that $\frac{q_{n+1}}{q_{k}} \leq C \rho^{k-n}$, with $\rho<1$, for $k \geq n+1$. By hypothesis, $a_{k+1} \leq A k^{p}$. It follows with the notation (5.2): $(A) \leq \frac{C}{q_{n+1}} \sum_{k=0}^{n} c_{k} \leq \frac{C n^{p+2} \ln n}{q_{n+1}}$ and for ( $B$ ), with a bound which doesn't depend on $\ell \geq n$ :

$$
\begin{aligned}
\frac{1}{q_{n+1}} \sum_{k=n+1}^{\ell-1} \frac{q_{n+1}}{q_{k}} \frac{q_{k+1}}{q_{k}} \ln q_{k+1} & \leq C \frac{1}{q_{n+1}} \sum_{j=0}^{\infty} \rho^{j}(j+n+1)^{p+1} \ln (j+n+1) \\
& \leq \frac{C n^{p+1} \ln n}{q_{n+1}}
\end{aligned}
$$

Proof of (5.11). To bound the sum in (5.11), we cover the square [ $1, q_{\Lambda}\left[\times\left[1, q_{\Lambda}[\right.\right.$ in $\mathbb{N} \times \mathbb{N}$ by rectangles $R_{r, s}=\left[q_{r}, q_{r+1}\left[\times\left[q_{s}, q_{s+1}[\right.\right.\right.$ for $r$ and $s$ varying between 0 and $\Lambda-1$ and then we bound the sum on each of these rectangles (minus the diagonal if $r=s$ ).

Distinguishing different cases according to the positions of $r$ and $s$ with respect to $n+1$ and $m+1$, we have, for $j \in\left[q_{r}, q_{r+1}\left[, k \in\left[q_{s}, q_{s+1}[, j \neq k\right.\right.\right.$.

$$
\frac{\left\|q_{n} j \alpha\right\|\left\|\left\|q_{m} k \alpha\right\|\right.}{|k-j| j k\|j \alpha\|\|k \alpha\|} \leq \frac{1}{q_{\max (r, n+1)} q_{\max (s, m+1)}} \frac{1}{|k-j|\|j \alpha\|\|k \alpha\|}
$$

By (5.4) and (5.5), using $\|(k-j) \alpha\| \leq\|j \alpha\|+\|k \alpha\|)$, we have

$$
\begin{aligned}
& \sum_{(j, k) \in R_{r, s}} \frac{1}{|k-j|\|j \alpha\|\|k \alpha\|} \\
& \leq \sum_{(j, k) \in R_{r, s}}\left(\frac{1}{|k-j|\|(k-j) \alpha\|\|j \alpha\|}+\frac{1}{|k-j|\|(k-j) \alpha\|\|k \alpha\|}\right) \\
& \leq q_{\max (r, s)+1} \ln \left(q_{\max (r, s)+1}\right) \sum_{t=0}^{\max (r, s)} c_{t} .
\end{aligned}
$$

It follows

$$
\begin{aligned}
\sum_{(j, k) \in R_{r, s}, j \neq k} \frac{\left\|q_{n} j \alpha\right\|\left\|q_{m} k \alpha\right\|}{|k-j| k j\|j \alpha\|\|k \alpha\|} & \leq \frac{q_{\max (r, s)+1}}{q_{\max (r, n+1)} q_{\max (s, m+1)}} \ln \left(q_{\max (r, s)+1}\right) \sum_{t=0}^{\max (r, s)} c_{t} \\
& \leq \frac{1}{q_{n+1}} \ln \left(q_{\Lambda+1}\right) \sum_{t=0}^{\Lambda} c_{t} \max _{k=1, \ldots, \Lambda} a_{k}^{\prime}
\end{aligned}
$$

The square [ $1, q_{\Lambda}\left[\times\left[1, q_{\Lambda}\left[\right.\right.\right.$ is covered by $\Lambda^{2}$ rectangles $R_{r, s}$ and the sums on these rectangles are bounded by the same quantity. It follows, with Hypothesis 3.2,

$$
\begin{aligned}
\sum_{1 \leq j, k<q_{\Lambda}, j \neq k} \frac{\left\|q_{n} j \alpha\right\|\left\|q_{m} k \alpha\right\|}{|k-j| k j\|j \alpha\|\|k \alpha\|} & \leq \Lambda^{2} \frac{C}{q_{n+1}} \ln \left(q_{\Lambda+1}\right) \sum_{t=0}^{\Lambda} c_{t} \max _{k=1, \ldots, \Lambda} a_{k}^{\prime} \\
& \leq \frac{C}{q_{n+1}} \Lambda^{2 p+5} \ln (\Lambda)^{2} .
\end{aligned}
$$

Proof of (5.12). Here we consider sums with three indices $i, j, k$. Though we do not write it explicitly, these sums are to be understood to be taken on non zero indices $i, j, k$ such that $i+j+k \neq 0$. We cover the set of indices by sets of the form

$$
R_{ \pm r, \pm s, \pm t}=\left\{(i, j, k): \pm i \in\left[q_{r}, q_{r+1}\left[, \pm j \in\left[q_{s}, q_{s+1}\left[, \pm k \in\left[q_{t}, q_{t+1}[ \}\right.\right.\right.\right.\right.\right.
$$

Distinguishing different cases according to the positions of $r, s$ and $t$ with respect to $n+1$, we get: if $(i, j, k) \in R_{ \pm r, \pm s, \pm t}$ and $n \leq m \leq \ell$,

$$
\begin{equation*}
\frac{\left\|q_{n} i \alpha\right\|\left\|\left\|q_{m} j \alpha\right\| \mid q_{\ell} k \alpha\right\|}{|i\|j\| k|} \leq \frac{1}{q_{\max (r, n+1)} q_{\max (s, n+1)} q_{\max (t, n+1)}} \tag{5.13}
\end{equation*}
$$

We have $\frac{1}{\|i \alpha\|\|\|j \alpha\|\| k \alpha \|} \leq \frac{1}{\|(i+j+k) \alpha\|}\left[\frac{1}{\|j \alpha\|\|k \alpha\|}+\frac{1}{\|i \alpha\|\| \| \alpha \|}+\frac{1}{\|i \alpha\|\|j \alpha\|}\right]$.
We then use (5.4) and (5.5) three times, sum over $R_{ \pm r, \pm s, \pm t}$ and get:

$$
\begin{aligned}
\sum_{(i, j, k) \in R_{ \pm r, \pm s, \pm t}} & \frac{1}{|i+j+k|\|i \alpha\|\|\|\alpha\|\| k \alpha \|} \\
& \leq\left(\sum_{v=0}^{3 \max (r, s, t)} c_{v}\right) \ln ^{2}\left(q_{\max (r, s, t)+1}\right)\left(q_{s+1} q_{t+1}+q_{r+1} q_{t+1}+q_{r+1} q_{s+1}\right)
\end{aligned}
$$

By (5.13) we then have:

$$
\begin{aligned}
& \sum_{R_{ \pm r, \pm s, \pm t}} \frac{\left\|q_{n} i \alpha\right\|\left\|q_{m} j \alpha\right\| \mid q_{q} k \alpha \|}{|i+j+k\|i\| j\|k \mid\| i \alpha\| \| j \alpha\| \| k \alpha \|} \\
& \quad \leq C \frac{\left(\sum_{v=0}^{3 \max (r, s, t)} c_{v}\right) \ln ^{2}\left(q_{\max (r, s, t)}\right)}{q_{\max (r, n+1)} q_{\max (s, n+1)} q_{\max (t, n+1)}}\left(q_{s+1} q_{t+1}+q_{r+1} q_{t+1}+q_{r+1} q_{s+1}\right) \\
& \quad \leq \frac{C}{q_{n+1}}\left(\sum_{v=0}^{3 \Lambda} c_{v}\right) \ln ^{2}\left(q_{\Lambda+1}\right)\left(\max _{k=1, \ldots, \Lambda} a_{k}^{\prime}\right)^{2} .
\end{aligned}
$$

One needs $8 \Lambda^{3}$ boxes $R_{ \pm r, \pm s, \pm t}$ to cover the set $\left\{-q_{\Lambda}<i, j, k<q_{\Lambda}, i+j+k \neq 0\right\}$. This implies for a constant $C$ :

$$
\sum_{-q_{\Lambda}<i, j, k<q_{\Lambda}, i+j+k \neq 0} \frac{\left\|q_{n} i \alpha\right\|\left\|q_{m} j \alpha\right\|| | q_{\ell} k \alpha \|}{|i+j+k\|i\| j||k|\|i \alpha\|\|j \alpha\|\|k \alpha\|} \leq \frac{C}{q_{n+1}} \Lambda^{3 p+8} .
$$

Proof of Proposition 3.5. By (5.1) we have $\left|\int \psi \varphi_{q_{n}} \mathrm{~d} \mu\right| \leq \sum_{j \neq 0}\left|\widehat{\varphi_{q_{n}}}(j) \| \widehat{\psi}(-j)\right| \leq$ $K \sum_{j \geq 1} \frac{\left\|q_{n} j \alpha\right\|}{j^{2}\|j \alpha\|}$ and (3.11) follows from (5.10): $\sum_{j \geq 1} \frac{\left\|q_{n} j \alpha\right\|}{j^{2}\|j \alpha\|} \leq C \frac{n^{p+2} \ln n}{q_{n+1}}$.

We prove now (3.12). With $L=q_{\Lambda}$, we have:

$$
\int \psi S_{L} \varphi_{q_{n}} S_{L} \varphi_{q_{m}} \mathrm{~d} \mu=\sum_{|j|,|k| \leq L, j \neq k} \widehat{\varphi_{q_{n}}}(-j) \widehat{\varphi_{q_{m}}}(k) \widehat{\psi}(j-k) .
$$

In what follows, the constant $C$ is equal to $V(\psi) V(\varphi)^{2}$ (up to a factor not depending on $\psi$ and $\varphi$ which may change).

Recall that, by (2.5), there is a constant $B$ such that $m \leq B \ln q_{m}, \forall m \geq 1$.
The functions $\psi, \varphi$ are real valued. By (5.11), it holds $\left|\int \psi S_{L} \varphi_{q_{n}} S_{L} \varphi_{q_{m}} \mathrm{~d} \mu\right| \leq$

$$
\begin{aligned}
\sum_{|j|,|k| \leq L, j \neq k}\left|\widehat{\varphi_{n}}(j)\left\|\widehat{\varphi_{q_{m}}}(k)\right\| \widehat{\psi}(j-k)\right| & \leq C \sum_{1 \leq j, k \leq L} \frac{\left\|q_{n} j \alpha\right\|\| \| q_{m} k \alpha \|}{|k-j| j k\|j \alpha\|\|k \alpha\|} \\
& \leq \frac{C}{q_{n+1}} \Lambda^{2 p+4}(\ln \Lambda)^{2} .
\end{aligned}
$$

Putting it together with the truncation error term (5.7) and replacing $q_{n+1}$ by $q_{n}$, we get

$$
\begin{equation*}
\left|\int_{X} \psi \varphi_{q_{n}} \varphi_{q_{m}} \mathrm{~d} \mu\right| \leq C\left[\frac{\Lambda^{2 p+4}(\ln \Lambda)^{2}}{q_{n}}+\left(\frac{q_{m}}{q_{\Lambda}}\right)^{\frac{1}{2}}\right], \quad \text { for } n \leq m \leq \Lambda . \tag{5.14}
\end{equation*}
$$

Recall that $\left(\frac{q_{m}}{q_{\Lambda}}\right)^{\frac{1}{2}} \leq \rho^{\frac{\Lambda-m}{2}}$. Let us take $\Lambda-m$ of order $2\left(\ln \frac{1}{\rho}\right)^{-1} \ln q_{n}$, i.e., such that the second term in the bracket of the RHS of (5.14) is of order $1 / q_{n}$. We have then
$\Lambda \leq \max \left(m, C_{1} \log q_{n}\right)$ and with Hypothesis 3.2 the first term in the bracket is less than

$$
\begin{aligned}
\frac{C_{1}}{q_{n}} \max \left(\left(\ln q_{n}\right)^{2 p+5}, m^{2 p+5}\right) & \leq \frac{C_{2}}{q_{n}} \max \left(\left(\ln q_{n}\right)^{2 p+5},\left(\ln q_{m}\right)^{2 p+5}\right) \\
& \leq \frac{C_{2}}{q_{n}}\left(\ln q_{m}\right)^{2 p+5} \leq C_{3} \frac{m^{2 p+5}}{q_{n}} .
\end{aligned}
$$

This shows (3.12) with $\theta_{2}=2 p+5$.
In the same way, (3.13) follows from (5.8) and (5.12).

## Appendix A. Proof of Proposition 2.5

The proof consists in several steps. To bound from below $d\left(n q_{j} \alpha, \mathbb{Z}\right)$, successively we code $n$ as an admissible word (Ostrowski's coding), reduce long words to short words, then interpret cardinals in terms of cylinders and invariant measure for a subshift. Finally we use a result of large deviations recalled in Lemma A.1.

For the reader's convenience, at each step we will consider first the simpler special case of the golden mean $\alpha=\frac{\sqrt{5}+1}{2}$ (the corresponding rotation number is $\left.\frac{\sqrt{5}-1}{2} \in\right] 0,1[$ ). Then the general case is treated between the signs " $\checkmark$ " and " $\Delta$ " and may be skipped if $\alpha$ is the golden mean.

When $\alpha$ is the golden mean, its partial quotients are equal to 1 and $\left(q_{n}\right)$ (the Fibonacci sequence with $\left.q_{-1}=0, q_{0}=1, q_{1}=1, \ldots\right)$ is almost a geometric sequence of ratio $\alpha$. We have

$$
\begin{gather*}
q_{n}=\frac{1}{5}\left[(2+\alpha) \alpha^{n}+(-1)^{n}(3-\alpha) \alpha^{-n}\right], n \geq 0  \tag{A.1}\\
\alpha^{n}+(-\alpha)^{-n} \in \mathbb{Z}, d\left(\alpha^{n}, \mathbb{Z}\right)=\alpha^{-n}, n \geq 1 \tag{A.2}
\end{gather*}
$$

$\diamond$ For a general quadratic number $\alpha$, the sequence $\left(a_{n}\right)$ is ultimately periodic: there are integers $n_{0}, p$ such that $a_{n+p}=a_{n}, \forall n \geq n_{0}$.

Let $A_{1}:=\left(\begin{array}{cc}0 & 1 \\ 1 & a_{n_{0}+1}\end{array}\right), A_{i}:=\left(\begin{array}{cc}0 & 1 \\ 1 & a_{n_{0}}+i\end{array}\right)\left(\begin{array}{cc}0 & 1 \\ 1 & a_{n_{0}}+i-1\end{array}\right) \ldots\left(\begin{array}{cc}0 & 1 \\ 1 & a_{n_{0}+1}\end{array}\right)$, for $i>1$.
From the recursive relation $\binom{q_{n}}{q_{n+1}}=\left(\begin{array}{cc}0 & 1 \\ 1 & a_{n+1}\end{array}\right)\binom{q_{n-1}}{q_{n}}$, between the denominators $\left(q_{n}\right)$ of $\alpha$, it follows, $\forall k \geq 1,\binom{q_{n_{0}+k p+m-1}}{q_{n_{0}}+k p+m}=A_{m} A_{p}^{k}\binom{q_{n_{0}-1}}{q_{n_{0}}}, m=1, \ldots, p$.

The matrix $A_{p}$ is a $2 \times 2$ matrix with determinant $(-1)^{p}$ and non negative integer coefficients (positive if $p>1$ ). It has two distinct eigenvalues $\lambda>1$ and $(-1)^{p} \lambda^{-1}$ (where $\lambda$ is a quadratic number) and it is diagonal in a basis of $\mathbb{R}^{2}$ with coordinates in $\mathbb{Q}[\lambda]$. We have $\lambda+(-1)^{p} \lambda^{-1} \in \mathbb{Z}$.

Without loss of generality we may suppose that $p$ is even (otherwise, we replace it by $2 p$ ). Therefore there are integers $r, s_{\ell}, t_{\ell}, u_{\ell}, v_{\ell}$ for $\ell \in\{0, \ldots, p-1\}$ such that

$$
\begin{equation*}
q_{n_{0}+k p+\ell}=\frac{1}{r}\left[\left(s_{\ell}+t_{\ell} \lambda\right) \lambda^{k}+\left(u_{\ell}+v_{\ell} \lambda\right) \lambda^{-k}\right], \forall k \geq 0 \tag{A.3}
\end{equation*}
$$

For every $\ell,\left(q_{n_{0}+k p+\ell}\right)_{k \geq 1}$ behaves like a geometric progression with ratio $\lambda$.
For the golden mean, (A.3) corresponds to (A.1) for $n$ even and $r=5$.

## Step 1. Ostrowski's coding, invariant measure for a subshift of finite type and counting

As recalled in Subsection 2.1, every $n<q_{m+1}$ is coded by an "admissible" word $b_{0} \ldots b_{m}$, with $b_{0} \in\left\{0,1, \ldots, a_{1}-1\right\}, b_{j} \in\left\{0,1, \ldots, a_{j+1}\right\}, j=1, \ldots, m$, where for two consecutive letters $b_{j-1}, b_{j}$, if $b_{j}=a_{j+1}$, then $b_{j-1}=0$.

For $\alpha$ the golden mean, a finite word $b_{0} \ldots b_{m}$ is admissible if it is composed of 0 's and 1's and two consecutive letters $b_{i}, b_{i+1}$ cannot be both 1 . We denote by $X$ the space of one-sided infinite admissible sequences, that is sequences of 0,1 without two consecutive 1's. For simplicity the letter $b$ will denote an admissible word, either finite or infinite. The context will make clear if $b$ is finite or not.

If $b=b_{0} \ldots b_{\ell}$ is an admissible word, we put $n_{b, \ell}:=\sum_{i=0}^{\ell} b_{i} q_{i}$.
When $\alpha$ is the golden mean, we use the sub-shift $(X, \sigma)$, where $\sigma=\sigma_{X}$ is the shift on $X$. Let $\mu$ be the $\sigma$-invariant probability measure on $X$ of maximal entropy. Let $C_{x_{0} \ldots x_{n}}$ denote the cylinder composed of sequences starting with $x_{0} \ldots x_{n}$. For $n \geq 1$, depending whether $x_{0}$ and $x_{n}$ are both equal to 1 , or only one of them, or none, we have

$$
\mu\left(C_{x_{0} \ldots x_{n}}\right)=\frac{1}{\alpha+2} \alpha^{-n}, \quad \frac{\alpha}{\alpha+2} \alpha^{-n} \quad \text { or } \quad \frac{\alpha^{2}}{\alpha+2} \alpha^{-n} .
$$

If $E \subset X$ is a union of cylinders of length $n$, its measure can be compared to the number of cylinders which compose it:

$$
\begin{equation*}
\frac{\alpha+2}{\alpha+1} \mu(E) \leq \alpha^{-n} \operatorname{Card}\{\text { cylinder } \mathcal{W} \text { of length } n: \mathcal{W} \subset E\} \leq(\alpha+2) \mu(E) \tag{A.4}
\end{equation*}
$$

$\diamond$ In the general case, let us consider the set of infinite admissible sequences corresponding to the Ostrowski expansions for the periodic part of the sequence $\left(a_{n}\right)$ :

$$
X:=\left\{x=\left(x_{i}\right)_{i \in \mathbb{N}} \text { such that } \forall i x_{i-1} x_{i} \neq u a_{n_{0}+i+1} \text { with } u \neq 0\right\} .
$$

The space $X$ is invariant under the action of $\sigma_{X}^{p}$ (because $\left(a_{n}\right)$ is $p$-periodic for $\left.n \geq n_{0}\right)$. We define an irreducible aperiodic sub-shift of finite type as follows: the state space of $Y$ is the set of words $x_{0} \ldots x_{p-1}$ of $X$, a transition between two such words $w_{1}$ and $w_{2}$ is allowed if the concatenation $w_{1} w_{2}$ is the beginning of length $2 p$ of a sequence in $X$.

From (A.3) we see that the exponential growth rate of the number of Ostrowski expansions of length at most $n_{0}+p k$ is $\ln \lambda$ (with respect to $k$ ). It is also the growth rate of the number of words of length $p k$ of $X$. As these words correspond to the words of length $k$ in $Y$, the topological entropy of $(Y, \sigma)$ is $\ln \lambda$ (where $\sigma=\sigma_{Y}$ is the shift to the left on $Y$ ). There is a unique invariant probability measure $\mu$ on $(Y, \sigma)$ with entropy $\ln \lambda$. This measure can be constructed as follows. Let $B$ be the matrix with entries 0 and 1 that gives the allowed transitions between elements of the alphabet of $Y$. As the topological entropy of $Y$ is the logarithm of the spectral radius of B, this spectral radius is $\lambda$. Let $U$ and $V$ be two positive vectors such that $B U=\lambda U,{ }^{t} B V=\lambda V,{ }^{t} U V=1$. The measure $\mu$ is the Markovian measure determined by its values on cylinders given by

$$
\mu\left(C_{y_{0} y_{1} \ldots y_{n}}\right)=V_{y_{0}} U_{y_{n}} \lambda^{-n},
$$

when $y_{0} y_{1} \ldots y_{n}$ is an admissible word (see [17, p. 21-23 and p. 166] for more details on this classical construction). As there are only finitely many products $V_{y_{0}} U_{y_{n}}$, there exists a constant $c^{\prime}>0$ such that, if a subset $E$ of $Y$ is a union of cylinders of length $n$, then

$$
\begin{equation*}
\frac{1}{c^{\prime}} \mu(E) \leq \operatorname{Card}\{\mathcal{W} \text { cylinders of length } n: \mathcal{W} \subset E\} \lambda^{-n} \leq c^{\prime} \mu(E) \tag{A.5}
\end{equation*}
$$

## Lemma of large deviations

We will use the following inequality of large deviations for irreducible Markov chains with finite state space (see [20, Theorem 3.3]):
Lemma A.1. Let A be finite union of cylinders. For every $\varepsilon \in] 0,1[$, there are two positive constants $R(\varepsilon), \xi(\varepsilon)$ depending on $A$ such that

$$
\begin{equation*}
\mu\left\{x \in X: \frac{1}{L} \sum_{k=0}^{L-1} \mathbf{1}_{A}\left(\sigma^{k} x\right) \leq \mu(A)(1-\varepsilon)\right\} \leq R(\varepsilon) e^{-\xi(\varepsilon) L}, \quad \forall L \geq 1 \tag{A.6}
\end{equation*}
$$

## Step 2. Reduction of the Ostrowski expansion to a "window"

By (2.3) and (2.5) we have, for a constant $\rho<1,\left\|q_{i} q_{j} \alpha\right\| \leq C \rho^{|j-i|}$. Hence, for $0 \leq j \leq \ell$, if $\kappa$ is such that $0 \leq j-\kappa \leq j+\kappa \leq \ell$ :

$$
\begin{equation*}
\left\|\sum_{i=0}^{\ell} b_{i} q_{i} q_{j} \alpha\right\|-\left\|\sum_{i=j-\kappa}^{j+\kappa-1} b_{i} q_{i} q_{j} \alpha\right\|\|\leq\| \sum_{i=j+\kappa}^{\ell} b_{i} q_{i} q_{j} \alpha\|+\| \sum_{i=0}^{j-\kappa-1} b_{i} q_{i} q_{j} \alpha \| \leq C \rho^{-\kappa} \tag{A.7}
\end{equation*}
$$

It means that $\left\|\sum_{i=0}^{\ell} b_{i} q_{i} q_{j} \alpha\right\|=\left\|n_{b, \ell} q_{j} \alpha\right\|$ is well approximated by $\left\|\sum_{i=j-\kappa}^{j+\kappa-1} b_{i} q_{i} q_{j} \alpha\right\|$ which depends on a word with indices belonging to a window around $j$, with a precision depending on the size of the window. This is valid for any irrational $\alpha$.

The quantity introduced in the next definition can be viewed as a function of an infinite word $b$ or of a finite word $b_{j-\kappa_{0}}, \ldots, b_{j+\kappa_{0}}$. We put

$$
\begin{equation*}
\Gamma(b, j):=\frac{1}{5} \sum_{i=j-\kappa_{0}}^{j+\kappa_{0}}(-1)^{i} b_{i}\left(\alpha^{j-i}+(-\alpha)^{i-j}\right) \alpha \tag{A.8}
\end{equation*}
$$

A simple computation shows that $\Gamma(b, j+1)=-\Gamma(\sigma b, j)$. Therefore we have:

$$
\begin{equation*}
\Gamma\left(\sigma^{k} b, \kappa_{0}\right)=(-1)^{k} \Gamma\left(b, k+\kappa_{0}\right) . \tag{A.9}
\end{equation*}
$$

Lemma A.2. Let $\alpha$ be the golden mean. For every $\delta>0$, there is $\kappa_{0}=\kappa_{0}(\delta)$ such that

$$
\begin{equation*}
d\left(n_{b, \ell} q_{j} \alpha-\Gamma(b, j), \mathbb{Z} / 5\right) \leq \delta, \text { if } j \geq \kappa_{0} . \tag{A.10}
\end{equation*}
$$

Proof. We can restrict the sum $n_{b, \ell} q_{j} \alpha=\sum_{i=0}^{\ell} b_{i} q_{i} q_{j} \alpha$ to the sum $\sum_{i=j-\kappa_{0}}^{j+\kappa_{0}} b_{i} q_{i} q_{j} \alpha$, since their distance modulo 1 is small for $\kappa_{0}$ big enough by (A.7).

By (A.1), we have $q_{i} q_{j}=\frac{1+\alpha}{5} \alpha^{i+j}+\frac{2-\alpha}{5}(-\alpha)^{-(i+j)}+\frac{(-1)^{i}}{5}\left(\alpha^{j-i}+(-\alpha)^{i-j}\right)$; hence:

$$
\sum_{i=j-\kappa_{0}}^{j+\kappa_{0}} b_{i} q_{i} q_{j} \alpha=\frac{1}{5} \sum_{i=j-\kappa_{0}}^{j+\kappa_{0}}\left[b_{i}(1+\alpha) \alpha^{i+j+1}+b_{i}(-1)^{i+j}(2-\alpha) \alpha^{1-(i+j)}\right]+\Gamma(b, j) .
$$

The distance to $\mathbb{Z}$ of the first sum above at right is small by (A.2).
The lemma shows that for the golden mean the distance to $\mathbb{Z} / 5$ of $\sum_{j=0}^{\ell} b_{i} q_{i} q_{j} \alpha$ is almost the distance to $\mathbb{Z} / 5$ of $\Gamma(b, j)$, which depends on the "short" word $b_{j-\kappa_{0}} \ldots b_{j+\kappa_{0}}$ (reduction to a window of width $2 \kappa_{0}$ of the "long" word $b_{0} \ldots b_{\ell}$ ) in such a way that its values, when $j$ varies, are the values of a fixed function computed for shifted words.
$\diamond$ The lemma extends to a general quadratic number. We need some notation.
For an integer $i$, we write $i=\underline{i}+p \eta_{i}+n_{0}$, where $\underline{i}$ is the class of $i-n_{0}$ modulo $p$ and $\eta_{i}$ the integer part of $\left(i-n_{0}\right) / p$. The classes $\bmod p$ are identified with the integers $0, \ldots, p-1$. With the notation introduced in (A.3), we put

$$
T(\underline{i}, \underline{j}):=\frac{\alpha}{r^{2}}\left(s_{\underline{i}}+t_{\underline{i}} \lambda\right)\left(u_{\underline{j}}+v_{\underline{j}} \lambda\right), U(\underline{i}, \underline{j}):=\frac{\alpha}{r^{2}}\left(u_{\underline{i}}+v_{\underline{i}} \lambda\right)\left(s_{\underline{j}}+t_{\underline{j}} \lambda\right) .
$$

Lemma A.3. Let $\delta \in] 0, \frac{1}{2 r}$. There is $\kappa_{0}=\kappa_{0}(\delta)$ such that, if $j \geq n_{0}+\kappa_{0} p$,

$$
\begin{equation*}
d\left(n_{b, \ell} q_{j} \alpha-\sum_{i=n_{0}+\left(\eta_{j}-\kappa_{0}\right) p}^{n_{0}+\left(\eta_{j}+\kappa_{0}\right) p-1} b_{i}\left[T(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\eta_{j}}+U(\underline{i}, \underline{j}) \lambda^{\eta_{j}-\eta_{i}}\right], \mathbb{Z} / r\right) \leq \delta \tag{A.11}
\end{equation*}
$$

Proof. Recall that ( $a_{n}$ ) is p-periodic for $n \geq n_{0}$. We consider indices $j \geq n_{0}$ and take sums on windows union of blocks of length $p$, hence of the form $n_{0}+m p, \ldots, n_{0}+q p-1$.

Using (A.3), the product $q_{n_{0}+k p+m} q_{n_{0}+k^{\prime} p+m^{\prime}}$ is equal to

$$
\begin{aligned}
& \frac{1}{r^{2}}\left[\left(s_{m}+t_{m} \lambda\right)\left(s_{m^{\prime}}+t_{m^{\prime}} \lambda\right) \lambda^{k+k^{\prime}}+\left(u_{m}+v_{m} \lambda\right)\left(u_{m^{\prime}}+v_{m^{\prime}} \lambda\right) \lambda^{-\left(k+k^{\prime}\right)}\right] \\
& \quad+\frac{1}{r^{2}}\left[\left(s_{m}+t_{m} \lambda\right)\left(u_{m^{\prime}}+v_{m^{\prime}} \lambda\right) \lambda^{k-k^{\prime}}+\left(u_{m}+v_{m} \lambda\right)\left(s_{m^{\prime}}+t_{m^{\prime}} \lambda\right) \lambda^{k^{\prime}-k}\right]
\end{aligned}
$$

Still using (A.3), we have

$$
\frac{s_{m}}{r} \frac{\left(s_{m^{\prime}}+t_{m^{\prime}} \lambda\right)}{r} \lambda^{k+k^{\prime}}=\frac{s_{m}}{r}\left(q_{n_{0}+\left(k^{\prime}+k\right) p+m^{\prime}}-\frac{1}{r}\left(u_{m^{\prime}}+v_{m^{\prime}} \lambda\right) \lambda^{-\left(k^{\prime}+k\right)}\right) .
$$

From this (and a similar equality) we obtain

$$
\begin{aligned}
& q_{n_{0}+k p+m} q_{n_{0}+k^{\prime} p+m^{\prime}} \alpha \\
& \quad-\frac{1}{r^{2}}\left[\left(s_{m}+t_{m} \lambda\right)\left(u_{m^{\prime}}+v_{m^{\prime}} \lambda\right) \lambda^{k-k^{\prime}}+\left(u_{m}+v_{m} \lambda\right)\left(s_{m^{\prime}}+t_{m^{\prime}} \lambda\right) \lambda^{k^{\prime}-k}\right] \alpha \\
& =\frac{s_{m}}{r} q_{n_{0}+\left(k^{\prime}+k\right) p+m^{\prime}} \alpha+\frac{t_{m}}{r} q_{n_{0}+\left(k^{\prime}+k+1\right) p+m^{\prime} \alpha} \\
& \quad-\left[\frac{s_{m}}{r^{2}}\left(u_{m^{\prime}}+\frac{v_{m^{\prime}}}{r^{2}} \lambda\right) \lambda^{-\left(k^{\prime}+k\right)} \alpha+\frac{t_{m}}{r^{2}}\left(u_{m^{\prime}}+v_{m^{\prime}} \lambda\right) \lambda^{-\left(k^{\prime}+k+1\right)} \alpha\right] .
\end{aligned}
$$

Since $d\left(q_{n_{0}+\left(k^{\prime}+k\right) p+m^{\prime}} \alpha, \mathbb{Z} / r\right) \leq d\left(q_{n_{0}+\left(k^{\prime}+k\right) p+m^{\prime}} \alpha, \mathbb{Z}\right) \leq C \lambda^{-\left(k^{\prime}+k\right)}$ by (2.3), the distance of the left side term above to $\mathbb{Z} / r$ is bounded by $C \lambda^{-\left(k^{\prime}+k\right)}$. It follows:

$$
d\left(q_{i} q_{j} \alpha-\left[T(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\eta_{j}}+U(\underline{i}, \underline{j}) \lambda^{\eta_{j}-\eta_{i}}\right], \mathbb{Z} / r\right) \leq C \lambda^{-\left(\eta_{j}+\eta_{i}\right)}, \quad \forall i, j \geq n_{0}
$$

Thus, using (A.7), for $\kappa_{0}$ large enough and if $j \geq n_{0}+\kappa_{0} p$, we have:

$$
\begin{array}{r}
d\left(n_{b, \ell} q_{j} \alpha-\sum_{i=n_{0}+\left(\eta_{j}-\kappa_{0}\right) p}^{n_{0}+\left(\eta_{j}+\kappa_{0}\right) p-1} b_{i}\left[T(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\eta_{j}}+U(\underline{i}, \underline{j}) \lambda^{\eta_{j}-\eta_{i}}\right], \mathbb{Z} / r\right) \\
\leq C \lambda^{-\kappa_{0} p}+C \sum_{i=n_{0}+\left(\eta_{j}-\kappa_{0}\right) p}^{n_{0}+\left(\eta_{j}+\kappa_{0}\right) p-1} \lambda^{-\eta_{j}-\eta_{i}} \leq C \lambda^{-\kappa_{0} p}+2 C \kappa_{0} \lambda^{\kappa_{0}-2 \eta_{j}} \leq \delta .
\end{array}
$$

## Step 3. From long words to short words

Lemma A.4. Let $1 \leq \ell_{0} \leq \ell_{1} \leq \ell$ be three integers and let $\Lambda: b_{0} \ldots b_{\ell} \mapsto b_{\ell_{0}} \ldots b_{\ell_{1}}$ be the "restriction" map from the set $\mathcal{L}$ of admissible words to shortened words. There is a constant $c>0$ such that, if $\mathcal{S}$ is the image of $\Lambda$, for any subset $\mathcal{P}$ of $\mathcal{S}$, we have

$$
\frac{\operatorname{Card}(\mathcal{P})}{\operatorname{Card}(\mathcal{S})} \leq c \frac{\operatorname{Card}\{w \in \mathcal{L}: \Lambda(w) \in \mathcal{P}\}}{\operatorname{Card}(\mathcal{L})}
$$

We can take $c=4$ for the golden mean, $c=u_{0}^{-2}$, with $u_{0}=\inf _{k>1} \frac{q_{k-1}-1}{q_{k}-1}$, in the general case.

Proof. The proof is given for the golden mean. The general case is analogous.
The ways of completing a short word into a long one depend only on the first letter $b_{\ell_{0}}$ and the last letter $b_{\ell_{1}}$ : if $b_{\ell_{0}} \neq 1$, any admissible beginning fits; if $b_{\ell_{0}}=1$, then only the admissible beginnings finishing by 0 fit; if $b_{\ell_{1}}=0$ then any admissible ending fits; if $b_{\ell_{1}}=1$, only endings with 0 as first letter fit.

The number of admissible words of length $r$ is $q_{r+1}$, the number of admissible words of length $r$ beginning (or ending) by 0 is $q_{r}$.

Let denote $\mathcal{S}_{i}, i=1, \ldots, 4$, the set of short words $b_{\ell_{0}} \ldots b_{\ell_{1}}$ such that $b_{\ell_{0}}=b_{\ell_{1}}=0$, $b_{\ell_{0}}=0$ and $b_{\ell_{1}}=1, b_{\ell_{0}}=1$ and $b_{\ell_{1}}=0, b_{\ell_{0}}=b_{\ell_{1}}=1$, respectively.

Depending on the set $\mathcal{S}_{i}$ to which $\Lambda(w)$ belongs, the cardinal of $\operatorname{Card} \Lambda^{-1}(\Lambda(w))$ is $D_{1}=q_{\ell_{0}} q_{\ell-\ell_{1}+1}, D_{2}=q_{\ell_{0}} q_{\ell-\ell_{1}}, D_{3}=q_{\ell_{0}-1} q_{\ell-\ell_{1}+1}$, or $D_{4}=q_{\ell_{0}-1} q_{\ell-\ell_{1}}$ respectively.

Since, $\frac{1}{2} \leq q_{r} / q_{r+1} \leq 1$, for all $r$, we have $D_{1}=\max _{i} D_{i}, D_{4}=\min _{i} D_{i}, D_{4} \leq$ $D_{1} \leq 4 D_{4}$ and finally

$$
\begin{aligned}
\operatorname{Card}(\mathcal{P}) & =\sum_{i=1}^{4} \operatorname{Card}\left(\mathcal{P} \cap \mathcal{S}_{i}\right)=\sum_{i=1}^{4} \frac{1}{D_{i}} \operatorname{Card}\left\{w \in \mathcal{L}: \Lambda(w) \in \mathcal{P} \cap \mathcal{S}_{i}\right\} \\
& \leq \frac{1}{D_{4}} \sum_{i=1}^{4} \operatorname{Card}\left\{w \in \mathcal{L}: \Lambda(w) \in \mathcal{P} \cap \mathcal{S}_{i}\right\} \\
& =\frac{1}{D_{4}} \operatorname{Card}\{w \in \mathcal{L}: \Lambda(w) \in \mathcal{P}\} \\
\operatorname{Card}(\mathcal{S}) & =\sum_{i=1}^{4} \operatorname{Card}\left(\mathcal{S}_{i}\right)=\sum_{i=1}^{4} \frac{1}{D_{i}} \operatorname{Card}\left\{w \in \mathcal{L}: \Lambda(w) \in \mathcal{S}_{i}\right\} \geq \frac{1}{D_{1}} \operatorname{Card}(\mathcal{L}) .
\end{aligned}
$$

## Step 4a. End of the proof of Proposition 2.5 when $\alpha$ is the golden mean

Let $\delta$ be a small positive number. Its value will be chosen later. It follows from (2.15) for $\ell$ big enough that, if $q_{j+1}^{-1}<\delta$ :

$$
\begin{equation*}
\operatorname{Card}\left\{n \in \left[1, q_{\ell+1}\left[: d\left(n q_{j} \alpha, \mathbb{Z} / 5\right) \leq 3 \delta\right\} \leq C_{1} \delta q_{\ell+1}, \quad \forall j \leq \ell .\right.\right. \tag{A.12}
\end{equation*}
$$

If $\kappa_{0}$ is big enough, from (A.10) in Lemma A.2, we have with $\Gamma(b, j)$ defined in (A.8):

$$
\begin{equation*}
d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / 5\right) \geq 3 \delta \Longrightarrow d(\Gamma(b, j), \mathbb{Z} / 5) \geq 2 \delta \Longrightarrow d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / 5\right) \geq \delta \tag{A.13}
\end{equation*}
$$

By taking $\kappa_{0}$ large enough, we can suppose $q_{\kappa_{0}+1}^{-1}<\delta$. By (A.12) (translated in terms of words) for each $j \in\left[\kappa_{0}, \ell\right]$, the proportion of words $b=b_{0} \ldots b_{\ell}$ of length $\ell+1$ for
which $d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / 5\right) \geq 3 \delta$, is smaller than $C_{1} \delta$. Therefore, if $\ell \geq \kappa_{0}$, we get

$$
\begin{equation*}
\operatorname{Card}\left\{b_{0} \ldots b_{\ell}: d(\Gamma(b, j), \mathbb{Z} / 5) \leq 2 \delta\right\} \leq C_{1} \delta q_{\ell}, \quad \forall j \in\left[\kappa_{0}, \ell\right] . \tag{A.14}
\end{equation*}
$$

But $\Gamma(b, j)$ depends only on the short word $b_{j-\kappa_{0}} \ldots b_{j+\kappa_{0}}$, part of the long word $b=b_{0} \ldots b_{\ell}$. It follows, using Lemma A. 4 that

$$
\begin{equation*}
\operatorname{Card}\left\{b_{j-\kappa_{0}} \ldots b_{j+\kappa_{0}}: d(\Gamma(b, j), \mathbb{Z} / 5) \leq 2 \delta\right\} \leq C_{2} \delta q_{2 \kappa_{0}+2}, \quad \forall j \in\left[\kappa_{0}, \ell\right] \tag{A.15}
\end{equation*}
$$

Putting $A_{\delta}:=\left\{b: d\left(\Gamma\left(b, \kappa_{0}\right), \mathbb{Z} / 5\right) \geq 2 \delta\right\}$, it follows from (A.15) and (A.4):

$$
\begin{aligned}
\mu\left(A_{\delta}^{c}\right) & \leq \alpha^{-2 \kappa_{0}-2} \operatorname{Card}\left\{b_{j-\kappa_{0}} \ldots b_{j+\kappa_{0}}: d(\Gamma(b, j), \mathbb{Z} / 5) \leq 2 \delta\right\} \\
& \leq C_{2} \delta q_{2 \kappa_{0}+2} \alpha^{-2 \kappa_{0}-2} \leq C_{3} \delta .
\end{aligned}
$$

Let $C_{4}$ be a constant $>C_{3}$ and $\varepsilon=C_{4} \delta$. Observe that we can chose $\ell$ large enough so that $\mu\left(A_{\delta}\right)\left(\ell-\kappa_{0}\right) \geq(1-\varepsilon) \ell$ : indeed, we have $\mu\left(A_{\delta}\right)-(1-\varepsilon)>0$ and by taking $\ell>\mu\left(A_{\delta}\right) \kappa_{0} /\left(\mu\left(A_{\delta}\right)-(1-\varepsilon)\right)$ we obtain the required inequality.

Now we use $\sum_{k=0}^{L-1} \mathbf{1}_{A_{\delta}}\left(\sigma^{k} b\right)=\operatorname{Card}\left\{k<L: d\left(\Gamma\left(\sigma^{k} b, \kappa_{0}\right), \mathbb{Z} / 5\right) \geq 2 \delta\right\}$ and (A.9). According (A.13) with $j=k+\kappa_{0}$ and Lemma A. 1 with $A=A_{\delta}$ and $\varepsilon=C_{4} \delta$ (we assume $\delta<C_{4}^{-1}$ ), there are two positive constants $R=R(\varepsilon), \xi=\xi(\varepsilon)$ such that

$$
\mu\left\{b \in X: \operatorname{Card}\left\{j \in\left[\kappa_{0}, L+\kappa_{0}[: d(\Gamma(b, j), \mathbb{Z} / 5) \geq 2 \delta\} \leq \mu\left(A_{\delta}\right)(1-\varepsilon) L\right\} \leq R(\varepsilon) e^{-\xi L}\right.\right.
$$

Using " $\Rightarrow$ " in (A.13), we have therefore, taking $L=\ell-\kappa_{0}$, for $\ell-\kappa_{0} \geq j \geq \kappa_{0}$,

$$
\begin{aligned}
\mu\left\{b \in X: \operatorname{Card}\left\{j \in \left[\kappa_{0}, \ell\left[: d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / 5\right) \geq \delta\right\} \leq \mu\left(A_{\delta}\right)(1-\varepsilon)\right.\right.\right. & \left.\left(\ell-\kappa_{0}\right)\right\} \\
& \leq R e^{-\xi\left(\ell-\kappa_{0}\right)}
\end{aligned}
$$

By (A.4), the previous inequality translated in terms of cardinal yields for a constant $C_{5}$ : $\operatorname{Card}\left\{b_{0} \ldots b_{\ell}: \operatorname{Card}\left\{j<\ell: d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / 5\right) \geq \delta\right\} \leq\left(1-C_{4} \delta\right)^{2} \ell\right\} \leq C_{5} e^{-\xi \ell} q_{\ell+1}$.
If $\delta$ is taken small enough to get $\left(1-\varepsilon_{0}\right) \leq\left(1-C_{4} \delta\right)^{2}$ and using that $e^{\xi \ell}$ is equivalent to a power of $q_{\ell+1}$ (because $\left(q_{\ell}\right)_{\ell}$ is equivalent to a geometric progression), the previous inequality shows (2.21) of Proposition 2.5.

## Step 4b. End of the proof of Proposition 2.5 for a general quadratic number

As for the golden number, we take a positive number $\delta$ whose value will be fixed later. By (2.15), if $\kappa_{0}$ is large enough, we have for some $C_{1}>0$

$$
\operatorname{Card}\left\{n \in \left[1, q_{\ell+1}\left[: d\left(n q_{j} \alpha, \mathbb{Z} / r\right) \leq 3 \delta\right\} \leq C_{1} \delta q_{\ell+1}, \forall j \in\left[n_{0}+\kappa_{0}, \ell\right] ;\right.\right.
$$

hence, in terms of admissible words $b_{0} \ldots b_{\ell}$, if $j \in\left[n_{0}+\kappa_{0}, \ell\right]$,

$$
\begin{equation*}
\operatorname{Card}\left\{b_{0} \ldots b_{\ell}: d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / r\right) \leq 3 \delta\right\} \leq C_{1} \delta q_{\ell+1} . \tag{A.16}
\end{equation*}
$$

Let $\Gamma_{j}, \Gamma_{\underline{j}}^{0}$ be the functions on $Y$

$$
\begin{aligned}
\Gamma_{j}(b) & :=\sum_{i=n_{0}+\left(\eta_{j}-\kappa_{0}\right) p}^{n_{0}+\left(\eta_{j}+\kappa_{0}\right) p-1} b_{i}\left[T(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\eta_{j}}+U(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\eta_{j}}\right], \\
\Gamma_{\underline{j}}^{0}(b) & :=\sum_{i=n_{0}}^{n_{0}+2 \kappa_{0} p-1} b_{i}\left[T(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\kappa_{0}}+U(\underline{i}, \underline{j}) \lambda^{\kappa_{0}-\eta_{i}}\right] .
\end{aligned}
$$

Remark that the sums on the right can be viewed as functions of $y$ through the $b_{i}$ 's. Letting $y_{k}:=b_{n_{0}+k p} \ldots b_{n_{0}+k p+p-1}$, we see that the sum inside the definition of $\Gamma_{j}$ is a function of $y_{\eta_{j}-\kappa_{0}} \ldots y_{\eta_{j}+\kappa_{0}-1}$.

Let $A_{\delta}$ be the subset of $Y$ defined by

$$
A_{\delta}:=\left\{y: d\left(\Gamma_{\underline{j}}^{0}(y), \mathbb{Z} / r\right) \geq 2 \delta, \text { for } \underline{j}=0, \ldots, p-1\right\} .
$$

By (A.11) in Lemma A.3, if $\kappa_{0}$ is sufficiently large, we have the implication

$$
\begin{equation*}
d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / r\right) \geq 3 \delta \Longrightarrow d\left(\Gamma_{j}(b), \mathbb{Z} / r\right) \geq 2 \delta . \tag{A.17}
\end{equation*}
$$

As $\underline{j}=\underline{j-p}, \eta_{j+p}=\eta_{j}+1$ and $\eta_{i+p}=\eta_{i}+1$, we obtain by $\eta_{j}-\kappa_{0}$ iterations:

$$
\begin{aligned}
& \sum_{i=n_{0}+\left(\eta_{j}-\kappa_{0}\right) p}^{n_{0}+\left(\eta_{j}+\kappa_{0}\right) p-1} b_{i} T(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\eta_{j}}=\sum_{i=n_{0}+\left(\eta_{j}-1-\kappa_{0}\right) p}^{n_{0}+\left(\eta_{j}-1+\kappa_{0}\right) p-1} b_{i+p} T(\underline{i+p}, \underline{j}) \lambda^{\eta_{i+p}-\eta_{j}} \\
& \quad=\sum_{i=n_{0}+\left(\eta_{j-p}-\kappa_{0}\right) p}^{n_{0}+\left(\eta_{j-p}+\kappa_{0}\right) p-1} b_{i+p} T(\underline{i}, \underline{j}) \lambda^{\eta_{i}+1-\eta_{j}}=\sum_{i=n_{0}+\left(\eta_{j-p}-\kappa_{0}\right) p}^{n_{0}+\left(\eta_{j-p}+\kappa_{0}\right) p-1} b_{i+p} T(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\eta_{j-p}} \\
& \quad=\cdots=\sum_{i=n_{0}+\left(\eta_{\left.j-\left(\eta_{j}-\kappa_{0}\right) p-\kappa_{0}\right) p}^{n_{0}+\eta_{j-\left(\eta_{j}-\kappa_{0}\right) p+\kappa_{0} p-1}} b_{i+\left(\eta_{j}-\kappa_{0}\right) p} T(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\eta_{j-\left(\eta_{j}-\kappa_{0}\right) p}} .\right.} .
\end{aligned}
$$

Since $\eta_{j-\left(\eta_{j}-\kappa_{0}\right) p}=\kappa_{0}$, the last quantity reduces to $\sum_{i=n_{0}}^{n_{0}+2 \kappa_{0} p-1} b_{i+\left(\eta_{j}-\kappa_{0}\right) p} T(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\kappa_{0}}$.
The same computation can be done for $\sum_{i=n_{0}+\left(\eta_{j}-\kappa_{0}\right) p}^{n_{0}+\left(\eta_{j}+\kappa_{0}\right) p-1} b_{i} U(\underline{i}, \underline{j}) \lambda^{\eta_{i}-\eta_{j}}$. Taking the sum for the $T$ 's and $U$ 's, we get

$$
\begin{equation*}
\Gamma_{j}(y)=\Gamma_{\underline{j}}^{0}\left(\sigma^{\eta_{j}-\kappa_{0}} y\right) \tag{A.18}
\end{equation*}
$$

From (A.17), (A.16) and (A.18), it follows, if $\ell \geq n_{0}+2 \kappa_{0} p$ and $j \geq n_{0}+\kappa_{0}$,

$$
\operatorname{Card}\left\{b_{0} \ldots b_{\ell}: d\left(\Gamma_{\underline{j}}\left(\sigma^{\eta_{j}-\kappa_{0}} y\right), \mathbb{Z} / r\right) \leq 2 \delta\right\} \leq C_{1} \delta q_{\ell+1}
$$

But $\Gamma_{\underline{j}}^{0}\left(\sigma^{\eta_{j}-\kappa_{0}} y\right)$ depends only on the short word $b_{n_{0}+\left(\eta_{j}-\kappa_{0}\right) p} \ldots b_{n_{0}+\left(\eta_{j}+\kappa_{0}\right) p-1}$, which is a sub-word of the "long" word $b_{0} \ldots b_{\ell}$. By Lemma A. 4 we obtain for constants
$C_{2}, C_{3}>0$ :

$$
\begin{equation*}
\operatorname{Card}\left\{b_{n_{0}+\left(\eta_{j}-\kappa_{0}\right) p} \ldots b_{n_{0}+\left(\eta_{j}+\kappa_{0}\right) p-1}: d\left(\Gamma_{\underline{j}}^{0}\left(\sigma^{\eta_{j}-\kappa_{0}} y\right), \mathbb{Z} / r\right) \leq 2 \delta\right\} \leq C_{2} \delta \lambda^{2 \kappa_{0}} \tag{A.19}
\end{equation*}
$$

Then, (A.19) and (A.5) imply that

$$
\begin{equation*}
\mu\left(A_{\delta}^{c}\right)=\mu\left\{y \in Y: d\left(\Gamma_{m}^{0}(y), \mathbb{Z} / r\right)<2 \delta, m=0, \ldots, p-1\right\} \leq C_{3} \delta . \tag{A.20}
\end{equation*}
$$

Now, we have

$$
\begin{gathered}
\sum_{k=0}^{n-1} \mathbf{1}_{A_{\delta}}\left(\sigma^{k} y\right)=\operatorname{Card}\left\{k<n: d\left(\Gamma_{m}^{0}\left(\sigma^{k} y\right), \mathbb{Z} / r\right) \geq 2 \delta, m=0, \ldots, p-1\right\}, \\
\Gamma_{m}^{0}\left(\sigma^{k} y\right)=\sum_{i=n_{0}+k p}^{n_{0}+\left(k+2 \kappa_{0}\right) p-1} b_{i}\left[T(\underline{i}, m) \lambda^{\eta_{i}-k-\kappa_{0}}+U(\underline{i}, m) \lambda^{k+\kappa_{0}-\eta_{i}}\right]
\end{gathered}
$$

and, if $j=\left(k+\kappa_{0}\right) p+m \in\left[n_{0}+\kappa_{0}, \ell\right]$ (i.e., $\left.\eta_{j}=k+\kappa_{0}, \underline{j}=m\right)$,

$$
d\left(\Gamma_{m}^{0}\left(\sigma^{k} y\right), \mathbb{Z} / r\right) \geq 2 \delta \Longrightarrow d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / r\right) \geq \delta
$$

In particular:

$$
\begin{aligned}
p \operatorname{Card}\left\{k<\eta_{\ell}-\kappa_{0}: d\left(\Gamma_{m}^{0}\left(\sigma^{k} y\right)\right.\right. & , \mathbb{Z} / r) \geq 2 \delta, m=0, \ldots, p-1\} \\
& \leq \operatorname{Card}\left\{j<\left(\eta_{\ell}-\kappa_{0}\right) p: d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / r\right) \geq \delta\right\}
\end{aligned}
$$

By Lemma A.1, for the Markov chain deduced from $Y$ with state space the set of words of length $2 \kappa_{0}$ in $Y$, we get from (A.6):

$$
\begin{array}{r}
\mu\left\{y: \operatorname{Card}\left\{j<\left(\eta_{\ell}-\kappa_{0}\right) p: d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / r\right) \geq \delta\right\} \leq \mu\left(A_{\delta}\right)(1-\varepsilon) p\left(\eta_{\ell}-\kappa_{0}\right)\right\} \\
\leq R e^{\xi\left(\kappa_{0}-\eta_{\ell}\right)}
\end{array}
$$

This can be translated in terms of cardinal using (A.5): the cardinal of

$$
\left\{y_{0} \ldots y_{\eta_{\ell}-\kappa_{0}}: \operatorname{Card}\left\{j<\left(\eta_{\ell}-\kappa_{0}\right) p: d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / r\right) \geq \delta\right\} \leq \mu\left(A_{\delta}\right)(1-\varepsilon) p\left(\eta_{\ell}-\kappa_{0}\right)\right\}
$$

is smaller than $C_{4} e^{-\xi\left(\eta_{\ell}-\kappa_{0}\right)} \lambda^{\eta_{\ell}-\kappa_{0}}$. It implies

$$
\begin{array}{r}
\operatorname{Card}\left\{b_{0} \ldots b_{\ell}: \operatorname{Card}\left\{j<\ell: d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / r\right) \geq \delta\right\} \leq \mu\left(A_{\delta}\right)(1-\varepsilon) p\left(\eta_{\ell}-\kappa_{0}\right)\right\} \\
\leq C_{5} e^{-\xi\left(\eta_{\ell}-\kappa_{0}\right)} \lambda^{\eta_{\ell}-\kappa_{0}}
\end{array}
$$

If $\eta_{\ell}>\left(\kappa_{0}+1\right) / \varepsilon$ (that is $\left.\ell \geq p\left(\kappa_{0}+2\right) / \varepsilon\right)$, then $p\left(\eta_{\ell}-\kappa_{0}\right) \geq(1-\varepsilon) \ell$ and, for some $C_{6}>0$, there are less than $C_{6} e^{-\xi \eta_{\ell}} \lambda^{\eta_{\ell}}$ words $b$ of length $\ell$ such that

$$
\begin{equation*}
\operatorname{Card}\left\{j \leq \ell: d\left(\sum_{i=0}^{\ell} b_{i} q_{i} q_{j} \alpha, \mathbb{Z} / r\right) \geq \delta\right\} \leq \mu\left(A_{\delta}\right)(1-\varepsilon)^{2} \ell \tag{A.21}
\end{equation*}
$$

By (A.20), for $\varepsilon_{0}>0$, we can choose $\varepsilon$ and $\delta$ such that $\mu\left(A_{\delta}\right)(1-\varepsilon)^{2}=1-\varepsilon_{0}$. On the other hand, since $c^{-1} q_{\ell+1} \leq \lambda^{\eta_{\ell}} \leq c q_{\ell+1}$ for some $c>0, C_{6} e^{-\xi \eta_{\ell}} \lambda^{\eta_{\ell}} \leq C_{7} q_{\ell+1}^{1-\zeta}$, for some positive constants $\zeta, C_{7}$. Finally, we have obtained (2.21) (in terms of number of admissible words):

$$
\operatorname{Card}\left\{b_{0} \ldots b_{\ell}: \operatorname{Card}\left\{j<\ell: d\left(n_{b, \ell} q_{j} \alpha, \mathbb{Z} / r\right) \geq \delta\right\} \leq\left(1-\varepsilon_{0}\right) \ell\right\} \leq C_{7} q_{\ell+1}^{1-\zeta}
$$

## Appendix B. Weighted orthogonal functions

Let $\left(g_{n}\right)$ be a sequence of orthogonal real functions in $L^{2}$ of a probability space $(X, \mu)$ and $\left(u_{n}\right)$ be a sequence of positive constants. By the Lebesgue dominated convergence theorem, if the functions $g_{n}$ are uniformly bounded, the following condition

$$
\begin{equation*}
\lim _{n} \frac{\sum_{k=1}^{N} u_{k}^{2}}{\left(\sum_{k=1}^{N} u_{k}\right)^{2}} \rightarrow 0 \tag{B.1}
\end{equation*}
$$

is necessary for

$$
\lim _{N} \frac{\sum_{k=1}^{N} u_{k} g_{k}(x)}{\sum_{k=1}^{N} u_{k}}=0, \text { for a.e. } x .
$$

to hold. Property (B.1) is satisfied if the following condition holds:

$$
\begin{equation*}
1 \leq u_{n} \leq n^{\gamma}, \forall n \geq 1, \text { with } 0 \leq \gamma<1 . \tag{B.2}
\end{equation*}
$$

Indeed we have $\frac{\sum_{k=1}^{N} u_{k}^{2}}{\left(\sum_{k=1}^{N} u_{k}\right)^{2}} \leq\left(\max _{k=1}^{N} u_{k}\right) \frac{\sum_{k=1}^{N} u_{k}}{\left(\sum_{k=1}^{N} u_{k}\right)^{2}} \leq \frac{\max _{k=1}^{N} u_{k}}{\left(\sum_{k=1}^{N} u_{k}\right)} \leq N^{\gamma-1} \rightarrow 0$.
But (B.1) and the result of Proposition B. 1 below can fail if the parameter $\gamma$ in (B.2) is taken $\geq 1$. Indeed, suppose that $\left\|g_{k}\right\|_{2}=1$, and let us take $u_{k}=k$ if $k$ is a power of 2 , else $u_{k}=1$.

Then, we have $1 \leq u_{k} \leq k, \sum_{k=1}^{2^{n}} u_{k}^{2} \geq \frac{4}{3} 2^{2 n}$ and $\sum_{k=1}^{2^{n}} u_{k}=2^{n+1}-(n+1)$, so that

$$
\frac{\sum_{k=1}^{2^{n}} u_{k}^{2}}{\left(\sum_{k=1}^{2^{n}} u_{k}\right)^{2}} \geq \frac{1}{3}\left(1-2^{-(n+1)}(n+1)\right)^{-2} \rightarrow \frac{1}{3} .
$$

Proposition B.1. Let $\left(g_{k}\right)_{k \geq 1}$ be a sequence of orthogonal functions in $L^{2}(X, \mu)$, bounded in $L^{2}$ norm. Under the condition

$$
\begin{equation*}
1 \leq u_{n} \leq n^{\gamma}, \text { with } 0 \leq \gamma<\frac{1}{2} \tag{B.3}
\end{equation*}
$$

it holds

$$
\begin{equation*}
\lim _{N} \frac{\sum_{k=1}^{N} u_{k} g_{k}(x)}{\sum_{k=1}^{N} u_{k}}=0, \text { for a.e. } x \tag{B.4}
\end{equation*}
$$

Proof. Setting $R_{N}(x):=\frac{\sum_{k=1}^{N} u_{k} g_{k}(x)}{\sum_{k=1}^{N} u_{k}}$, by orthogonality and the conditions on $u_{k}$, there is a constant $C$ such that $\int_{X}\left|R_{N}(x)\right|^{2} \mathrm{~d} \mu \leq C N^{2 \gamma-1}$, which implies $\sum_{n=1}^{\infty}\left\|R_{n}\right\|_{2}^{2}<+\infty$, if $p(1-2 \gamma)>1$.

As $1-2 \gamma>0$, we can choose $p$ such that $p(1-2 \gamma)>1$. We have then: $\lim _{n} R_{n^{p}}(x)=0$, for a.e. $x$. Therefore, it suffices to show that:

$$
\lim _{n} \sup _{\ell \in J_{n}}\left|R_{n^{p}+\ell}(x)-R_{n} p(x)\right|=0, \text { where } J_{n}=\left\{0,1, \ldots,(n+1)^{p}-n^{p}-1\right\} .
$$

For reals $A, C, B_{\ell}, D_{\ell}, \ell \in J_{n}$, with $C, D_{\ell}>0$, it holds:

$$
\max _{\ell \in J_{n}}\left|\frac{A+B_{\ell}}{C+D_{\ell}}-\frac{A}{C}\right| \leq \frac{\max _{\ell \in J_{n}}\left|B_{\ell}\right|+|A|}{C} .
$$

This implies, with

$$
\begin{array}{r}
A=\sum_{k=1}^{n^{p}} u_{k} g_{k}, B_{\ell}=\sum_{k=n^{p}+1}^{n^{p}+\ell} u_{k} g_{k}, C=\sum_{k=1}^{n^{p}} u_{k}, D_{\ell}=\sum_{k=n^{p}+1}^{n^{p}+\ell} u_{k}, \\
\max _{\ell \in J_{n}}\left|R_{n^{p}+\ell}(x)-R_{n^{p}}(x)\right| \leq \frac{\max _{\ell \in J}\left|\sum_{k=n^{p}+1}^{n^{p}} u_{k} g_{k}\right|}{\sum_{k=1}^{n^{p}} u_{k}}+\frac{\left|\sum_{k=1}^{n^{p}} u_{k} g_{k}\right|}{\sum_{k=1}^{n^{p}} u_{k}} . \tag{B.5}
\end{array}
$$

By a lemma of Rademacher-Mensov ([11, p. 156]), if $Y_{1}, \ldots, Y_{L}$ are mutually orthogonal functions in a probability space $(X, \mu)$ with finite variances $\sigma_{1}^{2}, \ldots, \sigma_{L}^{2}$, then

$$
\begin{equation*}
\mathbb{E}\left[\left(\max _{\ell=1}^{L}\left(\sum_{j=1}^{\ell} Y_{j}\right)\right)^{2}\right] \leq C(\log (4 L))^{2} \sum_{\ell=1}^{L} \sigma_{\ell}^{2} \tag{B.6}
\end{equation*}
$$

If we put $M_{n, p}:=\max _{\ell \in J_{n}}\left|\sum_{k=n^{p}}^{n^{p}+\ell} u_{k} g_{k}\right|$, then by (B.6) we have

$$
\begin{aligned}
\mathbb{E}\left(M_{n, p}^{2}\right) & \leq C\left(\log \left(4 p n^{p-1}\right)\right)^{2} \sum_{j=n^{p}}^{(n+1)^{p}-1} u_{j}^{2} \leq C\left(\log \left(4 p n^{p-1}\right)\right)^{2} \sum_{j=n^{p}}^{(n+1)^{p}-1} j^{2 \gamma} \\
& \leq C^{\prime}(\log n)^{2} n^{p-1} n^{2 p \gamma}=C^{\prime}(\log n)^{2} n^{p(2 \gamma+1)-1}
\end{aligned}
$$

It follows:

$$
\mathbb{E}\left[\left(\frac{\max _{\ell \in J_{n}}\left|\sum_{k=n^{p}}^{n^{p}+\ell} u_{k} g_{k}\right|}{\sum_{j=1}^{n^{p}} u_{j}}\right)^{2}\right] \leq C^{\prime} \frac{(\log n)^{2} n^{p(2 \gamma+1)-1}}{n^{2 p}}=C^{\prime}(\log n)^{2} n^{p(2 \gamma-1)-1} .
$$

Therefore, since $2 \gamma-1<0$, we have

$$
\sum_{n} \mathbb{E}\left[\left(\frac{\max _{\ell \in J_{n}}\left|\sum_{k=n^{p}}^{n^{p}+\ell} u_{k} g_{k}\right|}{\sum_{k=1}^{n^{p}} u_{k}}\right)^{2}\right]<+\infty
$$

so that $\lim _{n} \frac{\max _{\epsilon \in J_{n}}\left|\sum_{k=n}^{n P}{ }^{p} u_{k} g_{k}\right|}{\sum_{k=1}^{n D} u_{k}}=0$, a.e.

Both terms in the right side of (B.5) converge a.e. to 0 , which implies a.e.:

$$
\lim _{n} \max _{\ell \in J_{n}}\left|R_{n^{p}+\ell}(x)-R_{n^{p}}(x)\right| \rightarrow 0
$$

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