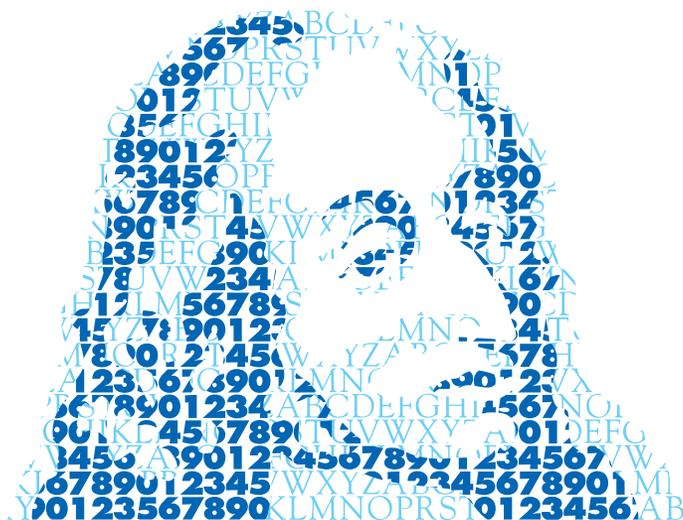


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# On 1-cocycles induced by a positive definite function on a locally compact abelian group

JORDAN FRANKS  
ALAIN VALETTE

## Abstract

For  $\varphi$  a normalized positive definite function on a locally compact abelian group  $G$ , let  $\pi_\varphi$  be the unitary representation associated to  $\varphi$  by the GNS construction. We give necessary and sufficient conditions for the vanishing of 1-cohomology  $H^1(G, \pi_\varphi)$  and reduced 1-cohomology  $\overline{H}^1(G, \pi_\varphi)$ . For example,  $\overline{H}^1(G, \pi_\varphi) = 0$  if and only if either  $\text{Hom}(G, \mathbb{C}) = 0$  or  $\mu_\varphi(1_G) = 0$ , where  $1_G$  is the trivial character of  $G$  and  $\mu_\varphi$  is the probability measure on the Pontryagin dual  $\hat{G}$  associated to  $\varphi$  by Bochner's Theorem. This streamlines an argument of Guichardet (see Theorem 4 in [7]).

*Sur les 1-cocycles induits par une fonction de type positif sur un groupe abélien localement compact*

## Résumé

Soit  $\varphi$  une fonction de type positif normalisée sur un groupe localement compact abélien  $G$ , et  $\pi_\varphi$  la représentation unitaire de  $G$  obtenue par construction GNS. Nous donnons des conditions nécessaires et suffisantes pour l'annulation de la 1-cohomologie  $H^1(G, \pi_\varphi)$  et de la 1-cohomologie réduite  $\overline{H}^1(G, \pi_\varphi)$ . Par exemple,  $\overline{H}^1(G, \pi_\varphi) = 0$  si et seulement si ou bien  $\text{Hom}(G, \mathbb{C}) = 0$  ou bien  $\mu_\varphi(1_G) = 0$ , où  $1_G$  est le caractère trivial de  $G$  et  $\mu_\varphi$  est la mesure de probabilité sur le dual de Pontryagin  $\hat{G}$  associée à  $\varphi$  par le théorème de Bochner. Cela simplifie un argument de Guichardet (Théorème 4 de [7]).

## 1. Introduction

The Gel'fand-Naimark-Segal construction (see [1]) provides a correspondence between positive definite functions  $\varphi$  on a locally compact

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*Keywords:* continuous 1-cohomology, cyclic representation, GNS construction, locally compact abelian group, positive definite function.

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group  $G$  and cyclic representations  $\pi_\varphi$  on Hilbert space. This allows one to construct a dictionary between the functional-analytic and algebro-geometric pictures of  $\varphi$  and  $\pi_\varphi$ . For example,  $\varphi$  is an extreme point in the cone  $\mathcal{P}(G)$  of positive definite functions on  $G$  if and only if  $\pi_\varphi$  is an irreducible representation; or, there exists a constant  $a > 0$  such that  $\varphi - a$  is again positive definite if and only if  $\pi_\varphi$  has nonzero fixed vectors (see [4]).

In view of their importance for rigidity questions and Kazhdan's property (T),<sup>1</sup> it seems natural to try to fit 1-cohomology and reduced 1-cohomology of  $\pi_\varphi$  in that dictionary. Two examples show that the answer will depend heavily on the underlying group  $G$ : if  $\varphi$  is the constant function 1, then  $\pi_\varphi$  is the trivial 1-dimensional representation  $1_G$  of  $G$ , and  $H^1(G, 1_G) = \text{Hom}(G, \mathbb{C})$ ; the vanishing of this group depends on the structure of the abelianized group  $G/[G, G]$  (e.g. if  $G$  is compactly generated, then  $\text{Hom}(G, \mathbb{C}) = 0$  if and only if  $G/[G, G]$  is compact). For a second example, assume that  $G$  is discrete, and let  $\varphi = \delta_1$  be the Dirac measure at the identity; then  $\pi_\varphi$  is the left regular representation  $\lambda_G$  of  $G$  on  $\ell^2(G)$ , and  $H^1(G, \lambda_G) = 0$  if and only if  $G$  is non-amenable with vanishing first  $\ell^2$ -Betti number (see [2]).

In this paper we consider the case where  $G$  is a locally compact abelian group. This assumption makes the question more tractable and gives us a powerful tool, Bochner's Theorem:  $\varphi$  is the Fourier transform of a positive Borel measure  $\mu_\varphi$  on the Pontryagin dual  $\hat{G}$  (see [5]). Without relying on the cohomological machinery available in the literature (see [8, 1]), we achieve by completely elementary means the results of this paper, namely, we show that the existence of nontrivial 1-cohomology is determined by two factors: the existence of non-trivial homomorphisms from  $G$  to  $\mathbb{C}$  and the behavior of  $\mu_\varphi$  near the trivial character  $1_G$ . This is all delineated in a precise way in Theorem 1.

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<sup>1</sup>Recall Shalom's results, see Theorems 0.2 and 6.1 in [10]: for a compactly generated group  $G$ , the group  $G$  has property (T) if and only if  $\overline{H}^1(G, \pi) = 0$  for every unitary representation  $\pi$  of  $G$ , if and only if  $H^1(G, \sigma) = 0$  for every unitary irreducible representation of  $G$ .

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## 2. Statement of results

For  $G$  a locally compact group and  $\pi$  a unitary representation of  $G$  on a Hilbert space  $\mathcal{H}_\pi$ , recall that the space of *1-cocycles* for  $\pi$  is

$$Z^1(G, \pi) = \{b : G \rightarrow \mathcal{H}_\pi : b \text{ continuous, } b(gh) = \pi(g)b(h) + b(g) \text{ for all } g, h \in G\}.$$

The space of *1-coboundaries* for  $\pi$  is:

$$B^1(G, \pi) = \{b \in Z^1(G, \pi) : \exists v \in \mathcal{H}_\pi \text{ such that } b(g) = \pi(g)v - v \text{ for every } g \in G\}.$$

The *1-cohomology* of  $\pi$  is then the quotient

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi).$$

Endow  $Z^1(G, \pi)$  with the topology of uniform convergence on compact subsets of  $G$ . The *reduced 1-cohomology* of  $\pi$  is the quotient of the space of 1-cocycles by the closure of the space of 1-coboundaries, i.e.

$$\overline{H}^1(G, \pi) = Z^1(G, \pi)/\overline{B^1(G, \pi)}.$$

From now on, let  $G$  be a locally compact *abelian* group,  $\varphi$  a positive definite function on  $G$ . Excluding the zero function, we may without loss of generality take  $\varphi$  to be normalized ( $\varphi(e) = 1$ ). Let  $\mu_\varphi$  be the probability measure on the Pontryagin dual  $\hat{G}$  provided by Bochner’s Theorem, i.e.  $\varphi(x) = \int_{\hat{G}} \xi(x) d\mu_\varphi(\xi)$  for  $x \in G$ . Let  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  be the cyclic representation of  $G$  associated to  $\varphi$  through the GNS construction, so that the cyclic vector  $\xi_\varphi \in \mathcal{H}_\varphi$  satisfies  $\langle \pi_\varphi(x)\xi_\varphi | \xi_\varphi \rangle = \varphi(x)$ . Let also  $\rho_\varphi$  be the representation of  $G$  on  $L^2(\hat{G}, \mu_\varphi)$  given by  $(\rho_\varphi(x)f)(\xi) = \xi(x)f(\xi)$  ( $\xi \in \hat{G}, f \in L^2(\hat{G}, \mu_\varphi)$ ).

If  $\lambda_G$  denotes the left regular representation of  $G$ , then  $(\widehat{\lambda_G(x)h})(\xi) = \int_{\hat{G}} h(x^{-1}g)\overline{\xi(g)}dg = \overline{\xi(x)}\hat{h}(\xi)$  for  $h \in L^1(G)$ . From Plancherel’s Theorem, it follows that the composition of the Fourier transform with conjugation is a unitary equivalence between the regular representation on  $L^2(G)$  and the unitary representation defined by  $(f(\xi) \mapsto \xi(x)f(\xi))$  on  $L^2(\hat{G})$ . This, together with Bochner’s Theorem, intuitively introduces  $\rho_\varphi$ , as well as the following proposition which we prove in Section 3.

**Proposition 1.** *The representations  $\pi_\varphi$  and  $\rho_\varphi$  are unitarily equivalent.*

We assume from now on that  $\varphi$  is not the constant function 1, so that  $\mu_\varphi$  is not the Dirac measure at the trivial character  $1_G$  of  $G$ . This is still equivalent to  $\mu_\varphi(1_G) < 1$ . Let  $\mu_\varphi^\perp$  be the probability measure on  $\hat{G}$  defined by  $\mu_\varphi = \mu_\varphi(1_G)\delta_{1_G} + (1 - \mu_\varphi(1_G))\mu_\varphi^\perp$ .

Let  $\pi_\varphi^0$  be the (trivial) subrepresentation of  $\pi_\varphi$  on the subspace  $\mathcal{H}_\varphi^0$  of  $\pi_\varphi$ -fixed vectors, and  $\pi_\varphi^\perp$  be the subrepresentation on the orthogonal complement, so that  $\pi_\varphi = \pi_\varphi^0 \oplus \pi_\varphi^\perp$ . A simple computation in the  $\rho_\varphi$ -picture shows that  $\mathcal{H}_\varphi^0 \neq 0$  if and only if  $\mu_\varphi(1_G) > 0$ , and in this case  $\mathcal{H}_\varphi^0 = \mathbb{C}\delta_{1_G}$ . Moreover, the map  $L^2(\hat{G} \setminus \{1_G\}, \mu_\varphi^\perp) \rightarrow L^2(\hat{G}, \mu_\varphi) : f \mapsto \frac{f}{\sqrt{1 - \mu_\varphi(1_G)}}$  is isometric and identifies  $\pi_\varphi^\perp$  with the restriction of  $\rho_\varphi$  to  $L^2(\hat{G} \setminus \{1_G\}, \mu_\varphi^\perp)$ .

Our main result is:

**Theorem 1.** *Let  $\varphi$  be a nonconstant, normalized positive definite function on a locally compact abelian group  $G$ .*

1) *Consider the following statements:*

- i)  $H^1(G, \pi_\varphi) = 0$ ;
- ii) *Both of the following properties are satisfied:*
  - a)  $\mu_\varphi(1_G) = 0$  or  $\text{Hom}(G, \mathbb{C}) = 0$ ;
  - b)  $1_G \notin \text{supp}(\mu_\varphi^\perp)$ .

*Then (ii)  $\Rightarrow$  (i), and the converse holds if  $G$  is  $\sigma$ -compact.*

2) *The following are equivalent:*

- i)  $\overline{H}^1(G, \pi_\varphi) = 0$ ;
- ii)  $\mu_\varphi(1_G) = 0$  or  $\text{Hom}(G, \mathbb{C}) = 0$ .

This result will be proved in Section 4. It is essentially equivalent to Theorem 4 in [7], but we emphasize the fact that our proof is direct and based on explicit construction of cocycles and coboundaries. Theorem 1 is sharp in the sense that the implication (i)  $\Rightarrow$  (ii) in Part 1 fails in general if  $G$  is not assumed  $\sigma$ -compact (see Example 1 below).

### 3. Proof of Proposition 1

**Lemma 1.** *The constant function  $1 \in L^2(\hat{G}, \mu_\varphi)$  is a cyclic vector for  $\rho_\varphi$ .*

**Proof:** Consider the associated  $L^1(G)$ -representation of  $\rho_\varphi$  acting on  $L^2(\hat{G}, \mu_\varphi^\perp)$  and given by  $\rho_\varphi(f) := \int_G f(x) \rho_\varphi(x) dx$ ,  $f \in L^1(G)$ . Then  $(\rho_\varphi(f).1)(\xi) = \int_G f(x) \xi(x) dx = \hat{f}(\bar{\xi})$ .

Denote by  $C_0(\hat{G})$  the space of continuous functions vanishing at infinity on  $\hat{G}$ , and recall that  $\hat{f} \in C_0(\hat{G})$  (the Riemann-Lebesgue Lemma). It is classical that the Fourier transform  $L^1(G) \rightarrow C_0(\hat{G})$  is a continuous algebra homomorphism with dense image (a consequence of Stone-Weierstrass). Now compose this homomorphism with the continuous inclusion  $C_0(\hat{G}) \rightarrow L^2(\hat{G}, \mu_\varphi) : h \mapsto h.1$ . Since continuous functions with compact support are dense in  $L^2(\hat{G}, \mu_\varphi)$ , this inclusion has dense image. Since the map  $L^1(G) \rightarrow L^2(\hat{G}, \mu_\varphi) : f \mapsto \rho_\varphi(f).1$  is the composition of the previous maps, it has dense image, meaning that 1 is a cyclic vector for the  $L^1(G)$ -representation  $\rho_\varphi$ . Then 1 is also a cyclic vector for the unitary  $G$ -representation  $\rho_\varphi$  by a Dirac sequence argument (see [4]).  $\square$

Observe that  $\langle \rho_\varphi(x).1 | 1 \rangle = \int_{\hat{G}} \xi(x) d\mu_\varphi(\xi) = \varphi(x)$ , so Proposition 1 follows from Lemma 1 and the uniqueness statement of the GNS construction.  $\square$

#### 4. Proof of Theorem 1

Since  $\pi_\varphi = \pi_\varphi^0 \oplus \pi_\varphi^\perp$ , we have  $H^1(G, \pi_\varphi) = H^1(G, \pi_\varphi^0) \oplus H^1(G, \pi_\varphi^\perp)$  and analogously for  $\overline{H}^1$ . As  $B^1(G, \pi_\varphi^0) = 0$ , we see that  $H^1(G, \pi_\varphi^0) = 0$  if and only if  $\overline{H}^1(G, \pi_\varphi^0) = 0$ , if and only if either  $\mu_\varphi(1_G) = 0$  or  $\text{Hom}(G, \mathbb{C}) = 0$ : this proves the implications (i)  $\Rightarrow$  (ii)(a) in part 1 of Theorem 1, and (i)  $\Rightarrow$  (ii) in part 2 of Theorem 1; moreover, it reduces the main result to:

**Theorem 2.** *Let  $\varphi$  be a nonconstant, normalized positive definite function on a locally compact abelian group  $G$ .*

- 1) *If  $1_G \notin \text{supp}(\mu_\varphi^\perp)$ , then  $H^1(G, \pi_\varphi^\perp) = 0$ . The converse holds if  $G$  is  $\sigma$ -compact.*
- 2)  *$\overline{H}^1(G, \pi_\varphi^\perp) = 0$ .*

**Remark:** As was pointed out by the referee of a previous version, when  $G$  is separable, combining Corollary 1 and Proposition 2 in [6], one gets an explicit description of  $Z^1(G, \pi_\varphi^\perp)$ : it identifies with the space of

measurable functions  $u$  on  $\hat{G}$  such that, for every  $g \in G$ , the integral  $\int_{\hat{G}} |\xi(g) - 1|^2 |u(\xi)|^2 d\mu_{\varphi}^{\perp}(\xi)$  is finite, and it tends to zero for  $g$  tending to the identity of  $G$  (in this picture,  $B^1(G, \pi_{\varphi}^{\perp})$  identifies with  $L^2(\hat{G}, \mu_{\varphi}^{\perp})$ ). We will, however, not need that result.

**Example 1.** *In Part 1 of Theorems 1 and 2, the converse implications are false when  $G$  is not assumed to be  $\sigma$ -compact. Indeed, let  $G$  be an uncountable abelian group with the discrete topology, and take  $\varphi = \delta_1$ . Then  $\pi_{\varphi}$  is the left regular representation  $\lambda_G$  on  $\ell^2(G)$ , while  $\mu_{\varphi} = \mu_{\varphi}^{\perp}$  is the Haar measure on the compact group  $\hat{G}$ . Since  $\mu_{\varphi}$  has full support, in particular  $1_G$  lies in its support. On the other hand  $H^1(G, \lambda_G) = 0$  by Proposition 4.13 in [3].*

To prove the implication “ $\Rightarrow$ ” in part 1 of Theorem 2, we will need:

**Lemma 2.** *Let  $F$  be a closed subset of  $\hat{G}$ , with  $1_G \notin F$ .*

- a) *There exists a regular Borel probability measure  $\nu_0$  on  $G$  such that the Fourier transform  $\widehat{\nu}_0$  vanishes on  $F$ .*
- b) *For every  $\varepsilon > 0$ , there exists a compactly supported regular Borel probability measure  $\nu$  on  $G$  such that  $|\widehat{\nu}| < \varepsilon$  on  $F$ .*

**Proof:** (a) See Section 1.5.2 in [9].

(b) Let  $\nu_0$  be a probability measure on  $G$  as in (a). Let  $\delta$  be a number  $0 < \delta < 1$ , to be determined later. Let  $C$  be a compact subset of  $G$  such that  $\nu_0(C) > 1 - \delta$ . Let  $\nu$  be the probability measure on  $G$  defined by  $\nu(B) = \frac{\nu_0(B \cap C)}{\nu_0(C)}$ , for every Borel subset  $B \subseteq G$ . By taking  $\delta$  small enough, the total variation distance  $|\nu_0 - \nu|(G)$  between  $\nu_0$  and  $\nu$  can be made arbitrarily small. For any finite signed measure  $\mu$  on  $G$ , we have the classical inequality  $|\int_G f(x) d\mu(x)| \leq \|f\|_{\infty} |\mu|(G)$ ; applied to  $\mu = \nu_0 - \nu$  and  $f(x) = \xi(x)$  with  $\xi \in \hat{G}$ , it gives  $|\widehat{\nu}_0(\xi) - \widehat{\nu}(\xi)| \leq |\nu_0 - \nu|(G)$ , so that  $\|\widehat{\nu}_0 - \widehat{\nu}\|_{\infty} < \varepsilon$  for  $\delta$  small enough.  $\square$

**Proof of “ $\Rightarrow$ ” in part 1 of Theorem 2:** We assume that  $1_G$  is not in the support of  $\mu_{\varphi}^{\perp}$ , and prove that  $H^1(G, \pi_{\varphi}^{\perp}) = 0$ . Let  $b \in Z^1(G, \pi_{\varphi}^{\perp})$  be a 1-cocycle. Expanding  $b(xy) = b(yx)$  using the cocycle relation, we get:

$$(1 - \pi_{\varphi}^{\perp}(x))b(y) = (1 - \pi_{\varphi}^{\perp}(y))b(x) \quad (x, y \in G).$$

In the realization of  $\pi_\varphi^\perp$  on  $L^2(\hat{G}, \mu_\varphi^\perp)$ , this gives:

$$(1 - \xi(x))b(y)(\xi) = (1 - \xi(y))b(x)(\xi) \quad (4.1)$$

almost everywhere in  $\xi$  (w.r.t.  $\mu_\varphi^\perp$ ). By Lemma 2, we can find a compactly supported probability measure  $\nu$  on  $G$  such that  $|1 - \hat{\nu}| \geq \frac{1}{2}$  on  $\text{supp}(\mu_\varphi^\perp)$ . Define an element  $v \in L^2(\hat{G}, \mu_\varphi^\perp)$  by  $v := \int_G b(y) d\nu(y)$ : since  $b$  is continuous and  $\nu$  is compactly supported, the integral exists (in the weak sense) in  $L^2(\hat{G}, \mu_\varphi^\perp)$ . Integrating (4.1) w.r.t.  $\nu$  in the variable  $y$ , we get:

$$(1 - \xi(x))v(\xi) = (1 - \hat{\nu}(\xi))b(x)(\xi) \quad (4.2)$$

almost everywhere in  $\xi$ . Since  $|1 - \hat{\nu}| \geq \frac{1}{2}$  on  $\text{supp}(\mu_\varphi^\perp)$ , the function  $w(\xi) := \frac{v(\xi)}{\hat{\nu}(\xi) - 1}$  belongs to  $L^2(\hat{G}, \mu_\varphi^\perp)$ , and by (4.2) its coboundary is exactly  $b$ .  $\square$

**Lemma 3.** *Let  $H$  be a locally compact group. Let  $(\sigma_n)_{n \geq 0}$  be a sequence of unitary representations of  $H$  without nonzero fixed vectors, with  $\sigma_n$  acting on a Hilbert space  $\mathcal{H}_n$ . Assume that, for each  $n \geq 0$ , there exists a unit vector  $\eta_n \in \mathcal{H}_n$  such that the series  $\sum_{n=0}^\infty \|\sigma_n(x)\eta_n - \eta_n\|^2$  converges uniformly on compact subsets of  $H$ . Set  $\sigma = \bigoplus_{n=0}^\infty \sigma_n$ . Then  $b(x) := \bigoplus_{n=0}^\infty (\sigma_n(x)\eta_n - \eta_n)$  defines a nonzero element in  $H^1(G, \sigma)$ .*

**Proof:** By assumption  $b(x)$  belongs to  $\bigoplus_{n=0}^\infty \mathcal{H}_n$  and the map  $H \rightarrow \bigoplus_{n=0}^\infty \mathcal{H}_n$ :  $x \mapsto b(x)$  is continuous. Let  $\eta \in \prod_{n=0}^\infty \mathcal{H}_n$  be defined as  $\eta = (\eta_n)_{n \geq 0}$ . Since  $b$  is the formal coboundary of  $\eta$ , we have  $b \in Z^1(H, \sigma)$ . To prove that  $b$  is not a coboundary, it suffices to show that the associated affine action  $\alpha(x)v = \sigma(x)v + b(x)$  on  $\bigoplus_{n=0}^\infty \mathcal{H}_n$  has no fixed point. But  $\alpha(x)v = v$  translates into  $\sigma_n(x)(v_n + \eta_n) = v_n + \eta_n$  for every  $x \in H$  and  $n \geq 0$ . Since  $\sigma_n$  has no nonzero fixed vector, we have  $v_n + \eta_n = 0$  so  $\|v_n\| = 1$ , which contradicts  $\sum_{n=0}^\infty \|v_n\|^2 < +\infty$ .  $\square$

**Proof of “ $\Leftarrow$ ” in part 1 of Theorem 2, assuming  $G$  to be  $\sigma$ -compact:** We assume that  $1_G$  is in the support of  $\mu_\varphi^\perp$ , and prove that  $H^1(G, \pi_\varphi) \neq 0$ . Let  $(K_n)_{n \geq 0}$  be an increasing sequence of compact subsets of  $G$ , with  $G = \bigcup_{n=0}^\infty K_n$ , and  $K_0 = \{1\}$ . Define a basis  $(U_k)_{k \geq 0}$  of open neighborhoods of  $1_G$  in  $\hat{G}$  by  $U_k = \{\xi \in \hat{G} : \max_{g \in K_k} |\xi(g) - 1| < 2^{-k}\}$  (observe that  $U_0 = \hat{G}$ ). Define a sequence  $(k_n)_{n \geq 0}$  inductively by  $k_0 = 0$  and  $k_{n+1} = \min\{k : k > k_n, \mu_\varphi^\perp(U_k) < \mu_\varphi^\perp(U_{k_n})\}$  (since  $\mu_\varphi^\perp\{1_G\} = 0$  and  $1_G$  is in the support of  $\mu_\varphi^\perp$ , this is well-defined). Set then  $C_n := U_{k_n} \setminus U_{k_{n+1}}$

for  $n \geq 0$ . For each  $n \geq 0$  let  $\mathcal{H}_n$  be the space of functions in  $L^2(\hat{G}, \mu_\varphi^\perp)$  which are  $\mu_\varphi^\perp$ -almost everywhere 0 on  $\hat{G} \setminus C_n$ . Then  $\mathcal{H}_n$  is a closed,  $\rho_\varphi$ -invariant subspace of  $L^2(\hat{G}, \mu_\varphi^\perp)$ . Denote by  $\sigma_n$  the restriction of  $\rho_\varphi$  to  $\mathcal{H}_n$ , so that  $L^2(\hat{G}, \mu_\varphi^\perp) = \oplus_{n=0}^\infty \mathcal{H}_n$  and  $\rho_\varphi = \oplus_{n=0}^\infty \sigma_n$ . Let  $\eta_n = \frac{1_{C_n}}{\sqrt{\mu_\varphi^\perp(C_n)}}$  be the normalized characteristic function of  $C_n$ . To appeal to Lemma 3, we still have to check that  $x \mapsto \sum_{n=0}^\infty \|\sigma_n(x)\eta_n - \eta_n\|^2$  converges uniformly on every compact subset  $K$  of  $G$ . Clearly we may assume  $K = K_\ell$ . For  $n \geq \ell$  and  $x \in K_\ell$  and  $\xi \in C_n$ , we have  $|\xi(x) - 1| < 2^{-k_n}$ , hence

$$\begin{aligned} \max_{x \in K_\ell} \sum_{n=\ell}^\infty \|\sigma(x)\eta_n - \eta_n\|^2 &= \max_{x \in K_\ell} \sum_{n=\ell}^\infty \frac{1}{\mu_\varphi^\perp(C_n)} \int_{C_n} |\xi(x) - 1|^2 d\mu_\varphi^\perp(\xi) \\ &\leq \sum_{n=\ell}^\infty 4^{-k_n} \leq \sum_{n=0}^\infty 4^{-n} = \frac{4}{3} \end{aligned}$$

and

$$\begin{aligned} \max_{x \in K_\ell} \sum_{n=0}^\infty \|\sigma(x)\eta_n - \eta_n\|^2 &\leq \left( \max_{x \in K_\ell} \sum_{n=0}^{\ell-1} \|\sigma(x)\eta_n - \eta_n\|^2 \right) + \frac{4}{3} \\ &\leq 4\ell + \frac{4}{3} < +\infty. \end{aligned}$$

So the result follows from Lemma 3.  $\square$

**Proof of part 2 of Theorem 2:** Let  $b \in Z^1(G, \pi_\varphi^\perp)$  be a 1-cocycle. We must show that  $b$  is a limit of 1-coboundaries (uniformly on compact subsets of  $G$ ). Since  $\mu_\varphi^\perp(1_G) = 0$ , by the regularity of  $\mu_\varphi^\perp$ , we can find a decreasing sequence of relatively compact open neighborhoods  $(V_n)_{n \geq 0}$  of  $1_G$ , such that  $\mu_\varphi^\perp(V_n) \rightarrow 0$  for  $n \rightarrow \infty$ .

Set  $\mathcal{H}_n := \{f \in L^2(\hat{G}, \mu_\varphi^\perp) : f = 0 \text{ a.e. on } V_n\}$ ; then  $\mathcal{H}_n$  is a closed,  $\rho_\varphi$ -invariant subspace, and the sequence  $(\mathcal{H}_n)_{n \geq 0}$  is increasing with dense union in  $L^2(\hat{G}, \mu_\varphi^\perp)$ . Let  $\rho_n$  denote the restriction of  $\rho_\varphi$  to  $\mathcal{H}_n$ , and let  $b_n$  denote the projection of  $b$  onto  $\mathcal{H}_n$ . Then  $b_n \in Z^1(G, \rho_n)$ , and  $\lim_{n \rightarrow \infty} b_n = b$  (uniformly on compact subsets of  $G$ ). Since  $1_G$  does not belong to the closed subset  $\hat{G} \setminus V_n$ , by Lemma 2 and by repeating the proof of the forward direction of part 1 of Theorem 2, we obtain that  $b_n$  is a coboundary for each  $n \geq 0$ .  $\square$

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