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Abstract

A classical model for three-phase capillary immiscible flows in a porous medium is considered. Capillarity pressure functions are found, with a corresponding diffusion-capillarity tensor being triangular. The model is reduced to a degenerate quasilinear parabolic system. A global existence theorem is proved under some hypotheses on the model data.

1. Introduction

We study the question of global solvability for the 2×2 quasi-linear parabolic system

$$u_t + f(u)_x = (B(u)u_x)_x, \quad x \in \Omega = \{x \in \mathbb{R} : |x| < 1\}, \quad 0 < t < T, \tag{1.1}$$

motivated by three-phase capillary flows in a petroleum reservoir. Here

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

and (1.1) is a short version of

$$\frac{\partial u_i}{\partial t} + \frac{\partial f_i(u)}{\partial x} = \frac{\partial}{\partial x}(B_{ij}(u)\frac{\partial u_j}{\partial x}).$$

To help readers gain intuition about the work, we explain how the reservoir flow equations can be reduced to (1.1).

We consider one-dimensional horizontal flows of three incompressible immiscible fluids formed in phases, say, oil, gas, and water [1]. The balance of masses is governed by the mass conservation equations

$$(\Phi u_i \rho_i)_t + (\rho_i v_i)_x = 0, \quad \rho_i = \text{const}, \tag{1.2}$$

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where Φ denotes porosity of the porous medium, u_i, ρ_i, and v_i are the saturation, density, and seepage velocity of the i–th phase. Since u_i denotes the portion of the unite pore volume filled with the i–th phase, one has

$$u_1 + u_2 + u_3 = 1. \tag{1.3}$$

The momentum equations are given in the form of Darcy's law

$$v_i = -k\lambda_i p_{ix}, \quad \lambda_i = \lambda_i(u_1, u_2), \tag{1.4}$$

where k stands for the absolute permeability, λ_i is the mobility of the i–th phase, and p_i is the pressure of the i–th phase.

The functions $p_{ij}(u_1, u_2)$, which define the pressure differences

$$p_1 - p_3 = p_{13}, \quad p_2 - p_3 = p_{23}, \tag{1.5}$$

are called capillary pressures.

Positive constants Φ, ρ_i, and k and the functions $\lambda_i(u_1, u_2)$, $(i = 1, 2, 3)$, and $p_{13}(u_1, u_2)$, $p_{23}(u_1, u_2)$ constitute the model data. Assume for simplicity that $k = \Phi = 1$.

Let us denote

$$\lambda = \sum_1^3 \lambda_i, \quad v = \sum_1^3 v_i, \quad f_i = \frac{\lambda_i}{\lambda}, \qquad i = 1, 2, 3. \tag{1.6}$$

$$\triangle := \{u : u \in \mathbb{R}^2, \quad 0 \le u_i \le 1, \quad u_1 + u_2 \le 1\}, \tag{1.7}$$

It follows from (1.2) and (1.3) that $v_x = 0$, so v depends on t only. Setting $v \equiv 1$ and eliminating the third phase, we obtain system (1.1) for the vector $u = (u_1, u_2)^T$, where $f(u) := (f_1, f_2)^T$ and the matrix B is given by

$$B_{11} = \frac{\lambda_1(\lambda_2 + \lambda_3)}{\lambda}\frac{\partial p_{13}}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial p_{23}}{\partial u_1}, \tag{1.8}$$

$$B_{12} = -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial p_{23}}{\partial u_2} + \frac{\lambda_1(\lambda_2 + \lambda_3)}{\lambda}\frac{\partial p_{13}}{\partial u_2}, \tag{1.9}$$

$$B_{21} = \frac{\lambda_2(\lambda_1 + \lambda_3)}{\lambda}\frac{\partial p_{23}}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial p_{13}}{\partial u_1}, \tag{1.10}$$

$$B_{22} = -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial p_{13}}{\partial u_2} + \frac{\lambda_2(\lambda_1 + \lambda_3)}{\lambda}\frac{\partial p_2}{\partial u_2}. \tag{1.11}$$

Up to now very little is known about the functions $p_{ij}(u)$ both theoretically and experimentally [3, 6]. The same is true for the mobility functions $\lambda_i(u)$, $i = 1, 2, 3$ [11]. The following properties are conventional [1]:

(i) The functions $\lambda_i(u)$, $p_{ij}(u)$ and the corresponding matrix B, calculated by the formulas (1.6) and (1.8)-(1.11), should be such that system (1.1) is parabolic in a sense.

(ii) The functions λ_i satisfy the conditions

$$\lambda_i \geq 0, \quad \lambda_i|_{u_i=0} = 0, \quad i \in \{1,2,3\}. \tag{1.12}$$

Due to (1.12), system (1.1) is not parabolic on the boundary of the triangle $\triangle$. Since B is degenerate and not diagonal, the well-known theory of parabolic equations cannot be applied to system (1.1). For example, in the theory of Ladyzenskaya, Solonnikov and Ural'tseva [7], the matrix B is nondegenerate and it is a scalar multiple of the identity matrix. Nevertheless, system (1.1) can be analyzed in the case of special equations of state. In [3], a numerical study of system (1.1) was performed in the case of an existing potential $p_c(u)$:

$$\frac{\partial p_c}{\partial u_i} = f_1 \frac{\partial p_{13}}{\partial u_i} + f_2 \frac{\partial p_{23}}{\partial u_i}, \quad i = 1,2.$$

In [4] and [5], three-phase flows were considered with a triangular capillary diffusion tensor B, i.e., with the conditions

$$B_{21} \equiv \frac{\lambda_2(\lambda_1+\lambda_3)}{\lambda} \frac{\partial p_{23}}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda} \frac{\partial p_{13}}{\partial u_1} = 0, \quad \frac{\partial B_{22}}{\partial u_1} = 0, \tag{1.13}$$

which mean that the first and third phases do not influence the diffusion of the second phase.

As is well known by reservoir engineers, mobilities and capillary pressures can be plotted as functions of the saturations only in the case of flows where just two phases are present [9, 6], that is, when $u \in \partial\triangle$. While there are some widely accepted models which artificially prescribe the mobility functions in the interior of $\triangle$, such as the one proposed by H. Stone [10], the same is no longer true for the functions representing the capillary pressures. Now, the constraints (1.13) amount to a linear hyperbolic system of partial differential equations for the capillary pressures p_{13} and p_{23}, whose coefficients involve the mobility functions, for which we set the boundary conditions

$$p_{13}|_{u_2=0} = \varphi_{13}(u_1), \quad p_{23}|_{u_1=0} = \varphi_{23}(u_2), \tag{1.14}$$

enforcing compatibility with the two-phase flow case, where the functions φ_{ij} can be obtained from two-phase flows experiments, as already mentioned. The result is a consistent recipe for defining the capillary pressures in the interior of the triangle of saturations $\triangle$. Here, we study in detail the case when the mobilities are linear functions of the corresponding phase saturation

$$\lambda_i = k_i u_i, \quad k_i = \text{const} > 0. \tag{1.15}$$

Below we establish by the group analysis methods that conditions (1.13) are a tool for interpolating two-phase systems to three-phase ones in the following sense. Equalities (1.14) imply that the capillary pressure p_{13} is prescribed for the two-phase system phase 1–phase 3 and the capillary pressure p_{23} is prescribed for the two-phase system phase 2–phase 3. Equations (1.13) can be used to recover the capillary pressures p_{ij} in the entire phase triangle Δ by the formulas

$$p_{13}(u_1, u_2) = \varphi_{13}(\xi) - \int_0^{u_2} \frac{k_0 k_2 k_3 u_2 \varphi_{23}'(u_2)}{k_2 u_2 + k_3(1-u_2)}\, du_2 + \text{const}, \tag{1.16}$$

$$p_{23}(u_1, u_2) = \int_0^{\xi} A(\xi)\varphi_{13}'(\xi)\, d\xi + \varphi_{23}(u_2) + \text{const}, \tag{1.17}$$

where

$$k_0 = \frac{k_3 - k_1}{k_1 k_3}, \quad \xi = \frac{u_1}{1-u_2}, \quad A(\xi) = \frac{\lambda_1}{\lambda_1 + \lambda_3}.$$

When the mobilities and the capillary pressures are given by (1.15)-(1.17), the vector f and the matrix B become

$$f = \begin{pmatrix} \frac{k_1 u_1}{\varepsilon u_1 + k_2 u_2 + k_3(1-u_2)} \\ \frac{k_2 u_2}{\varepsilon u_1 + k_2 u_2 + k_3(1-u_2)} \end{pmatrix}, \tag{1.18}$$

$$B = \begin{pmatrix} \frac{k_1 \xi (1-\xi)\varphi_{13}'(\xi)}{k_1 \xi + k_3(1-\xi)} & \xi(B_{11} - B_{22}) \\ 0 & \frac{k_3 k_2 u_2 (1-u_2)\varphi_{23}'(u_2)}{k_2 u_2 + k_3(1-u_2)} \end{pmatrix}. \tag{1.19}$$

The volume balance equation (1.3) reduces to the condition

$$u(x,t) \in \triangle, \quad \forall\, (x,t) \in Q := \Omega \times (0,T). \tag{1.20}$$

In short, the condition (1.20) reads

$$0 \le u_i(x,t) \le 1, \quad 0 \le u_1(x,t) \le 1 - u_2(x,t), \quad \forall\, (x,t) \in Q.$$

We study system (1.1) under the restriction (1.20), with f and B given by (1.18) and (1.19). Observe that the existence theorems of H. Amann [2] are also obtained for system (1.1) (without the restriction (1.20)) under the condition $B_{21} = 0$ but they are valid only for nondegenerate matrix B in the case when $\frac{\partial f_2}{\partial u_1} \equiv 0$ and under the assumption that some solution's norms are finite a priori.

2. **Global existence**

We consider system (1.1), (1.18), (1.19), (1.20) with the following initial and boundary conditions:

$$\sigma u_n + u = d \quad \text{for} \quad |x| = 1, \quad u|_{t=0} = u_0(x), \tag{2.1}$$

where $\sigma = \text{const} > 0$ and

$$u_n = \pm u_x, \quad d = d^{\pm} \quad \text{for} \quad x = \pm 1.$$

Observe that system (1.1) decouples when $k_1 = k_3$ because $f_2(u)$ is then independent of u_1 and the equation for u_2 takes the form

$$u_{2t} + f_2^0(u_2)_x = (B_{22}(u_2)u_{2x})_x, \quad f_2^0 \equiv \frac{k_2 u_2}{k_2 u_2 + k_3(1 - u_2)}. \tag{2.2}$$

However, condition (1.20) written for u_2 in the form

$$0 \le u_2(x,t) \le 1 - u_1(x,t), \tag{2.3}$$

prevents us from solving equation (2.2) for u_2 independently.

The solvability of the degenerate problem (1.1), (1.20), (2.1) with $k_1 = k_3$ was established in [5]. Here, we consider the general case when (possibly) $k_1 \neq k_3$, but the parameter

$$\varepsilon = k_1 - k_3$$

is assumed to be small.

It is also assumed that the vector functions $u_0(x)$ and $d(t)$ take values strictly inside a triangle $\Delta_\delta \subset \Delta$; i.e., for $x \in \Omega$, $0 \le t \le T$,

$$\text{dist}\{u_0(x), \partial\Delta\} \ge \delta, \quad \text{dist}\{d^+(t), \partial\Delta\} \ge \delta, \quad \text{dist}\{d^-(t), \partial\Delta\} \ge \delta, \tag{2.4}$$

for some $\delta > 0$. Specifically, the first of these inequalities means that all the three phases are initially present at each point of the porous sample; i.e., the physical system is not degenerate.

Concerning smoothness, we assume that

$$u_0 \in H^{2+\alpha}(\overline{\Omega}), \quad d^{\pm} \in H^{(1+\alpha)/2}([0,T]) \tag{2.5}$$

for some $\alpha \in (0,1)$.

Under the conditions

$$\varphi_{i3}'(s) > 0 \quad \text{for} \quad 0 < s < 1, \quad \varphi_{i3} \in C^3([0,T]), \tag{2.6}$$

imposed on the elements of B, system (1.1) is parabolic. Below is the main result.

Theorem 2.1. *Let conditions (2.3)-(2.6) be satisfied and the compatibility conditions*

$$\pm\sigma u_0'(\pm 1) + u_0(\pm 1) = d^{\pm}(0)$$

hold. Then problem (1.1), (1.20), (2.1) has a solution in the class

$$u \in H^{2+\alpha,1+\alpha/2}(\overline{Q}), \quad Q = \Omega \times (0,T),$$

if $|\varepsilon| \le \varepsilon_* < 1$*, where* ε_* *is a constant depending on* δ*,* T*, an the number* M *bounding the norms of the initial and boundary data:*

$$\|u_0\|_{H^{2+\alpha}(\overline{\Omega})} \le M, \quad \|d^{\pm}\|_{H^{(1+\alpha)/2}([0,T])} \le M. \tag{2.7}$$

Proof. Step 1. We perform the change of variables $(u_1, u_2) \to (\xi, u_2)$. Note that ξ is the relative phase saturation, since $\xi = u_1/(u_1 + u_3)$. Written in terms of the new variables, the original problem becomes

$$\xi_t + A_{11}(\xi, u_2)\xi_x + \varepsilon A_{12}(\xi, u_2)u_{2x} = \tag{2.8}$$

$$= (B_{11}(\xi)\xi_x)_x - \xi_x u_{2x}(B_{11} + B_{22})/(1 - u_2),$$

$$\xi|_{t=0} = \xi_0(x) \equiv \frac{u_{10}(x)}{1 - u_{20}(x)}, \quad \Big(\frac{\sigma\xi_n(1-u_2)}{1-d_2} + \xi - \xi^{\pm}\Big)\Big|_{|x|=1} = 0, \tag{2.9}$$

$$u_{2t} + \varepsilon A_{21}(\xi, u_2)\xi_x + A_{22}(\xi, u_2)u_{2x} = (B_{22}(u)u_{2x})_x, \tag{2.10}$$

$$u_2|_{t=0} = u_{20}(x), \quad (\sigma u_{2n} + u_2 - d_2)|_{|x|=1} = 0, \tag{2.11}$$

where

$$\lambda^2 A_{11} = k_1((k_2 - k_3)u_2 + k_3) - \varepsilon k_2 \xi u_2, \quad \lambda^2 A_{12} = -\xi(1-\xi)/(1-u_2),$$

$$\lambda^2 A_{21} = -k_2 u_2(1 - u_2), \quad \lambda^2 A_{22} = k_2 k_3 + \varepsilon k_2 \xi,$$

and

$$\xi^{\pm} \equiv \frac{d_1^{\pm}}{1 - d_2^{\pm}}.$$

The advantage of system (2.8)-(2.11) over (1.1) is that the former decouples with respect to the higher derivatives: the equation for ξ does not contain u_{2xx}, and the equation for u_2 does not contain ξ_{xx}. It should be stressed that A_{12} vanishes at $\xi = 0$ and $\xi = 1$, and A_{21} vanishes at $u_2 = 0$ and $u_2 = 1$.

Step 2. The indicated structural properties of system (2.8)-(2.11) guarantee that none of the three phases disappears. This fact in underlain by the following statement.

Lemma 2.2. *Let $u(x,t)$ be a smooth solution to the degenerate parabolic equation*

$$u_t + a_j(x,t,u,\nabla u)u_{x_j} + u(1-u)F(x,t) = (B_{ij}(u)u_{x_i})_{x_j} \tag{2.12}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with the initial and boundary conditions

$$u|_{t=0} = u_0(x), \quad (\sigma u_n + u - d(x,t))|_{x\in\partial\Omega} = 0.$$

Given $\delta \in (0,1)$, let

$$\delta \le u_0(x) \le 1-\delta, \quad \delta \le d(x,t) \le 1-\delta.$$

Then there is a constant $\delta_1 \in (0,1)$ depending on δ, T, and $\sup_Q |F(x,t)|$, such that

$$\delta_1 \le u(x,t) \le 1-\delta_1, \quad \forall (x,t) \in \overline{Q}, \quad Q = \Omega \times (0,T). \tag{2.13}$$

The Lemma is proved by proceeding to $v = \frac{1}{2}\ln u/(1-u)$ and considering the equivalent parabolic problem

$$v_t + \tilde{a}_j(x,t,v,\nabla v)v_{x_j} + F = (\tilde{B}_{ij}(v)v_{x_i})_{x_j} - \tilde{B}_{ij}(v)v_{x_i}v_{x_j}\frac{1-e^{-2v}}{1+e^{2v}}, \tag{2.14}$$

$$\Big(\sigma v_n - \frac{\psi(d_1)-\psi(v)}{u'(v)}\Big)|_{x\in\partial\Omega} = 0, \quad v|_{t=0} = \frac{1}{2}\ln\frac{u_0}{1-u_0},$$

where

$$\psi(s) = \frac{s}{1+s}, \quad d_1 = \frac{1}{2}\ln\frac{d}{1-d}.$$

The coefficients $\tilde{a}_j$ and $\tilde{B}_{ij}$ are calculated in terms of a_j and B_{ij} as a result of the substitution $u \to v$.

Obviously, estimates (2.13) are equivalent to the boundedness estimate for $|v|$. This estimate can be easily obtained by applying the maximum principle.

Applying Lemma 1 to equation (2.10) and, then, to equation (2.8) gives the estimates

$$\delta_1 \le \xi \le 1-\delta_1, \quad \delta_2 \le u_2 \le 1-\delta_2, \tag{2.15}$$

where δ_i depends on δ, T and J_i:

$$J_1 = \sup_Q |u_{2x}|, \quad J_2 = \sup_Q |\xi_x|.$$

Moreover, δ_1 depends on δ_2. Note that δ_i does not decrease with J_i.

Although inequalities (2.15) are not a priori estimates, they nevertheless suggest that system (2.8)-(2.11) is nondegenerate on smooth solutions. Thus, the well-known theory of [7] can be applied to it.

Step 3. Now we reduce the problem to one of finding a fixed point of a certain operator. Let ε be fixed. We choose an arbitrary number $r > 0$ and consider an arbitrary function $\zeta \in H^{1+\alpha,(1+\alpha)/2}(\overline{Q})$ such that

$$\|\zeta\|^{(1+\alpha)} \leq r, \tag{2.16}$$

where $\|\cdot\|^{(k+\alpha)}$ is the norm in $H^{k+\alpha,(k+\alpha)/2}(\overline{Q})$.

Substituting ζ for ξ in equation (2.10), we consider equation (2.10), treating it as quasilinear for u_2. In view of (2.15), we have

$$\delta_2 \leq u_2 \leq 1 - \delta_2, \tag{2.17}$$

where δ_2 depends on δ, T, and εr. Therefor, under the condition (2.16), equation (2.10) is nondegenerate and we can apply the results of [7]; i.e., for a given function ζ with the condition (2.16), there exists a unique solution $u_2 \in H^{2+\alpha,(2+\alpha)/2}(\overline{Q})$ that satisfies (2.17) and the estimate

$$\|u_2\|^{(2+\alpha)} \leq b_1(\varepsilon r, \delta, T, M). \tag{2.18}$$

Now we turn to the problem (2.8), (2.9), assuming that $u_2(x,t)$ is the function found above by solving the problem (2.10), (2.11). The solution ξ to the problem (2.8), (2.9) satisfies the estimate (2.15). Therefore, equation (2.8) can be viewed as a quasilinear nondegenerate parabolic equation for ξ. Consequently, $\xi \in H^{2+\alpha,(2+\alpha)/2}(\overline{Q})$ and

$$\|\xi\|^{(2+\alpha)} \leq b_2(b_1, \varepsilon r, \delta, T, M). \tag{2.19}$$

Thus, the operator $\zeta \to \xi$, where $\xi = \mathcal{A}_\varepsilon(\zeta)$, is defined in the space $H^{1+\alpha,(1+\alpha)/2}(\overline{Q})$. The estimate (2.19) means that $\mathcal{A}_\varepsilon$ is completely continuous. Note that the constants in the estimates (2.17) – (2.19) do not vary with r and ε if the product εr remains a constant. Obviously, the fixed points of $\mathcal{A}_\varepsilon$ are the solutions to the original problem.

Step 4. We apply the Schauder fixed-point theorem. Proceeding as at Step 3, we set $r = 1$ in the inequality (2.16). Then it follows from (2.17) – (2.19) that, for any $\varepsilon \in [0,1]$ we have the estimates

$$\delta_2^* \leq u_2 \leq 1 - \delta_2^*, \quad \|u_2\|^{(2+\alpha)} \leq b_1^*, \quad \|\xi\|^{(1+\alpha)} \leq c_1^*. \tag{2.20}$$

In particular, they mean that, for any $\varepsilon \in [0,1]$, the operator $\mathcal{A}_\varepsilon$ maps the ball $\|\zeta\|^{(1+\alpha)} \le 1$ into $\|\zeta\|^{(1+\alpha)} \le c_1^*$.

Now we consider the operator $\mathcal{A}_\varepsilon$ on the ball $\|\zeta\|^{(1+\alpha)} \le c_1^*+1 \equiv r_1$. Let ε_1 be chosen so that $\varepsilon_1 r_1 \le 1$. Then, for any $\varepsilon \in [0,\varepsilon_1]$, the inequalities (2.20) hold true. Thus, for any $\varepsilon \in [0,\varepsilon_1]$, the operator $\mathcal{A}_\varepsilon$ is completely continuous and maps the ball $\|\zeta\|^{(1+\alpha)} \le r_1$ into itself. Therefore, Theorem 1 is valid by the Schauder fixed-point argument. □

3. Capillary pressure functions

Here we perform derivation of the capillary pressure functions (1.16) and (1.17) which agree with the hypotheses (1.13) on the diffusion capillarity tensor B.

First, we apply the group symmetry analysis to the homogeneous system for $p_{ij}(u_1,u_2)$

$$B_{12}=0, \quad B_{22}=0,$$

which writes

$$A\frac{\partial p_{13}}{\partial u_1} = \frac{\partial p_{23}}{\partial u_1}, \quad \frac{\partial p_{23}}{\partial u_2} = A\frac{\partial p_{13}}{\partial u_2}, \quad A=A(\xi), \quad \xi = \frac{u_1}{1-u_2}. \tag{3.1}$$

Application of the algorithm of group calculation [8] shows that any one-parameter group admitted by (3.1) is defined with the infinitesimal operator

$$X = \zeta^1(u_1,u_2,p_{13},p_{23})\frac{\partial}{\partial u_1} + \zeta^2(\cdots)\frac{\partial}{\partial u_2} + \eta^1(\cdots)\frac{\partial}{\partial p_{13}} + \eta^2(\cdots)\frac{\partial}{\partial p_{23}},$$

where the functions ζ^i and η^i are subject to the restrictions

$$\zeta^1\frac{\partial A}{\partial u_1} + \zeta^2\frac{\partial A}{\partial u_2} + A(\frac{\partial \eta^1}{\partial p_{13}} + A\frac{\partial \eta^1}{\partial p_{23}}) = \frac{\partial \eta^2}{\partial p_{13}} + A\frac{\partial \eta^2}{\partial p_{23}},$$

$$\frac{\partial \eta^2}{\partial u_2} = A\frac{\partial \eta^1}{\partial u_2}, \quad \frac{\partial \eta^2}{\partial u_1} = A\frac{\partial \eta^1}{\partial u_1}$$

Observe, that

$$\frac{\partial A}{\partial u_1} = A'(\xi)\frac{1}{1-u_2}, \quad \frac{\partial A}{\partial u_2} = A'(\xi)\frac{u_1}{(1-u_2)^2}.$$

Hence, there is a one-parameter group with the operator

$$X = -\xi\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}, \quad \xi = \frac{u_1}{1-u_2}.$$

The meaning of this group is that system (3.1) is invariant under the change of variables $(u_1, u_2) \to (u_1^{'}, u_2^{'})$:

$$u_1^{'} = u_1 - \frac{au_1}{1-u_2}, \quad u_2^{'} = u_2 + a, \quad a \in \mathbb{R}.$$

One can verify easily that system (3.1) has a solution depending only on the variable ξ. Indeed, given a function $q_1(\xi)$, the functions

$$p_{13} = q_1(\xi), \quad p_{23} = \int A(\xi) q_1^{'}(\xi) d\xi$$

solve system (3.1).

Now we address the nonhomogeneous system (1.13) which writes

$$A\frac{\partial p_{13}}{\partial u_1} = \frac{\partial p_{23}}{\partial u_1}, \quad \frac{\partial p_2}{\partial u_2} = A\frac{\partial p_{13}}{\partial u_2} + \frac{\lambda B_{22}(u_2)}{\lambda_2(\lambda_1+\lambda_3)}, \quad A = \frac{\lambda_1}{\lambda_1+\lambda_3}. \tag{3.2}$$

We study these equations for $p_{i3}(u_1, u_2)$ in the case when the mobilities λ_i are linear functions:

$$\lambda_i = k_i u_i, \quad k_i = \text{const}.$$

The above analysis of the homogeneous system suggests to look for solutions in the form

$$p_{i3} = q_i(\xi) + Q_i(u_2), \quad \xi = \frac{u_1}{1-u_2} \equiv \frac{u_1}{u_1+u_3}. \tag{3.3}$$

It follows from (3.2) that the functions q_i and Q_i solve the system

$$q_2^{'}(\xi) = q_1^{'}(\xi)A(\xi), \quad A = \frac{k_1\xi}{(k_1-k_3)\xi+k_3}, \quad Q_1^{'}(u_2) = -\frac{k_0 B_{22}(u_2)}{1-u_2},$$

$$Q_2^{'}(u_2) = B_{22}(u_2)(\frac{1}{k_3(1-u_2)} + \frac{1}{k_2u_2}), \quad k_0 = \frac{k_3-k_1}{k_1k_3}. \tag{3.4}$$

Assume that the capillary pressure $p_{13}(u)$ is a given function of u_1 at the part of the boundary of the triangle Δ where $u_2 = 0$:

$$p_{13}|_{u_2=0} = \varphi_{13}(u_1).$$

Assume also that the capillary pressure $p_{23}(u)$ is a given function of u_2 at the edge where $u_1 = 0$ of the triangle Δ:

$$p_{23}|_{u_1=0} = \varphi_{23}(u_2).$$

It follows from (3.3) that

$$\varphi_{13}(u_1) = q_1(u_1) + Q_1(0), \quad \varphi_{23}(u_2) = q_2(0) + Q_2(u_2).$$

It is naturally to set

$$q_1(\xi) = \varphi_{13}(\xi), \quad Q_2(u_2) = \varphi_{23}(u_2).$$

Then the other functions $Q_1(u_2)$ and $q_2(\xi)$ are defined from (3.4) as follows:

$$q_2(\xi) = \int_0^\xi A(\xi)\varphi_{13}'(\xi)d\xi, \quad B_{22}(u_2) = \frac{k_2k_3u_2(1-u_2)\varphi_{23}'(u_2)}{k_2u_2 + k_3(1-u_2)},$$

$$Q_1'(u_2) = -\frac{k_0}{1-u_2}B_{22}(u_2).$$

Thus, we arrive at formulas (1.16) and (1.17) for the capillary pressures.

We call the procedure yielding formulas (1.16) and (1.17) *the method of physical interpolation* since these formulas define the capillary pressures p_{13} and p_{23} in Δ from their values when $u_2 = 0$ and $u_1 = 0$, respectively.

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