BLAISE PASCAL

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Abstract

This work presents an approach towards the representation theory of the braid groups B_n. We focus on finite-dimensional representations over the field of Laurent series which can be obtained from representations of infinitesimal braids, with the help of Drinfeld associators. We set a dictionary between representation-theoretic properties of these two structures, and tools to describe the representations thus obtained. We give an explanation for the frequent apparition of unitary structures on classical representations. We introduce new objects — varieties of braided extensions, infinitesimal quotients — which are useful in this setting, and analyse several of their properties. Finally, we review the most classical representations of the braid groups, show how they can be obtained by our methods and how this setting enriches our understanding of them.

Sur les représentations des groupes de tresses

Résumé

Ce travail présente une approche en direction d'une théorie des représentations des groupes de tresses B_n. Nous nous concentrons sur les représentations de dimensions finie sur le corps des séries de Laurent qui peuvent être obtenues à partir des représentations des tresses infinitésimales en utilisant des associateurs de Drinfeld. Nous établissons un dictionnaire entre les propriétés de théorie des représentations de ces deux structures, ainsi que des outils pour décrire les représentations ainsi obtenues. Nous expliquons l'apparition fréquente de structures unitaires préservées par les représentations classiques. Nous introduisons de nouveaux objets (variétés d'extensions tressées, quotients infinitésimaux) qui sont utiles dans ce cadre, et nous analysons plusieurs de leurs propriétés. Enfin, nous passons en revue les représentations les plus classiques des groupes de tresses, montrons comment elles peuvent être obtenues par nos méthodes et comment ce cadre enrichit la compréhension que l'on en a.

Contents

1. Introduction

1.1. Overview

This paper presents a uniform approach to the representations of the braid groups B_n, in order to get as close as possible to a "representation theory" for these finitely generated torsion-free groups. We focus here on the general setting of this approach, whose benefits include:

(1) Obtention of new representations of the braid groups (see [19, 20] and part 4).

(2) Natural explanations and simple proofs of the appearance of unitary structures on the classical representations of B_n (see section 3.2.2, appendix A and part 6).

(3) Determination of the irreducible components of the derived representations obtained by tensor products and Schur functors (see [20, 25, 24, 28, 29] and part 5).

(4) Determination of the algebraic hull of the braid groups inside classical representations (see [28, 29, 30] and section 5.6).

(5) A more simple picture of the different sorts of generic representations of the braid groups (see part 5 et 6).

(6) An arithmetic action of absolute Galois groups on classical representations of B_n (see [26]), using parts 3 and 5.

Most of these aspects are developped here, applied to several representations in [24, 28, 29, 30], and some of them are generalized to other kind of so-called Artin groups in [22, 25, 29, 27], or even to the generalized braid groups associated to complex reflection groups. This work is a slightly revised and enriched version of [23], which was unpublished and used as a basic ressource in most of the works quoted above.

1.2. **Motivations**

Artin's braid group on n strands B_n, as originally introduced in [1], is one of the finitely generated, infinite and torsion-free group which appears most often in mathematics. Concurrently, and in particular in the past thirty years, lots of linear (finite-dimensional, characteristic-zero) representations appeared in the most diverse contexts. Although it seems illusory to aim at a complete classification of its representations, there is thus a real need for some understanding of its representation theory. In particular it seems useful to start unifying as much as possible these approaches on some common ground — even if it entails a restriction on the range of representations which can be considered.

The main difficulty in the investigation of the representations of this kind of groups originates in the fact that they usually belong to families depending on transcendant parameters — contrary to the finite groups, whose representation theory can be studied over the field of algebraic numbers. Rather than study representations on a pure (algebraically closed) field it is thus useful to consider on this field additional structures, for instance to assume that it is (the algebraic closure of) the quotient field of a discrete valuation ring. Let $\Bbbk$ be an arbitrary field of characteristic 0 and $K = \Bbbk((h))$ be the field of formal series with coefficients in $\Bbbk$. The field K is of infinite transcendance degree over $\Bbbk$, and the field of matrix coefficients of a linear representation of B_n over K or its algebraic closure $\overline{K}$ is a finitely generated extension of $\Bbbk$.

We choose to investigate representations of B_n over $\Bbbk((h))$. It is well-known that such representations can be obtained from the monodromy of flat vector bundles over $X_n/\mathfrak{S}_n$, where $X_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \Leftrightarrow i \neq j\}$. Indeed, let $\pi : B_n \to \mathfrak{S}_n$ be the projection of the braid

group onto the symmetric group, $P_n = \operatorname{Ker} \pi$ the pure braids group; one has $P_n = \pi_1(X_n)$, $B_n = \pi_1(X_n/\mathfrak{S}_n)$. The algebraic variety X_n has good properties with respect to $P_n : X_n$ is an Eilenberg-MacLane space and is the complement of hypersurfaces in some projective space. Associated to it one finds a holonomy Lie algebra $\mathcal{T}_n$, which has been shown to be closely connected to the nilpotent completion of the pure braid group by Kohno in [15], and it can be shown ([16]) that the representations sufficiently close to the trivial action on the same vector space are monodromies of representations of the holonomy Lie algebra. The analogous object for B_n is the Hopf algebra $\mathfrak{B}_n$ defined as the semidirect product of the group algebra of $\mathfrak{S}_n$ and the envelopping algebra $\mathsf{U}\mathcal{T}_n$. The geometric properties of X_n imply that the defining relations of $\mathcal{T}_n$ are *homogeneous* with respect to its natural generators. This allows us to introduce a scalar parameter h inside any representation of $\mathcal{T}_n$ or $\mathfrak{B}_n$. The computation of the monodromy in terms of K.T. Chen's iterated integrals thus yields representations of B_n over the local ring of formal (and even convergent) power series with *complex* coefficients.

This linear structure associated to the group structure of B_n is far more tractable than its group algebra, and the monodromy operation commutes with direct sums and tensor products. It follows that one may hope to consider $\mathfrak{B}_n$ as the Lie algebra of B_n, in the sense that the representation theory of connected Lie groups essentially relies on its Lie algebra; in particular we would like to have at disposal a convenient dictionnary which avoids the explicit computation of the monodromy, in order to decide whether the representation is irreducible, to decompose tensor products, or decide invariance with respect to bilinear forms.

Transcendant monodromy however presents some major drawbacks. First of all it can only lead to representations over (finitely generated extensions of) the field of complex numbers $\mathbb{C}$. But above all it depends on the choice of a base point. This lack of symmetry raises problems in the study of the dual representations and the invariance with respect to bilinear forms. In order to solve these problems, we use Drinfeld associators. In his article [12], Drinfeld expressed the universal monodromy with respect to some "base point at infinity" in a very simple way, in terms of a formal power series in two non-commuting variables $\Phi_{KZ}(x,y)$. He then determined which were the algebraic equations that such a series $\Phi(x,y)$, with coefficients in any field $\Bbbk$ of characteristic 0, had to fulfill so that

the same formulas of universal monodromy satisfy the braid relations. Solutions of these equations are called (Drinfeld) associators, and Drinfeld proved in the same article that there exists an associator with coefficients in the field $\mathbb{Q}$ of rational numbers.

The use of associators enables us to remove the two a priori obstacles to our purpose — a contrario, the main obstruction to the generalization of several aspects of this work to other similar groups, such as generalized Artin groups, is due to the fact that analogues of Drinfeld associators have not been defined and extensively studied yet (see however [27] and the references there for Artin groups of type $I_2(m)$ and B). Drinfeld associators and their analogues may be seen as non-commutative versions of Chevalley's formal exponentiation (see [8]).

1.3. **Outline of the results**

In the first place (part 2) we recall the basic notions on braids, infinitesimal braids and associators, which will be of use to us in the sequel. The third part achieves the program of setting a dictionnary between representations of $\mathfrak{B}_n$ over $\Bbbk$ and representations of B_n over K. Beyond linear representations, we give a uniform explanation to the apparition of unitary strutures on "monodromy" representations. Let us assume $\Bbbk \subset \mathbb{C}$. Once we have obtained representations of B_n over the local ring $\Bbbk[[h]]$ of formal power series, representations over a pure field can be deduced by at least two means: either by forgetting the local structure on $\Bbbk((h))$ — whose algebraic closure is isomorphic to $\mathbb{C}$ — or by specialization in $h \in \Bbbk$. This last method may be used only in case the matrix coefficients of the representation are convergent power series. We show in appendix A that, up to a field automorphism, this situation can always be assumed. We then get unitary representations by this method.

In view of understanding representation-theoretic aspects, a first advantage of $\mathfrak{B}_n$ over B_n is that $\mathfrak{B}_n$ is a semidirect product, whereas the short exact sequence $1 \to P_n \to B_n \to \mathfrak{S}_n \to 1$ is not split. We thus define in part 4, for every representation of $\mathfrak{S}_n$, the variety of all representations of $\mathfrak{B}_n$ of which it is the restriction. This yields a systematic approach to the search for (irreducible) representations of B_n. We analyse in this part how much information on the corresponding representations of B_n is contained in this variety. In particular, we show how its factorization through classical quotients of the braid groups can be detected from the

original data. Moreover, a third standard manipulation of the local ring $\Bbbk[[h]]$, namely reduction modulo h, leads to representations of B_n over $\Bbbk$ which are iterated extensions of representations of $\mathfrak{S}_n$ — in fact, of the irreducible components of the original representation.

At the other end of the fundamental exact sequence, another operation is the restriction to the pure braid group of the representation (part 5). The replacement of P_n by the holonomy Lie algebra $\mathcal{T}_n$ enlights noteworthy phenomenons for a large class of representations — in particular the "generic" irreducibility of tensor products, or the irreducibility of the restriction to various subgroups of B_n. We study the consequences of these properties on the "infinitesimal quotients" of B_n, i.e. Hopf quotients of $\mathfrak{B}_n$ whose structure explains in particular the decomposition of tensor products. We moreover indicate (appendix B) how it is possible to get, in particularly auspicious situations, an explicit (matrix) description of representations of B_n from their infinitesimal version.

The last part in this work (part 6) reviews the most classical constructions of representations of the braid group: the Iwahori-Hecke and Birman-Wenzl-Murakami algebras, Yang-Baxter representations, and Long's generalization of Magnus induction. We show how our approach enforces the understanding we have on them. In particular we insist on the unitary structures which appear on the representations and also indicate the general properties of part 5 that they satisfy.

1.4. **Beyond**

We underline here a few directions of research which are suggested by the uniform approach settled here. We part them in two types. The first one is concerned with the analysis of standard algebraic or geometric structures, shown here to be relevant for the representation theory of the braid group.

Among them are the infinitesimal quotients defined in section 5.6, and the reductive Lie algebras associated to them. The work of decomposing the Lie algebra associated to representations of the Temperley-Lieb algebra and the Iwahori-Hecke algebra was done by the author in [24, 28], as well as the one associated to the Krammer representation ([29]) and more generally to the Birman-Wenzl-Murakami algebras ([30]), although only for generic values of the additional defining parameter. Another kind of structures are the varieties of braided extensions $\mathcal{V}^s(M)$ defined in section 4.1. The analysis of these varieties and the study of their relationship with

the space of extensions is another interesting task. Apart from the examples studied in part 4, the analysis was carried out for irreducible M (see [19, 20]). A special question of particular interest is whether all irreducible representations of all the generic cyclotomic Hecke algebras considered in section 6.1.1 can be deduced in some way from our approach — answering this question is a mainly computational but still delicate matter. A third kind of structure is given by the connection of "essentially pure" representations with the Deligne-Simpson problem (section 5.4), namely a description of which roots of the Kac-Moody algebras defined by W. Crawley-Boevey corresponds to (irreducible, essentially pure) representations of the braid group.

The second one is concerned with questions which are consequences of this work. A first one is to determinate the field of matrix coefficients for the representations $\widehat{\Phi}(\rho)$ studied here from the infinitesimal datas, and at least its transcendance degree. A second one is to find a criterium on ρ such that $\widehat{\Phi}(\rho)$ is faithful. For the time being, the approaches to faithfulness questions are based on thorough studies of special representations, such as the Burau representation or the representation of the Birman-Wenzl-Murakami algebra which was intensively studied and shown to be faithful, algebraically by D. Krammer and geometrically by S. Bigelow. Now that the linearity of the braid group is known, we need to find a general criterium for faithfulness. It is our hope that such a criterium could be expressed in terms of the infinitesimal datas — weak evidence in this direction can be found in section 4.2. However, while the delicate questions of whether the Hecke algebra representation and the Burau representation for $n = 4$ are faithful remain unsettled, it seems premature to state even vague conjectures. The third one is about the "inverse problem", that is, when does a representation R of B_n "come from" some representation ρ of $\mathfrak{B}_n$. This third question is first not to answer this very vague formulation, but to specify what is meant. Clearly, some representations of B_n cannot be isomorphic to some $\widehat{\Phi}(\rho)$, because their field of matrix coefficients is an algebraic extension of $\mathbb{Q}$ — however, it may still be possible to get them by specialization (e.g. the representations of the Iwahori-Hecke algebra at roots of unity). Some other representations cannot be equal to some $\widehat{\Phi}(\rho)$, but are twists of some $\widehat{\Phi}(\rho)$ by elements of $\mathrm{Gal}(K/\Bbbk)$ or other field morphisms (see the 4-dimensional example in section 6.1.1, or take any

$\widehat{\Phi}(\rho)$ and replace h by h^2). We then wonder how many (semisimple) representations of the braid group it is possible to get from representations of the form $\widehat{\Phi}(\rho)$ by an alternating use of these operations, namely manipulations of the field of matrix coefficients and representation-theoretic operations such as taking direct sums, sub-modules and quotients.

General Notations. By convention, all rings are commutative with unit, all algebras are associative, all representations are finite dimensional, and all Hopf algebras have antipodes. All the fields occuring in the text are of characteristic 0. Unless otherwise stated, n (the number of strands) and N (the dimension of the representation) are integers with $n \geq 2$, $N \geq 1$, $\Bbbk$ is an arbitrary field of characteristic 0. Whenever $\mathfrak{g}$ is a Lie algebra, $U\mathfrak{g}$ is its universal envelopping algebra. Whenever A is a ring, we shall denote by $A^\times$ its set of invertible elements, by $M_N(A)$ the set of $N \times N$ matrices and, if G is a group, by AG the group algebra of G over A. We let $\mathbb{G}_m$, $\mathbb{A}_r$ and μ_r for $r \geq 1$ be the algebraic varieties whose A-points are $A^\times$, A^r and the set of r-th roots of 1 in A. We use the notation $\mathrm{diag}(a_1, \ldots, a_N)$ to designate diagonal matrices in $M_N(A)$ whose diagonal coefficients are $a_1, \ldots, a_N$. We let $\overline{K}$ be the algebraic closure of the field K, and denote by $\mathrm{Gal}(L/K)$ the Galois group of an extension of L over K. Finally, if R is a representation, either of a group or of a Hopf algebra, we shall denote by $R^\vee$ the dual representation, and by $\mathbb{1}$ the trivial representation.

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2. **Preliminaries**

2.1. **Braids**

Let $n \geq 2$ be an integer. In the sequel, we shall denote by $\mathfrak{S}_n$ the symmetric group on n letters, and by s_i the transposition $(i\ i+1)$ for $1 \leq i < n$. The relations between the s_i's give a presentation, found by Moore in the end of the 19-th century, of the symmetric group :

$$< s_1, \ldots, s_{n-1} \ | \ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \ s_i s_j = s_j s_i \ |i-j| \geq 2, \ s_i^2 = 1 > .$$

The braid group on n strands B_n is defined by generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ (so-called braid relation), $\sigma_i \sigma_j = \sigma_j \sigma_i$ if

$|i - j| \geq 2$ (so-called locality relation). We let π be the (non-split) projection $B_n \to \mathfrak{S}_n$ given by $\pi(\sigma_i) = s_i$. The kernel P_n of π is by definition the pure (or colored) braid group, and is generated by the elements $\xi_{ij} = \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$ for $1 \leq i < j \leq n$. Other fundamental elements of B_n are $\gamma_n = (\sigma_1 \ldots \sigma_{n-1})^n$, $\delta_n = \sigma_{n-1}\ldots\sigma_2\sigma_1^2\sigma_2\ldots\sigma_{n-1}$. It is readily checked that γ_n and δ_n belong to P_n. By convention we let $\mathfrak{S}_1 = P_1 = B_1 = \{e\}$ (the trivial group), and we mention the obvious isomorphisms $P_2 \simeq \mathbb{Z} \simeq B_2$.

The following algebraic facts are known. See the book by Birman [5] and the references there for the most classical ones. The groups P_n and B_n are torsion free. For $n \geq 3$ the center of B_n is the same as the center of P_n, is infinite cyclic and generated by γ_n. Each B_n (resp. P_n) embeds into B_{n+1} (resp. P_{n+1}) by $\sigma_i \mapsto \sigma_i$ (resp. $\xi_{ij} \mapsto \xi_{ij}$); this will be refered to as the "usual inclusion" $B_n \subset B_{n+1}$. Hence each γ_r and δ_r, for $2 \leq r \leq n$, will be implicitly considered as an element of B_n. The elements $\xi_{1,n}, \xi_{2,n}, \ldots, \xi_{n-1,n}$ generate a free normal subgroup F_n in P_n, whose centralizer in P_n is the center of P_n, and the subgroup generated by $\sigma_1^2, \ldots, \sigma_{n-1}^2$ is "locally free": this means that the relations between these elements are generated by the obvious commutation relations $\sigma_i^2\sigma_j^2 = \sigma_j^2\sigma_i^2$ for $|i - j| \geq 2$ (see [10]). The elements $\delta_2, \ldots, \delta_n$ generate a free *abelian* subgroup in P_n. It is readily checked that $\gamma_n = \gamma_{n-1}\delta_{n-1} = \delta_{n-1}\gamma_{n-1}$, and $\delta_n = \sigma_{n-1}\delta_{n-1}\sigma_{n-1}$. The action by conjugation of B_{n+1} on its normal subgroup P_{n+1} restricts to an action of $B_n \subset B_{n+1}$ which leaves the free subgroup generated by $\xi_{1,n+1}, \ldots, \xi_{n,n+1}$ invariant. Hence induces an action (so-called Artin action) on a free group on n generators; this action is known to be faithful. By the usual inclusion $P_{n-1} \subset P_n$ we have $P_n = P_{n-1} \ltimes < \xi_{1,n}, \ldots, \xi_{n-1,n} >$, hence P_n is an iterated extension of free groups.

We proceed by recalling some well-known quotients of B_n. First note that all σ_i's belong to the same conjugacy class in B_n. Likewise, all $\xi_{i,j}$ belong to the same conjugacy class in B_n — but *not* in P_n. The quotient of B_n by its subgroup of commutators $[B_n, B_n]$ is infinite cyclic, generated by the image of any σ_i. A canonical isomorphism between $B_n/[B_n, B_n]$ and $\mathbb{Z}$ is given by the so-called length function l; one has $l(\sigma_i) = 1$ for $1 \leq i \leq n$. The quotient of B_n by the subgroup of commutators of P_n will be denoted by $\widetilde{\mathfrak{S}_n}$ and refered to as the *enhanced symmetric group*. This probably non-standard terminology is a translation — and a specialization

to type A — of Tits notion of *groupes de Coxeter étendus* (see [37]). The kernel of the projection $\widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$ is free abelian of rank $n(n-1)/2$, with free generators given by the images of the ξ_{ij}'s in $\widetilde{\mathfrak{S}}_n$. Finally, let $Hurw_n$ be Hurwitz' group, also called the braid group of the sphere; it is defined as the quotient of B_n by its normal subgroup generated by δ_n.

2.2. **Infinitesimal braids**

The Lie algebra of infinitesimal pure braids on n strands $\mathcal{T}_n$ is defined over $\mathbb{Q}$ by generators t_{ij} for $1 \leq i, j \leq n$ and (homogeneous) relations

$$
\begin{cases}
t_{ij} = t_{ji} & t_{ii} = 0 \\
[t_{ij}, t_{kl}] &=& 0 & \text{if } \#\{i,j,k,l\} = 4 \\
[t_{ij}, t_{ik} + t_{kj}] &=& 0
\end{cases}
$$

It is endowed with the natural grading defined by all the generators t_{ij} being of degree 1. The sum of all these generators is easily seen to be central in $\mathcal{T}_n$, and even to generate its center. A remarkable set of homogeneous elements of degree 1 in $\mathcal{T}_n$ is the following. We define, for every $1 \leq r \leq n$,

$$
Y_r = \sum_{1 \leq i < r} t_{ir} = \sum_{1 \leq u < v \leq r} t_{uv} - \sum_{1 \leq u < v \leq r-1} t_{uv}.
$$

thus $Y_1 = 0$ and $Y_2 = t_{12}$. Considering $\mathcal{T}_r$ for $r < n$ as embedded in $\mathcal{T}_n$ in the natural way, it is clear that each Y_r belongs to $\mathcal{T}_r$ and commutes to $\mathcal{T}_{r-1}$. It follows that the Y_r's commute. The elements t_{ij} and Y_r are to be seen as infinitesimal analogues (or, better, "residues") of the elements ξ_{ij} and δ_r of P_n. Likewise, the Lie subalgebra generated by $t_{1,n}, \ldots, t_{n-1,n}$ is shown to be free and an ideal of $\mathcal{T}_n$, which is a semi-direct product of $\mathcal{T}_{n-1}$ and this Lie algebra. It is shown in [9] that the centralizer of $\mathcal{F}_n$ is the center of $\mathcal{T}_n$ — in the same way that the centralizer of F_n in P_n is the center of P_n.

There exists a natural action of $\mathfrak{S}_n$ on $\mathcal{T}_n$ (hence also on $\mathsf{U}\mathcal{T}_n$) which preserves the grading: it is defined by $s.t_{ij} = t_{s(i)s(j)}$ for $s \in \mathfrak{S}_n$. The semi-direct product $\mathbb{Q}\mathfrak{S}_n \ltimes \mathsf{U}\mathcal{T}_n$ is a Hopf algebra denoted by $\mathfrak{B}_n$, and we call it the algebra of infinitesimal braids. The sum of the elements t_{ij} remains central in $\mathfrak{B}_n$. This algebra is naturally graded, with $\deg t_{ij} = 1$ and $\deg s = 0$ for $s \in \mathfrak{S}_n$. We denote by $\widehat{\mathfrak{B}_n}$ its completion with respect to this grading.

Let $\mathcal{L}$ be the (associative) algebra defined by generators s, Y, Y', t and relations

$$\begin{cases} s^2 = 1 & sYs = Y' - t & sts = t \\ [Y, Y'] = 0 & [Y + Y', t] = 0 \end{cases}$$

The algebra $\mathcal{L}$ is graded, with Y, Y' and t of degree 1, and s of degree 0. Consequences of the defining relations are that $Y + Y'$ is central in $\mathcal{L}$ and that $[Y', t] = -[Y, t]$, $[Y', [Y, t]] = -[Y, [Y, t]]$. It is easily checked that, for every $n \geq 3$, there exists a graded algebra morphism $\mathcal{L} \to \mathfrak{B}_n$ given by

$$\begin{array}{ccccc} s & \mapsto & (n-1 \ n) & t & \mapsto & t_{n-1,n} \\ Y & \mapsto & Y_{n-1} & Y' & \mapsto & Y_n \end{array}$$

This morphism naturally extends to the completions with respect to the grading, whence a morphism from the completion $\widehat{\mathcal{L}}$ of $\mathcal{L}$ to $\widehat{\mathfrak{B}_n}$.

2.3. Associators

In its fundamental article [12], Drinfeld defined, for every commutative $\mathbb{Q}$-algebra $\Bbbk$, and every $\lambda \in \Bbbk$, the set $\mathbb{Assoc}_\lambda(\Bbbk)$ of all formal series Φ in two non-commuting variables A and B, which satisfy the following relations:

$$\Delta(\Phi) = \Phi \hat{\otimes} \Phi \tag{2.1}$$

$$\Phi(B, A) = \Phi(A, B)^{-1} \tag{2.2}$$

$$e^{\lambda A} \Phi(C, A) e^{\lambda C} \Phi(B, C) e^{\lambda B} \Phi(A, B) = 1 \tag{2.3}$$

and

$$\Phi(t_{12}, t_{23} + t_{24}) \Phi(t_{13} + t_{23}, t_{34})$$
$$= \Phi(t_{23}, t_{34}) \Phi(t_{12} + t_{13}, t_{24} + t_{34}) \Phi(t_{12}, t_{23}) \tag{2.4}$$

where $C = -A - B$. Our λ is Drinfeld's $\mu/2$, i.e. $\mathbb{Assoc}_\lambda(\Bbbk)$ is Drinfeld's $M_\mu(\Bbbk)$ (see [12], §5) with $\mu = 2\lambda$. Equation (2.3) is called the *hexagonal relation*, equation (2.4) the *pentagonal relation* — these names are motivated by MacLane coherence conditions for monoidal categories. In relation (2.1), the symbol $\hat{\otimes}$ denotes the completed tensor product associated to the identification of the algebra $\Bbbk \ll A, B \gg$ of formal series in non-commuting variables A and B with the completed envelopping bigebra of the free Lie algebra on two generators. The equation (2.4) lies in

$\widehat{U\mathcal{T}}_4$. Finally, we note that equation (2.1) is equivalent to Φ being the exponential of a Lie series, let's say $\Phi = \exp \Psi$. In particular, let A, B and Z belong to some complete graded $\Bbbk$-algebra and have zero constant term; if Z commutes with A and B, then $\Psi(A + Z, B) = \Psi(A, B + Z) = \Psi(A + Z, B + Z) = \Psi(A, B)$, and

$$\Phi(A + Z, B) = \Phi(A, B + Z) = \Phi(A + Z, B + Z) = \Phi(A, B).$$

It follows that equation (2.3) reads, for $A + B + C = Z$ *central* i.e. commuting with A and B (hence with C),

$$e^{\lambda A}\Phi(C, A)e^{\lambda C}\Phi(B, C)e^{\lambda B}\Phi(A, B) = e^{\lambda Z}.$$

Also notice that, if $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ and $\mu \in \Bbbk$, then $\Phi(\mu A, \mu B) \in \mathbb{Assoc}_{\lambda\mu}(\Bbbk)$. Moreover, for every commutative $\mathbb{Q}$-algebra $\Bbbk$ and every $\lambda \in \Bbbk$, $\mathbb{Assoc}_\lambda(\Bbbk) = \mathbb{Assoc}_{-\lambda}(\Bbbk)$. The very first terms of an associator, as a formal series, are easy to find. In particular, the following result is well-known to specialists in the field. For convenience, if $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ denotes a graded algebra and $x, y \in \mathcal{A}$, we denote by $x \equiv y$ the relation $x - y \in \bigoplus_{n \geq 4} \mathcal{A}_n$ ("equality up to the order 3").

Proposition 2.1. *For any $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$, there exists $\alpha \in \Bbbk$ such that*
$$\Phi(A, B) \equiv 1 + \tfrac{\lambda^2}{6}[A, B] + \alpha\left([A, [A, B]] - [B, [B, A]]\right)$$

Proof. Because of (2.1), we can write $\Phi(A, B) = exp\Psi(A, B)$, with Ψ a Lie series in A and B with no constant term. Up to the order 2, $\Psi(A, B)$ takes the form $uA + vB$ with $u, v \in \Bbbk$, hence $\Phi(A, B)$ equals $1 + uA + vB$ plus higher terms. The pentagonal relation (2.4) implies $u = v = 0$, whence $\Psi(A, B)$ and $\Phi(A, B)$ have no linear term. Since there is only one Lie monomial in degree 2, namely $[A, B]$, there exists $u \in \Bbbk$ such that $\Psi(A, B)$ equals $u[A, B]$ plus higher terms, and $\Phi(A, B) = \exp \Psi(A, B)$ equals $1 + u[A, B]$ up to the order 2. Replacing $\Phi(A, B)$ by this value in (2.3) we get $u = \lambda^2/6$.

The space of homogeneous Lie polynomials of degree 3 is spanned by the two Lie monomials $[A, [A, B]]$ and $[B, [B, A]]$, thus there exists $\alpha, \beta \in \Bbbk$ such that $\Phi(A, B) \equiv 1 + \tfrac{\lambda^2}{6}[A, B] + \alpha[A, [A, B]] + \beta[B, [B, A]]$. Using equation (2.2), we get $\beta + \alpha = 0$. $\qquad\square$

The state of knowledge about these associators is roughly as follows. Drinfeld defined an explicit associator $\Phi_{KZ} \in \mathbb{Assoc}_{i\pi}(\mathbb{C})$ and proved by nonconstructive means, not only that $\mathbb{Assoc}_1(\mathbb{Q}) \neq \emptyset$, but also that

$\mathbb{Assoc}_1^\circ(\mathbb{Q}) \neq \emptyset$, where $\mathbb{Assoc}_\lambda^\circ(\Bbbk) = \{\Phi \in \mathbb{Assoc}_\lambda(\Bbbk) \mid \Phi(-A, -B) = \Phi(A, B)\}$ denotes the set of *even* associators — in particular, one can choose $\alpha = 0$ in the expansion of proposition 2.1. The associator Φ_{KZ} is not even, as show its first terms:

$$\Phi_{KZ}(A, B) \equiv 1 - \zeta(2)[A, B] + \zeta(3)\left([[A, B], B] - [A, [A, B]]\right)$$

where ζ is the Riemann zeta function. More generally, Le and Murakami [17] gave an explicit formula for its coefficients, which involves multiple zeta values.

Deep transcendance conjectures on these values lead many people to think that all the algebraic relations over $\mathbb{Q}$ between the multiple zeta values may be equivalent to those implied by equations (2.1-2.4). There now exists a rather clear algebraic picture of what these algebraic relations should be; assuming these relations to be the only ones, it is then possible to guess explicit — but still complicated — formulas for rational even associators. In particular, Jean Ecalle announced the existence of a conjectural *canonical* even rational associator.

In this paper we will not anticipate on the developpments in this field, and we will choose arbitrary associators in accordance with Drinfeld existence theorems. It should however be kept in mind that, although the formulas which involve even associators are non-explicit in the actual state of knowledge, it should not be so in a (rather) near future. In addition to explicit formulas, another unknown fact which could be of use to us concerns the *convergence* of these associators, when $\Bbbk$ is a complete topological field. It can be shown that $\Phi_{KZ}(A, B)$ is *not* universally convergent, since the coefficient of $A^{n-1}B$ (resp. AB^{n-1}) is $\zeta(n) \geq 1$. However, one easily shows, from Le and Murakami formulas, that $\Phi_{KZ}(A, B)$ converges at least for $||A|| \leq \frac{1}{4}$ and $||B|| \leq \frac{1}{4}$, whenever A and B lies in some Banach algebra. For general associators, we did not hear of any result concerning convergence. We do not know, in particular, whether there exists convergent $\Phi \in \mathbb{Assoc}_1(\mathbb{R})$. One only knows that these cannot be universally convergent (see [27]). In order to repair this problem, we prove in appendix A several approximation results which help us making the series converge. Further developments in the field may (or may not) make these tools less necessary.

2.4. **Drinfeld isomorphisms**

In the same paper [12], Drinfeld states that, for any $\Phi \in \mathbb{Assoc}_\lambda(\mathbb{k})$, there exists an homomorphism from $\mathbb{k}B_n$ to the completion $\widehat{\mathfrak{B}_n}$ of $\mathfrak{B}_n$, given by

$$\sigma_i \mapsto \Phi(t_{i,i+1}, Y_i)s_i e^{\lambda t_{i,i+1}} \Phi(Y_i, t_{i,i+1}).$$

In particular, $\sigma_1 \mapsto s_1 e^{\lambda t_{12}}$ and $\sigma_2 \mapsto \Phi(t_{23}, t_{12})s_2 e^{\lambda t_{23}} \Phi(t_{12}, t_{23})$. Note that equation (2.4) is not needed for the case $n = 3$. In order to lighten notations we identify here, whenever Φ is fixed, B_n with its image in $\widehat{\mathfrak{B}_n}$.

We recall that $\delta_n = \sigma_{n-1} \ldots \sigma_2 \sigma_1^2 \sigma_2 \ldots \sigma_{n-1}$, and make precise the elusive analogy between δ_n and Y_n.

Proposition 2.2. *If* $\lambda \in \mathbb{k}$ *and* Φ *satisfies (2.1), (2.2), (2.3) then, for all* $n \geq 2$, $\delta_n = e^{2\lambda Y_{n+1}}$.

Proof. We use induction on n. The assertion is clear for $n = 2$, since $\delta_2 = \sigma_1^2$. We now assume $\delta_n = e^{2\lambda Y_n}$. Let us set for convenience $t = t_{n,n+1}$, $s = (n\ n+1)$, $Y'_n = Y_{n+1} - t$, $\Phi_n = \Phi(t, Y_n)$ and $\Psi_n = \Phi(t, Y'_n)$. We have $sY_n s = Y'_n$ and $s\Phi_n s = \Psi_n$. By definition, $\delta_{n+1} = \sigma_n \delta_n \sigma_n$ equals $\Phi_n s e^{\lambda t} \Phi_n^{-1} e^{2\lambda Y_n} \Phi_n s e^{\lambda t} \Phi_n^{-1}$, hence

$$\delta_{n+1} = \Phi_n e^{\lambda t} \Psi_n^{-1} \left(e^{\lambda Y'_n}\right)^2 \Psi_n e^{\lambda t} \Phi_n^{-1}.$$

Because $Y_n + Y'_n + t = Y_n + Y_{n+1}$ commutes with t, Y_n et Y'_n, the hexagonal relation reads

$$\begin{aligned}
\Phi_n e^{\lambda t} \Psi_n^{-1} e^{\lambda Y'_n} &= e^{\lambda(Y_n + Y_{n+1})} e^{-\lambda Y_n} \Phi(Y'_n, Y_n) \\
e^{\lambda Y'_n} \Psi_n e^{\lambda t} \Phi_n^{-1} &= \Phi(Y_n, Y'_n) e^{-\lambda Y_n} e^{\lambda(Y_n + Y_{n+1})}
\end{aligned}$$

It follows that $\delta_{n+1} = e^{2\lambda(Y_n + Y_{n+1} - Y_n)} = e^{2\lambda Y_{n+1}}$, and we conclude by induction. $\square$

Let us notice that the image under Drinfeld isomorphism of $\sigma_{n-1} \in B_n$ is the image in $\widehat{\mathfrak{B}_n}$ of

$$\sigma = \Phi(t, Y)s e^{\lambda t} \Phi(Y, t) \in \widehat{\mathcal{L}}$$

by the morphism $\widehat{\mathcal{L}} \to \widehat{\mathfrak{B}_n}$ that we already defined. Using the expression of $\Phi(A, B)$ up to the order 3, with parameters λ and α, we now get the expression of σ up to the order 3.

Proposition 2.3. *If* $\Phi(A, B) \equiv 1 + \frac{\lambda^2}{6}[A, B] + \alpha([A, [A, B]] - [B, [B, A]])$, *then*

$$\sigma \equiv s\left(e^{\lambda t} + \frac{\lambda^2}{3}[Y, t] - \alpha[t, [t, Y]] + \frac{\lambda^3}{6}(t[Y, t] + [Y, t]t)\right)$$

Proof. One has

$$
\begin{aligned}
\Phi(t, Y)s &\equiv (1 + \tfrac{\lambda^2}{6}[t, Y] + \alpha[t, [t, Y]] - \alpha[Y, [Y, t]])s \\
&\equiv s(1 + \tfrac{\lambda^2}{6}[t, Y' - t] + \alpha[t, [t, Y' - t]] - \alpha[Y' - t, [Y' - t, t]]) \\
&\equiv s(1 + \tfrac{\lambda^2}{6}[t, Y'] + \alpha[t, [t, Y']] - \alpha[Y' - t, [Y', t]])
\end{aligned}
$$

and $[Y' - t, [Y', t]] = [Y', [Y', t]] - [t, [Y', t]] = -[Y', [Y, t]] + [t, [t, Y']] = [Y, [Y, t]] + [t, [t, Y']]$ hence

$$\Phi(t, Y)s \equiv s(1 + \frac{\lambda^2}{6}[Y, t] - \alpha[Y, [Y, t]])$$

Since $e^{\lambda t}$ equals $1 + \lambda t$ plus terms of order at least 2,

$$\Phi(t, Y)se^{\lambda t} \equiv s(e^{\lambda t} + \frac{\lambda^2}{6}[Y, t] - \alpha[Y, [Y, t]] + \frac{\lambda^3}{6}[Y, t]t)$$

Since $\Phi(Y, t) \equiv 1 + \frac{\lambda^2}{6}[Y, t] + \alpha[Y, [Y, t]] - \alpha[t, [t, Y]]$, we get the conclusion. $\square$

These morphisms commute with the "addition of a strand" in the sense that the usual inclusions $\Bbbk B_n \hookrightarrow \Bbbk B_{n+1}$, $\mathfrak{B}_n \hookrightarrow \widehat{\mathfrak{B}}_{n+1}$ along with the Drinfeld morphism $\Bbbk B_r \to \widehat{\mathfrak{B}}_r$ for $r = n, n + 1$ associated to the same $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ form a commutative square. If $n = 4$ there is in addition a section of the usual inclusion $B_3 \hookrightarrow B_4$ defined by $\sigma_3 \mapsto \sigma_1$, as well as a section of the usual inclusion $\mathfrak{B}_3 \hookrightarrow \mathfrak{B}_4$. This last one is defined by mapping $t_{i,4}$ to $t_{k,l}$ where $\{i, k, l\} = \{1, 2, 3\}$ and $\mathfrak{S}_4$ to $\mathfrak{S}_3$ by $s_1, s_3 \mapsto s_1$ and $s_2 \mapsto s_2$. The corresponding square involving Drinfeld morphisms also commutes, as can be easily checked.

3. Representation theory over $\Bbbk[[h]]$

Let $\Bbbk$ be a field of characteristic 0, $\Bbbk[h]$ the ring of polynomials in one indeterminate h over $\Bbbk$, $\Bbbk[[h]]$ its completion with respect to the h-adic topology, i.e. the ring of formal power series in h. We shall denote by $\Bbbk(h)$ and $\Bbbk((h))$ their field of fractions. The multiplicative group $\mathbb{G}_m(\Bbbk) = \Bbbk^\times$

acts on $\Bbbk[[h]]$ and $\Bbbk((h))$ by continuous automorphisms: $\alpha \in \mathbb{G}_m(\Bbbk)$ sends $f(h)$ on $f(\alpha h)$. Let A be a subring of $\Bbbk[[h]]$ containing $\Bbbk[h]$, such that the h-adic valuation of $\Bbbk[[h]]$ induces on A the structure of a discrete valuation ring with local parameter h. We call such an A a discrete valuation subring of $\Bbbk[[h]]$. We also assume that A is invariant under $\mathbb{G}_m(\Bbbk)$. These very strong conditions are fulfilled of course by $A = \Bbbk[[h]]$, but also by $A = \Bbbk\{h\}$, the ring of convergent power series, in case $\Bbbk$ is a complete topological field. Let K be the quotient field of A, and consider it as embedded in $\Bbbk((h))$. For any $x \in \Bbbk[[h]]$ and $v \in \Bbbk[[h]]^N$, we shall denote by $\overline{x} \in \Bbbk$ and $\overline{v} \in \Bbbk^N$ their reduction modulo h. By default, tensor products are taken over $\Bbbk$.

We study finite dimensional representations of a finitely generated group G over K. We first deal with the general case before specializing to $G = B_n$ and to representations obtained through associators. In case $\Bbbk$ has cardinality at most the continuum, the algebraic closure of $\Bbbk((h))$ is isomorphic to $\mathbb{C}$, so these representations can be seen (in a highly non-canonical way) as ordinary representations of G. However, their field of matrix coefficients, as finitely generated extensions of $\Bbbk$, have more structure. In particular their transcendence degree r has important meaning, namely that such a representation can be considered as an r-dimensional family of representations of G over $\Bbbk$.

3.1. **Lifting properties**

Our aim here is to establish properties of a representation of G over A or K from the study of its coefficients in h. In order to distinguish these coefficients from the matrix coefficients of the representation, and because this appellation has some geometrical meaning, we call them "infinitesimal data". It turns out that the $\Bbbk$-algebra generated by these infinitesimal data already contains an important part of the representation-theoretic information about the representation (see propositions 3.2, 3.4 and 3.5). In order to prove this, we first need to draw a parenthesis about idempotents in $GL_N(A)$. Notations: Whenever V is a $\Bbbk$-vector space, we shall denote by $V[[h]]$ the A-module of formal series with coefficients in V. If V is endowed with a $\Bbbk$-algebra structure, $V[[h]]$ inherits a $\Bbbk[[h]]$-algebra structure. For any $v \in V[[h]]$, we let $\overline{v} \in V \subset V[[h]]$ be its constant term.

3.1.1. Conjugation of idempotents and Hensel lemma

Let $S \in GL_N(A)$ be a symmetry, i.e. an element such that $S^2 = 1$. Because $\overline{S} \in GL_N(\Bbbk)$ is also a symmetry with the same trace, S is conjugated to $\overline{S}$ in $GL_N(K)$. This conjugation also holds in $GL_N(A)$ because of the following identity

$$(1 + \overline{S}S)S = S + \overline{S} = \overline{S}(\overline{S}S + 1).$$

Since $\overline{1 + \overline{S}S} = 2$, $1 + \overline{S}S$ is invertible and its inverse belongs to $GL_N(A)$. Because symmetries and projectors are linked by linear (rational) relations, the same result holds for idempotents. These two facts are particular instances of the following non-commutative version of Hensel lemma.

Proposition 3.1. *Let $\mathcal{A}$ be an associative $\Bbbk$-algebra with unit, $Q \in \Bbbk[X]$ a polynomial in one indeterminate, and $a \in \mathcal{A}[[h]]$. If $Q(a) = 0$ and $Q'(\overline{a}) \in \mathcal{A}^{\times}$, then a is conjugated to $\overline{a}$ in $\mathcal{A}[[h]]$ by a (non-commutative) polynomial in a and $\overline{a}$ with coefficients in $\Bbbk$.*

Proof. For any $n \geq 0$, we let $I_n = \sum_{r=0}^{n} a^{n-r}\overline{a}^r \in \mathcal{A}[[h]]$. In particular $I_0 = 1$, $I_1 = a + \overline{a}$. We have $I_n\overline{a} - \overline{a}^{n+1} = aI_n - a^{n+1}$. Let us write $Q(X) = \sum_{r=0}^{d} c_r X^r$ with $c_r \in \Bbbk$, and define $P = \sum_{r=0}^{d-1} c_{r+1} I_r$. Since

$$P\overline{a} = \sum_{r=0}^{d-1} c_{r+1} I_r \overline{a} = \sum_{r=0}^{d-1} c_{r+1} \left(\overline{a}^{r+1} + aI_r - a^{r+1} \right) = aP + Q(\overline{a}) - Q(a)$$

it follows that $P\overline{a} = aP$, and one easily checks $\overline{P} = Q'(\overline{a})$. $\qquad\square$

3.1.2. General setting

Let $\mathcal{A}$ be a $\Bbbk$-algebra with unit and G a group. We are interested in triples (R, ρ, N) such that N is a positive integer, $R : G \to GL_N(A)$ and $\rho : \mathcal{A} \to M_N(\Bbbk)$ are representations of G and $\mathcal{A}$ related by the following conditions:

(i) $\forall g \in G \quad R(g) \in \rho(\mathcal{A}) \otimes A$

(ii) $\overline{R(KG) \cap M_N(A)} = \rho(\mathcal{A})$

In the sequel, if the integer N is implicit we eventually drop the last index and consider *pairs* (R, ρ) satisfying (i) and (ii). The first condition implies that the coefficients in h of R (its infinitesimal data) belong to the

image of ρ; it also implies that the images of elements of the group algebra KG which belong to $M_N(A)$ have their reduction modulo h inside $\rho(\mathcal{A})$. Condition (ii) forces the reverse inclusion to be true.

We sometimes have to be more specific about this last condition. Let $\mathfrak{m} = hA$ be the maximal ideal of A. Let us assume that we are given a family $\mathcal{S}$ of generators for the $\Bbbk$-algebra $\mathcal{A}$, and a function $\mathsf{f} : \mathcal{A} \to KG$. Then consider the following condition on the pair (R, ρ).

$$(\text{ii})_{\mathsf{f}} \quad \forall j \in J \quad R(\mathsf{f}(a_j)) \in \rho(a_j) + M_N(\mathfrak{m})$$

It is clear that (i) and (ii)$_{\mathsf{f}}$ imply (i) and (ii). Conversely, if (R, ρ) satisfies (i) and (ii), then (R, ρ) satisfies (i) and (ii)$_{\mathsf{f}}$ for *some* f. The specification of the function f is useful in the study of intertwinners. For any triples (R_1, ρ_1, N_1) and (R_2, ρ_2, N_2), we shall denote by $\mathrm{Hom}_{\Bbbk}(\rho_1, \rho_2)$ the set of intertwinners of ρ_1 and ρ_2, and

$$\mathrm{Hom}_A(R_1, R_2) = \mathrm{Hom}_A(A^{N_1}, A^{N_2})^G,$$
$$\mathrm{Hom}_K(R_1, R_2) = \mathrm{Hom}_K(K^{N_1}, K^{N_2})^G.$$

We then have the following result.

Proposition 3.2. *Let (R_1, ρ_1, N_1) and (R_2, ρ_2, N_2) be triples satisfying (i) and (ii)$_{\mathsf{f}}$. Then $\mathrm{Hom}_A(R_1, R_2) = \mathrm{Hom}_{\Bbbk}(\rho_1, \rho_2) \otimes A$ and $\mathrm{Hom}_K(R_1, R_2) = \mathrm{Hom}_{\Bbbk}(\rho_1, \rho_2) \otimes K$.*

Proof. We identify $\mathrm{Hom}_A(A^{N_1}, A^{N_2})$ with $\mathrm{Hom}_{\Bbbk}(\Bbbk^{N_1}, \Bbbk^{N_2}) \otimes A$. Because of condition (i), $\mathrm{Hom}_A(R_1, R_2)$ contains $\mathrm{Hom}_{\Bbbk}(\rho_1, \rho_2) \otimes A$. We assume by contradiction that there exists $e \in \mathrm{Hom}_A(R_1, R_2)$ which does not belong to $\mathrm{Hom}_{\Bbbk}(\rho_1, \rho_2) \otimes A$. Since we already proved the reverse inclusion, and since A contains $\Bbbk[h]$, we may assume $\bar{e} \notin \mathrm{Hom}_{\Bbbk}(\rho_1, \rho_2)$. Then for every $a \in \mathcal{A}$ there exists $b \in AG$ and $r \geq 0$ such that $h^{-r} R_i(b) \in \rho_i(a) + M_{N_i}(\mathfrak{m})$ for $i \in \{1, 2\}$. Thus $eR_1(b) = R_2(b)e$ implies $\bar{e}\rho_1(a) = \rho_2(a)\bar{e}$, a contradiction. Finally, the identity

$$\mathrm{Hom}_K(R_1, R_2) = \mathrm{Hom}_A(R_1, R_2) \otimes K = (\mathrm{Hom}_{\Bbbk}(\rho_1, \rho_2) \otimes_{\Bbbk} A) \otimes_A K$$

proves the last assertion. $\square$

3.1.3. **Indecomposability and irreducibility**

Since every couple (R, ρ) satisfying (i) and (ii) satisfies (i) and (ii)$_{\mathsf{f}}$ for some f, proposition 3.2 proves in particular that $\mathrm{End}_A(R) = \mathrm{End}_{\Bbbk}(\rho) \otimes A$ and

$\operatorname{End}_K(R) = \operatorname{End}_{\Bbbk}(\rho) \otimes K$. Given such a pair (R, ρ), we would like to lift indecomposability, i.e. to relate the potential indecomposability of R and ρ. Here we have to be careful because, if ρ is indecomposable, then $\operatorname{End}_{\Bbbk}(\rho)$ is a local $\Bbbk$-algebra and, if $\operatorname{End}_{\Bbbk}(\rho)$ contains a non-invertible element $u \neq 0$, then $\operatorname{End}_K(R) = \operatorname{End}_{\Bbbk}(\rho) \otimes K$ is not local (indeed, $-u + h$ and $u + h$ are non-invertible elements with invertible sum), hence R is decomposable over K. The convenient definition here is to call R decomposable over A if A^N can be written as a direct sum of two $R(G)$-invariant (free) submodules. We then have

Proposition 3.3. *Let (R, ρ) be a pair satisfying (i) and (ii). Then ρ is indecomposable iff R is indecomposable over A.*

Proof. If ρ is decomposable, then R is decomposable because of condition (i). Now assume that R is decomposable. This means that there exists a non-trivial idempotent $p \in \operatorname{End}_A(R)$, then conjugated in $GL_N(A)$ to its constant term $\overline{p}$ by proposition 3.1. Thus $\overline{p}$ is a non-trivial idempotent in $\operatorname{End}_{\Bbbk}(\rho)$ by proposition 3.2, and ρ is decomposable. $\square$

By definition, a representation R over the local ring A is called (absolutely) irreducible if and only if it is (absolutely) irreducible over its quotient field K.

Proposition 3.4. *Let (R, ρ) be a pair satisfying (i) and (ii). Then ρ is irreducible iff R is irreducible.*

Proof. Let U be a proper $\mathcal{A}$-invariant subspace of $\Bbbk^N$. Then $U \otimes K \subset \Bbbk^N \otimes K = K^N$ is a proper subspace of K^N because $\dim_K U \otimes K = \dim_{\Bbbk} U$, and it is a G-invariant subspace because of property (i). Thus, if R is irreducible then ρ is irreducible. Conversely, let E be a proper and G-invariant subspace of K^N, and $E' = E \cap A^N$. Because of condition (i), A^N is a G-invariant lattice in K^N, and E' is G-invariant. Since E is proper, there exists $v \in E \setminus \{0\}$. Since v is non-zero there exists $r \in \mathbb{Z}$ such that $h^r v \in A^N$ and $\overline{h^r v} \neq 0$. It follows that $\overline{E'} \neq \{0\}$. Moreover, if $\overline{E'}$ were equal to $\Bbbk^N$, any $\Bbbk$-basis of $\Bbbk^N$ would lift to a K-basis of K^N in E, hence $E = K^N$. We thus proved that $\overline{E'}$ is a proper subspace of $\Bbbk^N$. Let us choose $a \in \mathcal{A}$, and prove $\rho(a)\overline{E'} \subset \overline{E'}$. Condition (ii) implies that there exists $b \in KG$ such that $R(b) \in M_N(A)$ and $\overline{R(b)} = \rho(a)$. Let $v_0 \in \overline{E'}$ and v be a lift of v_0 in $(v_0 + \mathfrak{m}^N) \cap E$. We have $R(b)v \in E \cap A^N = E'$, and $\overline{R(b)v} = \overline{R(b)}v_0 = \rho(a)v_0 \in \overline{E'}$. It follows that $\overline{E'}$ is a proper $\mathcal{A}$-invariant subspace of $\Bbbk^N$, and this concludes the proof. $\square$

When ρ is irreducible and $\Bbbk$ is algebraically closed, in which case ρ is absolutely irreducible, it is not clear a priori that R is absolutely irreducible, because $\Bbbk((h))$ is not algebraically closed. However, this is true:

Proposition 3.5. *Let (R, ρ) be a pair satisfying (i) and (ii). Then ρ is absolutely irreducible iff R is absolutely irreducible.*

Proof. Let us recall that ρ is absolutely irreducible if and only if it is surjective, and that R is absolutely irreducible if and only if $R : KG \to M_N(K)$ is surjective. If $R : KG \to M_N(K)$ is surjective, then for every $m \in M_N(\Bbbk) \subset M_N(K)$, there exists $b \in KG$ such that $R(b) = m$. Because of condition (i), this implies that m belongs to the image of ρ. Conversely, let us assume that ρ is surjective. There exists then $a_1, \ldots, a_{N^2} \in \mathcal{A}$ such that $\rho(a_1), \ldots, \rho(a_{N^2})$ form a K-basis of the K-vector space $M_N(K) \simeq K^{N^2}$. Let $b_1, \ldots b_{N^2}$ be the corresponding elements in KG given by (ii) such that $R(b_i) \in M_N(A)$ and $\overline{R(b_i)} = \rho(a_i)$ for $1 \leq i \leq N^2$. Because of condition (ii), the determinant of the family $R(b_1), \ldots, R(b_{N^2})$ is an element of A with invertible constant term, hence is invertible in K. This family then forms a K-basis of $M_N(K)$, thereby proving that R is absolutely irreducible. $\square$

3.2. **Representations of B_n**

Let $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ with $\lambda \in \Bbbk^\times$. This associator Φ defines a morphism from the group algebra $\Bbbk B_n$ to $\widehat{\mathfrak{B}_n}$, which is a Hopf algebra morphism for the canonical structures on $\Bbbk B_n$ and $\widehat{\mathfrak{B}_n}$. On the other hand, if $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$ is a representation of $\mathfrak{B}_n$ over $\Bbbk$, we may extend ρ to a representation ρ' of $\mathfrak{B}_n$ over $\Bbbk[h]$ by $\rho'(t_{ij}) = h\rho(t_{ij})$, hence to a representation ρ'' of $\widehat{\mathfrak{B}_n}$ over $\Bbbk[[h]]$. Let $\widehat{\Phi}(\rho)$ be the induced representation of B_n over $\Bbbk[[h]]$. Let A be a discrete valuation subring of $\Bbbk[[h]]$ such that for all $N > 0$ and all $\rho : \mathfrak{B}_n \to M_N(A)$, we have $\widehat{\Phi}(\rho)(B_n) \subset GL_N(A)$. For instance, if $\Bbbk = \mathbb{R}$ and $\Phi \in \mathbb{Assoc}_\lambda(\mathbb{R})$ is convergent, $A = \Bbbk\{h\}$ may be chosen. We study general properties of these functors $\widehat{\Phi}$, from the category of finite-dimensional $\mathfrak{B}_n$-modules to the category of representations of G over A.

3.2.1. Linear representations

Since the Drinfeld morphism $\Bbbk B_N \to \widehat{\mathfrak{B}_n}$ preserves coproducts and antipodes, the functor $\widehat{\Phi}$ commutes with tensor products and duals: for any representations ρ_1, ρ_2 of $\widehat{\mathfrak{B}_n}$,

$$\widehat{\Phi}(\rho_1 \otimes \rho_2) = \widehat{\Phi}(\rho_1) \otimes \widehat{\Phi}(\rho_2), \quad \widehat{\Phi}(\rho^\vee) = \widehat{\Phi}(\rho)^\vee.$$

Let $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$ be a representation of $\mathfrak{B}_n$, $R = \widehat{\Phi}(\rho)$. Then the pair (R, ρ) satisfies the properties (i) and (ii) of the previous section, for $\mathcal{A} = \mathfrak{B}_n$ and $G = B_n$. Moreover, it satifies (ii)$_f$ for a universal function f independent of the given pair. Indeed, let us choose as generators of $\mathfrak{B}_n$ the set $\mathcal{S} = \{s_i \mid 1 \leq i \leq n-1\} \cup \{t_{12}\}$. Then $R(\sigma_i) \in \rho(s_i) + hM_N(A)$ and $R(\sigma_1^2) \in 1 + 2\lambda h\rho(t_{12}) + h^2 M_N(A)$, so a convenient f is defined by $f(s_i) = \sigma_i$ and $f(t_{12}) = (\sigma_1^2 - 1)/2\lambda h$. Consequences of the above results can be condensed in the following theorem.

Theorem 3.6. *Let $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ with $\lambda \in \Bbbk^\times$. Then $\widehat{\Phi}$ preserves indecomposability, irreducibility and absolute irreducibility. It commutes with direct sums, tensor products and duals. Moreover, for any given representations $\rho_1 : \mathfrak{B}_n \to M_{N_1}(\Bbbk)$ and $\rho_2 : \mathfrak{B}_n \to M_{N_2}(\Bbbk)$, we have*

$$
\begin{aligned}
\mathrm{Hom}_A(\widehat{\Phi}(\rho_1), \widehat{\Phi}(\rho_2)) &= \mathrm{Hom}_\Bbbk(\rho_1, \rho_2) \otimes A, \\
\mathrm{Hom}_K(\widehat{\Phi}(\rho_1), \widehat{\Phi}(\rho_2)) &= \mathrm{Hom}_\Bbbk(\rho_1, \rho_2) \otimes K.
\end{aligned}
$$

In addition, note that $\widehat{\Phi}$ is exact, since the functor $\_ \otimes A$ from the category of finite dimensional vector spaces over $\Bbbk$ to the category of free A-modules is exact. Finally, $\widehat{\Phi}$ obviously commutes with the natural restrictions corresponding to the removal of the last strand — with a slight imprecision in the notations, we may write:

$$\mathrm{Res}_{B_{n-1}} \widehat{\Phi}(\rho) = \widehat{\Phi}\left(\mathrm{Res}_{\mathfrak{B}_{n-1}} \rho\right).$$

The determination of the field of matrix coefficients of $\widehat{\Phi}(\rho)$ is a far more delicate matter. A coarse lower bound for its transcendance degree over $\Bbbk$ is given by the trivial case $n = 2$: since B_2 is isomorphic to $\mathbb{Z}$, we only have to determine the field L of matrix coefficients of $\exp(hX)$ when $X \in M_N(\Bbbk)$ is known. At least if $\Bbbk$ is algebraically closed, a straightforward use of Jordan canonical form shows that, if the spectrum $Sp(X)$ of X equals $\{a_1, \ldots, a_r\}$, then $L = \Bbbk(e^{ha_1}, \ldots, e^{ha_r})$ if X is semisimple, and $L = \Bbbk(h, e^{ha_1}, \ldots, e^{ha_r})$ otherwise. In particular, writing $\dim_\mathbb{Q} S$ for the

dimension of the $\mathbb{Q}$-vector space spanned by $S \subset \mathbb{k}$, its transcendence degree over $\mathbb{k}$ is equal to $\dim_{\mathbb{Q}} Sp(X)$ if X is semisimple, and $\dim_{\mathbb{Q}} Sp(X)+1$ otherwise.

This results extends to general n if $\rho(\mathcal{T}_n)$ is commutative; in this case $\Phi(\rho(t_{i,i+1}), \rho(Y_i)) = 1$ for $1 \leq i \leq n-1$, $\widehat{\Phi}(\rho)$ does not depend on Φ and factorizes through $\widetilde{\mathfrak{S}_n}$. In the general case, it seems difficult to get a non-trivial upper bound for the transcendence degree; a thiner lower bound is given by taking into account the spectrum of the $\rho(Y_i)'s$, since $\widehat{\Phi}(\rho)(\delta_i) = \exp(2\lambda h \rho(Y_i))$. Unfortunately, these eigenvalues are usually rational linear combinations of $Sp(\rho(t_{12}))$ — see appendix B — so this may not improve the lower bound at all.

A last feature of this construction, and another consequence of the lifting properties is the following. Let $\mathcal{A}$ be one of the envelopping algebras $\mathsf{U}\mathcal{T}_n$, $\mathsf{U}\mathcal{F}_n$, $\mathsf{U}\mathcal{I}_n$, where we denote by $\mathcal{F}_n$ (resp. $\mathcal{I}_n$) the Lie subalgebra of $\mathcal{T}_n$ generated by the $t_{i,n}$ (resp. $t_{i,i+1}$) for $1 \leq i \leq n-1$, and G be the corresponding group P_n, F_n or I_n, where we denote by F_n (resp. I_n) the subgroup of P_n generated by $\xi_{i,n}$ (resp. σ_i^2) for $1 \leq i \leq n-1$. Let $\rho : \mathfrak{B}_n \to M_N(\mathbb{k})$ be a representation and consider $R = \widehat{\Phi}(\rho)$. It can be easily checked that

$$\widehat{\Phi}(\rho)(\xi_{i,j}) \in 1 + 2\lambda h \rho(t_{ij}) + h^2 M_N(A)$$

whence an immediate consequence of the lifting properties is the following.

Proposition 3.7. *Let* $\lambda \in \mathbb{k}^\times$, $\Phi \in \mathbb{Assoc}_\lambda(\mathbb{k})$ *and* $\rho : \mathfrak{B}_n \to M_N(\mathbb{k})$. *Let* $(H, \mathfrak{g})$ *be one of the following pairs:* $(P_n, \mathcal{T}_n)$, $(F_n, \mathcal{F}_n)$, $(I_n, \mathcal{I}_n)$. *Then* $\mathrm{Res}_H \widehat{\Phi}(\rho)$ *is indecomposable (resp. irreducible, absolutely irreducible) if and only if* $\mathrm{Res}_{\mathfrak{g}} \rho$ *is so.*

3.2.2. **Orthogonal, symplectic and unitary representations**

An immediate corollary of theorem 3.6 is that the functors $\widehat{\Phi}$ for $\Phi \in \mathbb{Assoc}_\lambda(\mathbb{k})$, $\lambda \in \mathbb{k}^\times$, preserve semisimplicity. It is thus a natural question to ask whether B_n-invariant bilinear or sesquilinear form can be detected at the infinitesimal level. This is all the more challenging that it appeared in the past twenty years, starting from the Squier form on the Burau representation, that the most classical representations of the braid group are naturally endowed with a "unitary" structure. In order to answer this question, we first need some notations.

Let L be a field of characteristic 0, N a positive integer and β a non-degenerate bilinear form on L^N. To any $x \in M_N(L)$ we associate its transpose $x^\dagger$ with respect to $\beta : \beta(xu, v) = \beta(u, xv)$. As usual, $x \in M_N(L)$ is called symmetric if $x^\dagger = x$, antisymmetric if $x^\dagger = -x$ and isometric if $x^\dagger x = 1$. If β is symmetric, we denote by $O_N(L, \beta)$ the group of isometries in $GL_N(L)$. If β is skew-symmetric, we denote it by $SP_N(L, \beta)$. If L is embedded in some larger field L', then β is trivially extended to a non-degenerate bilinear form β' on $(L')^N$. For the sake of simplicity, we write $O_N(L', \beta) = O_N(L', \beta')$ and $SP_N(L', \beta) = SP_N(L', \beta')$. Now assume that L admits an involutive non-trivial field automorphism ϵ. Given a non-degenerate symmetric bilinear form β on L^N, we make ϵ act on $M_N(L)$ coefficientwise and define
$$U_N^\epsilon(L, \beta) = \{x \in GL_N(L) \mid \epsilon x^\dagger = x^{-1}\}.$$
If β is implicit or is the standard bilinear form on L^N we simply write $U_N^\epsilon(L)$. In particular, the ordinary unitary group U_N is defined as $U_N^{z \mapsto \bar{z}}(\mathbb{C})$. Recall that the Squier form of the Burau representation (see [34]) sends B_n to $U_n^\epsilon(\mathbb{R}(q))$ where $\epsilon \in \mathrm{Gal}(\mathbb{R}(q)/\mathbb{R})$ is defined by $\epsilon(q) = q^{-1}$, and ordinary unitary representations are obtained by specializing at $q \in \mathbb{C}$ of modulus 1. On K, we will consider the field automorphism ϵ defined by $f(h) \mapsto f(-h)$, i.e. the only automorphism in $\mathrm{Gal}(K/\Bbbk)$ continuous for the h-adic topology such that $\epsilon(h) = -h$.

The conditions on the infinitesimal data in order for our construction to provide orthogonal, symplectic or unitary representations of B_n are the following ones.

Definition 3.8. A representation $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$ is said to be orthogonal (resp. symplectic) if there exists a symmetric (resp. skew-symmetric) non-degenerate bilinear form β on $\Bbbk^N$ such that $\rho(t_{12})^\dagger = -\rho(t_{12})$ and $\rho(\mathfrak{S}_n)$ is contained in $O_N(K, \beta)$ (resp. $SP_N(K, \beta)$). In case β is symmetric, ρ is said to be unitary if $\rho(t_{12})^\dagger = \rho(t_{12})$ and $\rho(\mathfrak{S}_n) \subset O_N(\Bbbk, \beta)$.

Note that the condition $\rho(t_{12})^\dagger = \pm\rho(t_{12})$ implies that $\rho(t_{ij})^\dagger = \pm\rho(t_{ij})$ for all $1 \leq i, j \leq n$, since $\rho(s)$ is isometric for all $s \in \mathfrak{S}_n$. Under these conditions, we have the following properties.

Proposition 3.9. *Let $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$ be orthogonal (resp. symplectic) with respect to β, and $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ with $\lambda \in \Bbbk$. Then $\widehat{\Phi}(\rho)$ factorizes through $O_N(K, \beta)$ (resp. $SP_N(K, \beta)$).*

Proof. It is sufficient to show that $\widehat{\Phi}(\rho)(\sigma_i)$ is isometric for all $1 \leq i < n$. For this, we use its explicit expression. Since $O_N(\Bbbk, \beta) \subset O_N(K, \beta)$ and $SP_N(\Bbbk, \beta) \subset SP_N(K, \beta)$, $\rho(s_i)$ is isometric. The same assertion for $\exp(\lambda h \rho(t_{ij}))$ is standard under this hypothesis. Now we have to show that $\Phi(hx, hy)$ for $x, y \in M_N(\Bbbk)$ is isometric as soon as $x^\dagger = -x$, $y^\dagger = -y$. This follows from the fact that Φ is the exponential of a Lie series Ψ, and that the set of skew-symmetric elements forms a Lie subalgebra of $M_N(\Bbbk) = \mathfrak{gl}_n(\Bbbk)$. Hence $\Psi(hx, hy)^\dagger = -\Psi(hx, hy)$ and $\Phi(hx, hy)^\dagger = \Phi(hx, hy)^{-1}$. $\qquad\square$

Proposition 3.10. *Let $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$ be unitary with respect to β, and $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ with $\lambda \in \Bbbk$. Then $\widehat{\Phi}(\rho)$ factorizes through $U_N^\epsilon(K, \beta)$, with $\epsilon : f(h) \mapsto f(-h)$.*

Proof. It is sufficient to show that each $\widehat{\Phi}(\rho)(\sigma_i)$ belong to $U_N^\epsilon(K, \beta)$. We have $\rho(s_i) \in O_N(\Bbbk, \beta) \subset U_N^\epsilon(K, \beta)$ and, since $\rho(t_{ij})^\dagger = \rho(t_{ij})$, then $\exp(\lambda h \rho(t_{i,i+1})) \in U_N^\epsilon(K, \beta)$. It remains to show that $\Phi(hx, hy)$ belongs to $U_N^\epsilon(K, \beta)$ as soon as x and y are selfadjoint. We know that Φ is the exponential of a Lie series Ψ. Let us introduce the outer Lie algebra automorphism $\tau : x \mapsto -x^\dagger$ of $\mathfrak{gl}_N(\Bbbk)$, trivially extended to $\mathfrak{gl}_N(K) = \mathfrak{gl}_N(\Bbbk) \otimes K$. One has $\tau(x) = -x$, $\tau(y) = -y$, and

$$-\Psi(hx, hy)^\dagger = \tau\Psi(hx, hy) = \Psi(h\tau x, h\tau y) = \Psi(-hx, -hy)$$

hence $\Psi(hx, hy)^\dagger = -\Psi(-hx, -hy)$ and $\Phi(hx, hy)^\dagger = \Phi(-hx, -hy)^{-1} = \epsilon\Phi(hx, hy)$ so that $\Phi(hx, hy) \in U_N^\epsilon(K, \beta)$. $\qquad\square$

In case $\Bbbk$ is a subfield of $\mathbb{R}$, the field of coefficients of $\widehat{\Phi}(\rho)$ is a finitely generated extension of $\Bbbk$ contained in $\mathbb{R}((h))$, not necessarily in the field of convergent power series $\mathbb{R}(\{h\})$. We show in the appendix how this representation may be twisted by field isomorphisms in order to get convergent coefficients. It is then possible to get ordinary unitary representations of B_n from representations of B_n into $U_N^\epsilon(K)$, by specialization in h small and real — we refer to appendix A for more details on this topic.

4. Variety of braided extensions

One of the major discomforts in the study of braid group representations is that the extension $1 \to P_n \to B_n \to \mathfrak{S}_n \to 1$ of the symmetric group is not split. In particular we can not use the well-known representation

theory of the symmetric group and study a restriction to the symmetric group of a given representation of B_n.

Things change drastically with the infinitesimal version $\mathfrak{B}_n$, which is a semi-direct product. It then appears an important algebraic variety associated to any $\Bbbk\mathfrak{S}_n$-module M. Its set of $\Bbbk$-points is formed by all representations of $\mathfrak{B}_n$ whose restriction to $\Bbbk\mathfrak{S}_n$ is M. For reasons that we shall make clear later, we call this variety the *variety of braided extensions* of the module M.

Notations: For $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_{>0}^r$ of size $\alpha_1 + \cdots + \alpha_r = n$, we let $\mathfrak{S}_{\underline{\alpha}}$ be the corresponding Young subgroup of $\mathfrak{S}_n$. In particular, $\mathfrak{S}_{2,n-2} \simeq \mathfrak{S}_2 \times \mathfrak{S}_{n-2}$.

4.1. Definition and elementary properties

Let M be a finite-dimensional $\Bbbk\mathfrak{S}_n$-module. We define

$$\mathcal{V}(M) = \{\rho : \mathfrak{B}_n \to \mathrm{End}_{\Bbbk}(M) \mid \mathrm{Res}_{\mathfrak{S}_n}\rho = M\}.$$

In order to have at disposal a more tractable definition of the same object, we introduce the following algebraic functions on $\mathrm{End}_{\Bbbk}(M)$: $C_1(x) = [x, (1\ 3).x + (2\ 3).x]$, $C_2(x) = [x, (1\ 3)(2\ 4).x]$, where we denote by $x \mapsto s.x$ the action of $s \in \mathfrak{S}_n$ on $\mathrm{End}_{\Bbbk}(M) = M \otimes M^\vee$. We let C_2 (resp. C_1) be zero if $n < 4$ (resp. $n < 3$).

Lemma 4.1. *The map $\rho \mapsto \rho(t_{12})$ defines a bijection from $\mathcal{V}(M)$ to the set $\{\tau \in \mathrm{End}_{\mathfrak{S}_{2,n-2}}(M) \mid C_1(\tau) = C_2(\tau) = 0\}$.*

Proof. Let $\mathcal{S}$ be the set introduced in the statement. If $\rho \in \mathcal{V}(M)$ and $\tau = \rho(t_{12})$, we have $C_1(\tau) = \rho([t_{12}, t_{13} + t_{23}])$, $C_2(\tau) = \rho([t_{12}, t_{34}])$ so $\tau \in \mathcal{S}$ and the map is well-defined. Since t_{12} and $\mathfrak{S}_n$ generate $\mathfrak{B}_n$ it is injective. Surjectivity is a consequence of Frobenius reciprocity law

$$\begin{aligned}
\mathrm{End}_{\mathfrak{S}_{2,n-2}}(M) &= \mathrm{Hom}_{\mathfrak{S}_{2,n-2}}\left(\mathbb{1}, \mathrm{Res}_{\mathfrak{S}_{2,n-2}}\mathrm{End}_{\Bbbk}(M)\right) \\
&\simeq \mathrm{Hom}_{\mathfrak{S}_n}\left(\mathrm{Ind}_{\mathfrak{S}_{2,n-2}}^{\mathfrak{S}_n}\mathbb{1}, \mathrm{End}_{\Bbbk}(M)\right).
\end{aligned}$$

Indeed, let $e_{\{i,j\}}$ for $1 \le i \ne j \le n$ be a basis of the induction of the trivial representation of $\mathfrak{S}_{2,n-2}$, such that the action of $\mathfrak{S}_n$ is given by $s.e_{\{i,j\}} = e_{\{s(i),s(j)\}}$. It is readily checked that the canonical isomorphism given by Frobenius reciprocity law is such that $\tau \in \mathrm{End}_{\mathfrak{S}_{2,n-2}}(M)$ is sent to a morphism φ_τ satisfying $\varphi_\tau(e_{\{1,2\}}) = \tau$, whence $\rho(t_{ij}) = \varphi_\tau(e_{\{i,j\}})$

defines a preimage of τ in $\mathcal{V}(M)$ — all the defining relations of $\mathcal{T}_n$ being consequences of $[t_{12}, t_{13} + t_{23}] = [t_{12}, t_{34}] = 0$ by $\mathfrak{S}_n$-invariance. $\qquad\square$

This correspondance thus endows $\mathcal{V}(M)$ with the structure of an affine algebraic variety over the field of definition of M. Since all ordinary representations of the symmetric group can be defined over $\mathbb{Q}$, we usually consider $\mathcal{V}(M)$ as an affine algebraic variety over $\mathbb{Q}$. As a set, that is when this algebraic structure is not considered, we simply denote by $\mathcal{V}(M)$ the set of $\Bbbk$-points of this algebraic variety. Moreover, we identify both sets under this correspondance, and denote an element of $\mathcal{V}(M)$ indifferently by $\rho : \mathfrak{B}_n \to \mathrm{End}_{\Bbbk}(M)$ or $\tau \in \mathrm{End}_{\Bbbk}(M)$. We let $N = \dim M$, $A = \Bbbk[[h]]$, $K = \Bbbk((h))$. Any $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ sends $\rho \in \mathcal{V}(M)$ to a K-point of the variety of N-dimensional representations of B_n. The images of $\mathcal{V}(M_1)$ and $\mathcal{V}(M_2)$ under this map are disjoint provided that M_1 is not isomorphic to M_2. Indeed, let χ be the character of $\mathfrak{S}_n$ associated to the $\mathfrak{S}_n$-module M. Then the trace of $\widehat{\Phi}(\rho)(\sigma)$ for $\sigma \in B_n$ belongs to A and its reduction modulo h is $\chi \circ \pi(\sigma)$, hence χ is determined by $\widehat{\Phi}(\rho) \in \mathrm{Hom}(B_n, GL_N(K))$ and so is M up to isomorphism. In accordance with the general conventions about variety of representations, we consider *absolutely* irreducible representations instead of irreducible ones. In case $\lambda \in \Bbbk^\times$, the elements in $\mathcal{V}(M)$ which correspond to these are precisely the absolutely irreducible representations of $\mathfrak{B}_n$, i.e. elements $\rho \in \mathcal{V}(M)$ which are surjective. We denote this subset (resp. open subvariety) of $\mathcal{V}(M)$ by $\mathcal{V}^s(M)$. There exists a scheme morphism $\mathcal{V}(M) \to \mathbb{A}_1$ given by the trace. All its fibers are isomorphic and for all $l \in \Bbbk$ and $\tau \in \mathcal{V}(M)$ we have $\tau + l \in \mathcal{V}(M)$. These fibers as well as the open subvariety $\mathcal{V}^s(M)$ are invariant under the action of the reductive group $G(M) = \mathrm{Aut}_{\mathfrak{S}_n}(M)$ and the schematic quotient $\mathcal{V}^s(M)//G(M)$ is the natural object of study for classification purposes.

The tensor product of representations of B_n admits as infinitesimal counterparts the morphisms $\mathcal{V}(M_1) \times \mathcal{V}(M_2) \to \mathcal{V}(M_1 \otimes M_2)$ given by $(\tau_1, \tau_2) \mapsto \tau_1 \otimes 1 + 1 \otimes \tau_2$, which restrict to maps $\mathcal{V}_0(M_1) \times \mathcal{V}_0(M_2) \hookrightarrow \mathcal{V}_0(M_1 \otimes M_2)$, where we let $\mathcal{V}_0(M)$ be the set of all $\tau \in \mathcal{V}(M)$ with zero trace. These maps are injective on the $\Bbbk$-points, and closed immersions at the algebraic level. The dual corresponds to the involution $\tau \mapsto -\tau$ on $\mathcal{V}(M)$. Also note that $\tau \in \mathcal{V}(M)$ has trace $\alpha \in \Bbbk$ iff the corresponding representation of B_n deduced from some $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ is such that (one

whence) every σ_i act with determinant $\exp(\lambda h\alpha)$. In view of the transcendance degree of the field of matrix coefficients, another important information encoded in $\mathcal{V}(M)$ is the dimension of the $\mathbb{Q}$-vector space spanned by the eigenvalues of $\tau \in \mathcal{V}(M)$ — i.e. the roots of its characteristic polynomial. As an illustration, let us look for monic polynomials $P \in \mathbb{Q}[X]$ of degree $N > 1$, such that the $\mathbb{Q}$-linear span of its roots has dimension N. First P cannot admit any rational root, for the coefficient of X^{N-1} in P would provide a rational linear relation. For the same reason, this coefficient cannot be zero. Finally, P of course has to be prime with its derivative P'. These conditions are not yet sufficient (see $P = (X-1)^4 - 2$). They nevertheless imply that this $\mathbb{Q}$-linear span is at least 2-dimensional. Let $\alpha_1, \ldots, \alpha_N \in \overline{\mathbb{Q}}$ be the roots of P and $L = \mathbb{Q}(\alpha_1, \ldots, \alpha_N)$. A sufficient additional condition is that $G = \mathrm{Gal}(L/\mathbb{Q})$ acts 2-transitively on the roots. It is indeed a classical fact that the permutation $\mathbb{Q}G$-module with basis $\{e_{\alpha_i} \mid 1 \le i \le N\}$ and action $g.e_{\alpha_i} = e_{g(\alpha_i)}$ then splits into two components, the invariant vector $e_{\alpha_1} + \cdots + e_{\alpha_N}$ and a $(N-1)$-dimensional *irreducible* supplement. The kernel of its natural projection onto the $\mathbb{Q}$-linear span of the roots do not contain the invariant vector (the coefficient of X^{N-1} is non-zero) and do not contain its supplement because the image has to be of dimension at least 2. Then this morphism is bijective and the $\mathbb{Q}$-linear span of the roots has dimension N. Other standard properties that can be read on $\mathcal{V}(M)$ are dealt with in the next section.

The study of $\mathcal{V}(M) = \mathcal{V}^s(M)$ for irreducible M was carried out in [20] and the result is exposed in [19]. For example, if M is the irreducible 2-dimensional representation of $\mathfrak{S}_3$, there exist matrix models of M such that $(1\ 2)$ acts as the diagonal matrix $\mathrm{diag}(1, -1)$, and the $\mathfrak{S}_{2,n-2}$-invariance condition implies that $\tau = \mathrm{diag}(a, b)$ for $a, b \in \Bbbk$. We then have $G(M) \simeq \mathbb{G}_m$ and $\mathcal{V}(M) = \mathcal{V}^s(M) = \mathcal{V}^s(M)//G(M) \simeq \mathbb{A}_2$. We recall from this earlier work that, even when M is irreducible, the variety $\mathcal{V}^s(M)$ may admit several irreducible components. As a concrete example for non-irreducible M, we now study the case where M is the sum of the 2-dimensional irreducible representation of $\mathfrak{S}_3$ and of the trivial one. Assuming that $\Bbbk$ contains square roots of 3, we use Young orthogonal models to write down matrices for s_1, s_2 and for the general form of

$\tau \in \mathrm{End}_{\mathfrak{S}_{2,1}}(M)$.

$$s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \tau = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & l \end{pmatrix}$$

The representation is absolutely irreducible iff $b \neq 0$ and $c \neq 0$. Note that two such τ's define isomorphic representations iff the products bc are the same. An easy calculation shows that, assuming $bc \neq 0$, $\tau \in \mathcal{V}(M)$ iff $2a - d - l = 0$. The spectrum of τ then is $\{2a - d, \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}\}$. In view of the field of matrix coefficients for the associated representations of B_n we note that, provided $\Bbbk$ is large enough, the parameters can be chosen so that these three eigenvalues are linearly independent over $\mathbb{Q}$. For instance, if $\Bbbk \supset \mathbb{Q}(\sqrt{2}, \sqrt{3})$ we may choose $a = 1$, $d = \sqrt{2}$, $bc = 1/\sqrt{2}$.

If the original $\Bbbk\mathfrak{S}_n$-module M is endowed with an $\mathfrak{S}_n$-invariant non-degenerate symmetric bilinear form β, we let M_β be the corresponding structure and $\mathcal{V}(M_\beta) = \{\tau \in \mathcal{V}(M) \mid \tau^\dagger = \tau\}$. Elements of $\mathcal{V}(M_\beta)$ give rise to unitary representation of B_n because of proposition 3.10. In the example above, with β being the standard scalar product on $\Bbbk^3$, $\tau \in \mathcal{V}(M_\beta)$ iff $b = c$. If $\Bbbk = \mathbb{R}$ note in addition that, for any $\tau \in \mathcal{V}(M)$ the set $G(M).\tau \cap \mathcal{V}(M_\beta)$ is non-empty as soon as $bc > 0$. By comparison, the elements $\tau \in \mathcal{V}(M)$ giving rise to orthogonal representations, i.e. such that $\tau^\dagger = -\tau$, form a family of smaller dimension: they have to fulfill the conditions $a = d = l = 0$, $b = -c$ and their eigenvalues can never be linearly independent over $\mathbb{Q}$.

We end this section by a rough geometric description of $\mathcal{V}(M)$ and the related varieties in case M is a $\mathbb{Q}\mathfrak{S}_n$-module with $\dim_{\mathbb{Q}} M = 4$ and $n = 3$. Our motivation is to illustrate how these varieties may depend on the original $\mathbb{Q}\mathfrak{S}_n$-module. We identify partitions of 3 and representations of $\mathfrak{S}_3$: in particular we respectively denote by $[3]$ and $[1^3]$ the trivial and the sign representation.

First consider $M = 2[2, 1]$. Then $\mathrm{End}_{\mathfrak{S}_{2,n-2}}(M) \simeq \mathbb{A}_8$. Let $p : \mathbb{A}_8 \to \mathbb{A}_6$ be the map $(y_1, \ldots, y_8) \mapsto (y_1 - y_2, y_3 - y_4, y_5, \ldots, y_8)$ and $S \subset \mathbb{A}_6$ be the vanishing variety of the polynomials $x_3 x_6 - x_4 x_5$, $x_5 x_1 - x_3 x_2$, $x_4 x_2 - x_1 x_6$. Then $\mathcal{V}(M) \simeq p^{-1}(S)$ and $\mathcal{V}^s(M) = \emptyset$. Let now $M = [2, 1] + [3] + [1^3] = [2, 1]^{\otimes 2}$. Again $\mathrm{End}_{\mathfrak{S}_{2,n-2}}(M) \simeq \mathbb{A}_8$. Let $p : \mathbb{A}_8 \to \mathbb{A}_6$ be the map $(y_1, \ldots, y_8) \mapsto (y_1 + y_2 - 2y_3, y_1 + y_2 - 2y_4, y_5, \ldots, y_8)$ and $S \subset \mathbb{A}_6$ be the vanishing variety of the polynomials $y_1 y_6, y_2 y_6, y_2 y_8, y_1 y_7$. Then $\mathcal{V}(M) \simeq$

$p^{-1}(S)$ and $\mathcal{V}^s(M) \simeq p^{-1}(S')$ where $S' \subset S \subset \mathbb{A}_6$ is $\{(0,0)\} \times \mathbb{G}_m^4$, hence $\mathcal{V}^s(M) \simeq \mathbb{A}_2 \times \mathbb{G}_m^4$. We have $G(M) \simeq \mathbb{G}_m^2$ and $\mathcal{V}^s(M)//G(M) \simeq \mathbb{A}_2 \times \mathbb{G}_m^2$. Let (x, y, u, v) be a $\Bbbk$-point in $\mathbb{A}_2 \times \mathbb{G}_m^2$. Then the spectrum of 4τ under this isomorphism is $\{3x + y \pm \sqrt{(x-y)^2 + u}, 3y + x \pm \sqrt{(x-y)^2 + v}\}$. If $\Bbbk$ is large enough (e.g. $\Bbbk \supset \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$) it is possible to get four eigenvalues linearly independent over $\mathbb{Q}$ (take $x = 1$, $y = 1 + \sqrt{3}$, $u = 1$, $v = 2$). Moreover, $2t_{12} + 2t_{13} + 2t_{23}$ acts by the scalar $3(x + y)$, which is not an integer linear combination of the eigenvalues of τ. It follows that the field of matrix coefficients of $R = \widehat{\Phi}(\rho)$ for $\Phi \in \mathbb{A}\mathrm{ssoc}_\lambda(\Bbbk)$, $\lambda \in \Bbbk^\times$ is larger than the field generated by the eigenvalues of $R(\sigma_1)$, since $R(\gamma_3)$ is then the exponential of $3\lambda h(x + y)$.

Let us now consider the morphism $\psi : \mathcal{V}([2,1])^2 \to \mathcal{V}(M)$. The intersection of the image of ψ with $\mathcal{V}^s(M)$ is projected in $\mathcal{V}^s(M)//G(M) \simeq \mathbb{A}_2 \times \mathbb{G}_m^2$ on quadruples (x, y, u, v) such that $x = y$, whence the corresponding eigenvalues for τ cannot be linearly independent over $\mathbb{Q}$. Likewise, for $M = [2,1] + 2[3]$ it is not possible to get linearly independent eigenvalues over $\mathbb{Q}$. More precisely, if $\tau \in \mathcal{V}^s(M)$ and R is a representation of B_3 corresponding to it, the spectrum $\{a, b, c, d\}$ of $R(\sigma_1)$ satisfy the algebraic relation $cd = a^4 b^6$. A similar result of course holds for $M = [2,1] + 2[1^3]$. Finally, in case M is a sum of 1-dimensional irreducible representations, one easily gets $\mathcal{V}^s(M) = \emptyset$. It follows that the only means to get by these methods a 4-dimensional linear irreducible representation of B_3 such that σ_1 has algebraically independent eigenvalues is to start with $M = [2,1] + [3] + [1^3]$. We shall use this result later (see section 6.1.1).

4.2. Quotients of the braid group

Let M be a $\Bbbk\mathfrak{S}_n$-module, $\tau \in \mathcal{V}^s(M)$ and $\rho : \mathfrak{B}_n \to \mathrm{End}_\Bbbk(M)$ be the corresponding representation with $\tau = \rho(t_{12})$. We choose once and for all $\Phi \in \mathbb{A}\mathrm{ssoc}_\lambda(\Bbbk)$ with $\lambda \in \Bbbk^\times$ and let $R = \widehat{\Phi}(\rho)$. We look for conditions on τ in order to characterize factorizations of R through the usual quotients $Hurw_n$, $\mathbb{Z} = B_n/[B_n, B_n]$ and $\widetilde{\mathfrak{S}_n}$. We also consider the map $\pi \times l$ from B_n to $\mathfrak{S}_n \times \mathbb{Z}$. We let T be the sum of the $\rho(t_{ij})$ for $1 \leq i < j \leq n$ and recall that $\overline{R}$ designates the reduction modulo h of R. We assume that $n \geq 3$ and state the results as a list of lemmas.

Lemma 4.2. *We have $tr(\tau) = 0$ iff $T = 0$ iff R factorizes through $B_n/Z(B_n)$.*

Proof. Indeed, this element T commutes with $\rho(\mathfrak{B}_n)$, hence is scalar because ρ is absolutely irreducible. It is then zero if and only if its trace is zero. But since all the $\rho(t_{ij})$'s are conjugated in $\mathfrak{B}_n$, it follows that $tr(T) = 0$ iff $tr(\tau) = 0$. The last assertion is a direct consequence of $R(\gamma_n) = \exp(2\lambda h T)$. $\qquad\square$

Lemma 4.3. *R factorizes through $Hurw_n$ iff $\rho(Y_n) = 0$ iff $\rho(Y_n)$ commutes with $\rho(\mathfrak{S}_n)$ and $tr(\tau) = 0$.*

Proof. The first part comes from $R(\delta_n) = \exp(2\lambda h \rho(Y_n))$. If $\rho(Y_n) = 0$, it of course commutes with $\rho(\mathfrak{S}_n)$. Conversely, for $n \geq 3$, if $\rho(Y_n)$ commutes with $\mathfrak{S}_n$, it also commutes with $\rho(\mathfrak{B}_n)$ since it commutes with $\rho(t_{12})$. It is then a scalar, and the equality $tr(\rho(Y_n)) = (n-1)tr(\tau)$ concludes the proof. $\qquad\square$

Lemma 4.4. *τ commutes with $\rho(\mathfrak{S}_n)$, i.e. $\rho(t_{ij}) = \tau$ for all $1 \leq i < j \leq n$, iff for all $\sigma \in B_n$ we have $R(\sigma) = \overline{R}(\sigma)z^{l(\sigma)}$ for some $z \in GL_N(A)$.*

Proof. If t commutes with $\rho(\mathfrak{S}_n)$, $R(\sigma_i) = \rho(s_i)\exp(\lambda h t)$ and the conclusion follows with $z = \exp(\lambda h t)$. Conversely, if $R(\sigma) = \overline{R}(\sigma)z^{l(\sigma)}$ for all $\sigma \in B_n$, $R(\sigma_1) = \rho(s_1)z$ implies $z = \exp(\lambda h t)$. Moreover, $R(\sigma_i) \in \overline{R}(\sigma_i)(1 + \lambda h \rho(t_{i,i+1})) + h^2 M_N(A)$ and $\overline{R}(\sigma_i)z \in \overline{R}(\sigma_i)(1 + \lambda h \tau) + h^2 M_N(A)$, hence $R(\sigma_i) = \overline{R}(\sigma_i)z$ implies $\rho(t_{i,i+1}) = \tau$ for $1 \leq i \leq n$. In particular $\rho(t_{23}) = \tau$, i.e. τ commutes with $\rho((1\ 3))$. Since it already commutes with $\rho(\mathfrak{S}_{2,n-2})$, it also commutes with all $\rho(\mathfrak{S}_n)$. $\qquad\square$

Lemma 4.5. *The following conditions are equivalent: (i) The image $\rho(\mathcal{T}_n)$ of $\mathcal{T}_n$ is commutative; (ii) $\rho([t_{12}, t_{23}]) = 0$; (iii) R factorizes through $\widetilde{\mathfrak{S}}_n$.*

Proof. Of course $i) \Rightarrow ii)$. Condition $i)$ means that $\rho([t_{ij}, t_{kl}]) = 0$ for all subsets $\{i, j, k, l\}$ of $\{1, \ldots, n\}$, such that $i \neq j$, $k \neq l$. If $\#\{i, j, k, l\} = 4$, this is part of the definition of $\mathcal{T}_n$; if $\#\{i, j, k, l\} = 2$ this only means $\rho([t_{ij}, t_{ij}]) = 0$, and if $\#\{i, j, k, l\} = 3$, this expression is conjugated to $\rho([t_{12}, t_{23}])$ by some $\rho(s)$ for $s \in \mathfrak{S}_n$. Hence $ii) \Rightarrow i)$. Now $i) \Rightarrow iii)$ because in that case $R(\sigma_i) = \rho(s_i)\exp(\lambda h \rho(t_{i,i+1}))$. Conversely, since $R(\sigma_1^2\sigma_2^2) - R(\sigma_2^2\sigma_1^2)$ belongs to $4\lambda^2 h^2 \rho([t_{12}, t_{23}]) + h^3 M_N(A)$, it follows that $iii) \Rightarrow ii)$. $\qquad\square$

Note that the hypothesis that ρ is absolutely irreducible was useful only for lemma 4.3. A characterization of the absolutely irreducible representations R which factorize through $\mathbb{Z}$ is very easy, since M must then

be 1-dimensional. Lemma 4.4 gives a characterization of a more subtle situation, namely when R factorizes through the non-surjective map $l \times \pi : B_n \to \mathbb{Z} \times \mathfrak{S}_n$. Note that such an R in particular factorizes through $\widetilde{\mathfrak{S}_n}$.

4.3. **Braided extensions**

4.3.1. **General construction**

Let A be a discrete valuation subring of $\Bbbk[[h]]$ and K be its quotient field. Let $R : B_n \to GL_N(A)$ be a representation of the braid group. As usual, we let $\overline{R}$ be its reduction modulo h, and we assume that $\overline{R}(\sigma_1^2) = 1$, i.e. that $\overline{R}$ factorizes through $\mathfrak{S}_n$. This condition is of course satisfied if $R = \widehat{\Phi}(\rho)$ for some $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$. More precisely, if $\rho \in \mathcal{V}^s(M)$, $\overline{R}$ corresponds to the $\Bbbk\mathfrak{S}_n$-module M. By conjugation in $GL_N(K)$ we can get other representations $S : B_n \to GL_N(A)$. We denote this relationship between R and S by $R \sim S$. It means that R and S are isomorphic as representations over K, but correspond to different B_n-invariant lattices in K^N. It is a classical fact that the simple modules appearing in the Jordan-Hölder decomposition of $\overline{R}$ and $\overline{S}$ are the same.

Proposition 4.6. *If R is irreducible and $\overline{R}$ is not irreducible, then there exists $S \sim R$ such that $\overline{S}$ does not factorize through the symmetric group.*

Proof. Under these hypothesis it is known (see [3, 4]) that there exists $S \sim R$ such that $\overline{S}$ is not semisimple. Since all representations of the symmetric group are semisimple, the conclusion follows. $\square$

More generally, let M be a $\Bbbk\mathfrak{S}_n$-module, and let $(M_i)_{i \in I}$ be its simple submodules. Let $\tau \in \mathcal{V}^s(M)$, $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ with $\lambda \in \Bbbk^\times$ such that $\widehat{\Phi}(\rho)(B_n) \subset GL_N(A)$. The relationship between the reductions modulo h of $\widehat{\Phi}(\rho)$ given by different choices of lattices in K^N on the one hand and B_n-extensions of the modules M_i on the other hand enter a general setting extensively studied by Bellaïche and Graftieaux in [3, 4]. This is the justification for the name we have chosen of "variety of braided extensions" of the given $\Bbbk\mathfrak{S}_n$-module M. A case of particular interest is when $M = M_1 \oplus M_2$, with M_1 and M_2 irreducible. Then any $\tau \in \mathcal{V}^s(M)$ allows us to define, following the original idea of Ribet [32] in the framework of algebraic number theory, elements in $\text{Ext}_{B_n}(M_1, M_2)$ and $\text{Ext}_{B_n}(M_2, M_1)$. We now explain this in more detail.

Assume $M = M_1 \oplus M_2$ with M_1 and M_2 being simple $\Bbbk\mathfrak{S}_n$-modules. Let $\tau \in \mathcal{V}^s(M)$ and $\rho : \mathfrak{B}_n \to \mathrm{End}_{\Bbbk}(M)$ be the associated representation of $\mathfrak{B}_n$. We choose $\Phi \in \mathbb{A}\mathrm{ssoc}_\lambda(\Bbbk)$ with $\lambda \in \Bbbk^\times$. For simplicity we assume that $A = \Bbbk[[h]]$. If, according to the decomposition $M = M_1 \oplus M_2$, we write $\rho(s_i)$ and $\rho(t_{i,i+1})$ for $1 \le i \le n$ in matrix form as follows

$$\rho(s_i) = \begin{pmatrix} s_i^{(1)} & 0 \\ 0 & s_i^{(2)} \end{pmatrix} \quad \rho(t_{i,i+1}) = \begin{pmatrix} \tau_{i,i+1}^{(11)} & \tau_{i,i+1}^{(12)} \\ \tau_{i,i+1}^{(21)} & \tau_{i,i+1}^{(22)} \end{pmatrix}$$

then, on the lattices $hM_1 \oplus M_2$ and $M_1 \oplus hM_2$, the reduction of $\widehat{\Phi}(\rho)(\sigma_i)$ modulo h reads

$$\begin{pmatrix} s_i^{(1)} & \lambda s_i^1 \tau_{i,i+1}^{(12)} \\ 0 & s_i^{(2)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s_i^{(1)} & 0 \\ \lambda s_i^{(2)} \tau_{i,i+1}^{(21)} & s_i^{(2)} \end{pmatrix}$$

respectively. Note that these expressions do *not* depend on the choice of $\Phi \in \mathbb{A}\mathrm{ssoc}_\lambda(\Bbbk)$. Moreover, since $\mathfrak{B}_n$ is generated by the s_i's and $t_{i,i+1}$'s, the fact that τ belongs to $\mathcal{V}^s(M)$ implies that there exists i and j such that $\tau_{i,i+1}^{(12)}$ and $\tau_{j,j+1}^{(21)}$ are non zero, hence that the elements obtained in $\mathrm{Ext}_{B_n}(M_1, M_2)$ and $\mathrm{Ext}_{B_n}(M_2, M_1)$ are non trivial. The most elementary case is the following fact:

Proposition 4.7. *Let B and C be two non-trivial $\Bbbk\mathfrak{S}_n$-module. Then*

$$\mathrm{Hom}_{\mathfrak{S}_{2,n-2}}(C, B) = 0 \Rightarrow \mathcal{V}^s(B \oplus C) = \emptyset, \ \mathrm{Ext}_{B_n}(B, C) = \mathrm{Ext}_{B_n}(C, B) = 0.$$

Proof. We assume $\mathrm{Hom}_{\mathfrak{S}_{2,n-2}}(C, B) = 0$. Since all $\Bbbk\mathfrak{S}_{2,n-2}$-modules are semisimple, we also have $\mathrm{Hom}_{\mathfrak{S}_{2,n-2}}(B, C) = 0$. The $\mathfrak{S}_{2,n-2}$-endomorphisms of $B \oplus C$ decompose as follows: $\mathrm{End}_{\mathfrak{S}_{2,n-2}}(B \oplus C)$ equals

$$\mathrm{End}_{\mathfrak{S}_{2,n-2}}(B) \oplus \mathrm{Hom}_{\mathfrak{S}_{2,n-2}}(B, C) \oplus \mathrm{Hom}_{\mathfrak{S}_{2,n-2}}(C, B) \oplus \mathrm{End}_{\mathfrak{S}_{2,n-2}}(C, C).$$

Then every $\tau \in \mathcal{V}(B \oplus C)$ leaves B and C invariant, hence can not belong to $\mathcal{V}^s(B \oplus C)$. Let us now consider an element of $\mathrm{Ext}_{B_n}(C, B)$, i.e. a representation of B_n on the vector space $B \oplus C$ which can be written in matrix form

$$\sigma \mapsto \begin{pmatrix} \sigma^B & \sigma^{CB} \\ 0 & \sigma^C \end{pmatrix}$$

Since B and C are representations of $\mathfrak{S}_n$, we have $(\sigma_1^B)^2 = 1$ and $(\sigma_1^C)^2 = 1$. Moreover, σ_1^2 commutes with $B_2 \times B_{n-2} \subset B_n$. Since the image of σ_1^2 has the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, this means that $\sigma^B x = x\sigma^C$ for every $\sigma \in B_2 \times B_{n-2}$,

hence that $x \in \mathrm{Hom}_{\mathfrak{S}_{2,n-2}}(C, B)$. By hypothesis we then have $x = 0$, hence $\sigma_1^2 = 1$. Since the braids σ_i^2 for $1 \leq i \leq n$ are conjugated in B_n, it follows that $\sigma_i^2 = 1$ for all $1 \leq i \leq n$ and the representation factorizes through $\mathfrak{S}_n$, hence is semi-simple and $\mathrm{Ext}_{B_n}(C, B) = 0$. Similarly, $\mathrm{Ext}_{B_n}(B, C) = 0$. $\qquad\square$

In general, elements in $\mathrm{Ext}_{B_n}(C, B)$ can then be constructed from elements in $\mathcal{V}^s(B \oplus C)$. Note that $\mathcal{V}^s(B \oplus C)$ is not a $\Bbbk$-vector space in general but only a cone, contrary to $\mathrm{Ext}_{B_n}(B, C)$. A natural question, which we leave open, is whether these elements generate $\mathrm{Ext}_{B_n}(B, C)$ as a $\Bbbk$-vector space.

4.3.2. Example: extensions between hooks

Let $\mathbb{1}$ be the trivial representation of $\mathfrak{S}_n$, E its permutation representation over $\Bbbk$. Then $E = \mathbb{1} \oplus U$, where U is an irreducible representation of $\mathfrak{S}_n$ corresponding to the partition $[n-1, 1]$ of n. Let $(e_1, \ldots, e_n)$ be a basis for E such that $s.e_i = e_{s(i)}$ for $s \in \mathfrak{S}_n$. The variety $\mathcal{V}^s(E)$ was studied in [20]. It is shown there that $\mathcal{V}^s(E)$ is of the form $\Bbbk^\times \tau$ for some non-zero $\tau \in \mathcal{V}^s(E)$, and that the corresponding irreducible representation of B_n deduced from any associator $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ is given by

$$\begin{cases} \sigma_k.e_k &= q e_{k+1} \\ \sigma_k.e_{k+1} &= q e_k \\ \sigma_k.e_r &= e_r \text{ if } r \notin \{k, k+1\} \end{cases}$$

with $q = e^{\alpha h}$, α depending on λ and on the choice of $\tau \in \mathcal{V}^s(E)$. By the general procedure described above, we get the following non-split indecomposable representations of B_n over $\Bbbk$:

$$\begin{cases} \sigma_k.f_k &= f_{k+1} + \alpha \frac{n-2}{n^2} v \\ \sigma_k.f_{k+1} &= f_k + \alpha \frac{n-2}{n^2} v \\ \sigma_k.f_r &= f_r - \frac{2\alpha}{n^2} v \end{cases}$$

$$\begin{cases} \sigma_k.g_k &= g_{k+1} - \frac{2\alpha}{n^2} w + \frac{\alpha}{n}(g_k + g_{k+1}) \\ \sigma_k.g_{k+1} &= g_k - \frac{2\alpha}{n^2} w + \frac{\alpha}{n}(g_k + g_{k+1}) \\ \sigma_k.g_r &= g_r - \frac{2\alpha}{n^2} w + \frac{\alpha}{n}(g_k + g_{k+1}) \end{cases}$$

with the following notations : $r \notin \{k, k+1\}$, $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ are some basis of E, $v = f_1 + \cdots + f_n$, $w = g_1 + \cdots + g_n$. The p-th exterior power $\Lambda^p U$ of U is irreducible under $\mathfrak{S}_n$ and corresponds to the hook

diagram $[n - p, 1^p]$. One has $\Lambda^p V = \Lambda^{p-1} U \oplus \Lambda^p U$. From these non-trivial elements in $\text{Ext}_{B_n}(\mathbb{1}, U)$ and $\text{Ext}_{B_n}(U, \mathbb{1})$ we then get non-trivial elements in $\text{Ext}_{B_n}(\Lambda^{p-1} U, \Lambda^p U)$ and $\text{Ext}_{B_n}(\Lambda^p U, \Lambda^{p-1} U)$. For the sake of concision, we introduce the Koszul differential on the exterior algebra $\Lambda^\bullet E$, defined by

$$\underset{\cdot}{(e_1 \wedge \cdots \wedge e_m)} = \sum_{j=1}^m (-1)^{j+1} e_1 \wedge \cdots \wedge \widehat{e_j} \wedge \cdots \wedge e_m.$$

If f is of the form $f_{i_1} \wedge \cdots \wedge f_{i_m}$ for some m such that $\{i_1, \ldots, i_m\} \cap \{k, k+1\} = \emptyset$, then the action of σ_k in the first case is given by

$$\begin{cases} \sigma_k.f & = & f - \frac{2\alpha}{n^2} v \wedge \underset{\cdot}{f} \\ \sigma_k.(f_k \wedge f) & = & f_{k+1} \wedge f - \frac{2\alpha}{n^2} v \wedge \underset{\cdot}{f} + \alpha \frac{n-2}{n^2} v \wedge f \\ \sigma_k.(f_{k+1} \wedge f) & = & f_k \wedge f - \frac{2\alpha}{n^2} v \wedge \underset{\cdot}{f} + \alpha \frac{n-2}{n^2} v \wedge f \\ \sigma_k.(f_k \wedge f_{k+1} \wedge f) & = & f_{k+1} \wedge f_k \wedge f + \alpha \frac{n-2}{n^2} v \wedge (f_k - f_{k+1}) \\ & & - \frac{2\alpha}{n^2} (f_{k+1} - f_k) \wedge v \wedge \underset{\cdot}{f} \end{cases}$$

In order to describe the action of σ_k in the second case, we introduce a permutation action of $\mathfrak{S}_n$ on the vector space $\Lambda^\bullet E$, defined by

$$s.(g_{i_1} \wedge \cdots \wedge g_{i_m}) = g_{s(i_1)} \wedge \cdots \wedge g_{s(i_m)} \text{ for } s \in \mathfrak{S}_n$$

. We then have

$$\sigma_k.g = s_k.g + \left(\frac{\alpha}{n}(g_k + g_{k+1}) - \frac{2\alpha}{n^2} w \right) \wedge \underset{\cdot}{g}$$

for all $g \in \Lambda^\bullet E$. Note that, since the restriction to $\mathfrak{S}_{2,n-2}$ of $\Lambda^p U$ and $\Lambda^q U$ has no isomorphic irreducible component for $|p - q| \geq 2$, it follows from proposition 4.7 that $\text{Ext}_{B_n}(\Lambda^p U, \Lambda^q U) = 0$ in these cases. The cohomology groups of B_n in its representation U were computed by Vassiliev in [38]. In particular, he there showed that $\text{Ext}_{B_n}(\mathbb{1}, U) = H^1(B_n, U) \simeq \Bbbk$ for $n \geq 3$.

5. Restriction to pure braids

Keeping in mind the short exact sequence $1 \to P_n \to B_n \to \mathfrak{S}_n \to 1$, we now focus on the restriction of representations of B_n to its normal subgroup of finite index P_n. Being considered as an iterated extension of free groups, P_n does not have a well-understood representation theory to build on. On the contrary, it seems hopeless to classify its representations — whatever this could mean. Nevertheless, as it is a normal subgroup of finite index, most irreducible representations of B_n are already irreducible

under the action of P_n. Moreover, its infinitesimal structure is a graded Lie algebra, that is a more elementary structure than the infinitesimal braid algebra $\mathfrak{B}_n$. It follows that studying the restriction to $\mathcal{T}_n$ helps in understanding the decomposition of semisimple representations into irreducible components. Since $\mathcal{T}_n$ is generated by its elements of degree 1, it is first a natural task to investigate the image of these under a representation ρ of $\mathfrak{B}_n$, and their role in irreducibility properties.

We show that the restriction of ρ to this subspace of homogeneous degree 1 elements of $\mathcal{T}_n$ is injective, unless the associated B_n-representation factorizes through one of the classical quotients of the braid group. We then focus on representation-theoretic properties which are *tannakian*, i.e. such that the collection of representations satisfying these forms a tannakian subcategory of the category of representations. For instance, semisimplicity is a tannakian property. In order to enlarge this setting, we define a notion of generically tannakian properties, making use of the natural $\mathbb{G}_m(\Bbbk)$-action on $\mathcal{T}_n$. Two generically tannakian properties naturally appear. One of them is essential purity. Essentially pure irreducible representations of $\mathfrak{B}_n$ restrict to irreducible representations of $\mathcal{T}_n$. As a matter of fact, they are already irreducible under the action of several Lie subalgebras of $\mathcal{T}_n$. This means that the associated representations of B_n are irreducible under the action of several subgroups. The other one is aggregation. This notion was first defined in [25], and is a generalization of the fact that tensor products of irreducible representations of a free group are "usually" irreducible.

Another interesting property of a representation ρ is when the eigenvalues of $\rho(t_{12})$ form a simplex over $\mathbb{Q}$. We conclude this part by studying the incidence of all these properties on the Hopf quotients of $\mathfrak{B}_n$ induced by ρ, and mention the connection with the algebraic hull of the pure braid group in the corresponding representations. For convenience, an associator $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$, $\lambda \in \Bbbk^\times$, integers $n \geq 2$, $N \geq 1$ are chosen once and for all.

5.1. **Linear independence**

Let $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$ and $R = \widehat{\Phi}(\rho)$, for some $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ and $\lambda \in \Bbbk^\times$. We showed in the last part that it is possible to characterize the potential factorization of R through $B_n/Z(B_n)$, $\mathfrak{S}_n \times \mathbb{Z}$, $Hurw_n$ or $\widetilde{\mathfrak{S}_n}$ in terms of ρ. We show here that, if R does not factorize through one of

these groups, then the $\Bbbk$-vector space spanned by $\rho(t_{ij})$ for $1 \leq i < j \leq n$ is $n(n-1)/2$-dimensional. In particular, since $B_2 = \widetilde{\mathfrak{S}}_2$, we assume $n \geq 3$.

We first need to recall several elementary facts about representations of the symmetric group. Le E be the $n(n-1)/2$-dimensional $\Bbbk\mathfrak{S}_n$-module with basis $e_{ij} = e_{\{i,j\}}$ and action $s.e_{ij} = e_{s(i),s(j)}$. It is the induced representation of the trivial representation of the Young subgroup $\mathfrak{S}_{2,n-2}$. Littlewood-Richardson rule shows that provided $n \geq 4$ it is the sum of three irreducible non-isomorphic components respectively corresponding to the partitions $[n]$, $[n-1,1]$ and $[n-2,2]$ whose dimensions are 1, $n-1$ and $n(n-3)/2$. If $n = 3$ there are only two irreducible components, corresponding to $[n]$ and $[n-1,1]$. In both cases, the first component is generated by the sum y of all the e_{ij}'s for $1 \leq i < j \leq n$, and the direct sum of the first two components admits as basis the y_i's defined for $1 \leq i \leq n$ as $y_i = \sum_{j \neq i} e_{ij}$.

Proposition 5.1. *If R does not factorize through $B_n/Z(B_n)$, $Hurw_n$ or $\widetilde{\mathfrak{S}}_n$, then the $\rho(t_{ij})$'s are linearly independent.*

Proof. We let L be the sub-$\mathfrak{S}_n$-module of $\mathrm{End}(M)$ linearly generated by the $\rho(t_{ij})$'s. We use the notations above and let $u_i = \varphi(y_i)$. There is a natural surjective $\mathfrak{S}_n$-morphism $\varphi : E \to L$ defined by $\varphi(e_{ij}) = t_{ij}$. Consequences of the characterizations in section 4.2 are the following ones. Lemma 4.2 implies $\varphi(y) \neq 0$. Lemma 4.3 implies that u_n cannot commute to every $s \in \mathfrak{S}_n$, hence there exists i, j such that $u_i - u_j \neq 0$. Lemma 4.4 implies that L has dimension at least 2. It follows that $\mathrm{Ker}\, \varphi$, as a $\mathfrak{S}_n$-subrepresentation of E, cannot contain the invariant vector of E and may contain at most one of the remaining irreducible components. One of them is generated by the elements $y_i - y_j$, hence the kernel cannot contain this one. It follows that, if φ were not surjective L would be of dimension n, generated by $u_1, \ldots, u_n$, and $n \geq 4$. We assume this and show that it would contradict lemma 4.5 — that is, the family $u_1, \ldots, u_n$ would be commutative.

Let $u_1 + \cdots + u_n = \mu \neq 0$. Since $\rho(t_{12})$ is a linear $\mathfrak{S}_{2,n-2}$-invariant linear combination of the u_i's, there exists $b, c \in \Bbbk$ such that $\rho(t_{12}) = b(u_1+u_2)+c\mu$, hence $\rho(t_{ij}) = b(u_i+u_j)+c\mu$ for all $1 \leq i < j \leq n$. It follows that $u_i = (n-2)bu_i+((n-1)c+b)\mu$ whence $b = 1/(n-2) \neq 0$ and u_i+u_j is a linear combination of $\rho(t_{ij})$ and μ. The infinitesimal braid relations and $\mathfrak{S}_n$-invariance then imply that $[u_i + u_j, u_k] = 0$ whenever $\#\{i,j,k\} = 3$. Choosing i, j such that $\#\{1, 2, i, j\} = 4$ one has $[u_i, u_j] = [u_i + u_1, u_j] -$

$[u_1, u_j] = -[u_1, u_j] = [u_1, u_2]$ and similarly $[u_j, u_i] = [u_1, u_2]$. By $\mathfrak{S}_n$-invariance it follows that $[u_i, u_j] = 0$ for all i, j, hence a contradiction. $\square$

5.2. Tannakian and generically tannakian properties

Let $\mathcal{A}$ be a graded Hopf algebra over $\Bbbk$. We shall study here special properties that representations of $\mathcal{A}$ may have, in order to apply this setting to the case where $\mathcal{A} = \mathsf{U}\mathcal{T}_n$ or $\mathcal{A} = \mathfrak{B}_n$. For a given property (P), we study in particular their stability by the standard operations in representation theory, namely taking sub-modules and quotients, dual and tensor products, and extension of the base field.

Definition 5.2. A property (P) is said to be *tannakian* if it is stable by taking sub-modules and quotients, dual and tensor products, and extension of the base field. In other words, if

1) If ρ satifies (P), then so does $\rho^{\vee}$, and all its quotients and sub-modules.

2) If $\rho^1, \ldots, \rho^r$ satifies (P), then so does $\rho^1 \otimes \cdots \otimes \rho^r$.

3) If ρ satifies (P), then $\rho \otimes L$ satifies (P) over any extension L of $\Bbbk$.

If a property (P) is tannakian, the full sub-category of the (left) $\mathcal{A}$-modules that satisfy (P) inherits the tannakian structure, hence corresponds to a pro-algebraic group scheme over $\overline{\Bbbk}$. In particular, the property of factorizing through some Hopf quotient of $\mathcal{A}$ is the basic example of a tannakian property.

We also define a larger class of properties, by relaxing condition 2). Since $\mathcal{A}$ is graded, it is naturally endowed with an action of $\mathbb{G}_m(\Bbbk)$ such that $\alpha \in \mathbb{G}_m(\Bbbk)$ sends a homogeneous element $x \in \mathcal{A}$ of degree r to $\alpha^r x$. If ρ is a representation of $\mathcal{A}$ and $\alpha \in \mathbb{G}_m(\Bbbk)$, we let ρ_α be the representation twisted by this action: $\rho_\alpha(x) = \rho(\alpha.x)$. In case $\mathcal{A} = \mathfrak{B}_n$ and $R = \widehat{\Phi}(\rho)$, $\widehat{\Phi}(\rho_\alpha)$ is deduced from R by the field automorphism $\alpha \in \mathbb{G}_m(\Bbbk) \subset \mathrm{Gal}(\Bbbk((h))/\Bbbk)$, this inclusion being defined by $\alpha.f(h) = f(\alpha h)$ for $f \in \Bbbk((h))$.

Definition 5.3. A property (P) is said to be *generically tannakian* if 1), 2') and 3) are satified, where

2') If $\rho^1, \ldots, \rho^r$ satifies (P), then so does $\rho^1_{\alpha_1} \otimes \cdots \otimes \rho^r_{\alpha_r}$ for generic $(\alpha_1, \ldots, \alpha_r) \in \mathbb{G}_m(\Bbbk)^r$.

In this definition, the genericity has to be understood with respect to the Zariski topology on $\mathbb{G}_m(\Bbbk) = \operatorname{Spec} \Bbbk[t^{-1}, t]$.

We end this section by studying properties of $\mathsf{U}\mathcal{T}_n$ and $\mathfrak{B}_n$ which are tannakian. All the proofs are standard. The first property under consideration is semisimplicity. We first need a lemma.

Lemma 5.4. *Let G be a finite group acting on a Lie algebra $\mathfrak{g}$ by automorphisms. A representation of $\mathsf{U}_G\mathfrak{g} = \Bbbk G \ltimes \mathsf{U}\mathfrak{g}$ is semi-simple if and only if its restriction to $\mathsf{U}\mathfrak{g}$ is semi-simple.*

Proof. Let $\rho : \mathsf{U}_G\mathfrak{g} \to M_N(\Bbbk)$ be a representation of $\mathsf{U}_G\mathfrak{g}$, and ρ' be its restriction to $\mathsf{U}\mathfrak{g}$. We first assume that ρ is irreducible. Let us choose an irreducible subrepresentation of ρ' and let U be the underlying subspace of $\Bbbk^N$. Then any vector space $\rho(s)U$ for $s \in G$ is ρ'-invariant and irreducible as a representation of $\mathsf{U}\mathfrak{g}$, and the sum of all of these is a non-zero ρ-invariant subspace of $\Bbbk^N$. Since ρ is irreducible, this subspace is $\Bbbk^N$ and ρ' is semisimple. If ρ is semisimple, the same conclusion of course holds. Conversely, we assume that ρ' is semisimple. Let ψ be a sub-representation of ρ and U be the underlying subspace. Let $i : U \hookrightarrow \Bbbk^N$ be the canonical inclusion. Since ρ' is semisimple, i admits a left inverse j as a $\mathsf{U}\mathfrak{g}$-module. Let $J = (\sum \psi(s) \circ j \circ \rho(s)^{-1})/(\#G)$, with the sum being taken over all elements of the finite group G. One has $J \circ i = \operatorname{id}_U$ and J is a morphism of $\mathsf{U}_G\mathfrak{g}$-modules. Then ρ is semisimple and the lemma is proved. $\square$

Proposition 5.5. *The property of being semisimple is tannakian for $\mathcal{A} = \mathsf{U}\mathcal{T}_n$ and $\mathcal{A} = \mathfrak{B}_n$.*

Proof. For every algebra $\mathcal{A}$, every sub-quotient of a semisimple module is semisimple. Condition 3) is moreover satified because $\Bbbk$ is perfect, being of characteristic 0. In case $\mathcal{A} = \mathsf{U}\mathcal{T}_n$, the dual is semisimple and condition 2) is satified because $\mathcal{A}$ is an enveloping algebra. The general case follows from lemma 5.4. $\square$

We call a representation ρ of $\mathfrak{B}_n$ unitary if is unitary with respect to some non-degenerate symmetric bilinear form.

Proposition 5.6. *The property of being unitary is tannakian.*

Proof. Since any unitary representation is semisimple, condition 1) is true. Condition 2) holds because, if ρ, ρ' are unitary with respect to β, β', then $\rho \otimes \rho'$ is unitary with respect to $\beta \otimes \beta'$, which is symmetric and non degenerate. Condition 3) is clear. $\square$

5.3. Aggregating representations

A less standard property is the property of aggregation. This is our first example of a generically tannakian property. We recall from [25] its definition and its main application.

Definition 5.7. An element in $M_N(\Bbbk)$ is called regular if it is diagonalizable with distinct eigenvalues. A representation ρ of $\mathcal{T}_n$ is said to be *aggregating* if $\rho(V)$ contains a regular element, where V is the vector space generated by the $\rho(t_{ij})$. A representation of $\mathfrak{B}_n$ is called aggregating if its restriction to $\mathsf{U}\mathcal{T}_n$ is aggregating.

Proposition 5.8. *The property of being aggregating is generically tannakian.*

It is sufficient to show this for $\mathsf{U}\mathcal{T}_n$. See [25] for a proof. A consequence of this property is the following :

Proposition 5.9. *(see* [25]*) Let ρ be a representation of $U\mathcal{T}_n$. If ρ is aggregating and irreducible, then it is absolutely irreducible. If $\rho^1, \ldots, \rho^r$ are aggregating representations of $\mathsf{U}\mathcal{T}_n$, then $\rho^1_{\alpha_1} \otimes \cdots \otimes \rho^r_{\alpha_r}$ is irreducible (resp. indecomposable) for generic values of $\alpha_1, \ldots, \alpha_r \in \mathbb{G}_m(\Bbbk)$ if and only if all the ρ^i's are so.*

5.4. Essential purity

In this section, we let $s = (1\ 2)$, $t = t_{12}$. As an associative algebra with unit, $\mathfrak{B}_2$ is generated by s and t, with relations $s^2 = 1$, $st = ts$.

Definition 5.10. A representation ρ of $\mathfrak{B}_2$ is said to be *essentially pure* if $\rho(s)$ is a polynomial in $\rho(t)$. A representation of $\mathfrak{B}_n$ is called essentially pure if its restriction to $\mathfrak{B}_2$ is so.

If $\rho(t)$ is diagonalizable, it means that $\rho(s)$ acts by a scalar on each sub-vector space $\mathrm{Ker}(\rho(t) - \alpha)$ for $\alpha \in \Bbbk$. Note that a representation ρ of $\mathfrak{B}_n$ is essentially pure if and only if *any* $\rho((i\ j))$ is a polynomial in $\rho(t_{ij})$.

This property has special incidences on the restrictions of a representation of B_n or $\mathfrak{B}_n$. The notations F_n, I_n, $\mathcal{F}_n$ and $\mathcal{I}_n$ were defined in section 3.2.1.

Proposition 5.11. *Let ρ be a representation of $\mathfrak{B}_n$, and $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ with $\lambda \in \Bbbk^\times$. If ρ is essentially pure, then the irreducible components of ρ*

(resp. $\widehat{\Phi}(\rho)$) are irreducible under the action of $\mathcal{T}_n$, $\mathcal{F}_n$ and $\mathcal{I}_n$ (resp. P_n, F_n and I_n).

Proof. If ρ is essentially pure, the image $\rho((i \; j))$ of each transposition is a polynomial in $\rho(t_{ij})$. We can suppose that ρ is irreducible. Since the transpositions generate $\mathfrak{S}_n$, its restriction to $\mathcal{T}_n$ is irreducible. Since the transpositions of the special form $(i \; n)$ (resp. $(i \; i+1)$) generate $\mathfrak{S}_n$, its restriction to $\mathcal{F}_n$ (resp. $\mathcal{I}_n$) is also irreducible. The consequences for $\widehat{\Phi}(\rho)$ follow from proposition 3.4. $\qquad\square$

Since F_n is a free group, it follows from this that every essentially pure representation of the braid group B_n yields a solution of the Deligne-Simpson problem on n-tuples, in the particular case where the first $n-1$ conjugacy classes are the same. Recall that this problem concerns the classification of n-tuples $(C_1, \ldots, C_n)$ of conjugacy classes in $GL_N(K)$ such that there exists a n-tuple $(X_1, \ldots, X_n) \in GL_N(K)^n$ satisfying $X_i \in C_i$ for $1 \le i \le n$, K^N is irreducible under $X_1, \ldots, X_n$, and $X_1 \ldots X_n = 1$. Here $X_i = R(\xi_{i,n})$ for $1 \le i \le n-1$ and $X_n = R((\xi_{1,n} \ldots \xi_{n-1,n})^{-1})$. Conversely, a simple solution of this problem in this particular case could be of great benefit for the representation theory of the braid group. An infinitesimal variant of this problem was considered and solved by specialists in the field, namely the classification of n-tuples $(c_1, \ldots, c_n)$ of conjugacy classes in $M_N(\Bbbk)$ such that there exists a n-tuple $(x_1, \ldots, x_n) \in M_N(\Bbbk)^n$ satisfying $x_i \in c_i$ for $1 \le i \le n$, $\Bbbk^N$ is irreducible under $x_1, \ldots, x_n$ and $x_1 + \cdots + x_n = 0$. Here $x_i = \rho(t_{i,n})$ for $1 \le i \le n-1$ and $x_n = -\rho(t_{1n} + \cdots + t_{n-1,n})$. Let us mention the solution recently given by Crawley-Boevey [11] in terms of representations of quivers and the associated Kac-Moody algebra. The solutions given by essentially pure irreducible representations of B_n correspond to a special kind of quivers, and it would be most interesting to decide which roots of the associated Kac-Moody algebra are concerned by this construction. Let us moreover notice that the so-called Katz rigidity index is very easy to compute from a given essentially pure $\mathfrak{B}_n$-representation, and gives an interesting invariant for such representations — for instance, the Burau representation leads to a rigid tuple in this sense, and it is the only one doing so among representations of the Iwahori-Hecke algebra of type A (see [21]).

Lemma 5.12. *The representation $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$ is essentially pure iff it is essentially pure over $\overline{\Bbbk}$.*

Proof. If ρ is essentially pure, it is in particular essentially pure over $\overline{\Bbbk}$. Conversely, let $P \in \overline{\Bbbk}[X]$ be such that $\rho(s) = P(\rho(t))$, and let L be smallest normal extension of $\Bbbk$ containing the coefficients of P. Since L is Galois and finite one has $\rho(s) = Q(\rho(t))$ with $Q = (\sum \sigma P)/(\#\mathrm{Gal}(L/\Bbbk)) \in \Bbbk[X]$, the sum being taken over all $\sigma \in \mathrm{Gal}(L/\Bbbk)$. $\qquad\square$

Proposition 5.13. *If $\rho^1, \ldots, \rho^r$ is a family of essentially pure represen-tations then, for generic $\alpha_1, \ldots, \alpha_r \in \mathbb{G}_m(\Bbbk)$, $\rho^\alpha = \rho^1_{\alpha_1} \otimes \cdots \otimes \rho^r_{\alpha_r}$ is essentially pure.*

Proof. Because of lemma 5.12, we may assume that $\Bbbk$ is algebraically closed. Let $s^i = \rho^i(s)$ and $t^i = \rho^i(t)$. Let d^i be the semisimple component of the endomorphism t^i. Then $t^\alpha = \rho^\alpha(t)$ is the sum of the elements $\alpha_i 1 \otimes \cdots \otimes t^i \otimes \ldots 1$ for $1 \leq i \leq r$, and the sum d^α of the elements $\alpha_i 1 \otimes \cdots \otimes d^i \otimes \ldots 1$ for $1 \leq i \leq r$ is its semisimple component. In particular $Sp(t^\alpha) = Sp(d^\alpha)$ and there exists a polynomial $Q_\alpha \in \Bbbk[X]$ depending on $\alpha_1, \ldots, \alpha_r$ such that $d^\alpha = Q_\alpha(t^\alpha)$. Since the representations ρ^i are essentially pure, there exist for all $1 \leq i \leq r$ polynomials $P_i \in \Bbbk[X]$ such that $s^i = P_i(t^i)$ — hence $s^i = P_i(d^i)$ since s^i is semisimple. It follows that there exist set-theoretic maps $\mathsf{f}_i : Sp(t^i) = Sp(d^i) \to Sp(s^i)$ such that $d^i v = \mu v$ implies $s^i v = \mathsf{f}_i(\mu)v$.

Let v_i, $1 \leq i \leq r$, be a non-zero vector in the underlying vector space of ρ^i, and $v = v_1 \otimes \cdots \otimes v_r$. There exists a set-theoretic map $\mathsf{f} = \prod \mathsf{f}_i : Sp(t^1) \times \cdots \times Sp(t^r) \to Sp(s^\alpha)$ such that $\forall i \; d^i v_i = \mu_i v_i$ implies $s^\alpha v = \mathsf{f}(\mu_1, \ldots, \mu_r)v$. For generic $(\alpha_1, \ldots, \alpha_r) \in \mathbb{G}_m(\Bbbk)$ we may assume that the condition $d^i v_i = \mu_i v_i$ for all $1 \leq i \leq r$ is equivalent to the condition $d^\alpha v = (\sum \alpha_i \mu_i)v$. Hence there exists a map $\mathsf{f}' : Sp(d^\alpha) \to Sp(s^\alpha)$ such that $d^\alpha v = \mu v \Rightarrow s^\alpha v = \mathsf{f}'(\mu)$. Since d^α is semisimple and $\Bbbk$ is algebraically closed, it follows that $s^\alpha = P(d^\alpha)$ for some $P \in \Bbbk[X]$, hence $s^\alpha = (P \circ Q_\alpha)(t^\alpha)$ and ρ^α is essentially pure. $\qquad\square$

It easily follows:

Proposition 5.14. *The property of being essentially pure is generically tannakian.*

5.5. Simplicial representations

We study here N-dimensional representations of B_n such that the image of σ_1 or, equivalently, of σ_1^2, is semisimple and its field of matrix coefficients

is a purely transcendental extension of $\Bbbk$. As we saw before, if $R = \widehat{\Phi}(\rho)$ for some $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$ with $\lambda \in \Bbbk^\times$ and $\Bbbk$ is algebraically closed, this means that the eigenvalues of $\rho(t_{12})$ have no $\mathbb{Q}$-linear relations between them — in particular, this condition only depends on the restriction to $\mathcal{T}_n$, hence fits our preoccupations in this part. Since a rescaling of R by multiplication of some scalar (equivalently, a rescaling of ρ by addition of some scalar) does not change the representation in a significant way, the convenient setting at the infinitesimal level is not linear algebra but affine geometry over $\mathbb{Q}$. In this framework, a simplex will refer to the structure of $\mathbb{Q}$-affine vector space of $\Bbbk$.

Definition 5.15. A representation $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$ is called *simplicial* if $\rho(t_{12})$ is diagonalizable over $\Bbbk$ and its spectrum forms a simplex.

We introduce in appendix B a representation-theoretic criterium ensuring that the interior of this simplex contains the eigenvalues of $2T/n(n-1)$, with T being the sum of the $\rho(t_{ij})$'s — thus illustrating the relevance of this geometric definition.

For any Hopf algebra $\mathcal{A}$ and representation ρ of $\mathcal{A}$, the derived representations of ρ are the representations of the form $\rho^{\otimes p} \otimes (\rho^\vee)^{\otimes q}$ for $p, q \geq 0$, their subrepresentations and quotients. Besides its influence on the field of matrix coefficient, the most interesting aspect of simpliciality is that it makes essential purity extend to derived representations.

Proposition 5.16. *Let ρ be simplicial and essentially pure. Then all its derived representations are essentially pure.*

Proof. Let $\mathcal{S} \subset \Bbbk$ be the spectrum of $\rho(t_{12})$. Essential purity means that there exists a set-theoretic map $\varepsilon : \mathcal{S} \to \{-1, 1\}$ such that, for all $\alpha \in S$, $v \in \Bbbk^N$, $\rho(t_{12})v = \alpha v \Rightarrow \rho(s) = \varepsilon(\alpha)v$. Let $r \geq 0$. We first show that $\rho^{\otimes r}$ is essentially pure. We let $t = t_{12}$, $s = (1\ 2)$ as before. Let $\underline{v} \in (\Bbbk^N)^{\otimes r}$ be a pure tensor formed by tensoring several eigenvectors for $t : n_1$ with eigenvalue $\alpha_1, \ldots, n_r$ with eigenvalue α_r. One has,

$$\rho^{\otimes r}(t)\underline{v} = \left(\sum_{i=1}^{\#S} n_i \alpha_i \right) \underline{v} \quad \rho^{\otimes r}(s)\underline{v} = \prod_{i=1}^{r} \varepsilon(\alpha_i)^{n_i} \underline{v}.$$

Since $\sum n_i = r$ and the α_i's are affine independent, the value of $\rho^{\otimes r}(t_{12})$ on $\underline{v}$ determines the n_i's as barycentric coordinates, hence the value of $\rho^{\otimes r}(s)$ on $\underline{v}$. Since the tensors like $\underline{v}$ form a basis of $(\Bbbk^N)^{\otimes r}$ which are eigenvectors

for $\rho^{\otimes r}(t_{12})$ it follows that $\rho^{\otimes r}(s)$ is a polynomial of $\rho^{\otimes r}(t_{12})$ and $\rho^{\otimes r}$ is essentially pure. The same more generally holds for $\hat{\rho} = \rho^{\otimes p} \otimes (\rho^*)^{\otimes q}$ for $p, q \geq 0$: the eigenvalues of $\hat{\rho}(t)$ are then of the form $\sum(n_i - m_i)\alpha_i$, with $\sum n_i = p$, $\sum m_i = q$, and uniquely determine the $(n_i - m_i)$'s, hence the parity of the $(n_i + m_i)$'s and the corresponding action $\prod \varepsilon(\alpha_i)^{n_i + m_i}$ of $\hat{\rho}(s)$. Since essential purity is generically tannakian and in particular satisfies condition 1), the conclusion follows. □

5.6. **Infinitesimal quotients and algebraic hulls**

Let $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$ be a representation and $\mathcal{H}_\rho = \rho(\mathcal{T}_n)$ the Lie subalgebra of $\mathfrak{gl}_N(\Bbbk)$ generated by the $\rho(t_{ij})$'s. The sum T of the $\rho(t_{ij})$'s belongs to the center of $\mathcal{H}_\rho$. The symmetric group acts on $\mathcal{H}_\rho$ by automorphisms and the Lie algebra morphism $\mathcal{T}_n \to \mathcal{H}_\rho$ extends to a surjective Hopf algebras morphism $\mathfrak{B}_n \to \mathcal{Q}_\rho$ where $\mathcal{Q}_\rho = \Bbbk\mathfrak{S}_n \ltimes \mathsf{U}\mathcal{H}_\rho$ is called the *infinitesimal quotient* of B_n associated to ρ. Note that, if ρ can be decomposed as $\rho_1 \oplus \cdots \oplus \rho_r$, then $\mathcal{H}_\rho$ naturally embeds into $\mathcal{H}_{\rho_1} \oplus \cdots \oplus \mathcal{H}_{\rho_r}$ by an $\mathfrak{S}_n$-equivariant Lie algebra morphism, hence $\mathcal{Q}_\rho$ embeds into $\mathcal{Q}_{\rho_1} \oplus \cdots \oplus \mathcal{Q}_{\rho_r}$ as a Hopf algebra.

The representation-theoretic conditions introduced above, when satisfied by ρ, have the following consequences on $\mathcal{H}_\rho$.

Proposition 5.17. *Let ρ be essentially pure and semisimple. Then $\mathcal{H}_\rho$ is a reductive Lie algebra and $Z(\mathcal{H}_\rho) = \rho(Z(\mathcal{T}_n)) = \Bbbk T$. In particular, if $\rho(t_{12}) \in \mathfrak{gl}_N(\Bbbk)$ has non-zero trace, then $\mathcal{H}_\rho$ has 1-dimensional center. Moreover, if ρ is aggregating then $\mathcal{H}_\rho$ is split over $\Bbbk$.*

Proof. The representation ρ induces a faithful representation ρ' of $\mathcal{H}_\rho$ on $\Bbbk^N$. Since ρ is essentially pure, every simple component of ρ is irreducible under the action of $\mathcal{T}_n$, hence ρ' is faithful and semisimple, and $\mathcal{H}_\rho$ is reductive.

One always has $\rho(Z(\mathcal{T}_n)) = \Bbbk T$. Since $\mathcal{H}_\rho$ is reductive, one has $\mathcal{H}_\rho = Z(\mathcal{H}_\rho) \times [\mathcal{H}_\rho, \mathcal{H}_\rho]$ and $\rho([\mathcal{T}_n, \mathcal{T}_n]) = [\mathcal{H}_\rho, \mathcal{H}_\rho]$. If $\mathcal{T}_n^1$ denotes the subspace generated by the $t'_{ij}s$ inside $\mathcal{T}_n$ one has $\mathcal{T}_n = \mathcal{T}_n^1 \times [\mathcal{T}_n, \mathcal{T}_n]$ as vector spaces and $\mathfrak{S}_n$-modules, and $\mathcal{H}_\rho = Z(\mathcal{H}_\rho) \times [\mathcal{H}_\rho, \mathcal{H}_\rho]$ is also a decomposition as $\mathfrak{S}_n$-modules because ρ is essentially pure hence $\mathfrak{S}_n$ acts trivially on $Z(\mathcal{H}_\rho)$. We denote by p the corresponding $\mathfrak{S}_n$-equivariant projection $\mathcal{H}_\rho \twoheadrightarrow Z(\mathcal{H}_\rho)$. It follows that $Z(\mathcal{H}_\rho) \subset p \circ \rho(\mathcal{T}_n^1)^{\rho(\mathfrak{S}_n)}$ hence it is a quotient of the $\mathfrak{S}_n$-module $\mathcal{T}_n^1$. Because of the decomposition of this latter semisimple

$\mathfrak{S}_n$-module one has $p \circ \rho(\mathcal{T}_n^1)^{\rho(\mathfrak{S}_n)} = p \circ \rho((\mathcal{T}_n^1)^{\mathfrak{S}_n}) = p(\Bbbk T)$. Hence $\Bbbk T \subset Z(\mathcal{H}_\rho) \subset p(\Bbbk T)$ with $\dim \Bbbk T \geq \dim(p(\Bbbk T))$ and the conclusion follows. Since the trace of $\rho(T)$ is $\frac{n(n-1)}{2}$-times the trace of $\rho(t_{12})$, if the latter is non-zero then $\Bbbk T$ has dimension 1.

If ρ is aggregating, there exists $x \in \mathcal{T}_n$ such that $u = \rho(x)$ is a regular element in $\mathfrak{gl}_N(\Bbbk)$. It is then a regular element of the Lie algebra $\mathcal{H}_\rho$ in the traditional sense and a Cartan subalgebra of $\mathcal{H}_\rho$ is given by $\mathcal{C}_\rho = \{y \in \mathcal{H}_\rho \mid \exists m \ (\mathrm{ad}\ u)^m y = 0\}$. But since $\mathcal{H}_\rho$ is reductive, then $\mathcal{C}_\rho$ is commutative. It follows that $\mathcal{C}_\rho \subset M_N(\Bbbk)$ is a set of polynomials in u, hence all elements in $\mathcal{C}_\rho$ are diagonalizable and $\mathcal{H}_\rho$ is split over $\Bbbk$. $\square$

As before in this part, we are interested here in whether the irreducible components of a representation of $\mathcal{Q}_\rho$ are already irreducible under the action of $\mathcal{H}_\rho$. Let $\rho = \rho_1 \oplus \cdots \oplus \rho_r$ be a semisimple, essentially pure and *simplicial* representation of $\mathfrak{B}_n$. It easily follows from propositions 5.16 and 5.17 that the decomposition of any tensor product of $\rho_1, \ldots, \rho_r$ and their duals only depends on the decomposition of the reductive Lie algebra $\mathcal{H}_\rho$ into simple factors. For applications of this situation to the most classical representations of the braid group, see [24, 28, 29].

These aspects are closely connected to the algebraic hull of $R(P_n)$, where $R = \widehat{\Phi}(\rho)$, inside $GL_N(K)$. We first need a lemma.

Lemma 5.18. *Let Γ be a Zariski-closed subgroup of $GL_N(K)$ and $\mathrm{Lie}\Gamma$ its Lie algebra over K. For all $x \in M_N(A)$, if $\exp(hx) \in \Gamma$ then $x \in \mathrm{Lie}\Gamma$.*

Proof. Let $a_1, \ldots, a_r$ be polynomial functions on $M_N(K)$ with coefficients in A such that $\Gamma = \{m \in GL_N(K) \mid \forall i \in [1,r] \ a_i(m) = 0\}$. Since Γ is a subgroup of $GL_N(K)$, one has $a_i(\exp(nhx)) = 0$ for all $i \in [1,r]$ and $n \in \mathbb{Z}$. Let $Q_i = a_i(\exp(\alpha h x)) \in (\Bbbk[\alpha])[[h]]$. Since $\Bbbk$ has caracteristic 0, $\mathbb{Z}$ is Zariski-dense in $\Bbbk$ hence $Q_i = 0$ and $\exp(\alpha h x)$ is a $\Bbbk[[\alpha]]$-point of Γ. It follows from [8] t. 2 §8 prop. 4 that $hx \in \mathrm{Lie}\Gamma$ hence $x \in \mathrm{Lie}\Gamma$. $\square$

Recall that F_n is a normal subgroup of P_n such that $P_n/F_n \simeq P_{n-1}$ and the centralizer of F_n in P_n is $Z(P_n)$. Similarly, $\mathfrak{F}_n$ is an ideal of $\mathcal{T}_n$, $\mathcal{T}_n/\mathfrak{F}_n \simeq \mathcal{T}_{n-1}$ and its centralizer in $\mathcal{T}_n$ is the center of $\mathcal{T}_n$. We let $\check{\mathcal{H}}_\rho = \rho(\mathfrak{F}_n)$.

Proposition 5.19. *Let H and $\check{H}$ be the Zariski closure of $R(P_n)$ and $R(F_n)$ inside $GL_N(K)$, and $\mathfrak{h}$, $\check{\mathfrak{h}}$ be their Lie algebras. Then $\mathcal{H}_\rho \otimes K \subset \mathfrak{h}$ and $\check{\mathcal{H}}_\rho \otimes K \subset \check{\mathfrak{h}}$.*

Proof. It easily follows from the definition of $\widetilde{\Phi}$ that $\widetilde{\Phi}(\xi_{ij}) = \Phi_{ij}e^{2\lambda t_{ij}}\Phi_{ij}^{-1}$ with $\Phi_{ij} \in \exp\widehat{\mathcal{T}}_n$, hence $\widetilde{\Phi}(\xi_{ij}) = \exp\widetilde{\psi}_{ij}$ where $\widetilde{\psi}_{ij} \in \mathcal{T}_n$ and $\widetilde{\psi}_{ij} = 2\lambda t_{ij}$ plus higher terms. In particular $R(\xi_{ij}) = \exp h\psi_{ij}$ with $\psi_{ij} \in \rho(\mathcal{T}_n) \otimes A$ and $\overline{\psi_{ij}} = 2\lambda\rho(t_{ij})$. It follows from the lemma that $\psi_{ij} \in \mathfrak{h}$. Now the K-Lie algebra generated by the ψ_{ij}'s is contained in $\mathcal{H}_\rho \otimes K$, and has dimension at least $\dim\mathcal{H}_\rho$ since the $\overline{\psi_{ij}}$'s generate $\mathcal{H}_\rho$. It follows $\mathcal{H}_\rho \otimes K \subset \mathfrak{h}$.

Because $\mathcal{F}_n$ is an ideal of $\mathcal{T}_n$ one has $\widetilde{\psi}_{ij} \in \widehat{\mathcal{F}}_n$ and one gets $\check{\mathcal{H}}_\rho \otimes K \subset \check{\mathfrak{h}}$ by the same argument. $\square$

In fact we proved more, namely that the same result holds when P_n and F_n are replaced by the subgroups generated by the ξ_{ij}^k's, and by the ξ_{in}^k's, for some fixed integer k. As a consequence of this, we get

Proposition 5.20. *Let G be a finite index subgroup of B_n, Γ the Zariski closure of $R(G)$ and $\mathfrak{h}$ the Lie algebra of Γ. Then $\mathcal{H}_\rho \otimes K \subset \mathfrak{h}$. The same holds for any subgroup of B_n containing ξ_{ij}^k, $1 \leq i < j \leq n$, for some fixed non-zero integer k.*

Proof. If G has finite index, there exists $k \in \mathbb{Z} \setminus \{0\}$ such that G contains the subgroup $P_n^{(k)}$ of B_n generated by the ξ_{ij}^k, $1 \leq i < j \leq n$. Then $R(G) \supset R(P_n^{(k)})$ and the Lie algebra of the Zariski closure of $R(P_n^{(k)})$ contains $\mathcal{H}_\rho \otimes K$ by the above remark, hence $\mathcal{H}_\rho \otimes K \subset \mathfrak{h}$. $\square$

The last subgoups of interest are the right-angled group $C_n^{(k)}$ generated by the elements $\sigma_1^{2k}, \ldots, \sigma_{n-1}^{2k}$, for some non-zero integer k.

Proposition 5.21. *For all $k \in \mathbb{Z} \setminus \{0\}$, let Γ_k be the Zariski closure of $R(C_n^k)$ and $\mathfrak{h}_k$ its Lie algebra. If $\mathcal{H}_\rho$ is generated by the elements $\rho(t_{i,i+1})$ for $1 \leq i \leq n-1$ then $\mathcal{H}_\rho \otimes K \subset \mathfrak{h}_k$*

Proof. We have $\widetilde{\Phi}(\sigma_i^{2k}) = \exp\widetilde{\psi}_i$ with $\widetilde{\psi}_i \in \widehat{\mathcal{T}}_n$ and $\widetilde{\psi}_i = 2k\lambda t_{i,i+1}$ plus higher terms. Hence $R(\sigma_i^{2k}) = \exp h\psi_i$ with $\overline{\psi_i} = 2k\lambda\rho(t_{i,i+1})$ and $\psi_i \in \mathfrak{h}_k$. Let $\mathcal{J}$ be the Lie algebra generated by the ψ_i. One has $\mathcal{J} \subset \mathfrak{h}_k$, $\mathcal{J} \subset \mathcal{H}_\rho \otimes K$ and $\mathcal{J}$ has dimension $\dim\mathcal{H}_\rho \otimes K$ since the elements $\overline{\psi_i}$ generate $\mathcal{H}_\rho$. $\square$

6. Basic constructions

In this last part we review the basic constructions of representations of the braid groups which appeared in the last decades. We show how these

can be dealt with in our terms, and how our approach – in particular our uniform approach to unitary representations – may simplify or improve either the constructions or the proof of their main properties. Incidentally, we shall get three different proofs of the unitarizability of the Burau representation: as a representation of the Iwahori-Hecke algebra (see 6.1.1), of the Birman-Wenzl-Murakami algebra (see 6.2.2), and as a product of Long's induction (see 6.3). For the sake of simplicity, we let $A = \Bbbk[[h]]$, $K = \Bbbk((h))$, and $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$, $\lambda \in \Bbbk^\times$ is chosen once and for all.

6.1. **Small representations**

Let $R : B_n \to GL_N(\Bbbk)$ be an irreducible representation of B_n. To any such representation are attached several integer invariants. Besides the dimension N, the most classical ones are the cardinality of the spectrum and the rank of $R(\sigma_1) - 1$. A systematic step-by-step search for irreducible representations of B_n can thus be carried out by imposing small values on these parameters. First note that, letting $R = \widehat{\Phi}(\rho)$, $\tau = \rho(t_{12})$, $R(\sigma_1) \in GL_N(K)$ is semisimple if and only if $\tau \in M_N(\Bbbk)$ is semisimple. We now review what happens when one of these parameters is "small".

6.1.1. **Small spectrum**

The first parameter we consider is the cardinality of the spectrum of $R(\sigma_1)$. We assume that $R(\sigma_1)$ is diagonalizable and let $\mathcal{S} = \{a_1, \ldots, a_r\} \subset K^\times$ be this spectrum. Then R factorizes through the K-algebra $H_n(a_1, \ldots, a_r)$ defined as the quotient of KB_n by the relation $(\sigma_1 - a_1)(\sigma_1 - a_2) \ldots (\sigma_1 - a_r) = 0$. If $\mathcal{S}$ has cardinality 1, then $H_n(a_1) = K$ and R factorizes through a morphism $B_n \to \mathbb{Z} \to GL_1(K)$. If $R = \widehat{\Phi}(\rho)$ the irreducible representation ρ must factorize through a morphism $\mathfrak{B}_n \to \Bbbk\mathfrak{S}_2 \ltimes \Bbbk$. We may thus assume $r \geq 2$.

Iwahori-Hecke algebras.   If $\mathcal{S} = \{a_1, a_2\}$ has cardinality 2, the algebra $H_n(a_1, a_2)$ is known as the Iwahori-Hecke algebra of type A_{n-1} with parameters a_1, a_2. It is well-known that this algebra is isomorphic to $K\mathfrak{S}_n$ as soon as $a_1/a_2 \in K$ is not a root of unity and is a square. In particular, if a_1 and a_2 are algebraically independent over $\mathbb{Q}$ and a_1/a_2 is a square, then $H_n(a_1, a_2) \simeq K\mathfrak{S}_n$ and, in case $R = \widehat{\Phi}(\rho)$, the spectrum of $\tau = \rho(t_{12})$ has to be of cardinality 2. We assume that $Sp(\tau) = \{u, v\} \subset \Bbbk$ with

$u \neq v$. Then $s = \rho(s_1)$ acts as a scalar on each eigenspace of τ, otherwise $R(\sigma_1)$ would admit three distinct eigenvalues; these scalars are distinct otherwise ρ would be reducible. We can thus assume that $\tau(x) = ux$ (resp. $\tau(x) = vx$) iff $s(x) = x$ (resp. $s(x) = -x$), up to tensoring by the sign representation of $\mathfrak{S}_n$. It follows that there exists $\alpha, \beta \in \Bbbk$ such that $\tau = \alpha + \beta s$.

Conversely, let M be an irreducible representation of $\mathfrak{S}_n$. For every $\alpha, \beta \in \Bbbk$ an element $\tau \in \mathcal{V}(M)$ can be defined by $\tau = \alpha + \beta s$, with s denoting the action of s_1 on M. If we let ρ be the corresponding representation of $\mathfrak{B}_n$, then $\widehat{\Phi}(\rho)(\sigma_1)$ is diagonalizable over K with two eigenvalues $\{e^{\lambda h(\alpha+\beta)}, -e^{\lambda h(\alpha-\beta)}\}$, which are algebraically independent as soon as $\beta \neq 0$, and it follows that R factorizes through the Hecke algebra. The cases when M corresponds to the partition $[n-1, 1]$ or $[2, 1^{n-2}]$ lead to the reduced Burau representation. Note that the existence of a unitary structure on the Hecke algebra representations (for parameters of modulus 1 and not roots of unity) can be seen in the following way. If $\Bbbk \subset \mathbb{R}$, there is a canonical euclidean structure on M for which the $\mathfrak{S}_n$-action is orthogonal. In particular, each transposition is orthogonal and involutive, hence selfadjoint, and τ is selfadjoint. This gives then rise to a unitary representation of $\mathfrak{B}_n$. Moreover, these representations are simplicial, essentially pure if $\beta \neq 0$, and the elements Y_n are sent to so-called Jucys-Murphy elements of the symmetric group. It follows that these representations are also aggregating. We determined the algebraic hull of B_n in these representations in [28].

Cubic Hecke algebras. For generic values of a, b, c – in particular when $a, b, c \in K$ are algebraically independent over $\Bbbk$ – the algebras $H_n(a, b, c)$ for $n \leq 5$ are now known to be finite-dimensional and semi-simple (see [31]). This is related to the fact that they appear as deformations of group algebras of exceptional finite complex reflections groups. We first study in some detail the case $n = 3$.

The finite complex reflection group called G_4 in Shephard and Todd classification (see [33]) admits the presentation $< S, T \mid S^3 = 1, T^3 = 1, STS = TST >$. It is obviously a quotient of B_3 through the map $\sigma_1 \mapsto S$, $\sigma_2 \mapsto T$, and has order 24. Let j be a 3-root of 1. This finite group admits three 1-dimensional irreducible representations X_ω for

$\omega \in \{1, j, j^2\}$ defined by $S, T \mapsto \omega$, three 2-dimensional irreducible representations U_ω characterized by the fact that S and T act with eigenvalues $\{1, j, j^2\} \backslash \{\omega\}$, and one 3-dimensional irreducible representation V. On the representation V, S and T act with eigenvalues $\{1, j, j^2\}$. It follows from [6] that KG_4 is isomorphic to its "cyclotomic Hecke algebra" $H_3(a, b, c)$ for generic a, b, c and in particular for a, b, c algebraically independent over $\Bbbk$. The irreducible representations of KG_4 uniquely deform into irreducible representations X_ω, U_ω and V of $H_3(a, b, c)$ with $\omega \in \{a, b, c\}$. More precisely, a computation of the discriminant shows that $H_3(a, b, c)$ is absolutely semi-simple if and only if

$$a.b.c.(c-b).(a-c).(b-a).(c^2-cb+b^2).(b^2-ba+a^2).(a^2+bc).(b^2+ac).(c^2+ab) \neq 0.$$

A first description of V was obtained in [6]. We obtained a symmetric model in [20]: σ_1 acts as the diagonal matrix with entries (a, b, c), and σ_2 acts as the following matrix :

$$\begin{pmatrix} \dfrac{(b+c)\,b\,c}{(a-b)\,(a-c)} & \dfrac{c\,(a\,c+b^2)}{(a-b)\,(a-c)} & \dfrac{b\,(a\,b+c^2)}{(a-b)\,(a-c)} \\[2ex] \dfrac{c\,(b\,c+a^2)}{(b-a)\,(b-c)} & \dfrac{(a+c)\,a\,c}{(b-a)\,(b-c)} & \dfrac{a\,(a\,b+c^2)}{(b-a)\,(b-c)} \\[2ex] \dfrac{b\,(b\,c+a^2)}{(c-a)\,(c-b)} & \dfrac{a\,(a\,c+b^2)}{(c-a)\,(c-b)} & \dfrac{(a+b)\,a\,b}{(c-a)\,(c-b)} \end{pmatrix}$$

The representations X_ω, U_ω arise from Hecke algebra representations, namely the trivial and reduced Burau representations. It follows from [7] that all these representations, including V, can be obtained by monodromy over the configuration space associated to G_4 from the corresponding G_4-representations. A natural question is to ask whether these representations arise as $\widehat{\Phi}(\rho)$ for some ρ. For all irreducible representations except V, this is a consequence of the study of Hecke algebra representations. It is also true for V, due to our work on $\mathcal{V}^s([2,1] \oplus [3])$ in section 4.1. It follows that every finite-dimensional representation can be obtained as a direct sum of representations of the form $\widehat{\Phi}(\rho)$, each twisted by some element in $\mathrm{Gal}(K/\Bbbk)$. Note however that some non-irreducible representations cannot be obtained directly as $\widehat{\Phi}(\rho)$ for some ρ. This is in particular the case for the representation $U_a \oplus U_b \oplus U_c$: the determinant of σ_1 should be equal to $(abc)^2$ and have -1 as constant term at the same time.

For every $n \geq 2$, there exists a finite-dimensional quotient of $H_n(a,b,c)$ known as the Birman-Wenzl-Murakami algebra. From the knowledge of the representation theory of G_4, it is easy to determine from the infinitesimal data whether the corresponding monodromy representation factorize through this algebra or not, and all representations of this algebra can be obtained by our method – we refer to [24] for details on these two points, and only mention that the corresponding representations of $\mathfrak{B}_n$ are simplicial, essentially pure and aggregating.

Important representations of the Birman-Wenzl-Murakami algebra are the Krammer representations — essentially the only known faithful representation of the braid groups. The algebraic hull of the braid groups inside them was computed in [29].

Quartic relations and beyond. In the same vein, $H_3(a,b,c,d)$ is a deformation of the group algebra of the finite complex reflection group called G_8. It is finite-dimensional, and isomorphic to KG_8 as soon as $a,b,c,d \in K$ are algebraically independent over $\Bbbk$ and $abcd$ admits a square root in K. For a detailed study of $H_3(1,u,v,w)$ we refer to the work of Broué and Malle [6]. The irreducible representations of G_8 of dimension less than 4 are not of interest for us, since they can be obtained by specialization from the cyclotomic Hecke algebra of G_4. The rest is formed by two 4-dimensional irreducible representations. The corresponding two irreducible representations of $H_3(a,b,c,d)$ can be distinguished by the scalar action of $(\sigma_1\sigma_2)^3$, and can be deduced from each other through the action of $\mathrm{Gal}(L/\Bbbk(a,b,c,d))$, where $L \subset K$ is the minimal (quadratic) extension of $\Bbbk(a,b,c,d)$ containing $\sqrt{abcd}$. These representations can be obtained by our method. Indeed, we proved in section 4.1 that there exists $\tau \in \mathcal{V}^s([2,1] + [3] + [1^3])$ such that the eigenvalues of τ can be chosen linearly independent over $\mathbb{Q}$, hence the corresponding eigenvalues of σ_1 are algebraically independent. Moreover, the determinant of σ_1 has all its square roots inside $\Bbbk[[h]]$ — hence its companion representation is deduced by using some element in $\mathrm{Gal}(K/\Bbbk)$. This example shows however that we definitely have to take into account these Galois actions: there is no way to get this other representation directly as $\widehat{\Phi}(\rho)$ for some ρ. The algebra $H_3(a,b,c,d,e)$ is again a deformation of the group algebra of a finite complex reflections group called G_{16}. However, this process does not go further: the quotient of B_3 by the relation $\sigma_1^r = 1$ for $r \geq 6$ is infinite.

6.1.2. **Small dimension**

Small-dimensional irreducible representations of B_n were studied by Formanek and Sysoeva (see [14, 35, 36]). In the infinitesimal setting, we denote by M a $\Bbbk\mathfrak{S}_n$-module, and ask how small can the dimension of M be with $\mathcal{V}^s(M) \neq \emptyset$. If M decomposes as a sum of 1-dimensional representations, then M should itself be of dimension 1 under this condition (see proposition 4.7). This happens in particular if M is of dimension at most $n-2$. If M is of dimension at most $n-1$, then the action of $\mathfrak{S}_n$ is the standard $(n-1)$-dimensional representation, or its tensorization by the sign representation and elements in $\mathcal{V}^s(M)$ correspond to the reduced Burau representation. If M has dimension n then M is, up to tensorization by the sign representation, the sum of the standard representation plus a 1-dimensional representation. This 1-dimensional representation must be the trivial one by proposition 4.7. The analysis of $\mathcal{V}^s([n-1,1]\oplus\mathbb{1})$ carried out in [20] shows that we get the irreducible representation of $\widetilde{\mathfrak{S}_n}$ described in section 4.3.2. These observations are infinitesimal analogues of the results of Formanek and Sysoeva. The other representations obtained in their work do not fit our approach, since the eigenvalues of $R(\sigma_1)$ are then algebraic over $\Bbbk$: they can be obtained only after specialization of the field of matrix coefficients — for instance, one of them is an $(n-2)$-dimensional representation of the Hecke algebra with parameters roots of unity.

6.1.3. **Small rank**

A central ingredient in Formanek's and Sysoeva's approach is the analysis of the irreducible representations R such that $R(\sigma_1) - 1$ has rank 1. Then $R(\sigma_1) - 1$ is either semisimple or nilpotent. Let $R = \widehat{\Phi}(\rho)$, $\tau = \rho(t_{12})$ for $\rho : \mathfrak{B}_n \to M_N(\Bbbk)$. In case $R(\sigma_1) - 1$ is semisimple it factorizes through the Hecke algebra of type A, and this general situation was already studied. In this particular case, since τ is then a linear combination of $\rho(s_1)$ and 1, one easily shows that $R(\sigma_1) - 1$ has rank one iff R is the reduced Burau representation. If it is nilpotent then $R(\sigma_1) - 1 \in \rho(s_1) - 1 + hM_N(A)$ implies that ρ is the trivial action of $\mathfrak{S}_n$ on $\Bbbk^N$. It thus contradicts our assumption that R is irreducible for $N \geq 2$. Besides this, it also implies that $R(\sigma_1) - 1 \in \lambda\rho(s_1)\tau h + h^2 M_N(A)$, hence that τ is nilpotent and of rank 1. A richer infinitesimal analogue of this analysis is then to ask what

happens when τ is a nilpotent transvection. The answer is that any such representation R will factorize through $\widetilde{\mathfrak{S}_n}$. Since it is equivalent to the fact that $\rho([t_{12}, t_{23}]) = 0$, this is an immediate consequence of the following result.

Lemma 6.1. *Let $\rho : \mathfrak{B}_3 \to M_N(\Bbbk)$ be a representation such that $\rho(t_{12})$ is a nilpotent transvection. Then $\rho([t_{12}, t_{23}]) = 0$.*

Proof. For all $1 \le i < j \le n$, there exists $v_{ij} \in \Bbbk^N \setminus \{0\}$ and a non-zero linear form $\varphi_{i,j}$ on $\Bbbk^N$ such that $\rho(t_{ij})(x) = \varphi_{ij}(x)v_{ij}$ for all $x \in \Bbbk^N$ and $\varphi_{ij}(v_{ij}) = 0$. Let us denote by $U \subset \Bbbk^N$ the $\Bbbk$-vector space spanned by the v_{ij}'s. It is invariant under the action of $\mathcal{T}_3$. Moreover, for all $x \in \Bbbk^N$ and $s \in \mathfrak{S}_3$,

$$\rho(st_{ij}s^{-1})(x) = \varphi_{ij}\left(\rho(s^{-1})(x)\right)\rho(s)(v_{ij})$$
$$= \rho(t_{s(i)s(j)})(x) = \varphi_{s(i)s(j)}(x)v_{s(i)s(j)}$$

and it follows that U is invariant under $\mathfrak{B}_3$. Now U is of dimension at most 3. This dimension cannot be 0 because $v_{12} \ne 0$. If it has dimension 1, then v_{12} and v_{23} are proportional, $\varphi_{12}(v_{12}) = 0$ implies $\varphi_{12}(v_{23}) = 0$ and similarly $\varphi_{23}(v_{12}) = 0$, then $\rho([t_{12}, t_{23}]) = 0$. If U has dimension 3, i.e. v_{12}, v_{13} and v_{23} are linearly independent, then $\rho([t_{12}, t_{13} + t_{23}]) = 0$ implies $\varphi_{12}(x)\varphi_{23}(v_{12}) = 0$ for all $x \in \Bbbk^N$, hence $\varphi_{23}(v_{12}) = 0$. Similarly, $\rho([t_{23}, t_{12} + t_{13}]) = 0$ implies $\varphi_{12}(v_{23}) = 0$, hence $\rho([t_{12}, t_{23}]) = 0$. The only remaining case is $\dim U = 2$. If $\mathfrak{S}_3$ acts on U by the trivial or sign action then U would be of dimension 1. If not, $\rho(s_1)$ would admit two distinct eigenvalues and $\rho(t_{12})$, commuting to $\rho(s_1)$, would be semisimple hence zero on U. Then $0 = \rho(t_{12})v_{23} = \varphi_{12}(v_{23})v_{12}$ implies $\varphi_{12}(v_{23}) = 0$ and similarly $\varphi_{23}(v_{12}) = 0$. This concludes the proof. $\square$

6.2. **Yang-Baxter representations**

6.2.1. **Preliminaries**

All the material here is standard. We recall that $\Bbbk$ has characteristic zero. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk$, and $K_\mathfrak{g}$ its Killing form. Since $\mathfrak{g}$ is semisimple, this bilinear symmetric invariant form is nondegenerate. Let $(e_\lambda)_{\lambda \in \Lambda}$ be a basis for $\mathfrak{g}$, and $(e^\lambda)_{\lambda \in \Lambda}$ the dual basis of $\mathfrak{g}$ with respect to $K_\mathfrak{g}$: $K_\mathfrak{g}(e_\lambda, e^\mu) = \delta_{\lambda,\mu}$. The Casimir element $C = \sum e_\lambda e^\lambda \in \mathsf{U}\mathfrak{g}$ does not depend on the choice of the basis (e_λ) and is central in $\mathsf{U}\mathfrak{g}$. We let

Δ be the coproduct of the Hopf algebra $\mathsf{U}\mathfrak{g}$, and introduce $c = \Delta(C) - C \otimes 1 - 1 \otimes C \in (\mathsf{U}\mathfrak{g})^{\otimes 2}$, $\tau = c \otimes 1 \in (\mathsf{U}\mathfrak{g})^{\otimes n}$. With respect to the action of $\mathfrak{S}_n$ on $(\mathsf{U}\mathfrak{g})^{\otimes n}$ by permutation of the factors, τ is $\mathfrak{S}_2 \times \mathfrak{S}_{n-2}$-invariant (because $\mathsf{U}\mathfrak{g}$ is cocommutative), and we define $\tau_{ij} = \tau_{ji}$ for $1 \leq i < j \leq n$ by the characteristic property $\tau_{s(1),s(2)} = s.\tau$ for $s \in \mathfrak{S}_n$. It is well known that c commutes to $\Delta(\mathfrak{g}) \subset (\mathsf{U}\mathfrak{g})^{\otimes 2}$ and that there exists a $\mathfrak{S}_n$-equivariant algebra morphism $\mathsf{U}\mathcal{T}_n \to (\mathsf{U}\mathfrak{g})^{\otimes n}$ defined by $t_{ij} \mapsto \tau_{ij}$.

As a consequence of this, there is an algebra morphism $\mathfrak{B}_n \to \Bbbk\mathfrak{S}_n \ltimes (\mathsf{U}\mathfrak{g})^{\otimes n}$. Since $\mathfrak{g}$ is semisimple, $\mathfrak{g}^n$ is semisimple and every representation of $(\mathsf{U}\mathfrak{g})^{\otimes n} = \mathsf{U}(\mathfrak{g}^n)$ is completely reducible. Moreover, every irreducible representation of $\mathfrak{g}^n$ has the form $V_1 \otimes \cdots \otimes V_n$ where the V_i's are irreducible $\mathfrak{g}$-modules. By lemma 5.4, every representation of $\Bbbk\mathfrak{S}_n \ltimes (\mathsf{U}\mathfrak{g})^{\otimes n}$ is completely reducible. A special kind of irreducible representation of $\Bbbk\mathfrak{S}_n \ltimes (\mathsf{U}\mathfrak{g})^{\otimes n}$ are of the type $V \otimes \cdots \otimes V$ for V an irreducible representation of $\mathfrak{g}$. Since c commutes to $\Delta(\mathfrak{g})$, the diagonal action of $\mathfrak{g}$ on any $(\mathsf{U}\mathfrak{g})^{\otimes n}$-module (resp. $\Bbbk\mathfrak{S}_n \ltimes (\mathsf{U}\mathfrak{g})^{\otimes n}$-module) commutes with the action of $\mathcal{T}_n$ (resp. $\mathfrak{B}_n$).

6.2.2. **Semisimplicity and unitarity**

Let us consider n *self-dual* irreducible $\mathfrak{g}$-modules $V_1, \ldots, V_n$, for which we fix an isomorphism $V_i \simeq V_i^\vee$. Let β_i for $1 \leq i \leq n$ be the non-degenerate $\mathfrak{g}$-invariant bilinear form on V_i defined by the composite $V_i \otimes V_i \simeq V_i \otimes V_i^\vee \to \mathbb{1}$, where $\mathbb{1}$ designates the trivial representation of $\mathfrak{g}$ and the last map is the canonical contraction. The form β_i is either symmetric or skew-symmetric, depending on whether the $\mathfrak{g}$-module V_i is orthogonal ($\mathbb{1} \hookrightarrow S^2 V_i$) or symplectic ($\mathbb{1} \hookrightarrow \Lambda^2 V_i$). We define a non-degenerate bilinear form β on $V_1 \otimes \cdots \otimes V_n$ as the product of the forms β_i. It is easily seen that $g_1 \otimes \cdots \otimes g_n \in (\mathsf{U}\mathfrak{g})^{\otimes n}$ is selfadjoint with respect to β as soon as there are an even number of $g_i \in \mathfrak{g}$ while the other ones belong to $\Bbbk$. In particular, each τ_{ij} is selfadjoint with respect to β and, if $V_1 = \cdots = V_n$, the action of $\mathfrak{S}_n$ is obviously isometric. This implies that the action of $\mathcal{T}_n$ (resp. $\mathfrak{B}_n$) is semisimple.

If the form β is symmetric and $V_1 = \cdots = V_n = V$, then the action of $\mathfrak{B}_n$ is unitary. This happens for instance if V is the adjoint representation of $\mathfrak{g}$, in which case each β_i is a multiple of the Killing form and is symmetric, or if n is even. However the most interesting situation is when $\Bbbk \subset \mathbb{R}$, but then β is not in general positive definite.

In one case however we can get unitary representations of $\mathfrak{B}_n$ on $V^{\otimes n}$ for any field of characteristic zero. Let α be the canonical symmetric nondegenerate bilinear form on $\Bbbk^m$ and $\mathfrak{g} = \mathfrak{so}_m(\Bbbk)$, $V = \Bbbk^m$. The forms β_i coincide with α, and β is the canonical bilinear form on $V^{\otimes n} = (\Bbbk^m)^{\otimes n} = \Bbbk^{mn}$. It follows that the action of $\mathfrak{B}_n$ and the corresponding action of B_n are unitary. This explains and proves the existence of a unitary structure on the representations of the Birman-Wenzl-Murakami algebra, since they appear in this way. More generally, if each representation V_i of $\mathfrak{g}$ is faithful and $\Bbbk = \mathbb{R}$, this situation happens exactly when the Lie algebra $\mathfrak{g}$ defined over $\mathbb{R}$ is compact; indeed, β_i can be chosen to be positive definite iff the image of $\mathfrak{g}$ in V_i is a compact Lie algebra.

6.2.3. **Other properties**

By its very definition, the eigenvalues of τ on $V^{\otimes n}$ are closely related to the eigenvalues of the Casimir operator on $V \otimes V$. In particular, if the decomposition of $V \otimes V$ as a $\mathfrak{g}$-module is multiplicity-free and the Casimir operator acts with distinct eigenvalues on each irreducible component, then the action of $\mathfrak{B}_n$ on $V^{\otimes n}$ is essentially pure. More generally, the eigenvalues of the action of each Y_r are related to the eigenvalues of the action of the Casimir operator on $V^{\otimes r}$. We note the following.

Lemma 6.2. *If $[\tau_{12}, \tau_{23}] = 0$ then $\mathfrak{g}$ is commutative.*

Proof. By extension of the base field, we can assume $\overline{\Bbbk} = \Bbbk$, hence there exists a basis $(e_\lambda)_{\lambda \in \Lambda}$ of $\mathfrak{g}$ which is self-dual with respect for the Killing form and $c = 2 \sum e_\lambda \otimes e_\lambda$. Then $[\tau_{12}, \tau_{23}]$ is the sum over all $\lambda, \mu \in \Lambda$ of the elements $e_\lambda \otimes [e_\lambda, e_\mu] \otimes e_\mu$ and the conclusion follows. $\square$

In particular such a representation will in general not factorize through the enhanced symmetric group. In case $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$, every representation is self-dual, so the actions of $\mathcal{T}_n$ will be semisimple. We recall from [25] the following.

Proposition 6.3. *If $V_1, \ldots, V_n$ are irreducible representations of $\mathfrak{sl}_2(\Bbbk)$, then the $\Bbbk$-vector space of highest weight vectors of $V_1 \otimes \cdots \otimes V_n$ is an aggregating representation of $\mathcal{T}_n$.*

6.3. **Artin action and Long induction**

6.3.1. **Artin action**

We denote by $f_1, \ldots, f_n$ given free generators of the free group L_n. The well-known faithful Artin action of B_n on L_n is given by making σ_i for $1 \leq i < n$ act on L_n by $f_i \mapsto f_{i+1}$, $f_{i+1} \mapsto f_{i+1}^{-1} f_i f_{i+1}$, and $f_j \mapsto f_j$ for $j \notin \{i, i+1\}$. A theorem of Artin says that this identifies B_n with the subgroup of automorphisms of L_n which preserve the set of conjugacy classes of $\{f_1, \ldots, f_n\}$ and the product $f_1 \ldots f_n$. The geometric meaning of this is that B_n can be identified with a set of diffeomorphisms of the disc minus n punctures, whose fundamental group is a free group on n generators. This action enables one to form the semi-direct product $B_n \ltimes L_n$, which embeds in B_{n+1} through the inclusion $B_n \subset B_{n+1}$ and the map $f_i \mapsto \xi_{i,n+1}$, thus identifying L_n and F_{n+1}. In particular, every representation of B_{n+1} restricts to a representation of $B_n \ltimes L_n \simeq B_n \ltimes F_{n+1} \subset B_{n+1}$. Also note that $L_n \rtimes B_n$ can be mapped onto $\mathbb{Z} \times B_n$ by sending each f_i to 1. In particular, every (irreducible) representation of B_n and any non-zero scalar α yield a (irreducible) representation of $B_n \ltimes L_n$. In general every representation R of $B_n \ltimes L_n$ can be rescaled by any non-zero scalar α in order to get a new representation R^α, letting $R^\alpha(\sigma) = R(\sigma)$ for $\sigma \in B_n$, $R^\alpha(f_r) = \alpha R(f_r)$. We define here an infinitesimal analogue of Artin action.

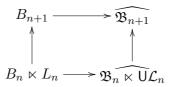Let $\mathcal{L}_n$ be the free Lie algebra on the abstract generators $g_1, \ldots, g_n$, considered as the Lie algebra of the pro-nilpotent completion of L_n. We make $s \in \mathfrak{S}_n$ act as the automorphism defined by $g_i \mapsto g_{s(i)}$ for $1 \leq i \leq n$, and t_{ij} as the derivation defined by

$$t_{ij}.g_k = 0 \text{ if } k \notin \{i, j\}$$
$$t_{ik}.g_k = [g_k, g_i]$$

This leads to a well-defined action by automorphisms of $\mathfrak{B}_n$ on $\mathsf{U}\mathcal{L}_n$, in the sense that grouplike elements in $\mathfrak{B}_n$ act as automorphisms and primitive elements in $\mathfrak{B}_n$ act as derivations, because the relation $[t_{ij}, t_{ik} + t_{kj}](g_i) = 0$ means $[[g_i, g_j], g_k] - [[g_i, g_k], g_j] - [g_i, [g_j, g_k]] = 0$, which is a version of the Jacobi identity. As a consequence of this, the semi-direct product $\mathfrak{B}_n \ltimes \mathsf{U}\mathcal{L}_n$ has a well-defined Hopf algebra structure, and the map $g_k \mapsto t_{k,n+1}$ together with the natural inclusion $\mathfrak{B}_n \subset \mathfrak{B}_{n+1}$ induce a Hopf algebra inclusion $\mathfrak{B}_n \ltimes \mathsf{U}\mathcal{L}_n \subset \mathfrak{B}_{n+1}$, thus identifying $\mathcal{L}_n$ with $\mathcal{F}_{n+1}$. The natural map $L_n \rtimes B_n \to \mathbb{Z} \times B_n$ corresponds to the following. Let x be an indeterminate, and make $\mathfrak{B}_n$ act trivially on the 1-dimensional space $\Bbbk x$ (i.e.

$s.x = x$ for $s \in \mathfrak{S}_n$, $t_{ij}.x = 0$). Now, $\Bbbk x$ can be considered as a (commutative) 1-dimensional Lie algebra and the Lie algebra morphism $\mathcal{L}_n \to \Bbbk x$ defined by $g_i \mapsto x$ is $\mathfrak{B}_n$-equivariant, hence leads to the awaited mapping $\mathsf{U}\mathcal{L}_n \rtimes \mathfrak{B}_n \to \Bbbk[x] \times \mathfrak{B}_n$. Here we let $\Bbbk[x]$ be the ring of polynomials in x and we identify it with the universal envelopping algebra of $\Bbbk x$.

We next apply the associator Φ. The image of $\xi_{i,n+1} \in P_{n+1}$ in $\widehat{\mathfrak{B}_{n+1}}$ under the associated Drinfeld morphism belongs to $\mathfrak{B}_n \ltimes \widehat{\mathsf{U}\mathcal{F}_{n+1}}$. Indeed, $\xi_{i,n+1}$ is a conjugate of σ_n^2 by elements of B_n, and the image of σ_n^2 is $\Phi(t_{n,n+1}, Y_{n+1})e^{2\lambda t_{n,n+1}}\Phi(Y_{n+1}, t_{n,n+1})$, hence belongs to $\widehat{\mathsf{U}\mathcal{F}_{n+1}}$. As a result, we get the following commutative diagram with all arrows injective.

$$
\begin{array}{ccc}
B_{n+1} & \longrightarrow & \widehat{\mathfrak{B}_{n+1}} \\
\uparrow & & \uparrow \\
B_n \ltimes L_n & \longrightarrow & \widehat{\mathfrak{B}_n \ltimes \mathsf{U}\mathcal{L}_n}
\end{array}
$$

6.3.2. Long induction

In [18], D.D. Long generalizes the classical Magnus construction to get (unitary) 1-parameter families of representations of B_n from a (unitary) representation of $F_n \ltimes B_n$. In particular, it is possible to deduce from given representations of B_{n+1}, or even representations of B_n, richer representations of B_n. We refer to [18] for the geometric interpretation of this construction, and recall the algebraic construction.

Let I be the augmentation ideal of $\Bbbk L_n$, and R a representation of $L_n \ltimes B_n$ on some $\Bbbk$-vector space V of finite dimension m. There are then well-defined actions of B_n on I and V, and the associated representation R^+ of B_n on the nm-dimensional $\Bbbk$-vector space $I \otimes_{\Bbbk L_n} V$ is given by making B_n act simultaneously on both factors. From the above observations, it follows that every representation R of B_{n+1} (resp. B_n), by restriction to $B_n \ltimes F_n$ and rescaling (resp. by the morphism $B_n \ltimes F_n \to B_n \times \mathbb{Z}$) leads to a family of B_n-representation $(R^\alpha)^+$. Using geometric means and a theorem of Deligne and Mostow, it is shown in [18] that, if $\Bbbk = \mathbb{C}$ and R is unitary, then for generic $\alpha \in \mathbb{C}$ of modulus 1 the representation $(R^\alpha)^+$ is unitary with respect to some non-degenerate hermitian form. This hermitian form is not explicitly given at the algebraic level, and comes from a Poincaré duality pairing.

We now give an infinitesimal analogue of this construction. Apart from linearizing the situation, it has the advantage that the infinitesimal bilinear form is explicitely described. Let ρ be a representation of $\mathsf{U}\mathcal{L}_n \rtimes \mathfrak{B}_n$, and ρ^α the representation deduced from ρ by the $\mathfrak{B}_n$-invariant twisting $g_i \mapsto g_i + \alpha$. We denote by V the underlying $\Bbbk$-vector space. A linear $\mathfrak{B}_n$-action ρ^+ is defined on V^n by making $s \in \mathfrak{S}_n$ act as

$$s.(v_1, v_2, \ldots, v_n) = (s.v_{s^{-1}(1)}, s.v_{s^{-1}(2)}, \ldots, s.v_{s^{-1}(n)})$$

and, denoting $x^+ = \rho^+(x)$, $t_{ij}^+ v = \tilde{v}$ with

$$\begin{cases} \tilde{v}_k &= t_{ij}v_k \text{ if } k \notin \{i, j\} \\ \tilde{v}_i &= t_{ij}v_i + g_j v_i - g_j v_j \\ \tilde{v}_j &= t_{ij}v_j + g_i v_j - g_i v_i \end{cases}$$

where $v = (v_1, \ldots, v_n)$, $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n)$. It is easily checked that ρ^+ is a representation of $\mathfrak{B}_n$. From the previous observations, we then get representations $(\rho^\alpha)^+$ of $\mathfrak{B}_n$ from any representation ρ of $\mathfrak{B}_{n+1}$ or $\mathfrak{B}_n$.

6.3.3. **Infinitesimal forms**

We assume that a representation ρ of $\mathfrak{B}_n \ltimes \mathsf{U}\mathcal{L}_n$ on a $\Bbbk$-vector space V is fixed, and that V is endowed with a non-degenerate bilinear form $(\ |\ )$ such that $\rho(s)$ is isometric for every $s \in \mathfrak{S}_n$. By abuse of notation, we shall write $tv = \rho(t)v$ (resp. $t^+\underline{v} = \rho^+(t)\underline{v}$) for every $t \in \mathfrak{B}_n$. We introduce on V the twisted bilinear forms $\beta_i(v, v') = (g_i v | v')$. We shall denote by $\underline{v} = (v_1, \ldots, v_n)$ and $\underline{v}' = (v_1', \ldots v_n')$ elements of V^n. We define on V^n the bilinear form

$$(\underline{v}, \underline{v}') = \sum_{i=1}^n (g_i v_i | v_i') = \sum_{i=1}^n \beta_i(v_i, v_i').$$

Last, we decompose each t_{ij}^+ into two endomorphisms, $t_{ij}^+ = d_{ij} + m_{ij}$, with

$$\begin{aligned} d_{ij}\underline{v} &= (t_{ij}v_1, \ldots, t_{ij}v_i, \ldots, t_{ij}v_j, \ldots, t_{ij}v_n) \\ m_{ij}\underline{v} &= (0, \ldots, g_j(v_i - v_j), \ldots, g_i(v_j - v_i), \ldots, 0) \end{aligned}$$

We need a technical lemma.

Lemma 6.4.   1) *For all $s \in \mathfrak{S}_n$, $\rho^+(s)$ is isometric.*

   2) *If $\rho(t_{ij})^\dagger = \pm\rho(t_{ij})$ then, for all $\underline{v}, \underline{v}' \in V^n$,*
     $(d_{ij}\underline{v}, \underline{v}') = \pm(\underline{v}, d_{ij}\underline{v}') + ([g_i, g_j]v_j | v_j') + ([g_j, g_i]v_i | v_i').$

3) *If $\rho(g_i)^\dagger = \pm\rho(g_i)$, then*
$$(m_{ij}\underline{v}, \underline{v}') = \pm(\underline{v}, m_{ij}\underline{v}') - ([g_i, g_j]v_j|v_j') - ([g_j, g_i]v_i|v_i').$$

Proof. For $s \in \mathfrak{S}_n$, $\rho(s)^\dagger\rho(s) = 1$ hence, for all $i \in [1, n]$, and $v, v' \in V$, $\beta_i(sv, sv') = (g_i sv, sv') = (sg_{s^{-1}(i)}v|sv') = \beta_{s^{-1}(i)}(v, v')$ and

$$(s^+\underline{v}, s^+\underline{v}') = \sum_{i=1}^{n} \beta_{s(i)}(sv_i, sv_i') = \sum_{i=1}^{n} \beta_i(v, v') = (\underline{v}, \underline{v}')$$

so we proved 1). We let $\Delta_{ij} = ([g_i, g_j]v_j|v_j') + ([g_j, g_i]v_i|v_i')$. Since $t_{ij}.g_r = 0$ for $r \notin \{i, j\}$ and $t_{ij}.g_j = [g_j, g_i]$ we have, for all $v, v' \in V$,

$$\begin{array}{rcl}
\beta_r(t_{ij}v, v') & = & (g_r t_{ij}v, v') = (t_{ij}g_r v, v') \\
\beta_i(t_{ij}v, v') & = & (t_{ij}g_i v|v') + ([g_j, g_i]v|v') \\
\beta_j(t_{ij}v, v') & = & (t_{ij}g_j v|v') + ([g_i, g_j]v|v')
\end{array}$$

It follows, if $\rho(t_{ij})^\dagger = \pm\rho(t_{ij})$, that $\beta_r(t_{ij}v, v') = \pm\beta_r(v, t_{ij}v')$ for $r \notin \{i, j\}$, and $(d_{ij}\underline{v}, \underline{v}') = (\underline{v}, d_{ij}\underline{v}') + \Delta_{ij}$ so 2) is proved. The last assertions are consequences of the following easy computations, where $u = v_i, w = v_j$, $u' = v_i', w' = v_j'$:

$$\begin{array}{l}
(m_{ij}\underline{v}, \underline{v}') = (g_i g_j u|u') - (g_i g_j w, u') + (g_j g_i w|w') - (g_i g_j u|w') \\
(\underline{v}, m_{ij}\underline{v}') = (g_i u|g_j u') - (g_j w|g_i u') + (g_j w, g_i w') - (g_i u|g_j w')
\end{array}$$

$\square$

In addition to this lemma, note that the (skew-)symmetricity of (,) depends on the (skew-)symmetricity of (|) and the (anti-)selfadjointness of the $\rho(g_i)$'s. This immediately leads to the following result, where a representation ρ of $\mathcal{L}_n$ is said to be orthogonal (resp. symplectic, unitary) with respect to some non-degenerate bilinear form β if, in accordance with proposition 3.8, β is symmetric (resp. skew-symmetric, symmetric) and $\rho(g_i)^\dagger = -\rho(g_i)$ (resp. $\rho(g_i)^\dagger = -\rho(g_i)$, $\rho(g_i)^\dagger = \rho(g_i)$).

Proposition 6.5. *If the restrictions of ρ to $\mathfrak{B}_n$ and $\mathcal{L}_n$ are symplectic (resp. orthogonal, unitary) then ρ^+ is orthogonal (resp. symplectic, unitary).*

Note that, for generic values of α (more precisely for $-\alpha \notin Sp(g_i)$), the form (,) is non-degenerate. In the case where $\Bbbk \subset \mathbb{R}$ and the restrictions of ρ are unitary, for a symmetric form (|) which is positive definite, (,) is also positive definite for $\alpha \in \mathbb{R}$ large enough. We then recover unitary representations of B_n.

Appendix A: Formal and real unitarity

Let $\Bbbk$ be one of the topological fields $\mathbb{R}$, $\mathbb{Q}_p$, or one of their finite extensions. In particular, it is endowed with a natural non-trivial valuation. A power series f in $\Bbbk[[h]]$ is called convergent if it has a non-zero radius of convergence with respect to the given valuation. We let $\Bbbk\{h\}$ be the ring of convergent power series, and by $\Bbbk(\{h\})$ its quotient field – that is, the set of Laurent series $f \in \Bbbk((h))$ such that $h^r f$ is a convergent power series for some integer r. We let $K = \Bbbk((h))$, $K^* = \Bbbk(\{h\})$ and, forgetting the given topology on $\Bbbk$, we endow these fields with the ordinary h-adic topology.

The purpose of this appendix is to prove approximation results in the framework of representation theory. More precisely, we work in the following scope. Let $R : G \to GL_N(K)$ be a representation of a *finitely generated* group G. Because of this finite generation hypothesis, the image of G in fact lies in $GL_N(L)$, for L a subfield of K which is finitely generated over $\Bbbk$. Let us assume next that there exists a subfield L^* of K^* which is isomorphic to L. The representation R^* deduced from R by this isomorphism is convergent, and the two representations R and R^* are essentially equivalent – they are conjugated by an element in $\mathrm{Gal}(K/\mathbb{Q})$. Here we prove more. Assuming that L contains $\Bbbk(h)$, we prove the existence of $\Bbbk(h) \subset L^* \subset K^*$ such that L^* is isomorphic to L as an extension of $\Bbbk(h)$ – this means that this isomorphism is the identity on $\Bbbk(h)$. We also say that L is isomorphic to L^* over $\Bbbk(h)$. Then R and R^* are conjugated by an element in $\mathrm{Gal}(K/\Bbbk(h))$.

We next deal with a more subtle question, involving unitarity. We let now $\Bbbk = \mathbb{R}$, and ϵ be the automorphism of K defined by $f(h) \mapsto f(-h)$. This is the only continuous automorphism of K such that $\epsilon(h) = -h$. It leaves K^* invariant – we say that K^* is an ϵ-invariant subfield of K. Using the notations of section 3.2.2, we denote by $U_N^\epsilon(K)$ the group $\{x \in GL_N(K) \mid x^{-1} =^t \epsilon(x)\}$, and assume that we are given a representation $R : G \to U_N^\epsilon(K)$. Again, because G is finitely generated, $R(G)$ lies in $U_N^\epsilon(L)$ for L some finitely generated subfield of K, which may be assumed to contain $\Bbbk(h)$ and to be ϵ-invariant. We prove that there exists a finitely generated subfield L^* of K^*, containing $\Bbbk(h)$ and ϵ-invariant, such that $U_N^\epsilon(L) \simeq U_N^\epsilon(L^*)$. We then explain how, after a convenient rescaling of the local parameter h, we get from this a convergent representation such

that specialization in h leads to representations of G into the ordinary unitary group U_N.

We first prove the following result.

Theorem 6.6. *Let L be a subfield of K which contains $\Bbbk(h)$ and is finitely generated over $\Bbbk$. Then there exists a subfield L^* of K^* which contains $\Bbbk(h)$ and is isomorphic to L over $\Bbbk(h)$.*

Moreover, for any finite family $a_1, \ldots, a_t \in L$ and $m \geq 0$, then the isomorphism $L \to L^*$ can be chosen such that each a_i is congruent to its image modulo h^m.

Proof. Let $\Bbbk(h) \subset L_0 \subset L$ be a maximal purely transcendental extension of $\Bbbk(h)$ contained in L. Since L is finitely generated over $\Bbbk$, its transcendence degree over $\Bbbk(h)$ is finite, whence there exists $f_1, \ldots, f_r \in K$ algebraically independent over $\Bbbk(h)$ such that $L_0 = \Bbbk(h)(f_1, \ldots, f_r)$. The field L is by definition an algebraic extension of L_0, and a finite extension because L is finitely generated over $\Bbbk$. The field K^* is an extension of $\Bbbk(h)$ of infinite transcendence degree – for instance, if $\Bbbk = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}_p$, the family $\{e^{h^d}\}_{d>0}$ is algebraically independent over $\Bbbk(h)$ and its elements belong to K^*. On the other hand, if $g_1, \ldots, g_r \in K^*$ are algebraically independent over $\Bbbk(h)$, so are $g_1 + P_1, \ldots, g_r + P_r$ for $P_1, \ldots, P_r$ arbitrary elements of $\Bbbk(h)$. It follows from these two facts that a family $g_1, \ldots, g_r$ of convergent power series, algebraically independent over $\Bbbk(h)$, can be chosen as close to $f_1, \ldots, f_r$ as we want with respect to the h-adic topology. Let us now introduce the abstract field $\tilde{L}_0 = \Bbbk(h)(Y_1, \ldots, Y_r)$, where $Y_1, \ldots, Y_r$ are indeterminates. Every family $g = (g_1, \ldots, g_r)$ as above yield an embedding of $\tilde{L}_0$ into K^*, through $Y_i \mapsto g_i$. We denote by L_0^g the image subfield of K^*. The extensions $\tilde{L}_0$, L_0 and L_0^g of $\Bbbk(h)$ are by definition isomorphic, and the above considerations show that there exists a family $g_1, \ldots, g_r$ such that $L_0^g \subset K^*$, so this proves the result in case L is a purely transcendantal extension. In order to prove the general case, we make use of the additional fact that $(g_1, \ldots, g_r)$ may be chosen arbitrarily near to $(f_1, \ldots, f_r)$. All families $g_1, \ldots, g_r$ chosen below are assumed to be algebraically independent over $\Bbbk(h)$.

Since L is finite and separable as an extension of L_0, the primitive element theorem yield $\alpha \in K$ such that $L = L_0(\alpha)$. Because L_0 contains $\Bbbk(h)$, we can and will assume $\alpha \in h\Bbbk[[h]]$. Let $P \in L_0[X]$ be a minimal polynomial of α over L_0. By not requiring P to be monic, we may assume

$P \in \Bbbk[h, f_1, \ldots, f_r]$. The field L is then isomorphic to $L_0[X]/(P)$. Since L_0 is isomorphic to $\tilde{L}_0$ over $\Bbbk(h)$, there exists $\tilde{P} \in \tilde{L}_0[X]$ such that P (resp. its derivative P') is deduced from $\tilde{P}$ (resp. $\tilde{P}'$) through the substitution $Y_i \mapsto f_i$. Similarly, the substitution $Y_i \mapsto g_i$ leads to polynomials P_g and P'_g in $L_0^g[X]$, such that

$$L \simeq L_0[X]/(P) \simeq \tilde{L}_0[X]/(\tilde{P}) \simeq L_0^g[X]/(P_g)$$

as extensions of $\Bbbk(h)$. In order to map $L_0^g[X]/(P_g)$ into K^*, it is then sufficient to find convergent $g_1, \ldots, g_r$ such that P_g admit a root in K^*. Because of Artin's theorem (see Artin [2] th. 1.2), it is sufficient for this that P_g admits a root in $h\Bbbk[[h]]$. Since P is a minimal polynomial for α over L_0, and L_0 has characteristic 0, $P(\alpha) = 0$ and $P'(\alpha) \neq 0$. Since $P'(\alpha) \in \Bbbk[[h]]$, there exists then $s \geq 0$, $\beta \in \Bbbk \setminus \{0\}$ such that $P'(\alpha) = h^s \beta$ modulo h^{s+1}. Choosing $g_1, \ldots, g_r \in \Bbbk\{h\}$ close enough to $f_1, \ldots, f_r$ so that $P_g(\alpha) = 0$ modulo h^{2s} and $P'_g(\alpha) = h^s \beta$ modulo h^{s+1}, we have $P_g(\alpha) \in P'_g(\alpha)^2 h\Bbbk[[h]]$. Hensel's lemma then asserts (see [13], theorem 7.3) the existence of $\gamma \in \Bbbk[[h]]$ such that $P_g(\gamma) = 0$ and $\gamma - \alpha \in P'_g(\alpha)h\Bbbk[[h]] \subset h\Bbbk[[h]]$, whence we conclude that $\gamma \in h\Bbbk[[h]]$ and $L \simeq L_0^g[X]/(P_g) \simeq L_0^g(\gamma) \subset K^*$ as extensions of $\Bbbk(h)$.

Now consider a family $a_1, \ldots, a_t$ as in the statement, let $f_1, \ldots, f_r$ and α be as above. There exist polynomials $Q_1, \ldots, Q_t$ with coefficients in $\Bbbk(h)$ such that $a_i = Q_i(f_1, \ldots, f_r, \alpha)$. By continuity of these polynomials, there exists $M_0 \in \mathbb{N}$ such that $f_i \equiv g_i$ and $\alpha \equiv \beta$ modulo h^{M_0} implies $a_i \equiv Q_i(g_1, \ldots, g_r, \beta)$ modulo h^m. By requiring the $g_1, \ldots, g_r$ chosen above to be congruent modulo h^{M_1} for some $M_1 \geq M_0$, one may assume that $s \geq M_0$ hence $\gamma \equiv \alpha$ modulo h^{M_0}. Then Artin's approximation theorem says that the corresponding root β of P_g in K^* can be chosen arbitrarily close to γ, and the conclusion follows. $\qquad\square$

Recall that we denote by ϵ the automorphism of K defined by $f(h) \mapsto f(-h)$. We will need the following proposition only for $\Bbbk = \mathbb{R}$, although it can be proved in a more general context.

Proposition 6.7. *Let L be an ϵ-invariant finitely generated subfield of K containing $\Bbbk(h)$. Then there exists an ϵ-invariant subfield L^* of K^* containing $\Bbbk(h)$ and a field isomorphism $\Omega : L \to L^*$ such that $\epsilon \circ \Omega = \Omega \circ \epsilon$.*

Moreover, for any finite family $a_1, \ldots, a_t \in L$ and $m \geq 0$, then this isomorphism $L \to L^*$ can be chosen such that each a_i is congruent to its image modulo h^m.

Proof. Let $L^\epsilon = \{x \in L \mid \epsilon(x) = x\}$ be the set of elements of L which are fixed by ϵ. It obviously contains $\Bbbk(h^2)$ and is contained in $K^\epsilon = \Bbbk((h^2))$. Then L is a quadratic extension of L^ϵ, $L = L^\epsilon \oplus hL^\epsilon$ as a L^ϵ-vector space and $L \simeq L^\epsilon[X]/(X^2 - h^2)$ as a field. We let Φ be the canonical field isomorphism $\Bbbk((h^2)) \to K$ defined by $f(h^2) \mapsto f(h)$, and let $\Lambda = \Phi(L^\epsilon)$. Note that Φ and Φ^{-1} send convergent series to convergent series.

We have $L \simeq \Lambda_+ = \Lambda[X]/(X^2 - h)$ and the action of ϵ on Λ_+ is the non-trivial element of the Galois group $\mathrm{Gal}(\Lambda_+/\Lambda)$. The theorem claims that there exists a subfield Λ^* of K^*, which contains $\Bbbk(h)$ and is isomorphic to Λ over $\Bbbk(h)$. Thus there exists a field isomorphism between the field extensions Λ^+/Λ and Λ_+^*/Λ^*; it sends non-trivial elements in $\mathrm{Gal}(\Lambda/\Lambda_+)$ to non-trivial elements in $\mathrm{Gal}(\Lambda^*/\Lambda_+^*)$. We finally let $L_-^* = \Phi^{-1}(\Lambda^*)$ and $L^* = L_-^*(h) \subset K^*$. The composite of this isomorphisms is a field isomorphism $\Omega : L \to L^*$, and it is easily checked that $\Omega \circ \epsilon = \epsilon \circ \Omega$. We summarize the situation by the following commutative diagram in the category of field extensions. All vertical unlabelled arrows represent the only non-trivial element in the Galois group of the corresponding quadratic extensions.

$$
\begin{array}{ccccccccc}
L/\Bbbk(h^2) & \longrightarrow & L/L^\epsilon & \overset{\Phi}{\longrightarrow} & \Lambda_+/\Lambda & \longrightarrow & \Lambda_+^*/\Lambda^* & \overset{\Phi^{-1}}{\longrightarrow} & L^*/L_-^* \\
\epsilon \downarrow & & \downarrow & & \downarrow & & \downarrow & & \epsilon \downarrow \\
L/\Bbbk(h^2) & \longrightarrow & L/L^\epsilon & \overset{\Phi}{\longrightarrow} & \Lambda_+/\Lambda & \longrightarrow & \Lambda_+^*/\Lambda^* & \overset{\Phi^{-1}}{\longrightarrow} & L^*/L_-^*
\end{array}
$$

For the last part, $a_i \in L = L^\epsilon \oplus hL^\epsilon$ can be decomposed as $a_i = a_i^+ + ha_i^-$. Then the isomorphism $\Lambda \to \Lambda^*$ can be chosen such that the $a_i^\pm$'s equal their images modulo h^m and the conclusion follows. $\qquad\square$

We finally use this proposition in the set-up of the introduction. Let $R : G \to U_N^\epsilon(K)$ be a representation of the finitely generated group G into the formal unitary group $U_N^\epsilon(K)$ with $\Bbbk = \mathbb{R}$. Our purpose is to deduce from this non-trivial representations of G into the unitary group U_N. Let L be the smallest ϵ-invariant subfield of $K = \mathbb{R}((h))$ containing $\mathbb{R}(h)$ and the coefficients of $R(g)$ for $g \in G$. Because G is finitely generated as a group, and ϵ has finite order, L is finitely generated over $\mathbb{R}$ and we can apply the proposition to this field. Let $L^* \subset K^* = \mathbb{R}(\{h\})$ and $\Omega : L \to L^*$ be the field and field isomorphism given by the proposition. We extend $\Omega : L \to L^*$ coefficientwise to a group isomorphism $GL_N(L) \to GL_N(L^*)$.

Since L contains the coefficients of R, R factorizes through $U_N^\epsilon(L) \subset GL_N(L)$, and we have

$$
\begin{aligned}
U_N^\epsilon(L^*) &= \{x \in GL_N(L^*) \mid x^{-1} = {}^t\epsilon(x)\} \\
&= \{\Omega(y) \mid y \in GL_N(L),\ \Omega(y)^{-1} = {}^t\epsilon \circ \Omega(y)\} \\
&= \{\Omega(y) \mid y \in GL_N(L),\ \Omega(y^{-1}) = \Omega({}^t\epsilon(y))\} \\
&= \Omega(U_N^\epsilon(L)).
\end{aligned}
$$

hence $\Omega \circ \rho : G \to U_N^\epsilon(M)$ is a representation of G into $U_N^\epsilon(\mathbb{R}(\{h\}))$. Let c be the automorphism of $\mathbb{C}((h))$ defined by $f(h) \mapsto f(ih)$, and η the automorphism $f \mapsto \bar{f}$ induced by the complex conjugation of the coefficients. These two automorphisms leave $\mathbb{C}(\{h\})$ invariant, and it is readily checked that ϵ and η coincide on $c(\mathbb{R}((h))) = \mathbb{R}((ih))$. In particular, if we let $J = c(L^*) \subset \mathbb{C}(\{h\})$, then

$$
c \circ \Omega \circ \rho : G \to U_N^\epsilon(K) = U_N^\eta(K) \subset U_N^\eta(\mathbb{C}(\{h\}))
$$

hence, by specialization in h real and close to 0, we get morphisms $G \to U_N$, i.e. *unitary representations of G* in the ordinary sense.

Appendix B: Combinatorial aspects

For a given $\Bbbk$-algebra $\mathcal{A}$, a *chain* is a filtration of algebras $\Bbbk = \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n = \mathcal{A}$. A representation ρ of $\mathcal{A}$ is called multiply semisimple (MSS) with respect to this chain if the restriction of ρ to each $\mathcal{A}_r$, $1 \le r \le n$, is semisimple. For a group G, a chain is a filtration of subgroups $\{e\} \subset G_1 \subset \cdots \subset G_n = G$, and a representation ρ is said to be (MSS) with respect to this chain if and only if, as a $\Bbbk G$-module, it is (MSS) with respect to the corresponding filtration of the group algebra.

Under this assumption one can introduce the Bratteli diagram of a representation ρ. We consider this diagram as an oriented graph Γ with two distinguished vertices, the vertex $\mathbf{O}$ corresponding to the trivial representation of $\Bbbk = \mathcal{A}_1$, and the vertex ρ. We choose by convention that no edge ends at $\mathbf{O}$. The *level* of a vertex is by definition 1 plus its distance to $\mathbf{O}$ in the underlying non-oriented graph. A vertex of level r thus corresponds to an irreducible representation of $\mathcal{A}_r$. We say that such a Bratteli diagram is *multiplicity free* if every two vertices are connected by at most one edge.

B.1 Infinitesimal representations

Bratteli diagrams for essentially pure representations. We expose a combinatorial device to deal with an irreducible essentially pure $\mathfrak{B}_n$-representation ρ. We moreover assume that the (MSS) condition holds. Note that this condition is automatically satisfied if ρ is aggregating or unitary. One can then define its Bratteli diagram with respect to the chain

$$\Bbbk \simeq \mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \cdots \subset \mathfrak{B}_n$$

or to the chain

$$\Bbbk \simeq \mathsf{U}\mathcal{T}_1 \subset \mathsf{U}\mathcal{T}_2 \subset \cdots \subset \mathsf{U}\mathcal{T}_n.$$

Under our assumption of essential purity, it is clear that one gets the same Bratteli diagram Γ, choosing one chain or another. We let T_r be the sum of all t_{ij}'s in $\mathcal{T}_r$, so that $Y_r = T_r - T_{r-1}$.

We associate to each vertex of level r in Γ the action $T_r(\rho')$ of T_r on the corresponding $\mathfrak{B}_r$-representation ρ': it belongs to $\mathrm{End}(\rho')$ as a $\mathfrak{B}_r$-module, since T_r belongs to the center of $\mathfrak{B}_r$. We also associate to each edge $p \to q$ with q of level r corresponding to the representation ρ' the action $Y_r(p \to q) \in \mathrm{End}_{\mathbb{C}}(\rho')$ on the components of (the representation corresponding to) q which are isomorphic to p. We then get a *natural coloring* of the sets of vertices and edges of the Bratteli diagram.

Multiplicity free diagrams. We now assume that the Bratteli diagram Γ of ρ is multiplicity free. Let $\rho_1, \ldots, \rho_r$ be the irreducible components of the restriction of ρ to $\mathfrak{B}_{n-1}$, and $Z_n(\rho)$ (resp. $Z_{n-1}(\rho_i)$) the action of $Z_n = 2T_n/n(n-1)$, (resp. $Z_{n-1} = 2T_{n-1}/(n-1)(n-2)$) on ρ (resp. ρ_i). If $Z_n(\rho)$ and $Z_{n-1}(\rho_i)$ are *scalars*, then the following relations hold

$$(\dim \rho)Z_n(\rho) = tr(Z_n) = \sum_{i=1}^r tr(Z_{n-1}(\rho_i)) = \sum_{i=1}^r (\dim \rho_i)Z_{n-1}(\rho_i)$$
$$\dim \rho = \sum_{i=1}^r \dim \rho_i$$

i.e. $(Z_n(\rho), \dim \rho)$ is obtained as the barycentre of the massive points $(Z_{n-1}(\rho_i), \dim \rho_i)$. It follows that from a Bratteli diagram and a set of colors on the vertices of level 2 (given by the natural coloring) one can build a *formal coloring* of the sets of vertices and edges by scalars, defining $Z_r(\rho)$ by the above formula, $T_r(\rho) = r(r-1)Z_r(\rho)$, and $Y_r(p \to q) = T_r(q) - T_{r-1}(p)$ for p of level $r < n$. It is clear that the formal and natural colorings are the same as soon as every T_r acts as a scalar on each vertex of level r — in particular, if $\Bbbk$ is considered as a $\mathbb{Q}$-affine space, the value

of $Z_n(\rho)$ then lies inside the convex hull of the spectrum of $\rho(t_{12})$. We say that a coloring of such a graph is *injective* if the map which to a path from **O** to ρ associate the corresponding n-tuple of colors of the vertices (or, equivalently, the $(n-1)$-tuple of colors of the edges) is injective. We then have

Proposition 6.8. *Let us suppose ρ as above. If the formal coloring of its Bratteli diagram is injective, then ρ is aggregating and the natural coloring is the same as the formal coloring.*

Proof. We prove this by induction on n. The case $n = 2$ is obvious. Let us suppose $n \geq 3$, and denote by $\rho_1, \ldots, \rho_r$ the irreducible components of the restriction of ρ to $\mathfrak{B}_{n-1}$. The coloring of the ρ_i's is injective, then they are aggregating and the natural coloring of their diagram is the same as their formal coloring. In particular their natural coloring is a coloring by scalars and is injective. It easily follows (see [25]) that ρ is aggregating, hence is absolutely irreducible as stated in proposition 5.9. Then T_n acts as a scalar hence the natural and formal coloring are the same. We conclude by induction. □

As a corollary, the diagram Γ together with given colors on the level 2 vertices is sufficient data in this case to determine whether the corresponding monodromy representation R is aggregating, or simplicial. It also enables one to check whether R factors through $Hurw_n$, $\mathbb{Z}$ or $\widetilde{\mathfrak{S}}_n$. Indeed, the action of Y_n is readily determined from the coloring, and R factors through $\widetilde{\mathfrak{S}}_n$ iff it factors through $\mathbb{Z}$ iff Γ is a segment — because we assumed ρ to be essentially pure. Note that the eigenvalues of the $\rho(t_{ij})'s$, the dimension of ρ and the Bratteli diagram of R with respect to $B_1 \subset \cdots \subset B_n$ are also determined by these data.

B.2 Braid representations over an algebraic closed field

We denote by K an algebraically closed field of characteristic zero. Let R be a (MSS) representation of B_n with respect to the chain $B_1 \subset \cdots \subset B_n$, such that the corresponding Bratteli diagram Γ is multiplicity free. In the same vein as before, we get a "natural" coloring of the vertices (resp. of the edges) from the action of the γ_r's (resp. the δ_r's) — since K is algebraically closed, $R(\gamma_n)$ is a scalar by Schur's lemma, and $R(\delta_n) = R(\gamma_n)/R(\gamma_{n-1})$ is split.

We want to know how much information one can get from Γ and the natural coloring of the vertices of level 2. First notice that, if no two level-2 vertices have the same natural coloring, it implies that $R(P_n) = R(B_n)$. Indeed, since $\delta_2 = \sigma_1^2$, it means that $R(\sigma_1)$ is a polynomial in $R(\sigma_1^2)$, hence every $R(\sigma_i)$ is a polynomial in $R(\sigma_i^2)$. We now try to recover the natural coloring from these data. For this purpose, let us introduce $z_n(R) = R(\gamma_n)^{\frac{1}{n(n-1)}} \in K^\times/\mu_\infty(K)$. Since $\gamma_n = (\sigma_1 \dots \sigma_{n-1})^n$,

$$\det R(\gamma_n) = (\det R(\sigma_1))^{n(n-1)} = R(\gamma_n)^{\dim R}$$

and $z_n(R)^{\dim R}$ equals $\det R(\sigma_1)$ modulo $\mu_\infty(K)$.

Let us denote by $R_1, \dots, R_r$ the vertices of level $n-1$ in Γ. Since $\det R(\sigma_1)$ is the product of the $\det R_i(\sigma_1)$ for $1 \le i \le r$, it follows that

$$z_n(R) = \prod_{i=1}^{r} z_{n-1}(R_i)^{\frac{\dim R_i}{\dim R}}$$

and the natural coloring can be recovered up to roots of unity.

Now assume that the natural coloring of Γ is known, as well as the representations $R_1, \dots, R_r$. Assuming this coloring known, we do not need the algebraic closeness assumption anymore. For a complete description of R, the only missing piece is $R(\sigma_{n-1})$. Since σ_{n-1} commutes to B_{n-2}, it suffices to determine the action of σ_{n-1} on each vector space $\mathrm{Hom}_{B_{n-2}}(\check{R}, R)$, for $\check{R}$ a vertex of level $n-2$ in Γ. The equation $\delta_n = \sigma_{n-1}\delta_{n-1}\sigma_{n-1}$ can be rewritten as $(\sigma_{n-1}\delta_{n-1})^2 = (\delta_{n-1}\sigma_{n-1})^2 = \gamma_n$. Depending on the complexity of Γ, this yields to a sometimes very tractable set of equations on each space $\mathrm{Hom}_{B_{n-2}}(\check{R}, R)$. Applications of this method can be found in [20]. Its interest heavily depends on the knowledge we have on Γ, the roots of unity involved in its coloring, and the spectrum of σ_1.

As we have seen before, these data are known if $R = \Phi(\rho)$ with $\Phi \in \mathbb{Assoc}_\lambda(\Bbbk)$, $\lambda \neq 0$ and ρ an essentially pure irreducible representation of $\mathfrak{B}_n$. This combinatorial approach has thus the advantage of avoiding the (intricated) calculation of $\Phi(x, y)$ for $x = \rho(t_{i,i+1})$, $y = \rho(Y_i)$ and nevertheless getting an explicit (matrix) description of R.

References

[1] E. Artin – "Theorie der Zöpfe", *Abhandlungen Hamburg* **4** (1925), p. 47–72.

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[2] M. Artin – "On the solutions of analytic equations", *Invent. Math.* **5** (1968), p. 277–291.

[3] J. Bellaïche – "A propos d'un lemme de Ribet", *Rend. Semin. Mat. Univ. Padova* **109** (2003), p. 45–62.

[4] J. Bellaïche & P. Graftieaux – "Représentations sur un anneau de valuation discrète complet", *Math. Ann.* **334** (2006), p. 465–488.

[5] J. Birman – *Braids, links, and mapping class groups, Annals of Mathematics Studies* **82**, Princeton University Press and University of Tokyo Press, Princeton, Tokyo, 1975.

[6] M. Broué & G. Malle – "Zyklotomische Heckealgebren", *Astérisque* **212** (1993), p. 119–189.

[7] M. Broué, G. Malle & R. Rouquier – "Complex reflection groups, braid groups, Hecke algebras", *J. Reine Angew. Math.* **500** (1998), p. 127–190.

[8] C. Chevalley – *Théorie des groupes de Lie, vol. 2-3*, Hermann, Paris, 1961.

[9] F. Cohen & S. Prassidis – "On injective homomorphisms for pure braid groups, and associated Lie algebras", *J. Algebra* **298** (2006), p. 363–370.

[10] D. Collins – "Relations among the squares of the generators of the braid group", *Invent. Math.* **117** (1994), p. 525–529.

[11] W. Crawley-Boevey – "On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero", *Duke Math. J.* **118** (2003), p. 339–352.

[12] V. Drinfeld – "On quasitriangular quasi-Hopf algebras and a group closely connected with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$", *Leningrad Math. J.* **2** (1991), p. 829–860.

[13] D. Eisenbud – *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, Heidelberg, 1995.

[14] E. Formanek – "Braid group representations of low degree", *Proc. London Math. Soc.* **73** (1996), p. 279–322.

[15] T. Kohno – "Série de Poincaré-Koszul associée aux groupes de tresses pures", *Inven. Math.* **82** (19!5), p. 57–76.

[16] _____, "Linear representations of braid groups and classical Yang-Baxter equations", *Contemp. Math.* **78** (1988), p. 339–363.

[17] T. Le & J. Murakami – "The universal Vassiliev-Kontsevitch invariant for framed oriented links", *Compositio Math.* **102** (1996), p. 41–64.

[18] D. Long – "Constructing representations of the braid groups", *Comm. in analysis and geometry* **2** (1994), p. 217–238.

[19] I. Marin – "On KZ-systems which are irreducible under the action of the symmetric group", *C. R. Acad. Sci. Paris Sér. I* **333** (2001), p. 517–522.

[20] _____, "Représentations linéaires des tresses infinitésimales", Thèse de l'université Paris XI-Orsay, 2001.

[21] _____, "Caractérisations de la représentation de Burau", *Expo. Math.* **21** (2003), p. 263–278.

[22] _____, "Infinitesimal Hecke algebras", *Comptes Rendus Mathématiques Série I* **337** (2003), p. 297–302.

[23] _____, "On the representation theory of braid groups", *Preprint Université d'Evry* (2003).

[24] _____, "Quotients infinitésimaux du groupe de tresses", *Ann. Inst. Fourier (Grenoble)* **53** (2003), p. 1323–1364.

[25] _____, "Irréductibilité générique des produits tensoriels de monodromies", *Bull. Soc. Math. Fr.* **132** (2004), p. 201–232.

[26] _____, "Caractères de rigidité du groupe de Grothendieck-Teichmüller", *Compositio Math.* **142** (2006), p. 657–678.

[27] _____, "Monodromie algébrique des groupes d'Artin diédraux", *J. Algebra* **303** (2006), p. 97–132.

[28] _____, "L'algèbre de Lie des transpositions", *J. Algebra* **310** (2007), p. 742–774.

[29] _____, "Sur les représentations de Krammer génériques", *Ann. Inst. Fourier (Grenoble)* **57** (2007), p. 1883–1925.

[30] _____, "Braids inside the Birman-Wenzl-Murakami algebra", *Algebraic Geometric Topology* **10** (2010), p. 1865–1886.

APPENDIX B: COMBINATORIAL ASPECTS

[31] _____, "The cubic Hecke algebra on at most 5 strands", *J. Pure Applied Algebra* **216** (2012), p. 2754–2782.

[32] K. RIBET – "A modular construction of unramified extensions of $\mathbb{Q}(\mu_p)$", *Invent. Math.* **34** (1976), p. 151–162.

[33] G. SHEPHARD & J. TODD – "Finite unitary reflection groups", *Canad. J. Math.* **6** (1954), p. 274–304.

[34] C. SQUIER – "The Burau representation is unitary", *Proc. Am. Math. Soc.* **90** (1984), p. 199–202.

[35] I. SYSOEVA – "On irreducible representations of braid groups", Ph.D. thesis, University of Pennsylvania, 1999.

[36] _____, "Dimension n representations of the braid groups on n strings", *J. Algebra* **243** (2001), p. 518–538.

[37] J. TITS – "Normalisateurs de tores I : Groupes de Coxeter étendus", *J. Algebra* **4** (1966), p. 96–116.

[38] V. VASSILIEV – "Complements of discriminants of smooth maps : topology and applications", *Translation of mathematical monographs* **98** (1992).

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