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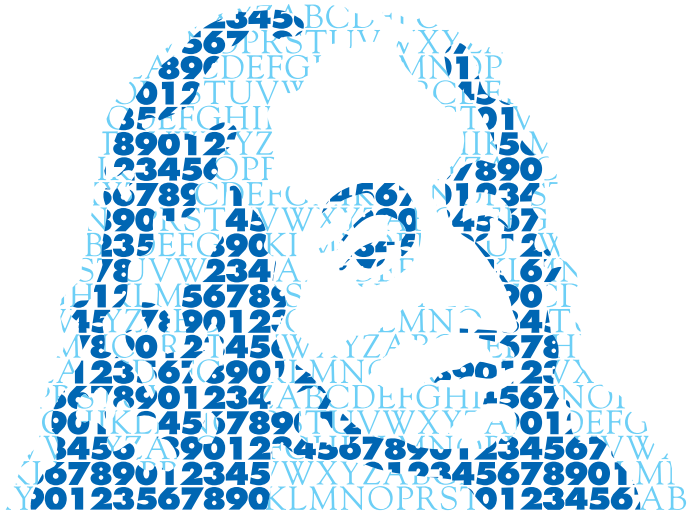

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Volume 23, n° 2 (2016), p. 151-169.

<http://ambp.cedram.org/item?id=AMBP_2016__23_2_151_0>

Publication éditée par le laboratoire de mathématiques de l'université Blaise-Pascal, UMR 6620 du CNRS Clermont-Ferrand — France

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Annales mathématiques Blaise Pascal **23**, 151-169 (2016)

Reduced $L_{q,p}$-Cohomology of Some Twisted Products

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Abstract

Vanishing results for reduced $L_{q,p}$-cohomology are established in the case of twisted products, which are a generalization of warped products. Only the case $q \leq p$ is considered. This is an extension of some results by Gol′dshtein, Kuz′minov and Shvedov about the L_p-cohomology of warped cylinders. One of the main observations is the vanishing of the "middle-dimensional" cohomology for a large class of manifolds.

La cohomologie $L_{q,p}$ réduite de quelques produits twistés

Résumé

On établit des résultats d'annulation de la cohomologie $L_{q,p}$ réduite pour les produits twistés, une généralisation des produits tordus dans le cas de $q \leq p$. Le résultats obtenus sont des généralisations de certains résultats par Gol′dshtein, Kuz′minov et Shvedov sur la cohomologie L_p des cylindres tordus. Une des observations principales est la trivialité de la cohomolgie en dimension "moyenne" pour une large classe de variétés.

1. Introduction

The *$L_{q,p}$-cohomology* $H^k_{q,p}(M)$ of a Riemannian manifold (M, g) is defined to be the quotient of the space of closed p-integrable differential k-forms by the exterior differentials of q-integrable k-forms. The quotient space of $H^k_{q,p}(M)$ by the closure of zero is called the *reduced $L_{q,p}$-cohomology* $\overline{H}^k_{q,p}(M)$. If $p = q$ then $L_{q,p}$-cohomology is usually referred to as simply L_p-cohomology and the index p is used instead of p, p in all the notations.

The second author was partially supported by the State Maintenance Program for the Leading Scientific Schools and Junior Scientists of the Russian Federation.
Keywords: differential form, reduced $L_{q,p}$-cohomology, twisted cylinder.
Math. classification: 58A10, 58A12.

A *twisted product* $X \times_h Y$ of two Riemannian manifolds (X, g_X) and (Y, g_Y) is the direct product manifold $X \times_g Y$ endowed with a Riemannian metric of the form

$$g := g_X + h^2(x, y)g_Y, \tag{1.1}$$

where $h : X \times Y \to \mathbb{R}$ is a smooth positive function (see [4]). If X is a half-interval $[a, b[$ then the twisted product $X \times_h Y$ is called a *twisted cylinder*.

We refer to an m-dimensional Riemannian manifold (M, g_M) as an *asymptotic twisted product* (respectively, as an *asymptotic twisted cylinder*) if, outside an m-dimensional compact submanifold, it is bi-Lipschitz equivalent to a twisted product (respectively, to a twisted cylinder).

In this paper, we prove some vanishing results for the (reduced) $L_{q,p}$-cohomology of twisted cylinders $[a, b) \times_h N$ for a positive smooth function $h : [a, b) \times N \to \mathbb{R}$ in the case where the base N is a closed manifold and $p \geq q > 1$.

If in (1.1) the function h depends only on x then we obtain the familiar notion of a *warped product* (see [1]). Twisted products were the object of recent investigations [2, 3, 5, 6, 10, 14]. The reduced $L_{q,p}$-cohomology of warped cylinders $[a, b) \times_h N$, i.e., of product manifolds $[a, b) \times N$ endowed with a warped product metric

$$g = dt^2 + h^2(t)g_N,$$

where g_N is the Riemannian metric of N and $h : [a, b) \to \mathbb{R}$ is a positive smooth function, was studied by Gol′dshtein, Kuz′minov, and Shvedov [7], Kuz′minov and Shvedov [12, 13] (for $p = q$), and Kopylov [11] for $p, q \in [1, \infty)$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{\dim N + 1}$.

The main results of this paper are technical. Here we mention a "universal" consequence of the main results on the vanishing of the "middle-dimensional" cohomology:

Let N be a closed smooth n-dimensional Riemannian manifold. If $p \geq q > 1$ and $\frac{n}{p}$ is an integer then $\overline{H}^{\frac{n}{p}}_{q,p}([0, \infty) \times_h N) = 0$.

In particular, if $1 < q \leq 2$ and n is even then $\overline{H}^{\frac{n}{2}}_{q,2}([0, \infty) \times_h N) = 0$.

The result was not known even for L_2-cohomology. It does not depend on the type of the warped Riemannian metric. Of course, the result leads to the vanishing of the "middle-dimensional" cohomology for asymptotic twisted cylinders.

2. Basic Definitions

In this section, we recall the main definitions and notations. In what follows, we tacitly assume all manifolds to be oriented. Let M be a smooth Riemannian manifold. Denote by $\mathcal{D}^k(M) := C_0^\infty(M, \Lambda^k)$ the space of all smooth differential k-forms with compact support contained in $M \setminus \partial M$ and designate as $L^1_{loc}(M, \Lambda^k)$ the space of locally integrable differential k-forms. Denote by $L^p(M, \Lambda^k)$ the Banach space of locally integrable differential k-forms endowed with the norm $\|\theta\|_{L^p(M,\Lambda^k)} := (\int_M |\theta|^p dx)^{\frac{1}{p}} < \infty$ (as usual, we identify forms coinciding outside a set of measure zero).

Definition 2.1. We call a differential $(k+1)$-form $\theta \in L^1_{loc}(M, \Lambda^{k+1})$ *the weak exterior derivative* (or *differential*) of a differential k-form $\phi \in L^1_{loc}(M, \Lambda^k)$ and write $d\phi = \theta$ if

$$\int_M \theta \wedge \omega = (-1)^{k+1} \int_M \phi \wedge d\omega$$

for any $\omega \in \mathcal{D}^{n-k-1}(M)$.

Remark 2.2. Note that the orientability of M is not substantial since one may take integrals over orientable domains on M instead of integrals over M.

We then introduce an analog of Sobolev spaces for differential k-forms, the space of q-integrable forms with p-integrable weak exterior derivative:

$$\Omega^k_{q,p}(M) = \left\{ \omega \in L^q(M, \Lambda^k) \mid d\omega \in L^p(M, \Lambda^{k+1}) \right\},$$

This is a Banach space for the graph norm

$$\|\omega\|_{\Omega^k_{q,p}(M)} = \left(\|\omega\|^2_{L^q(M,\Lambda^k)} + \|d\omega\|^2_{L^p(M,\Lambda^{k+1})} \right)^{1/2}.$$

The space $\Omega^k_{q,p}(M)$ is a reflexive Banach space for any $1 < q, p < \infty$. This can be proved using standard arguments of functional analysis.

Denote by $\Omega^k_{q,p,0}(M)$ the closure of $\mathcal{D}^k(M)$ in the norm of $\Omega^k_{q,p}(M)$. We now define our basic ingredients (for three parameters r, q, p).

Definition 2.3. Put

(a) $Z^k_{p,r}(M) = \mathrm{Ker}[d : \Omega^k_{p,r}(M) \to L^r(M, \Lambda^{k+1})]$.

(b) $B^k_{q,p}(M) = \mathrm{Im}[d : \Omega^{k-1}_{q,p}(M) \to L^p(M, \Lambda^k)]$.

Lemma 2.4. *The subspace $Z^k_{p,r}(M)$ does not depend on r and is a closed subspace in $L^p(M,\Lambda^k)$.*

Proof. The lemma is in fact [9, Lemma 2.4 (i)]. However, we now repeat the proof for the reader's convenience. Note that $Z^k_{p,r}(M)$ is a closed subspace in $\Omega^k_{p,r}(M)$ because it is the kernel of the bounded operator d. It is also a closed subspace of $L^p(M,\Lambda^k)$ since $\|\alpha\|_{\Omega^k_{p,r}(M)} = \|\alpha\|_{L^p(M,\Lambda^k)}$ for any $\alpha \in Z^k_{p,r}(M)$. □

This allows us to use the notation $Z^k_p(M)$ for all $Z^k_{p,r}(M)$. Note that $Z^k_p(M) \subset L^p(M,\Lambda^k)$ is always a closed subspace but this is in general not true for $B^k_{q,p}(M)$. Denote by $\overline{B}^k_{q,p}(M)$ its closure in the L^p-topology. Observe also that since $d \circ d = 0$, one has $\overline{B}^k_{q,p}(M) \subset Z^k_p(M)$. Thus,

$$B^k_{q,p}(M) \subset \overline{B}^k_{q,p}(M) \subset Z^k_p(M) = \overline{Z}^k_p(M) \subset L^p(M,\Lambda^k).$$

Definition 2.5. Suppose that $1 \le p,q \le \infty$. The *$L_{q,p}$-cohomology* of (M,g) is defined as the quotient

$$H^k_{q,p}(M) := Z^k_p(M)/B^k_{q,p}(M)\,,$$

and the *reduced $L_{q,p}$-cohomology* of (M,g) is, by definition, the space

$$\overline{H}^k_{q,p}(M) := Z^k_p(M)/\overline{B}^k_{q,p}(M)\,.$$

Since $B^k_{q,p}$ is not always closed, the $L_{q,p}$-cohomology is in general a (non-Hausdorff) semi-normed space, while the reduced $L_{q,p}$-cohomology is a Banach space. Considering only the forms equal to zero on some neighborhood (depending on the form) of a subset $A \subset M$ and taking closures in the corresponding spaces, we obtain the definition of the relative spaces $L^p(M,A,\Lambda^k)$ and $\Omega_{q,p}(M,A)$ and the *relative nonreduced* and *reduced cohomology spaces* $H^k_{q,p}(M,A)$ and $\overline{H}^k_{q,p}(M,A)$.

Similarly, one can define the $L_{q,p}$-cohomology with compact support (interior cohomology) $H^k_{q,p;0}(M,A)$ and $\overline{H}^k_{q,p;0}(M,A)$. The interior reduced cohomology is dual to the reduced cohomology:

Theorem 2.6 ([9])**.** *Let (M,g) be an oriented m-dimensional Riemannian manifold. If $1 < p,q < \infty$ then $\overline{H}^k_{q,p}(M)$ is isomorphic to the dual of $\overline{H}^{m-k}_{p',q';0}(M)$, where $\frac{1}{p'} + \frac{1}{p} = \frac{1}{q'} + \frac{1}{q} = 1$.*

Therefore, if M is complete then $\overline{H}^k_p(M) = \overline{H}^k_{p;0}(M)$ and $\overline{H}^k_p(M)$ is isomorphic to dual of $\overline{H}^{m-k}_{p'}(M)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ [9].

Below $|X|$ stands for the volume of a Riemannian manifold (X, g); the notation $|\omega|_X$ means the modulus of a differential form on (X, g).

3. $L_{q,p}$-Cohomology and Smooth Forms

It follows from the results of [8] that, under suitable assumptions on p, q, the $L_{q,p}$-cohomology of a Riemannian manifold can be expressed in terms of smooth forms.

Introduce the notations:

$$C^\infty L^p(M, \Lambda^k) := C^\infty(M, \Lambda^k) \cap L^p(M, \Lambda^k);$$
$$C^\infty \Omega^k_{q,p}(M) := C^\infty(M, \Lambda^k) \cap \Omega^k_{q,p}(M).$$

Theorem 3.1 ([8, Theorem 12.8 and Corollary 12.9])**.** *Let (M, g) be an m-dimensional Riemannian manifold and suppose that $p, q \in (1, \infty)$ satisfy $\frac{1}{p} - \frac{1}{q} \le \frac{1}{m}$. Then the cohomology $H^*_{q,p}(M)$ can be represented by smooth forms.*

More precisely, any closed form in $Z^k_p(M)$ is cohomologous to a smooth form in $L^p(M)$. Furthermore, if two smooth closed forms $\alpha, \beta \in C^\infty(M) \cap Z^k_p(M)$ are cohomologous modulo $d\Omega^{k-1}_{q,p}(M)$ then they are cohomologous modulo $dC^\infty\Omega^{k-1}_{q,p}(M)$.

Similarly, any reduced cohomology class can be represented by a smooth form.

In what follows, unless otherwise specified, we always assume that $p, q \in (1, \infty)$ and $\frac{1}{p} - \frac{1}{q} \le \frac{1}{\dim M}$.

4. The Homotopy Operator

From now on, $C^h_{a,b}N$ is the twisted cylinder $I \times_h N$, that is, the product of a half-interval $I := [a, b)$ and a closed smooth n-dimensional Riemannian manifold (N, g_N) equipped with the Riemannian metric $dt^2 + h^2(t, x)g_N$, where $h : I \times N \to \mathbb{R}$ is a smooth positive function.

Every differential form on $I \times N$ admits a unique representation of the form $\omega = \omega_A + dt \wedge \omega_B$, where the forms ω_A and ω_B do not contain dt

(cf. [7]). It means that ω_A and ω_B can be viewed as one-parameter families $\omega_A(t)$ and $\omega_B(t)$, $t \in I$, of differential forms on N. Given a form ω defined on $I \times N$ and numbers $c, t \in [a, b)$, consider the form $\int_c^t \omega = \int_c^t \omega_B(\tau) d\tau$ on N and the form $S_c\omega$ on $[a, b) \times N$, $(S_c\omega)(t) = \int_c^t \omega$ for all $t \in [a, b)$. The domains of of these operators will be specified below.

The modulus of a form ω of degree k on $C^h_{a,b}N$ is expressed via the moduli of $\omega_A(t)$ and $\omega_B(t)$ on N as follows:

$$|\omega(t,x)|_M = \left[h^{-2k}(t,x)|\omega_A(t,x)|^2_N + h^{-2(k+1)}(t,x)|\omega_B(t,x)|^2_X\right]^{1/2} \quad (4.1)$$

Consequently,

$$\begin{aligned}&\|\omega\|_{L^p(C^h_{a,b}N,\Lambda^k)}\\ &= \left[\int_a^b \int_N (h^{2(\frac{n}{p}-k)}(t,x)|\omega_A(t,x)|^2_N + h^{2(\frac{n}{p}-k+1)}(t,x)|\omega_B(t,x)|^2_N)^{\frac{p}{2}} dxdt\right]^{\frac{1}{p}}.\end{aligned} \quad (4.2)$$

In the particular case of $\omega = \omega_A$, call the form ω *horizontal*. If ω is a horizontal form then

$$\|\omega\|_{L^p(C^h_{a,b}N,\Lambda^k)} = \left[\int_a^b \int_N |\omega(t)|^p h^{n-kp}(t,x) dxdt\right]^{1/p}. \quad (4.3)$$

Put

$$f_{k,p}(t) = \min_{x\in N}\{h^{\frac{n}{p}-k}(t,x)\}$$

and

$$F_{k,p}(t) = \max_{x\in N}\{h^{\frac{n}{p}-k}(t,x)\}.$$

Remarks 4.1.

(1) Suppose that $k = \frac{n}{p}$ is an integer. Then $F_{n/p,p}(t) = f_{n/p,p}(t) \equiv 1$. For example, if n is even and $p = 2$ then $F_{n/2,2}(t) = f_{n/2,2}(t) \equiv 1$.

(2) For warped products (h depends only on x), $f_{k,p}(t) = F_{k,p}(t) = h^{\frac{n}{p}-k}(t)$.

Let

$$\pi : I \times N \to N, \quad \pi(t,x) = x$$

be the natural projection. For $c \in [a,b)$, put $N_c := \{c\} \times N$. Let $i_t : N \to N_t \subset [a,b) \times N$ and $i_c : N \to N_c \subset [a,b) \times N$ be the natural immersions.

Every smooth k-form ω on $C^h_{a,b}N$ satisfies the homotopy relations

$$d_N \left(\int_c^t \omega \right) + \int_c^t d\omega = i_t^* \omega - i_c^* \omega, \tag{4.4}$$

$$dS_c\omega + S_c d\omega = \omega - \pi^* i_c^* \omega. \tag{4.5}$$

Here d_N stands for the exterior derivative on N and d designates the exterior derivative on $[a,b) \times N$.

The homotopy relations cannot be used automatically for $\omega \in \Omega^k_{q,p}(C^h_{a,b}N)$ because of the problem of the existence of traces on submanifolds. However, by Theorem 3.1, we can take only smooth forms in all considerations concerning both reduced and nonreduced $L_{q,p}$-cohomology.

For the reader's convenience, we repeat the classical proofs of (4.4) and (4.5).

Using the representation $\omega = \omega_A + dt \wedge \omega_B$, we have

$$d\omega = d(\omega_A(t) + dt \wedge \omega_B(t)) = dt \wedge \frac{\partial \omega_A}{\partial t}(t) + d_N\omega_A(t) - dt \wedge d_N\omega_B(t),$$

$$d_N \left(\int_c^t \omega \right) = d_N \left(\int_c^t \omega_B(\tau) d\tau \right) = \int_c^t d_N\omega_B(\tau) d\tau,$$

$$\int_c^t d\omega = \int_c^t (d\omega)_B(\tau) d\tau = \int_c^t \left(\frac{\partial \omega_A}{\partial t}(\tau) \right) d\tau - \int_c^t d_N\omega_B(\tau) d\tau.$$

Hence,

$$d_N \left(\int_c^t \omega \right) + \int_c^t d\omega = \int_c^t \left(\frac{\partial \omega_A}{\partial t}(\tau) \right) d\tau = \omega_A(t) - \omega_A(c) = i_t^*\omega - i_c^*\omega.$$

Similarly,

$$S_c\omega = \int_c^t \omega_B(\tau) d\tau$$

$$dS_c\omega = d \left(\int_c^t \omega_B(\tau) d\tau \right) = dt \wedge \omega_B(t) + \int_c^t d_N\omega_B(\tau) d\tau,$$

$$S_c d\omega = \int_c^t d\omega = \int_c^t \left(\frac{\partial \omega_A}{\partial t}(\tau) \right) d\tau - \int_c^t d_N\omega_B(\tau) d\tau.$$

Therefore,

$$\begin{aligned} dS_c\omega + S_c d\omega &= dt \wedge \omega_B(t) + \int_c^t \left(\frac{\partial \omega_A}{\partial t}(\tau)\right) d\tau \\ &= dt \wedge \omega_B(t) + \omega_A(t) - \omega_c(t) \\ &= \omega - i_c^*\omega. \end{aligned}$$

Lemma 4.2. *Suppose that g is a function locally integrable on $[a,b)$, $\int_a^b |g(t)|dt = \infty$, and $\omega \in C^\infty L_p^k(C_{a,b}^h N)$. Then every set of full measure on $[a,b)$ contains a sequence $\{t_j\}$ that converges to b and is such that $g(t_j) \neq 0$ and $\|i_{t_j}^*\omega\|_{L^p(N,\Lambda^k)} = o([f_{k,p}(t_j)]^{-1}|g(t_j)|^{1/p})$ as $j \to \infty$.*

Proof. Equality (4.3) implies:

$$\begin{aligned} \int_a^b \|i_t^*\omega\|^p_{L^p(N,\Lambda^k)} f^p_{k,p}(t)dt &\le \int_a^b \int_N |\omega_A(t)|^p \, h^{n-kp}(t,x)dxdt \\ &\le \|\omega\|^p_{L^p(C^h_{a,b}N,\Lambda^k)} \\ &< \infty, \end{aligned}$$

which yields the lemma. □

Lemma 4.3. *Suppose that $\omega \in L_p^k(C_{a,b}^h N)$, $c \in [a,b)$, $p \ge q > 1$. Then the form $\int_c^t \omega$ is defined for each $t \in [a,b)$ and belongs to $L^q(N,\Lambda^{k-1})$. Moreover, the norm of the linear operator $\int_c^t : L_p^k(C_{a,b}^h N) \to L_q^{k-1}(C_{a,b}^h N)$ satisfies the inequality*

$$\left\| \int_c^t \right\| \le |N|^{\frac{1}{q}-\frac{1}{p}} \left| \int_c^t f_{k-1,p}^{-p'}(\tau)d\tau \right|^{1/p'}.$$

The same holds for $c = b$ if

$$\int_a^b f_{k-1,p}^{-p'}(\tau)d\tau < \infty.$$

Proof. By Hölder's inequality

$$\begin{aligned}
&\left\|\int_c^t \omega\right\|_{L^q(N,\Lambda^{k-1})}\\
&\le |N|^{\frac{1}{q}-\frac{1}{p}} \left\|\int_c^t \omega\right\|_{L^p(N,\Lambda^{k-1})}\\
&= |N|^{\frac{1}{q}-\frac{1}{p}} \left(\int_N \left|\int_c^t \omega_B(\tau)d\tau\right|_N^p dx\right)^{1/p}\\
&\le |N|^{\frac{1}{q}-\frac{1}{p}} \left(\int_N \left|\int_c^t |\omega_B(\tau)|_N d\tau\right|^p dx\right)^{1/p}\\
&\le |N|^{\frac{1}{q}-\frac{1}{p}} \left[\int_N \left|\int_c^t h^{-(\frac{n}{p}-k+1)p'}(x,\tau)d\tau \int_c^t |\omega_B(\tau)|_N^p\, h^{n-kp+p}(\tau,x)d\tau\right| dx\right]^{1/p}\\
&\le |N|^{\frac{1}{q}-\frac{1}{p}} \left[\int_N \left|\int_c^t f_{k-1,p}^{-p'}(\tau,x)d\tau \int_c^t |\omega_B(\tau)|_N^p\, h^{n-kp+p}(\tau,x)d\tau\right| dx\right]^{1/p}\\
&\le |N|^{\frac{1}{q}-\frac{1}{p}} \left|\int_c^t f_{k-1,p}^{-p'}(\tau)d\tau\right| \|\omega\|_{L^p(C^h_{a,b}N,\Lambda^k)}.
\end{aligned}$$

This proves the lemma. □

5. The Main Results

5.1. Absolute reduced $L_{q,p}$-cohomology

Using the results of the previous section, we prove

Theorem 5.1. *Let N be a closed smooth n-dimensional Riemannian manifold and let $p \ge q > 1$. If*

$$I_{a,b} := \int_a^b F_{k,p}^p(t)dt = \infty; \quad J_{\delta_0,b} := \int_{\delta_0}^b f_{k,p}^p(\tau)\left(\int_a^\tau F_{k,p}^p(t)dt\right)^{-1} d\tau = \infty$$

for some $\delta_0 \in [a,b)$ then $\overline{H}^k_{q,p}(C^h_{a,b}N) = 0$.

Remark 5.2. The condition "$J_{\delta_0,b} = \infty$ *for some* $\delta_0 \in [a,b)$" is in fact equivalent to "$J_{\delta_0,b} = \infty$ *for every* $\delta_0 \in [a,b)$". In this connection, below we sometimes write J_b instead of $J_{\delta_0,b}$.

Proof. Suppose that $\omega \in L^k_p(C^h_{a,b}N)$ is weakly differentiable and $d\omega = 0$. Therefore, $\omega \in \Omega^k_{p,p}(C^h_{a,b}N)$. By Theorem 3.1, we may assume that ω is a smooth form. If $\int_{\delta_0}^b f^p_{k,p}(\tau)\left(\int_a^\tau F^p_{k,p}(t)dt\right)^{-1} d\tau = \infty$ for $\delta_0 \in (a,b)$ then the function $g(\tau) = f^p_{k,p}(\tau)\left(\int_a^\tau F^p_{k,p}(t)dt\right)^{-1}$ is not integrable on intervals of the form (c,b), $\delta_0 \le c < b$. By Lemma 4.2, there exists a sequence $\{\tau_j\} \subset (\delta_0, b) \subset (a,b)$ such that $\tau_j \to b$ and

$$\|i^*_{\tau_j}\omega\|_{L^p(N,\Lambda^k)} = o\left(\left[\int_a^{\tau_j} F^p_{k,p}(t)dt\right]^{-1/p}\right) \quad \text{as } j \to \infty.$$

Consider the form

$$\omega_j = \begin{cases} \omega & \text{on } C^h_{a,\tau_j}N, \\ \pi^* i^*_{\tau_j}\omega & \text{on } C^h_{\tau_j,b}N. \end{cases}$$

It is easy to verify that ω_j is weakly differentiable, belongs to $\Omega^k_{q,p}(C^h_{a,b}N)$, and satisfies $d\omega_j = 0$. Indeed, if $u \in \mathcal{D}^{n-k-1}(C^h_{a,b}N)$ then, denoting the restrictions of the corresponding forms to C^h_{a,τ_j} and $C^h_{\tau_j,b}N$ by the same symbols for simplicity and using (4.5), we get

$$\begin{aligned}\int_{C^h_{a,b}N} \omega_j \wedge du &= \int_{C^h_{a,\tau_j}} \omega \wedge du + \int_{C^h_{\tau_j,b}} (\omega - dS_{\tau_j}\omega) \wedge du \\ &= (-1)^{k+1}\int_{C^h_{a,\tau_j}} d\omega \wedge u + (-1)^{k+1}\int_{C^h_{\tau_j,b}} d(\omega - dS_{\tau_j}\omega) \wedge u = 0,\end{aligned}$$

which implies that $d\omega_j = 0$.

We infer

$$\|\omega_j\|^p_{L^p(C^h_{a,b}N,\Lambda^k)} \le \int_a^{\tau_j} F^p_{k,p}(t)dt\ \|i^*_{\tau_j}\omega\|^p_{L^p(N,\Lambda^k)} + \|\omega\|^p_{L^p(C^h_{\tau_j,b}N,\Lambda^k)}.$$

Therefore, $\|\omega_j\|_{L^p(C^h_{a,b}N,\Lambda^k)} \to 0$ as $j \to \infty$; moreover, the form $\omega - \omega_j$ is equal to 0 on $[\tau_j, b) \times N$. Hence, $S_{\tau_j}(\omega - \omega_j) \in \Omega_{q,p}(C^h_{a,b}N)$ and $dS_{\tau_j}(\omega - \omega_j) = \omega - \omega_j$. Thus, the cocycle ω is zero in the reduced cohomology $\overline{H}^k_{q,p}(C^h_{a,b}N)$. □

From Remark 4.1 and Theorem 5.1 we obtain

Theorem 5.3. *Let N be a closed smooth n-dimensional Riemannian manifold. If $p \geq q > 1$, $b = \infty$ and $\frac{n}{p}$ is an integer then $\overline{H}^{\frac{n}{p}}_{q,p}(C^h_{a,\infty}N) = 0$.*

In particular, if $1 < q \leq 2$ and n is even then $\overline{H}^{\frac{n}{2}}_{q,2}(C^h_{a,\infty}N) = 0$.

Proof. In this case, $I_{a,\infty} = J_{a,\infty} = \infty$, and by Theorem 5.1

$$\overline{H}^{\frac{n}{p}}_{q,p}(C^h_{a,\infty}N) = 0. \qquad \square$$

Remark 5.4. For a warped cylinder $C^h_{a,b}N$, we have $f^p_{k,p}(\tau) = F^p_{k,p}(\tau) = h^{n-kp}(\tau)$. Therefore, if

$$\int_a^b h^{n-kp}(t)dt = \infty$$

then

$$\begin{aligned} J_{\delta_0,b} &= \int_{\delta_0}^b h^{n-kp}(\tau)\left(\int_a^\tau h^{n-kp}(t)\,dt\right)^{-1} d\tau \\ &= \int_{\delta_0}^b \frac{d}{d\tau}\log\left(\int_a^\tau h^{n-kp}(t)dt\right) d\tau \\ &= \lim_{\tau \nearrow b}\left\{\log\left(\int_a^\tau h^{n-kp}(t)dt\right) - \log\left(\int_a^{\delta_0} h^{n-kp}(t)dt\right)\right\} = \infty, \end{aligned}$$

and the result of Theorem 5.1 coincides with the corresponding result in [7].

5.2. **Relative reduced $L_{q,p}$-cohomology**

Here we prove a sufficient vanishing condition for $\overline{H}^k_{q,p}(C^h_{a,b}N, N_a)$, where $N_a = \{a\} \times N$.

Theorem 5.5. *Let N be a closed smooth n-dimensional Riemannian manifold. Assume that $p \geq q > 1$,*

$$\tilde{I}_{a,b} := \int_a^b f^{-p'}_{k-1,p}(t)dt = \infty,$$

and the integral

$$A_{\delta_0,b} := \int_{\delta_0}^b \frac{F^p_{k-1,p}(\tau)}{f^{pp'}_{k-1,p}(\tau)}\left(\int_a^\tau f^{-p'}_{k-1,p}(t)dt\right)^{-1}\left|\log\left(\int_a^\tau f^{-p'}_{k-1,p}(t)\,dt\right)\right|^{-p} d\tau$$

is finite for some $\delta_0 \in [a, b)$. Then $\overline{H}^k_{q,p}(C^h_{a,b}N, N_a) = 0$.

Proof. Suppose that $\omega \in C^\infty\Omega^k_{p,p}(C^h_{a,b}N, N_a)$ and $d\omega = 0$. The form ω is the limit in $\Omega^k_{p,p}(C^h_{a,b}N)$ of a sequence ω_j of smooth forms each of which is equal to zero on some neighborhood of N_a.

As above, for $a \le c < d \le b$ denote by $C^h_{c,d}N$ the product $[c,d)\times N$ with the metric induced from $C^h_{a,b}N$. For every $e \in (a,b)$, Lemma 4.3 implies that the operator $S_a : \Omega^k_{p,p}(C^h_{a,b}N) \to \Omega^{k-1}_{q,p}(C^h_{a,e}N)$ is bounded. Hence, $S_a\omega_j \to S_a\omega$ in $\Omega^{k-1}_{q,p}(C^h_{a,e}N)$. Each of the forms $S_a\omega_j$ vanishes on some neighborhood of N_a. Therefore, $S_a\omega \in \Omega^{k-1}_{q,p}(C^h_{a,e}N, N_a)$ for all $e \in (a,b)$. Then the fact that $i_a^*\omega = 0$ and relation (4.5) give the equality $dS_a\omega = \omega$.

Consider the functions

$$I(\tau) = \int_a^\tau f_{k-1,p}^{-p'}(t)dt,$$
$$\varphi(\tau) = \log|\log I(\tau)|,$$
$$\varphi_\delta(\tau) = \begin{cases} 1 & \text{if } \tau \le \delta, \\ \min(1, \max(1+\varphi(\delta)-\varphi(\tau), 0)) & \text{if } \tau \ge \delta. \end{cases}$$

The function $\varphi(t)$ is defined for τ sufficiently close to b and $\varphi_\delta(t)$ also exists only for δ that are sufficiently close to b.

Since $d(\varphi_\delta S_a\omega) = d\varphi_\delta \wedge S_a\omega + \varphi_\delta\omega$, it follows that

$$\begin{aligned}\|d(\varphi_\delta S_a\omega) - \omega\|_{L^p(C^h_{a,b}N,\Lambda^k)} &= \|d(\varphi_\delta S_a\omega) - \omega\|_{L^p(C^h_{\delta,b}N,\Lambda^k)} \\ &\le \|d\varphi_\delta \wedge S_a\omega\|_{L^p(C^h_{\delta,b}N,\Lambda^k)} + \|(\varphi_\delta - 1)\omega\|_{L^p(C^h_{\delta,b}N,\Lambda^k)}.\end{aligned}$$

Since $|\varphi_\delta - 1| \le 1$, we have

$$\|(\varphi_\delta - 1)\omega\|_{L^p(C^h_{\delta,b}N,\Lambda^k)} \le \|\omega\|_{L^p(C^h_{\delta,b}N,\Lambda^k)}.$$

By (4.2) and Lemma 4.3 for $p = q$, for δ sufficiently close to b we infer

$$\begin{aligned}\|d\varphi_\delta \wedge S_a\omega\|^p_{L^p(C^h_{\delta,b}N,\Lambda^k)} &= \int_\delta^b h^{n-kp+p}(\tau,x)\left|\frac{d\varphi_\delta}{d\tau}\right|^p \left|\int_a^\tau \omega\right|^p_N dx\, d\tau \\ &\le \int_\delta^b F^p_{k-1,p}(\tau)\left|\frac{d\varphi_\delta}{d\tau}\right|^p \left(\int_a^\tau f_{k-1,p}^{-p'}(t)dt\right)^{\frac{p}{p'}} d\tau\, \|\omega\|^p_{L^p(N,\Lambda^k)} \\ &\le \int_\delta^b \frac{F^p_{k-1,p}(\tau)}{f^{pp'}_{k-1,p}(\tau)}\left(\int_a^\tau f_{k-1,p}^{-p'}(t)dt\right)^{-1}\left|\log\left(\int_a^\tau f_{k-1,p}^{-p'}(t)\,dt\right)\right|^{-p} d\tau\, \|\omega\|^p_{L^p(N,\Lambda^k)}.\end{aligned}$$

By hypothesis, the last quantity vanishes as $\delta \to b$. Thus, $d(\varphi_\delta S_a\omega) \to \omega$ as $\delta \to b$. The function φ_δ is equal to zero in some neighborhood of b. Hence, $\varphi_\delta S_a\omega \in \Omega^k_{q,p}(C^h_{a,b}N, N_a)$. This shows that $\overline{H}^k_{q,p}(C^h_{a,b}N, N_a) = 0$. □

Remark 5.6. The paper [7] contains the following assertion for a warped cylinder $C^h_{a,b}N$ [7, Theorem 2]:

$$\textit{If } \int_a^b h^{-(\frac{n}{p}-k+1)p'}(t)dt < \infty \quad \textit{then } \overline{H}^k_{p,p}(C^h_{a,b}N, N_a) = 0.$$

Suppose that $\int\limits_a^b h^{-(\frac{n}{p}-k+1)p'}(t)dt < \infty$. We infer

$$\begin{aligned}\int_\delta^b \frac{h^{-(\frac{n}{p}-k+1)p'}(\tau)}{\left(\int_a^\tau h^{-(\frac{n}{p}-k+1)p'}(t)dt\right)\left|\log\left(\int_a^\tau h^{-(\frac{n}{p}-k+1)p'}(t)\,dt\right)\right|^p}\,d\tau \\ = \frac{1}{p-1}\int_\delta^b \frac{d}{d\tau}\left[\log\left(\int_a^\tau h^{-(\frac{n}{p}-k+1)p'}(t)\,dt\right)\right]^{1-p} d\tau \\ = \frac{1}{p-1}\left[\log\left(\int_a^\delta h^{-(\frac{n}{p}-k+1)p'}(t)dt\right)\right]^{1-p} \to 0 \text{ as } \delta \to b.\end{aligned}$$

Thus, Theorem 5.5 generalizes this assertion to twisted cylinders $C^f_{a,b}N$ with N closed.

Remark 4.1 and Theorem 5.5 imply

Theorem 5.7. *Let N be a closed smooth n-dimensional Riemannian manifold. If $p \geq q > 1$, $b = \infty$, and $\frac{n}{p}$ is an integer then $\overline{H}^{\frac{n}{p}+1}_{q,p}(C^h_{a,b}N, N_a) = 0$.*

In particular, if $1 < q \leq 2$ and n is even then $\overline{H}^{\frac{n}{2}+1}_{q,2}(C_{a,b}N, N_a) = 0$.

Proof. In this case, $A_{\delta_0,b} = \int_{\delta_0}^b (\tau - a)^{-1} \left(\log(\tau - a)\right)^{-p} d\tau < \infty$ for any a, b and, by Theorem 5.5, $\overline{H}^k_{q,p}(C^h_{a,b}N, N_a) = 0$. □

6. **Asymptotic Twisted Cylinders**

Definition 6.1. We refer to a pair (M, X) consisting of an m-dimensional manifold M and an m-dimensional compact submanifold X with boundary as an *asymptotic twisted cylinder* $AC^h_{a,b}\partial X$ if $M \setminus X$ is bi-Lipschitz diffeomorphically equivalent to the twisted cylinder $C^h_{a,b}\partial X$.

Theorem 5.5 readily implies

Theorem 6.2. *Let $(M, X) = AC^h_{a,b}\partial X$ be an asymptotic twisted cylinder with $\dim M = \dim X = m = n + 1$. Assume that $p \geq q > 1$,*

$$\int_a^b f_{k-1,p}^{-p'}(t)dt = \infty,$$

and the integral

$$A_{\delta_0,b} := \int_{\delta_0}^b \frac{F^p_{k-1,p}(\tau)}{f^{pp'}_{k-1,p}(\tau)} \left(\int_a^\tau f_{k-1,p}^{-p'}(t)dt\right)^{-1} \left|\log\left(\int_a^\tau f_{k-1,p}^{-p'}(t)\,dt\right)\right|^{-p} d\tau$$

is finite for some $\delta_0 \in (a, b)$.

Then $\overline{H}^k_{q,p}(M, X) = 0$.

Proof. Note that bi-Lipschitz diffeomorphisms preserve L_p and L_q. Moreover, extension by zero gives a topological isomorphism between the relative spaces $W_{r,s}(C^h_{a,b}\partial X, (\partial X)_a)$ and $W_{r,s}(M, X)$ for all r, s. This gives topological isomorphisms

$$H^*_{r,s}(M, X) \cong H^*_{r,s}(C^h_{a,b}\partial X, (\partial X)_a); \;\; \overline{H}^*_{r,s}(M, X) \cong \overline{H}^*_{r,s}(C^h_{a,b}\partial X, (\partial X)_a)$$

for all r, s. The theorem now follows from Theorem 5.5. □

To obtain a version of this theorem for $\overline{H}^k_{q,p}(M)$, we will need the exact sequences of a pair for $L_{q,p}$-cohomology.

Recall that an exact sequence

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0 \tag{6.1}$$

of cochain complexes of vector spaces is called an *exact sequence of Banach complexes* if A, B, C are Banach complexes and all mappings φ^k, ψ^k are bounded linear operators.

A short exact sequence (6.1) of Banach complexes induces an exact sequence in cohomology

$$\ldots \longrightarrow H^{k-1}(C) \xrightarrow{\partial^{k-1}} H^k(A) \xrightarrow{\varphi^{*k}} H^k(B) \xrightarrow{\psi^{*k}} H^k(C) \longrightarrow \ldots$$

with all operators ∂, φ^*, ψ^* bounded and a sequence in reduced cohomology

$$\ldots \longrightarrow \overline{H}^{k-1}(C) \xrightarrow{\overline{\partial}^{k-1}} \overline{H}^k(A) \xrightarrow{\overline{\varphi}^{*k}} \overline{H}^k(B) \xrightarrow{\overline{\psi}^{*k}} \overline{H}^k(C) \longrightarrow \ldots, \tag{6.2}$$

where all operators $\overline{\partial}$, $\overline{\varphi}^*$, $\overline{\psi}^*$ are bounded and the composition of any two consecutive morphisms is equal to zero. Sequence (6.2) is in general not exact but exactness at its particular terms can be guaranteed by some additional assumptions. For example, we have (see [7, Theorem 1 (1)]):

Proposition 6.3. *Given an exact sequence* (6.1) *of Banach complexes, if* $H^k(C)$ *is separated and* $\dim \partial^{k-1}(H^{k-1}(C)) < \infty$ *then the sequence*

$$\overline{H}^{k-1}(C) \xrightarrow{\overline{\partial}^{k-1}} \overline{H}^k(A) \xrightarrow{\overline{\varphi}^{*k}} \overline{H}^k(B) \xrightarrow{\overline{\psi}^{*k}} \overline{H}^k(C)$$

is exact.

Let $(M, X) = AC^h_{a,b}\partial X$ be an asymptotic twisted cylinder.
Consider the short exact sequence of Banach complexes

$$0 \to C^\infty\Omega_{\mathbb{P}}(M, X) \xrightarrow{j} C^\infty\Omega_{\mathbb{P}}(M) \xrightarrow{i} C^\infty\Omega_{\mathbb{P}}(X) \to 0. \tag{6.3}$$

In all the three complexes of (6.3),

$$C^\infty\Omega^l_{\mathbb{P}} = \begin{cases} C^\infty\Omega^l_{q,q} & \text{if } l \le k-1; \\ C^\infty\Omega^l_{q,p} & \text{if } l = k; \\ C^\infty\Omega^l_{p,p} & \text{if } l \ge k+1; \end{cases}$$

i is the natural embedding, and j is the restriction of forms. The cohomology of $C^\infty\Omega_{\mathbb{P}}(X)$ is simply the de Rham cohomology of X.

Theorem 6.4. *Let* $(M, X) = AC^h_{a,b}\partial X$ *be an asymptotic twisted cylinder with* $\dim M = \dim X = m = n+1$. *Assume that* $p \ge q > 1$, $H^k(X) = 0$,

$$\int_a^b f^{-p'}_{k-1,p}(t)dt = \infty,$$

and the integral

$$A_{\delta_0,b} := \int_{\delta_0}^b \frac{F^p_{k-1,p}(\tau)}{f^{pp'}_{k-1,p}(\tau)} \left(\int_a^\tau f^{-p'}_{k-1,p}(t)dt\right)^{-1} \left|\log\left(\int_a^\tau f^{-p'}_{k-1,p}(t)\,dt\right)\right|^{-p} d\tau$$

is finite for some $\delta_0 \in (a, b)$.

Then $\overline{H}^k_{q,p}(M) = 0$.

Proof. Using Theorem 3.1 and Proposition 6.3, we obtain the exact sequence of reduced cohomology spaces

$$\overline{H}^k_{q,p}(M, X) \longrightarrow \overline{H}^k_{q,p}(M) \longrightarrow 0. \tag{6.4}$$

By Theorem 6.2, $\overline{H}^k_{q,p}(M,X)=0$ if $p\geq q>1$, $\int_a^b f_{k-1,p}^{-p'}(t)dt=\infty$, and $A_{\delta_0,b}<\infty$ for some $\delta_0\in(a,b)$. Hence, $\overline{H}^k_{q,p}(M)=0$. □

Theorem 5.7 and the exact sequence (6.4) readily imply

Theorem 6.5. *Let $(M,X)=AC^h_{a,b}\partial X$ be an asymptotic twisted cylinder with $\dim M=\dim X=m$. If $p\geq q>1$, $\frac{m-1}{p}$ is an integer, and $H^{\frac{m-1}{p}+1}(X)=0$ then $\overline{H}^{\frac{m-1}{p}+1}_{q,p}(M)=0$.*

In particular, if $1<q\leq 2$, m is odd, and $H^{\frac{m-1}{2}+1}(X)=0$ then $\overline{H}^{\frac{m-1}{2}+1}_{q,2}(M)=0$.

Using the duality theorem (Theorem 2.6), we now reformulate this result for cohomology with compact support (interior cohomology) in the case of $p\geq q>1$. Consider the dual exponents $p'=p/(p-1)$ and $q'=q/(q-1)$.

Theorem 6.6. *Let $(M,X)=AC^h_{a,b}\partial X$ be an asymptotic twisted cylinder with $\dim M=\dim X=m$. If $p\geq q>1$, $\frac{m-1}{q}$ is an integer, and $H^{\frac{m-1}{q'}+1}(X)=0$ then $\overline{H}^{\frac{m-1}{q}}_{q,p;0}(M)=0$.*

In particular, if $q\geq 2$, m is odd, and $H^{\frac{m-1}{2}}(X)=0$ then $\overline{H}^{\frac{m-1}{2}}_{q,2;0}(M)=0$.

Proof. We have $q'\geq p'>1$. Note that

$$\frac{m-1}{q}=(m-1)\left(1-\frac{1}{q'}\right)=m-\left(\frac{m-1}{q'}+1\right).$$

By Theorem 6.5, $H^{\frac{m-1}{q'}+1}_{p',q',0}(M)=0$ and, by the Hölder–Poincaré duality for $L_{q,p}$-cohomology (Theorem 2.6),

$$\overline{H}^{\frac{m-1}{q}}_{q,p,0}(M)=\overline{H}^{\frac{m-1}{q'}+1}_{p',q'}(M)=0.$$ □

Recall that if a manifold Y is complete then $\overline{H}^k_p(Y)=\overline{H}^k_{p;0}(Y)$ for all k (see [9]). We have:

Corollary 6.7. *Let $(M, X) = AC^h_{a,b}\partial X$ be an asymptotic twisted cylinder with $\dim M = \dim X = m$ and let M be a complete manifold.*

If $p > 1$, $\frac{m-1}{p}$ is an integer, and $H^{\frac{m-1}{p}+1}(X) = 0$ then $\overline{H}_{p'}^{\frac{m-1}{p'}}(M) = \overline{H}_{p}^{\frac{m-1}{p}+1}(M) = 0$.

In particular, if m is odd and $H^{\frac{m-1}{2}+1}(X) = 0$ then $\overline{H}_{2}^{\frac{m-1}{2}}(M) = \overline{H}_{2}^{\frac{m-1}{2}+1}(M) = 0$.

Remark 6.8. Observe that, in Theorems 6.5 and 6.6 and Corollary 6.7, for the usual de Rham cohomology, we have $H^*(X) = H^*(M)$.

Unfortunately, our methods only make it possible to make conclusions mainly about warped cylinders and not about twisted cylinders.

The examples below can be considered in a straightforward manner with the use of Theorems 5.3 and 5.7. The results are new even for L_p-cohomology ($p = q$). Similar observations also hold for asymptotic twisted cylinders.

Below the symbols C_1, C_2 stand for positive constants.

(A) Suppose that $a \geq 0$, $b = \infty$, and $C_1 e^{s_1 t} \leq h(t, x) \leq C_2 e^{s_2 t}$ ($s_2 \geq s_1 \geq 0$). Then
  (1) $\overline{H}^k_{q,p}(C^h_{a,\infty}N) = 0$ in each of the following cases:
    (1a) $p \geq q > 1$, $k = \frac{n}{p}$;
    (1b) $p \geq q > 1$, $s_1 = s_2$, $k < \frac{n}{p}$;
  (2) $\overline{H}^k_{q,p}(C^h_{a,\infty}N, N_a) = 0$ in each of the following cases:
    (2a) $p \geq q > 1$, $k = \frac{n}{p} + 1$;
    (2b) $p \geq q > 1$, $k > \frac{n}{p} + 1 + \frac{1}{pp's_1}$, $s_1 = s_2$.

(B) Suppose that $a \geq 1$, $b = \infty$, and $C_1 t^{s_1} \leq h(t, x) \leq C_2 t^{s_2}$ with $s_2 \geq s_1 \geq 0$. Then
  (1) $\overline{H}^k_{q,p}(C^h_{a,\infty}N)$ is zero in each of the following cases:
    (1a) $p \geq q > 1$, $k = \frac{n}{p}$;
    (1b) $p \geq q > 1$, $s_1 = s_2$, $k < \frac{n}{p}$;
    (1c) $p \geq q > 1$, $s_1 = s_2$, $\frac{n}{p} < k \leq \frac{n}{p} + \frac{1}{ps_1}$.
  (2) $\overline{H}^k_{q,p}(C^h_{a,\infty}N, N_a)$ is zero in each of the following cases:
    (2a) $p \geq q > 1$, $s_1 = s_2$, $\frac{n}{p} + 1 - \frac{1}{p's_1} \leq k < \frac{n}{p} + 1$;
    (2b) $p \geq q > 1$, $k = \frac{n}{p} + 1$;

Acknowledgements

The authors are grateful to the anonymous referee for valuable remarks and comments which substantially improved the exposition of the paper.

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