ANNALES MATHÉMATIQUES

BLAISE PASCAL

Zayd Hajjej

Uniform polynomial observability of time-discrete conservative linear systems

Volume 23, n° 1 (2016), p. 53-73.

<http://ambp.cedram.org/item?id=AMBP_2016__23_1_53_0>

Publication éditée par le laboratoire de mathématiques de l'université Blaise-Pascal, UMR 6620 du CNRS Clermont-Ferrand — France

Annales mathématiques Blaise Pascal **23**, 53-73 (2016)

Uniform polynomial observability of time-discrete conservative linear systems

Zayd Hajjej

Abstract

In this paper we study time semi-discrete approximations of a class of polynomially observable infinite dimensional systems. By using a method based on the resolvent estimate, we derive uniform polynomial observability inequalities within a class of solutions of the time-discrete problem in which the high frequency components have been filtered. We also present an application of our result to stabilization problems.

Observabilité polynomiale uniforme des systèmes linéaires conservatifs semi-discrets en temps

Résumé

Dans cet article nous étudions la semi-discrétisation en temps des systèmes de dimension infinie qui sont polynomialement observables. En utilisant une méthode basée sur l'estimation de la résolvante, nous obtenons des inégalités d'observabilité polynomiale uniformes pour les solutions filtrées du problème semi-discret en temps. Nous présentons également des applications de notre résultat aux problèmes de stabilisation.

1. Introduction

Let H be a Hilbert space endowed with the norm $\|.\|_H$ and let $A_0 : D(A_0) \subset H \to H$ be a self-adjoint positive operator with A_0^{-1} compact in H. For $\alpha \geq 0$, we introduce the Hilbert spaces $H_\alpha = D(A_0^\alpha)$, with the norm $\|z\|_\alpha = \|A_0^\alpha z\|_H$. The space $H_{-\alpha}$ is the dual of H_α with respect to the pivot space H. Let us define $X = H_{\frac{1}{2}} \times H,\ X_\alpha = H_{\frac{\alpha}{2}+\frac{1}{2}} \times H_{\frac{\alpha}{2}}$.

We consider the following abstract system:

$$\ddot{w} + A_0 w(t) = 0,\ \ w(0) = w_0,\ \dot{w}(0) = w_1. \tag{1.1}$$

Keywords: Observability inequality, Time discretization, Filtering.
Math. classification: 93B07, 93C55, 65M06.

Here and henceforth, a dot $(\cdot)$ denotes differentiation with respect to the time t. Such systems are often used as models of vibrating systems (e.g., the wave equation).

Assume that U is another Hilbert space equipped with the norm $\|.\|_U$. We identify U with its dual. We denote by $\mathfrak{L}(U,H)$ the space of bounded linear operators from U to H, endowed with the classical operator norm. For all $(w_0,w_1)\in X_1$, the initial value problem (1.1) admits a unique solution satisfying

$$w\in C([0,+\infty[;H_1)\cap C^1([0,+\infty[;H_{\frac{1}{2}})\cap C^2([0,+\infty[;H).$$

Let $B_0\in\mathfrak{L}(U,H_{-\frac{1}{2}})$ be an observation operator and define the output function

$$\phi(t)=B_0^*\dot{w}(t). \tag{1.2}$$

In order to give a sense to (1.2), we make the assumption that B_0^* is an admissible observation operator in the following sense (see [18]).

Definition 1.1. The operator B_0^* is an admissible observation operator for systems (1.1)-(1.2) if for every $T>0$ there exists a constant $k_T>0$ such that

$$\int_0^T\|\phi(t)\|_U^2\,dt\le k_T\,\|(w_0,w_1)\|_X^2\,,\quad\forall(w_0,w_1)\in X_1. \tag{1.3}$$

Note that if B_0 is bounded in H, i.e., if it can be extended such that $B_0\in\mathfrak{L}(U,H)$, then B_0 is obviously an admissible observation operator. However, in applications, this is often not the case, and the admissibility condition is then a consequence of a suitable "hidden regularity" property (see [12]) of the solutions of the evolution equation (1.1).

The polynomial observability property can be formulated as follows:

Definition 1.2. System (1.1)-(1.2) is polynomially observable in time T if there exists $C_T>0$ such that

$$\int_0^T\|\phi(t)\|_U^2\,dt\ge C_T\,\|(w_0,w_1)\|_{X_{-\alpha}}^2\,,\ \forall(w_0,w_1)\in X_1, \tag{1.4}$$

where $\alpha>0$.

Moreover, (1.1)-(1.2) is said to be polynomially observable if it is polynomially observable in some time $T>0$.

Let us denote by $(\lambda_j)_{j\in\mathbb{N}^*}$ the increasing sequence formed by the eigenvalues of $A_0^{\frac{1}{2}}$ and by $(\varphi_j)_{j\in\mathbb{N}^*}$ a corresponding sequence of eigenvectors, forming an orthonormal basis of H.

If we set $z(t) = \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix}$, then the problem (1.1) and (1.2) may be rewritten as

$$\begin{cases} \dot{z}(t) = Az(t), \\ z(0) = z_0 = (w_0, w_1), \end{cases} \tag{1.5}$$

and

$$y(t) = B^* z(t). \tag{1.6}$$

where $A : X_1 \subset X \to X$ and $B : U \to X_{-1}$ are defined by

$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B_0 \end{pmatrix}, \quad B^* = (0 \ B_0^*).$$

We note that A is skew-adjoint on X with compact resolvent (see [17]) and $B^* \in \mathfrak{L}(X_1, U)$, i. e., there exists a constant C_B such that

$$\|B^* z\|_U \leq C_B \|Az\|_X, \ \ \forall z \in X_1.$$

As the exact observability, the polynomial observability holds if a "weakened" Hautus test holds. More precisely, we have the following result (see Appendix for the proof).

Proposition 1.3. *The system* (1.1)-(1.2) *is polynomially observable if there exist constants* $\beta, \alpha > 0$ *such that*

$$\|(i\omega I - A)z\|^2_{X_{-\alpha}} + \|B^* z\|^2_U \geq \beta \, \|z\|^2_{X_{-\alpha}} \quad \forall \, z \in X_1, \, \forall \, \omega \in \mathbb{R}. \tag{1.7}$$

Our goal in this paper is to prove uniform polynomial observability inequalities for time-discrete systems as a direct consequence of those corresponding to the time-continuous ones.

Let us first present a natural discretization of the continuous system. For any $\Delta t > 0$, we denote by z^k and y^k respectively the approximations of the solution z and the output function y of system (1.1)-(1.2) at time $t_k = k\Delta t$ for $k \in \mathbb{N}$. Consider the following implicit midpoint time discretization of system (1.5):

$$\begin{cases} \frac{z^{k+1} - z^k}{\Delta t} = A\left(\frac{z^k + z^{k+1}}{2}\right), \quad k \in \mathbb{N}, \\ z^0 = z_0. \end{cases} \tag{1.8}$$

The output function of (1.8) is given by

$$y^k = B^* z^k, \quad k \in \mathbb{N}. \tag{1.9}$$

Note that (1.8)-(1.9) is a discrete version of (1.5)-(1.6).

Taking into account that A is skew-adjoint, it is easy to show that $\|z^k\|_X$ is conserved in the discrete time variable $k \in \mathbb{N}$, i.e., $\|z^k\|_X = \|z^0\|_X$. Consequently the scheme under consideration is stable and its convergence (in the classical sense of numerical analysis) is guaranteed in an appropriate functional setting.

The uniform polynomial observability problem for system (1.8) is formulated as follows: To find a positive constant c_T, independent of Δt, such that the solutions z^k of system (1.8) satisfy:

$$c_T \left\|z^0\right\|^2_{X_{-\alpha}} \leq \Delta t \sum_{k=0}^{\lfloor \frac{T}{\Delta t} \rfloor} \|y^k\|^2_U, \tag{1.10}$$

for all initial data z^0 in an appropriate class. We recall that y^k is given by (1.9) and z^k is the solution of (1.8).

Remark that (1.10) is a discrete version of (1.4).

Note that this type of observability inequalities appears naturally when dealing with stabilization problems. For numerical approximation processes, it is important that these inequalities hold uniformly with respect to the discretization parameter Δt to recover uniform stabilization properties.

The numerical approximation of observability has been intensively studied in the literature (see, for instance, [7] and the references therein). It is by now well-known that discretization processes may create high frequency spurious solutions which might lead to non-uniform observability properties. Several remedies have been proposed to overcome this difficulty: Tychonoff regularization in [9] and filtering of high frequencies in [11]. We refer to the paper [8] for more details and extensive references. For stability results this phenomenon was underlined, for example, in [15] where a viscous finite-difference space semi-discretization of a damped wave equation has been studied. Let us mention the works [6, 5] based on properties of the continuous system where convergent variational approximations of an exact control are build. For fully discrete approximations schemes, we mention the work [14] , where the uniform controllability of a fully discrete approximation scheme of the 1d wave equation is analyzed, and also the recent work [7], where exact observability issues were discussed for abstract models. Let us also cite the paper [8], where exponential stabilization properties were studied. For time semi-discrete approximations of polynomial observability, the only work we are aware of is [10], and the present work that seems to be the first one deriving uniform polynomial observability in the case of unbounded observation. We note that this issue is still open in the space semi-discrete case.

In the sequel, we are interested in understanding under which assumptions inequality (1.10) holds uniformly with respect to Δt. One expects to do it so that, when letting $\Delta t \to 0$, one recovers the observability property of the continuous model.

Though our paper is inspired from [7]. But here, we only need a weaker version of observability in which the observed norm is weaker than $\|.\|_X$.

We first need to introduce some notations.

Notation. Since A is skew-adjoint with compact resolvent, its spectrum is discrete and $\sigma(A) = \{i\mu_j : j \in \mathbb{Z}^*\}$, where

$$\mu_j = \begin{cases} \lambda_j & \text{if } j \in \mathbb{N}^*, \\ -\lambda_{-j} & \text{if } (-j) \in \mathbb{N}^*. \end{cases}$$

If we set $\varphi_{-j} = \varphi_j$, for all $j \in \mathbb{N}^*$, then an orthonormal family of eigenvectors $(\Phi_j)_{j\in\mathbb{Z}^*}$ of A is given by

$$\Phi_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{i\mu_j}\varphi_j \\ \varphi_j \end{pmatrix} \quad \forall\, j \in \mathbb{Z}^*.$$

Moreover, we define

$$\mathcal{C}_s = \text{span}\{\Phi_j : \text{ the corresponding } i\mu_j \text{ satisfies } |\mu_j| \le s\}. \tag{1.11}$$

The main result of this paper reads as follows:

Theorem 1.4. *Assume that (A, B) satisfy* (1.7) *and* $B^* \in \mathfrak{L}(X_1, U)$. *Then, for any* $\delta > 0$, *there exists a time* T_δ *such that for any* $T > T_\delta$, *there exists a positive constant* $C = C_{T,\delta}$, *independent of* Δt, *such that for* Δt *small enough, the solution* z^k *of* (1.8) *satisfies*

$$C_{T,\delta}\|z^0\|^2_{X_{-\alpha}} \le \Delta t \sum_{k\in(0,T/\Delta t)} \|B^* z^k\|^2_U, \quad \forall z^0 \in \mathcal{C}_{\delta/\Delta t}. \tag{1.12}$$

The outline of this paper is as follows.

In Section 2, we prove Theorem 1.4. Our proof is mainly based on the resolvent estimate (1.7). In Section 3 we present some consequences of Theorem 1.4 to stabilization of time semi-discrete damped models. We end the paper by giving some applications of our main result.

2. Proof of Theorem 1.4

Before getting into the proof of Theorem 1.4, we recall some properties of the discrete Fourier transform at scale Δt (see [16]), that will be used in the sequel.

Definition 2.1. Given any sequence $(u^k) \in l^2(\Delta t\mathbb{Z})$, we define its Fourier transform as :

$$\hat{u}(\tau) = \Delta t \sum_{k\in\mathbb{Z}} u^k \exp(-i\tau k\Delta t), \qquad \tau\Delta t \in [-\pi, \pi] .$$

For any function $v \in L^2((-\pi/\Delta t, \pi/\Delta t))$, we define the inverse Fourier transform at scale $\Delta t > 0$:

$$\tilde{v}^k = \frac{1}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} v(\tau)\exp(i\tau k\Delta t)d\tau, \quad k \in \mathbb{Z}.$$

According to this definition,

$$\tilde{\hat{u}} = u, \qquad \hat{\tilde{v}} = v,$$

and the Parseval identity holds

$$\frac{1}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} |\hat{u}|^2 d\tau = \Delta t \sum_{k\in\mathbb{Z}} |u^k|^2. \tag{2.1}$$

Proof of Theorem 1.4. Taking the initial data $z_0 = \Phi_j$, then simple formal calculations give

$$\begin{aligned} z^{k+1} &= (I - \frac{\Delta t}{2}A)^{-1}(I + \frac{\Delta t}{2}A)z^k \\ &= e^{i\alpha_j\Delta t} z^k, \end{aligned}$$

where $e^{i\alpha_j\Delta t} = \frac{1+\frac{\Delta t}{2}i\mu_j}{1-\frac{\Delta t}{2}i\mu_j}$.

We get

$$\begin{cases} \cos(\alpha_j \Delta t) = \dfrac{1 - \frac{(\Delta t)^2}{4}\mu_j^2}{1 + \frac{(\Delta t)^2}{4}\mu_j^2}, \\ \sin(\alpha_j \Delta t) = \dfrac{\Delta t \mu_j}{1 + \frac{(\Delta t)^2}{4}\mu_j^2}. \end{cases} \tag{2.2}$$

Then

$$\tan(\alpha_j \Delta t) = \frac{\Delta t \mu_j}{1 - \frac{(\Delta t)^2}{4}\mu_j^2}.$$

By (2.2), we have $\alpha_j \Delta t \in]0, \frac{\pi}{2}[$.
Consequently

$$\begin{aligned} \alpha_j &= \frac{1}{\Delta t} \arctan\left(\frac{\Delta t \mu_j}{1 - \frac{(\Delta t)^2}{4}\mu_j^2} \right) \\ &= \frac{2}{\Delta t} \arctan(\frac{\mu_j \Delta t}{2}). \end{aligned}$$

Now, expand $z_0 \in \mathcal{C}_{\delta/\Delta t}$ as $z_0 = \sum_{|\mu_j| \leq \delta/\Delta t} a_j \Phi_j$. We explicitly compute the solution z^k as

$$z^k = \sum_{|\mu_j| \leq \delta/\Delta t} a_j e^{i\alpha_j k \Delta t} \Phi_j.$$

We have, for any $k \in \mathbb{N}$

$$\|z^0\|^2_{X_{-\alpha}} = \|z^k\|^2_{X_{-\alpha}} = \sum_{|\mu_j| \leq \delta/\Delta t} |a_j|^2 \mu_j^{-2\alpha}, \tag{2.3}$$

and

$$\frac{z^k + z^{k+1}}{2} = (I - \frac{\Delta t}{2} A)^{-1} z^k.$$

By using (2.3), we obtain that for any k,

$$\left\| \frac{z^0 + z^1}{2} \right\|^2_{X_{-\alpha}} = \left\| \frac{z^k + z^{k+1}}{2} \right\|^2_{X_{-\alpha}} = \sum_{|\mu_j| \leq \delta/\Delta t} \frac{|a_j|^2}{1 + \frac{(\Delta t)^2}{4}\mu_j^2} \mu_j^{-2\alpha}. \tag{2.4}$$

Since $|\mu_j| \leq \delta/\Delta t$, the last equality gives

$$\left\| \frac{z^0 + z^1}{2} \right\|^2_{X_{-\alpha}} \geq \frac{1}{1 + (\frac{\delta}{2})^2} \|z^0\|^2_{X_{-\alpha}}. \tag{2.5}$$

The proof of Theorem 1.4 is based on the following lemmas.

Lemma 2.2. *Set* $\chi \in H^1(\mathbb{R})$ *and* $\chi^k = \chi(k\Delta t)$*. Then, the solution* z^k *of* (1.8) *satisfies*

$$\begin{aligned}
&2\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}+\chi^k}{2}\right)^2\left\|B^*\left(\frac{z^k+z^{k+1}}{2}\right)\right\|_U^2\\
&\geq \left\|\frac{z^0+z^1}{2}\right\|_{X_{-\alpha}}^2\left[\frac{1}{2}\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}+\chi^k}{2}\right)^2 - a_2\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2\right]\\
&\qquad -\frac{(\Delta t)^4}{8}\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2\left\|B^*\left(\frac{z^{k+1}-z^k}{\Delta t}\right)\right\|_U^2
\end{aligned} \tag{2.6}$$

where

$$a_2 = \left(1+\frac{\delta^2}{4}\right)^2 + \frac{\delta^2}{16}(\Delta t)^2.$$

Lemma 2.3. *Let* $\chi(t) = \varphi(t/T)$ *with* $\varphi \in H^2(0,1)\cap H_0^1(0,1)$*, extended by zero outside* $(0,1)$*. The following estimates hold:*

$$\left|\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}+\chi^k}{2}\right)^2 - T\|\varphi\|_{L^2(0,1)}^2\right| \leq 2T\Delta t\|\varphi\|_{L^2(0,1)}\|\dot{\varphi}\|_{L^2(0,1)},$$

$$\left|\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2 - \frac{1}{T}\|\dot{\varphi}\|_{L^2(0,1)}^2\right| \leq \frac{2}{T}\Delta t\|\dot{\varphi}\|_{L^2(0,1)}\|\ddot{\varphi}\|_{L^2(0,1)}.$$

In this article, we give only the proof of Lemma 2.2. For the proof of Lemma 2.3, see (2.22) in [7].

Proof of Lemma 2.2. Let $g^k = \chi^k z^k$, and

$$f^k = \frac{g^{k+1}-g^k}{\Delta t} - A\left(\frac{g^{k+1}+g^k}{2}\right). \tag{2.7}$$

It is easy to check that (see (2.10) in [7])

$$f^k = \left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)\left(I-\frac{(\Delta t)^2}{4}A^2\right)\left(\frac{z^k+z^{k+1}}{2}\right) \tag{2.8}$$

and

$$\hat{f}(\tau) = \Big(i\frac{2}{\Delta t}\tan\Big(\frac{\tau\Delta t}{2}\Big)I - A\Big)\hat{g}(\tau)\exp\Big(i\frac{\tau\Delta t}{2}\Big)\cos\Big(\frac{\tau\Delta t}{2}\Big). \tag{2.9}$$

Using (2.4) and (2.8), we get

$$\|f^k\|^2_{X_{-\alpha}} \leq \left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2 \left\|\frac{z^0+z^1}{2}\right\|^2_{X_{-\alpha}} \left(1+\frac{\delta^2}{4}\right)^2. \tag{2.10}$$

Let

$$G(\tau) = \hat{g}(\tau)\exp(i\frac{\tau\Delta t}{2})\cos(\frac{\tau\Delta t}{2}).$$

It is obvious that $G(\tau) \in C_{\delta/\Delta t}$. In view of (2.9), applying the resolvent estimate (1.7) to $G(\tau)$, integrating on τ from $-\pi/\Delta t$ to $\pi/\Delta t$, it holds

$$\int_{-\pi/\Delta t}^{\pi/\Delta t} \|\hat{f}(\tau)\|^2_{X_{-\alpha}} d\tau + \int_{-\pi/\Delta t}^{\pi/\Delta t} \|B^*G(\tau)\|^2_U d\tau \geq \int_{-\pi/\Delta t}^{\pi/\Delta t} \|G(\tau)\|^2_{X_{-\alpha}}. \tag{2.11}$$

We note that

$$\tilde{G}^k = \frac{g^k+g^{k+1}}{2}, \quad \text{i.e. } G(\tau) = \left(\widehat{\frac{g^k+g^{k+1}}{2}}\right)(\tau).$$

By applying Parseval's identity (2.1) to (2.11), we obtain

$$\Delta t\sum_{k\in\mathbb{Z}}\|f^k\|^2_{X_{-\alpha}} + \Delta t\sum_{k\in\mathbb{Z}}\left\|B^*\left(\frac{g^k+g^{k+1}}{2}\right)\right\|^2_U \geq \Delta t\sum_{k\in\mathbb{Z}}\left\|\frac{g^k+g^{k+1}}{2}\right\|^2_{X_{-\alpha}}. \tag{2.12}$$

Now we estimate the terms in (2.12).

— *Estimation of* $\Delta t\sum_{k\in\mathbb{Z}}\|f^k\|^2_{X_{-\alpha}}$. In view of (2.10), we obtain

$$\Delta t\sum_{k\in\mathbb{Z}}\|f^k\|^2_{X_{-\alpha}} \leq \left(1+\frac{\delta^2}{4}\right)^2\left\|\frac{z^0+z^1}{2}\right\|^2_{X_{-\alpha}} \Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2. \tag{2.13}$$

— *Estimation of* $\Delta t \sum_{k\in\mathbb{Z}} \left\| B^*\left(\frac{g^k+g^{k+1}}{2}\right)\right\|_U^2$. Since

$$\frac{g^k+g^{k+1}}{2} = \frac{\chi^k+\chi^{k+1}}{2}\frac{z^k+z^{k+1}}{2} + \frac{\Delta t}{2}\frac{\chi^{k+1}-\chi^k}{\Delta t}\frac{z^{k+1}-z^k}{2}, \qquad (2.14)$$

using

$$\|a+b\|^2 \leq 2\left(\|a\|^2+\|b\|^2\right),$$

we deduce that

$$\left\|B^*\left(\frac{g^k+g^{k+1}}{2}\right)\right\|_U^2 \leq 2\left(\frac{\chi^k+\chi^{k+1}}{2}\right)^2 \left\|B^*\left(\frac{z^k+z^{k+1}}{2}\right)\right\|_U^2 + 2\frac{(\Delta t)^4}{16}\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2 \left\|B^*\left(\frac{z^{k+1}-z^k}{\Delta t}\right)\right\|_U^2,$$

and then

$$\Delta t\sum_{k\in\mathbb{Z}}\left\|B^*\left(\frac{g^k+g^{k+1}}{2}\right)\right\|_U^2 \leq 2\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^k+\chi^{k+1}}{2}\right)^2\left\|B^*\left(\frac{z^k+z^{k+1}}{2}\right)\right\|_U^2 + \frac{(\Delta t)^5}{8}\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2\left\|B^*\left(\frac{z^{k+1}-z^k}{\Delta t}\right)\right\|_U^2. \qquad (2.15)$$

— *Estimation of* $\Delta t \sum_{k\in\mathbb{Z}} \left\|\frac{g^k+g^{k+1}}{2}\right\|_{X_{-\alpha}}^2$. In view of (2.4) and (2.14), we get

$$\left\|\frac{g^k+g^{k+1}}{2}\right\|_{X_{-\alpha}}^2 \geq \frac{1}{2}\left(\frac{\chi^k+\chi^{k+1}}{2}\right)^2\left\|\frac{z^k+z^{k+1}}{2}\right\|_{X_{-\alpha}}^2 - \left(\frac{\Delta t}{2}\right)^2\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2\left\|\frac{z^{k+1}-z^k}{2}\right\|_{X_{-\alpha}}^2$$

$$\geq \frac{1}{2}\left(\frac{\chi^k+\chi^{k+1}}{2}\right)^2 \left\|\frac{z^0+z^1}{2}\right\|^2_{X_{-\alpha}}$$
$$-\left(\frac{\Delta t}{2}\right)^4\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2 \left\|A\left(\frac{z^{k+1}+z^k}{2}\right)\right\|^2_{X_{-\alpha}}$$
$$\geq \frac{1}{2}\left(\frac{\chi^k+\chi^{k+1}}{2}\right)^2 \left\|\frac{z^0+z^1}{2}\right\|^2_{X_{-\alpha}}$$
$$-\left(\frac{\delta(\Delta t)}{4}\right)^2\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2 \left\|\frac{z^0+z^1}{2}\right\|^2_{X_{-\alpha}}, \quad (2.16)$$

where we used

$$\|a+b\|^2 \geq \frac{1}{2}\|a\|^2-\|b\|^2,$$

and

$$\left\|A\left(\frac{z^{k+1}+z^k}{2}\right)\right\|^2_{X_{-\alpha}} \leq \left(\frac{\delta}{\Delta t}\right)^2 \left\|\frac{z^0+z^1}{2}\right\|^2_{X_{-\alpha}}.$$

Using (2.16), we get

$$\Delta t\sum_{k\in\mathbb{Z}}\left\|\frac{g^k+g^{k+1}}{2}\right\|^2_{X_{-\alpha}}$$
$$\geq \frac{\Delta t}{2}\left\|\frac{z^0+z^1}{2}\right\|^2_{X_{-\alpha}}\sum_{k\in\mathbb{Z}}\left(\frac{\chi^k+\chi^{k+1}}{2}\right)^2$$
$$-\left(\frac{\delta^2(\Delta t)^3}{16}\right)\left\|\frac{z^0+z^1}{2}\right\|^2_{X_{-\alpha}}\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2. \quad (2.17)$$

Applying (2.13), (2.15) and (2.17) to (2.12), we end the proof of Lemma 2.2. ☐

Lemma 2.3 shows that the coefficient of $\|(z^0+z^1)/2\|^2_{X_{-\alpha}}$ in (2.6) tends to

$$K_{T,\delta,\varphi} = \frac{1}{4}T\|\varphi\|^2_{L^2(0,1)} - \frac{1}{2}\left((1+\frac{\delta^2}{4})^2+2C_B^2\frac{\delta^4}{16}\right)\frac{1}{T}\|\dot{\varphi}\|^2_{L^2(0,1)},$$

when $\Delta t\to 0$.

Note that $K_{T,\delta,\varphi}$ is an increasing function of T tending to $-\infty$ when $T\to 0^+$ and to $+\infty$ when $T\to+\infty$. Let $T_{\delta,\varphi}$ be the unique positive solution of $K_{T,\delta,\varphi} = 0$. Then, for any time $T > T_{\delta,\varphi}$, choosing a positive $K_{T,\delta}$ such that

$$0 < K_{T,\delta} < K_{T,\delta,\varphi},$$

there exists $\Delta t_0 > 0$ such that for any $\Delta t < \Delta t_0$ the following holds:

$$K_{T,\delta}\Big\|\frac{z^0+z^1}{2}\Big\|^2_{X_{-\alpha}} \leq \left\|\frac{z^0+z^1}{2}\right\|^2_{X_{-\alpha}} \left(\frac{1}{2}\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}+\chi^k}{2}\right)^2 - a_2\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2\right). \quad (2.18)$$

Besides, using again Lemma 2.3, we have

$$\begin{aligned}
&(\Delta t)^4\Delta t\sum_{k\in\mathbb{Z}}\left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2 \Big\|B^*\Big(\frac{z^{k+1}-z^k}{\Delta t}\Big)\Big\|^2_U \\
&\leq (\Delta t)^4 C_B^2\Big(\frac{1}{T}\|\dot\varphi\|^2_{L^2(0,1)}+\frac{2}{T}\Delta t\|\dot\varphi\|_{L^2(0,1)}\|\ddot\varphi\|_{L^2(0,1)}\Big)\Big\|A^2\Big(\frac{z^{k+1}+z^k}{2}\Big)\Big\|^2_X \\
&\leq C_B^2\delta^4\Big(\frac{1}{T}\|\dot\varphi\|^2_{L^2(0,1)}+\frac{2}{T}\Delta t\|\dot\varphi\|_{L^2(0,1)}\|\ddot\varphi\|_{L^2(0,1)}\Big)\Big\|\frac{z^0+z^1}{2}\Big\|^2_X \\
&\leq C_B^2\delta^4\Big(\frac{1}{T}\|\dot\varphi\|^2_{L^2(0,1)}+\frac{2}{T}\Delta t\|\dot\varphi\|_{L^2(0,1)}\|\ddot\varphi\|_{L^2(0,1)}\Big)\Big\|z^0\Big\|^2_X.
\end{aligned}$$

This last term tends to

$$H_{T,\delta,\varphi} = \frac{C_B^2\delta^4\Big\|z^0\Big\|^2_X\|\dot\varphi\|^2_{L^2(0,1)}}{T}$$

as $\Delta t \to 0$.

Since $\lim\limits_{T\to+\infty} H_{T,\delta,\phi} = 0$, then there exists a positive constant M such that

$$T > M \Longrightarrow 0 < H_{T,\delta,\phi} < c\|z^0\|^2_{X_{-\alpha}},$$

where c is a very small positive constant.

Consequently, there exists $\Delta t_1 > 0$ such that for any $\Delta t < \Delta t_1$, we get

$$C_B^2\delta^4\left(\frac{1}{T}\|\dot\varphi\|^2_{L^2(0,1)}+\frac{2}{T}\Delta t\|\dot\varphi\|_{L^2(0,1)}\|\ddot\varphi\|_{L^2(0,1)}\right)\Big\|z^0\Big\|^2_X \leq c\|z^0\|^2_{X_{-\alpha}},$$

which implies

$$(\Delta t)^4 \Delta t \sum_{k\in\mathbb{Z}} \left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2 \left\| B^* \left(\frac{z^{k+1}-z^k}{\Delta t}\right) \right\|_U^2 \le c\|z^0\|_{X_{-\alpha}}^2, \tag{2.19}$$

for any $\Delta t < \Delta t_1$.

This inequality combined with (2.18) give

$$\begin{aligned} \left\|\frac{z^0+z^1}{2}\right\|_{X_{-\alpha}}^2 & \left[\frac{1}{2}\Delta t \sum_{k\in\mathbb{Z}} \left(\frac{\chi^{k+1}+\chi^k}{2}\right)^2 - a_2 \Delta t \sum_{k\in\mathbb{Z}} \left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2\right] \\ & - \frac{(\Delta t)^4}{8} \Delta t \sum_{k\in\mathbb{Z}} \left(\frac{\chi^{k+1}-\chi^k}{\Delta t}\right)^2 \left\| B^* \left(\frac{z^{k+1}-z^k}{\Delta t}\right) \right\|_U^2 \\ & \ge K_{T,\delta} \left\|\frac{z^0+z^1}{2}\right\|_{X_{-\alpha}}^2 - \frac{c}{8}\|z^0\|_{X_{-\alpha}}^2. \end{aligned}$$

Using (2.5) and Lemma 2.2, we get the existence of a positive constant $C_{T,\delta}$ such that

$$C_{T,\delta}\|z^0\|_{X_{-\alpha}}^2 \le \Delta t \sum_{k\in(0,T/\Delta t)} \left\| B^* \left(\frac{z^{k+1}+z^k}{2}\right) \right\|_U^2,$$

which yileds (1.12). □

Remark 2.4. Theorem 1.4 improves the result in [10] where the same result was stated but for some class of operators A. More precisely the uniform polynomial observability holds for time discrete systems, when there exists a positive constant $\gamma > 0$ such that the eigenvalues $(\mu_j)_j$ satisfy the gap condition: $\mu_{j+N} - \mu_j \ge N\gamma$ for $N = 1$ or $N = 2$.

3. Stabilization properties

This section is mainly based on the article [10], in which stabilization properties are derived for abstract linear damped systems. Below, we assume that A_0 is self-adjoint, positive operator with A_0^{-1} compact, and that $B_0 \in \mathfrak{L}(U, H)$.

3.1. **The continuous setting**

Consider the following damped wave type equations:

$$\ddot{w}(t)+A_0w(t)+B_0B_0^*\dot{w}(t)=0,\ t\geq 0\ \ (w(0),\dot{w}(0))=(w_0,w_1)\in X. \quad (3.1)$$

The energy of solutions of (3.1) is defined by

$$E(t)=\frac{1}{2}\left\{\|\dot{w}(t)\|_H^2+\|A_0^{\frac{1}{2}}w(t)\|_H^2\right\} \quad (3.2)$$

satisfies the dissipation law

$$\frac{dE}{dt}(t)=-\|B_0^*\dot{w}(t)\|_U^2, \qquad \forall t\geq 0. \quad (3.3)$$

System (3.1) is said to be polynomially stable if there exists positive constants C and γ such that for all $t>0$ and for all $(w_0,w_1)\in X_1$ we have

$$E(t)\leq \frac{C}{t^\gamma}\|(w_0,w_1)\|_{X_1}^2. \quad (3.4)$$

It is by now well-known (see [3]) that this property holds if the observability inequality (1.4) holds for solutions of (1.1), or if (1.7) is satisfied.

3.2. **The time semi-discrete setting**

We now assume that system (1.1)-(1.2) is observable in the sense of (1.4), or that (1.7) holds for some $\alpha,\beta>0$. Then, combining Theorem 1.4 and the results in [10], we get:

Theorem 3.1. *Let B_0 be a bounded operator in $\mathfrak{L}(U,H)$, and assume that (1.7) is satisfied. Then the time semi-discrete systems*

$$\begin{cases} \frac{z^{k+1}-z^k}{\Delta t}=A\left(\frac{z^k+\tilde{z}^{k+1}}{2}\right)-BB^*\left(\frac{z^k+\tilde{z}^{k+1}}{2}\right), & k\in\mathbb{N},\\ \frac{z^{k+1}-\tilde{z}^{k+1}}{\Delta t}=(\Delta t)^2A^2z^{k+1}, & k\in\mathbb{N},\\ z^0=z_0=(w_0,w_1), \end{cases}$$

are polynomially stable, uniformly with respect to the time discretization parameter $\Delta t>0$: there exist two positive constants C_1 and γ_1 independent of $\Delta t>0$ such that for any $\Delta t>0$ we have

$$\|z^k\|_X^2\leq \frac{C_1}{(1+t_k)^{\gamma_1}}\|z^0\|_{X_1}^2,\ \ \forall\ z^0\in X_1,\ \ \forall\ k\geq 0.$$

Notice that z, A and B are the same defined in the introduction. We refer to [10] and the references therein for more precise statements.

4. **Applications**

4.1. **A 2-D wave equation in a square**

Consider the square $\Omega = (0,\pi) \times (0,\pi)$ and let $\Gamma_0 = \{0\} \times (0,\pi)$. We consider the following initial and boundary value problem:

$$\begin{cases} \ddot{w} - \Delta w = 0, & x \in \Omega,\ t > 0, \\ w = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

with the output

$$\phi(t) = \frac{\partial [G\dot{w}]}{\partial \nu} \,|_{\Gamma_0}, \tag{4.2}$$

where $G = (-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$.

System (4.1)-(4.2) can be written in form (1.5)-(1.6) if we introduce the following notation:

$$X = L^2(\Omega) \times H^{-1}(\Omega),\ \ X_1 = D(A) = H_0^1(\Omega) \times L^2(\Omega),\ \ U = L^2(\Gamma_0),$$
$$Az = A(w,v)^T = (v, \Delta w)^T,\ \ \forall\, (w,v) \in D(A),$$
$$B^*(u,v)^T = \frac{\partial [Gv]}{\partial \nu} \,|_{\Gamma_0},\ \ \forall\, (u,v) \in D(A).$$

One can easily check that, with the above choice of the spaces and operators, we have that A is skew-adjoint with compact resolvent. Moreover the operator B^* is admissible (see [2]).

It is well known that this system is not exactly observable since the "geometric optics" condition is violated [4]. Now, we verify (1.7) for $\alpha = 1$.

It is easy to see that the normalized eigenvectors of A

$$\Phi_{m,l}(x_1,x_2) = \left(\frac{\sqrt{2}}{\pi} \sin(mx_1)\sin(lx_2), i\frac{\sqrt{2}}{\pi}\sqrt{m^2+l^2}\sin(mx_1)\sin(lx_2) \right)$$

for $m, l = 1, 2, ...$, form an orthonormal basis for X. The corresponding eigenvalue for $\Phi_{m,l}(x_1,x_2)$ is

$$\mu_{l,m} = iw_{l,m} = i\sqrt{m^2+l^2}.$$

Taking $z = \Phi_{m,l}(x_1, x_2)$ for fixed m and l. A simple calculations show that

$$\|(iwI - A)z\|^2_{H^{-1}(\Omega)\times(L^2(\Omega))'} = \frac{(w - w_{l,m})^2}{w^2_{l,m}},$$

$$\|B^* z\|^2_{L^2(\Gamma_0)} = \frac{m^2}{\pi(m^2 + l^2)},$$

and

$$\|z\|^2_{H^{-1}(\Omega)\times(L^2(\Omega))'} = \frac{1}{w^2_{l,m}}.$$

Notice that $(L^2(\Omega))'$ is the dual of $L^2(\Omega)$ with respect to the pivot space $H^{-1}(\Omega)$. It is clear that (1.7) holds with $\beta = \frac{1}{\pi}$.

Now, we expand z as

$$z = \sum_{m,l\geq 1} a_{m,l}\Phi_{m,l}(x_1, x_2).$$

Due to the orthogonality of the families $(\sin(mx_1))_{m\geq 1}$ and $(\sin(lx_2))_{l\geq 1}$ in $L^2(0, \pi)$, we get

$$\|(iwI - A)z\|^2_{H^{-1}(\Omega)\times(L^2(\Omega))'} = \sum_{m,l\geq 1} |a_{m,l}|^2 \frac{(w - w_{l,m})^2}{w^2_{l,m}},$$

$$\|B^* z\|^2_{L^2(\Gamma_0)} = \frac{1}{\pi}\sum_{m,l\geq 1} |a_{m,l}|^2 \frac{m^2}{m^2 + l^2},$$

and

$$\|z\|^2_{H^{-1}(\Omega)\times(L^2(\Omega))'} = \sum_{m,l\geq 1} \frac{|a_{m,l}|^2}{w^2_{l,m}}.$$

As above, (1.7) is verified with $\beta = \frac{1}{\pi}$. Consequently, the system (4.1)-(4.2) is polynomially observable. Then, we introduce the following time semi-discrete approximation scheme:

$$\begin{cases} \frac{z^{k+1} - z^k}{\Delta t} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \left(\frac{z^k + z^{k+1}}{2}\right), \quad k \in \mathbb{N}, \\ z^0 = \begin{pmatrix} w_0 \\ v_0 \end{pmatrix}. \end{cases} \tag{4.3}$$

Now, according to Theorem 1.4 we have:

Proposition 4.1. *For any $\delta > 0$, there exists a time T_δ such that for any $T > T_\delta$, there exists a positive constant $C = C_{T,\delta}$, independent of Δt, such that for Δt small enough, the solution z^k of (4.3) satisfies*

$$C_{T,\delta}\|z^0\|^2_{H^{-1}(\Omega)\times(L^2(\Omega))'} \leq \Delta t \sum_{k\in(0,T/\Delta t)} \|B^* z^k\|^2_{L^2(\Gamma_0)}, \quad \forall z^0 \in \mathcal{C}_{\delta/\Delta t}.$$

4.2. A 2-D plate equation in a square

Let Ω be the square $\Omega = (0,\pi)\times(0,\pi)$ and $\Gamma_0 = \{0\}\times(0,\pi)$. We consider the following initial and boundary value problem:

$$\begin{cases} \ddot{w} + \Delta^2 w = 0, & x \in \Omega,\ t > 0, \\ w = \Delta w = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4.4}$$

with the output

$$\phi(t) = \frac{\partial[G\ddot{w}]}{\partial\nu}\ |_{\Gamma_0}, \tag{4.5}$$

where G is defined as above.

Hence it is written in the form (1.5)-(1.6) with the following choices:

$$X = H_0^1(\Omega)\times H^{-1}(\Omega),\ \ X_1 = D(A) = V \times H_0^1(\Omega),\ \ U = L^2(\Gamma_0),$$

where

$$V = \{w \in H_0^1(\Omega)/\Delta w \in H_0^1(\Omega)\},$$

$$Az = A(w,v)^T = (v, -\Delta^2 w)^T,\ \ \forall\ (w,v) \in D(A),$$

$$B^*(u,v)^T = \frac{\partial[Gv]}{\partial\nu}\ |_{\Gamma_0},\ \ \forall\ (u,v) \in D(A).$$

In this setting, A is a skew-adjoint unbounded operator with compact resolvent on the Hilbert space X, and the operator B^* is admissible (see [1]).

As above, this system is not exactly observable. Now, we verify (1.7) for $\alpha = 1$.

It is easy to see that the normalized eigenvectors of A

$$\Phi_{m,l}(x_1,x_2) = \left(\frac{\sqrt{2}}{\pi(m^2+l^2)}\sin(mx_1)\sin(lx_2), i\frac{\sqrt{2}}{\pi}(m^2+l^2)\sin(mx_1)\sin(lx_2)\right)$$

for $m, l = 1, 2, ...$, form an orthonormal basis for X. The corresponding eigenvalue for $\Phi_{m,l}(x_1, x_2)$ is

$$\mu_{l,m} = iw_{l,m} = i(m^2 + l^2).$$

Taking $z = \Phi_{m,l}(x_1, x_2)$ for fixed m and l. A simple calculations show that

$$\|(iwI - A)z\|^2_{H^{-1}(\Omega)\times(H_0^1(\Omega))'} = \frac{(w - w_{l,m})^2}{w_{l,m}^4},$$

$$\|B^* z\|^2_{L^2(\Gamma_0)} = \frac{m^2}{\pi},$$

and

$$\|z\|^2_{H^{-1}(\Omega)\times(H_0^1(\Omega))'} = \frac{1}{w_{l,m}^4}.$$

Here $(H_0^1(\Omega))'$ is the dual space of $H_0^1(\Omega)$ with respect to the pivot space $H^{-1}(\Omega)$.

It is clear that (1.7) holds with $\beta = \frac{1}{\pi}$. Now, if we expand z as

$$z = \sum_{m,l\geq 1} a_{m,l}\Phi_{m,l}(x_1, x_2),$$

we get

$$\|(iwI - A)z\|^2_{H^{-1}(\Omega)\times(H_0^1(\Omega))'} = \sum_{m,l\geq 1} |a_{m,l}|^2 \frac{(w - w_{l,m})^2}{w_{l,m}^4},$$

$$\|B^* z\|^2_{L^2(\Gamma_0)} = \sum_{m,l\geq 1} |a_{m,l}|^2 \frac{m^2}{\pi},$$

and

$$\|z\|^2_{H^{-1}(\Omega)\times(H_0^1(\Omega))'} = \sum_{m,l\geq 1} \frac{|a_{m,l}|^2}{w_{l,m}^4}.$$

Hence (1.7) is verified with $\beta = \frac{1}{\pi}$. Consequently, the system (4.4)-(4.5) is polynomially observable.

Then, we introduce the following time semi-discrete approximation scheme:

$$\begin{cases} \frac{z^{k+1} - z^k}{\Delta t} = \begin{pmatrix} 0 & I \\ -\Delta^2 & 0 \end{pmatrix} \left(\frac{z^k + z^{k+1}}{2}\right), \quad k \in \mathbb{N}, \\ z^0 = \begin{pmatrix} w_0 \\ v_0 \end{pmatrix}. \end{cases} \tag{4.6}$$

As an application of Theorem 1.4, we get:

Proposition 4.2. *For any $\delta > 0$, there exists a time T_δ such that for any $T > T_\delta$, there exists a positive constant $C = C_{T,\delta}$, independent of Δt, such that for Δt small enough, the solution z^k of* (4.6) *satisfies*

$$C_{T,\delta} \|z^0\|^2_{H^{-1}(\Omega)\times(H^1_0(\Omega))'} \le \Delta t \sum_{k\in(0,T/\Delta t)} \|B^* z^k\|^2_{L^2(\Gamma_0)}, \quad \forall z^0 \in \mathcal{C}_{\delta/\Delta t}.$$

Appendix

In this section, we prove Proposition 1.3. Since A is skew-adjoint on X, by Stone's theorem A is the generator of a unitary group $\mathbb{T}(t)$. Let us denote $z(t) = \mathbb{T}(t)z_0$. By a simple adaptation of the proof of [13, Lemma 5.3], we have that, if (1.7) holds then for all $\chi \in C^1_{comp}(\mathbb{R})$,

$$\int_{\mathbb{R}} \|\mathbb{T}(t)z_0\|^2_{X_{-\alpha}} \left(\chi^2(t) - \frac{1}{\beta}\dot{\chi}^2(t)\right) dt \le \frac{1}{\beta}\int_{\mathbb{R}} \left\|C\mathbb{T}(t)z_0\right\|^2_U \chi^2(t)\, dt. \quad (4.7)$$

Taking $\chi(t) = \varphi(t/T)$ with $\varphi \in C^\infty_{comp}(]0,1[)$, we have

$$\int_{\mathbb{R}} \left\|C\mathbb{T}(t)z_0\right\|^2_U \chi^2(t)\, dt \le \|\varphi\|^2_\infty \int_0^T \left\|C\mathbb{T}(t)z_0\right\|^2_U dt$$

and since $(\mathbb{T}(t))_{t\in\mathbb{R}}$ is unitary group:

$$\int_{\mathbb{R}} \|\mathbb{T}(t)z_0\|^2_{X_{-\alpha}} \left(\chi^2(t) - \frac{1}{\beta}\dot{\chi}^2(t)\right) dt = \|z_0\|^2_{X_{-\alpha}} I_T$$

with

$$I_T = \int_{\mathbb{R}} \left(\varphi^2(t/T) - \frac{1}{\beta T^2}\dot{\varphi}^2(t/T)\right) dt = T\int_{\mathbb{R}} \varphi^2(t)\, dt - \frac{1}{\beta T}\int_{\mathbb{R}} \dot{\varphi}^2(t)\, dt.$$

For $\varphi \ne 0$ and T large enough, $I_T > 0$ so that (4.7) implies (1.4) with $C_T = \frac{\beta I_T}{\|\varphi\|^2_\infty}$. □

Acknowledgments

The author expresses his gratitude to the referee for his remarks which allowed us to improve the first version of the paper.

References

[1] K. Ammari, M. Tucsnak & G. Tenenbaum – "A sharp geometric condition for the boundary exponential stabilizability of a square plate by moment feedbacks only", in *Control of coupled partial differential equations*, Internat. Ser. Numer. Math., vol. 155, Birkhäuser, Basel, 2007, p. 1–11.

[2] K. Ammari – "Dirichlet boundary stabilization of the wave equation", *Asymptot. Anal.* **30** (2002), no. 2, p. 117–130.

[3] K. Ammari & M. Tucsnak – "Stabilization of second order evolution equations by a class of unbounded feedbacks", *ESAIM Control Optim. Calc. Var.* **6** (2001), p. 361–386 (electronic).

[4] C. Bardos, G. Lebeau & J. Rauch – "Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary", *SIAM J. Control Optim.* **30** (1992), no. 5, p. 1024–1065.

[5] N. Cîndea, S. Micu & M. Tucsnak – "An approximation method for exact controls of vibrating systems", *SIAM J. Control Optim.* **49** (2011), no. 3, p. 1283–1305.

[6] N. Cîndea & A. Münch – "A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations", *Calcolo* **52** (2015), no. 3, p. 245–288.

[7] S. Ervedoza, C. Zheng & E. Zuazua – "On the observability of time-discrete conservative linear systems", *J. Funct. Anal.* **254** (2008), no. 12, p. 3037–3078.

[8] S. Ervedoza & E. Zuazua – "Uniformly exponentially stable approximations for a class of damped systems", *J. Math. Pures Appl. (9)* **91** (2009), no. 1, p. 20–48.

[9] R. Glowinski, C. H. Li & J.-L. Lions – "A numerical approach to the exact boundary controllability of the wave equation. I. Dirichlet controls: description of the numerical methods", *Japan J. Appl. Math.* **7** (1990), no. 1, p. 1–76.

[10] Z. Hajjej – "Uniformly polynomially stable approximations for a class of second order evolution equations", *Palest. J. Math.* **2** (2013), no. 2, p. 312–329.

[11] J. A. Infante & E. Zuazua – "Boundary observability for the space semi-discretizations of the 1-D wave equation", *M2AN Math. Model. Numer. Anal.* **33** (1999), no. 2, p. 407–438.

[12] J.-L. Lions – *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1 contrôlabilité exacte*, Recherches en Mathématiques Appliquées, vol. 8, Masson, Paris, 1988, With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[13] L. Miller – "Controllability cost of conservative systems: resolvent condition and transmutation", *J. Funct. Anal.* **218** (2005), no. 2, p. 425–444.

[14] A. Münch – "A uniformly controllable and implicit scheme for the 1-D wave equation", *M2AN Math. Model. Numer. Anal.* **39** (2005), no. 2, p. 377–418.

[15] A. Münch & A. F. Pazoto – "Uniform stabilization of a viscous numerical approximation for a locally damped wave equation", *ESAIM Control Optim. Calc. Var.* **13** (2007), no. 2, p. 265–293 (electronic).

[16] L. N. Trefethen – "Finite difference and spectral methods for ordinary and partial differential equations", available at `http://people.maths.ox.ac.uk/trefethen/pdetext.html`, 1988.

[17] M. Tucsnak & G. Weiss – *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009.

[18] G. Weiss – "Admissible observation operators for linear semigroups", *Israel J. Math.* **65** (1989), no. 1, p. 17–43.

Zayd Hajjej
Département de mathématique
Faculté des Sciences de Monastir
Université de Monastir
5019 Monastir, Tunisia
hajjej.zayd@gmail.com