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ABSTRACT In this paper, we define a number of closely related isomorphisms. On one side of these isomorphisms sit a number of algebras generalizing the Hecke and affine Hecke algebras, which we call the “Hecke family”; on the other, we find generalizations of KLR algebras in finite and affine type A, the “KLR family.”

We show that these algebras have compatible isomorphisms generalizing those between Hecke and KLR algebras given by Brundan and Kleshchev. This allows us to organize a long list of algebras and categories into a single system, including (affine/cyclotomic) Hecke algebras, (affine/cyclotomic) q -Schur algebras, (weighted) KLR algebras, category \mathcal{O} for \mathfrak{gl}_N and for the Cherednik algebras for the groups $\mathbb{Z}/e\mathbb{Z} \wr S_n$, and give graded presentations of all of these objects.

1. INTRODUCTION

Fix a field \mathbb{k} and an element $q \neq 1, 0 \in \mathbb{k}$. Let e be the multiplicative order of q . In this paper, we discuss isomorphisms between two different families of algebras constructed from this data.

One of these families is ultimately descended from Erich Hecke, though it is a rather distant descent. It’s not clear he would recognize these particular progeny. The other family is of a more recent vintage. While the first hint of its existence was the nilHecke algebra acting on the cohomology of the complete flag variety, it was not written in full generality until the past decade in work of Khovanov, Lauda and Rouquier [11, 12, 18].

In the spirit of other families in representation theory, one can think of Hecke family as being *trigonometric* and the KLR family as *rational*. However, a common phenomenon in mathematics is the existence of an isomorphism between trigonometric and rational versions of an object after suitable completion; the “ur-isomorphism” of this type is between the associated graded of the K-theory of a manifold and its cohomology. Such an isomorphism has been given for completions of non-degenerate and degenerate affine Hecke algebras by Lusztig in [13]. Another similar isomorphism is

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KEYWORDS. Hecke algebra, Khovanov-Lauda-Rouquier algebra, Cherednik algebra.

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given in [6] for Yangians and quantum affine algebras. In this paper, we will define isomorphisms with a similar flavor between the algebras in the Hecke and KLR families. These isomorphisms are between certain special completions; before discussing the specific examples, we cover some generalities on this type of completion in Section 2.

In both cases, these families have somewhat complicated family trees. Every one depends on a choice of a rank, which we will denote n throughout. In the diagrammatics, this will always correspond to a number of strands. On the Hecke side, we will always have a dependence on a parameter q , which we will sometimes want to deform to qe^h with a h a formal parameter. On the KLR side, we will not see an explicit family of algebras as we vary q , but the underlying Dynkin diagram used in the definition of these algebras will depend on h .

Like blood types, there are two complementary ways that they can become complicated. The simplest case, our analogue of blood type O, is the affine Hecke algebra (on the Hecke side) and the KLR algebra of the Dynkin diagrams \hat{A}_e/A_∞ (on the KLR side). The two complications we can add are like the type A and type B antigens on our red blood cells. Since “type A/B” already have established connotations in mathematics, we will instead call these types W and F:

- algebras with the type W complication are “weighted”⁽¹⁾: these include affine q -Schur algebras [8] (on the Hecke side) and weighted KLR algebras [22].
- algebras with the type F complication are “framed”: these include cyclotomic Hecke algebras [2] and the \hat{A}_e/A_∞ tensor product categorifications from [23]. These are analogs of the passage from Lusztig to Nakajima quiver varieties.
- finally, both of these complications can be present simultaneously, giving type WF. The natural object which appears in the Hecke family is the category \mathcal{O} of a Cherednik algebra $\mathbb{Z}/e\mathbb{Z} \wr S_n$ [7], though in a guise in which has not been seen previously. On the KLR side, the result is a steadied quotient of a weighted KLR algebra for the Crawley-Boevey quiver of a dominant weight of type \hat{A}_e/A_∞ (see Definition 6.8 and [22, § 3.1]).

Our main theorem is that in each type, there are completions of these Hecke- and KLR-type algebras that are isomorphic.

Since a great number of different algebras of representation theoretic interest appear in this picture, it can be quite difficult to keep them all straight. For the convenience of the reader, we give a table in Figure 1, placing all the algebras and categories which appear in this picture in their appropriate type. Note that many of the items listed below (such as Ariki–Koike algebras, or cyclotomic q -Schur and quiver Schur algebras) are not the most general family members of that type, but rather special cases. We’ll ultimately focus on the category of representations of a given algebra, so we have not distinguished between Morita equivalent algebras.

On the KLR side, the diagrammatic formulation we give matches the original definition of these algebras (with the exception of quiver Schur algebras, which are shown to be Morita equivalent to certain reduced steadied quotients in [22, Th. 3.9]). For the Hecke side, typically our description is a bit different from the definitions readers will be used to, and we have listed the result in this paper or another which gives the relation.

REMARK 1.1. All of the algebras on the Hecke side of this list have degenerate analogues, and we could have written this paper, like [3] with parallel sets of formulas in the degenerate and non-degenerate cases. We avoided doing this because of length,

⁽¹⁾The referee has suggested that “wraith” in reference to the ghost strands which appear might be more appropriate. You might very well think that; the author couldn’t possibly comment.

Type	KLR side	Hecke side
O	KLR algebra R [11, 18]	affine Hecke algebra $\widetilde{\mathcal{H}}$ of type A (Thm. 3.4)
W	weighted KLR algebra [22], quiver Schur algebra [19]	affine q -Schur algebra $\mathcal{S}(n, m)$ (Thm. 4.9)
F	cyclotomic KLR algebras [11], algebras $T^\lambda(h, \mathbf{z})$ categorifying tensor products for type A/\hat{A} [23]	cyclotomic Hecke (Ariki–Koike) algebras (Prop. 5.7), category \mathcal{O} for \mathfrak{gl}_N ($e = \infty$) ([23, 9.11])
WF	reduced steadied quotients \mathbb{T}^θ categorifying Uglov Fock spaces [24], cyclotomic quiver Schur algebras [19]	category \mathcal{O} for a Cherednik algebra with $\mathbb{Z}/e\mathbb{Z} \wr S_n$ ([24, Thm. A]), cyclotomic q -Schur algebras (Prop. 6.6)

FIGURE 1. The algebras of interest.

because the correspondence between degenerate and non-degenerate formulas is easy to work out (just replace multiplication by $q^{\pm 1}$ by addition of ± 1), and our ultimate goal is to apply our results to the Cherednik category \mathcal{O} in [24], which only uses the non-degenerate case.

REMARK 1.2. Very closely related (and in many cases, Morita equivalent) algebras were introduced by Maksimau and Stroppel [16]; they use the terms “Hecke family” and “KLR family” exactly as above. The main difference between the approaches in these papers is that this paper emphasizes not Schur algebras as those working in the field understand them, but certain Morita equivalent algebras we find more convenient to work with, whereas [16] work more directly with the Schur algebra.

Type O. We’ll first consider the simplest case of this isomorphism. In essence, this is just a rewriting of the approach in [18, § 3.2], but for applications in [24], we require a small generalization of those results, and it will serve to illustrate our techniques for the sections on other types. The two algebras we consider are:

- the affine Hecke algebra $\widetilde{\mathcal{H}}(\mathfrak{q})$ of S_n with parameter qe^h , considered as a $\mathbb{k}[[h]]$ -algebra. Note that this deformation only makes sense if \mathbb{k} has characteristic 0.
- the KLR algebra $R(h)$ of rank n for $\widehat{\mathfrak{sl}}_e$ attached to the polynomials $Q_{i+1,i}(u, v) = u - v + h$, also considered over $\mathbb{k}[[h]]$.

These algebras are defined in [3, (4.1–5)] and [3, (1.6–15)] respectively; here we consider them with the addition of an h -adic deformation. This deformation is very important since it allows us to compare affine Hecke algebras with q at a root of unity with those for generic q . For KLR algebras, this corresponds to comparing KLR algebras for \widehat{A}_e and A_∞ (as in [22, Ex. 2.25]). By the usual idempotent lifting arguments (see, for example, [25, Lem. 2.2]), the Grothendieck group of projective modules for $R(h)$ is the same as that for the usual KLR algebra R with h set to 0; thus, $R(h)$ -modules categorify the algebra $U^+(\widehat{\mathfrak{sl}}_\infty)$ or $U^+(\widehat{\mathfrak{sl}}_e)$ by [12, Thm. 8].

THEOREM 1.3. *There is a $\mathbb{k}[[h]]$ -algebra isomorphism $\widehat{\mathcal{H}}(\mathfrak{q}) \cong \widehat{R}(h)$.*

The characteristic 0 assumption may look peculiar to experts in the field; the Hecke algebra over a field of characteristic p has similar deformations coming from deforming the parameter q (though e^h does not make sense here), but it’s not clear how to match other deformations of the Hecke algebra with the simplest deformations of the KLR

algebra. A different deformation of the KLR algebra defined by Hu and Mathas [9] is compatible with more general deformations of the Hecke algebra, in particular with the deformation of $\mathbb{F}_p[S_n]$ to $\mathbb{Z}_p[S_n]$. Since our primary applications will be to Hecke algebras and related structures of characteristic 0, this hypothesis is no problem for us. In general, we'll prove our results in parallel with the undeformed Hecke algebra (and related structures) in arbitrary characteristic, and with the exponentially deformed Hecke algebra in characteristic 0.

One isomorphism between type O completions was implicitly constructed by Brundan and Kleshchev in [3] and for a related localization by Rouquier in [18, § 3.2.5] for $h = 0$. Unfortunately, it is not clear how to extend these isomorphisms to the deformed case, so instead we construct an isomorphism which is different even after the specialization $h = 0$. This isomorphism still has a similar flavor to those previously defined; in brief, we use a general power series of the form $1 + y + \dots$ (in particular e^y) where Brundan and Kleshchev or Rouquier use $1 + y$.

We will also generalize this theorem in a small but useful way: in fact there is a natural class of completions of the Hecke algebra that correspond with the KLR algebra for a larger Lie algebra \mathfrak{G}_U . Here we consider an arbitrary finite subset $U \subset \mathbb{k} \setminus \{0\}$, given a graph structure connecting u and u' if $qu = u'$, and let \mathfrak{G}_U be the associated Kac–Moody algebra.

This definition is the same as the “type A graphs” in [18, § 3.2.5], but we do not impose a connectedness assumption. The most important case is when U is the e th roots of unity, so U is an e -cycle, but having a more general statement will be useful in an analysis of the category \mathcal{O} for a cyclotomic rational Cherednik algebra given in [24]. A more direct proof of this equivalence using the Dunkl–Opdam subalgebra is now given in [21]. The generalization of Theorem 1.3 to this case (Proposition 3.10) gives an alternate approach (and graded version) of the theorem of Dipper and Mathas [5] that Ariki–Koike algebras for arbitrary parameters are Morita equivalent to a tensor product of such algebras with q -connected parameters.

The technique we use for this isomorphism and all others considered in this paper is a variation on that used by Rouquier in [18, § 3.26]. We construct an isomorphism between completions of the polynomial representations of $\tilde{\mathcal{H}}(\mathbf{q})$ and $R(h)$, and then match the operators given by these algebras. This requires considerably less calculation than confirming the relations of the algebras themselves. It also has the considerable advantage of easily generalizing to other types.

In Maksimau and Stroppel's framework [16], these are the cases which are “no level, not Schur.”

Type W. The first variation we introduce is “weightedness.” This is a similar change of framework in both the Hecke and KLR families, though it is not easy to see from the usual perspective on the Hecke algebra. This algebra can be considered as the span of strand diagrams with number of strands equal to the rank of the algebra, and a crossing corresponding to $T_i + 1$ or $T_i - q$, depending on conventions. In this framework, we can introduce a generalization of the Hecke algebra which allows “action at a distance” where certain interactions between strands occur at a fixed distance from each other rather than when they cross. To see the difference between these, compare the local relations (1a–1c) with (12a–12f). We have already introduced this concept in the KLR family as *weighted KLR algebras* [22], but the idea of incorporating it into the Hecke algebra seems to be new. Note that wKLR algebras are defined for any Cartan datum, but as usual, we will only consider those attached to the quiver structures on sets U (which are always unions of finite and affine type A).

The main result in this case is that we obtain a graded KLR type algebra Morita equivalent to the affine Schur algebra after completion; after this preprint had appear on the arXiv, Miemietz and Stroppel [17] showed a direct isomorphism of the completed affine Schur algebra with a quiver Schur algebra from [19]. When $e = \infty$, these algebras are Morita equivalent to the type O algebras, and thus they still categorify the algebra $U^+(\mathfrak{sl}_\infty)$. When $e < \infty$, the category of representations is larger, and corresponds to the passage from $U^+(\widehat{\mathfrak{sl}}_e)$ to $U^+(\widehat{\mathfrak{gl}}_e)$.

Thus, in Maksimau and Stroppel’s framework [16], these are the cases which are “no level, Schur” (though again, we should emphasize that our algebra only match theirs up to Morita equivalence in the Schur cases).

Type F. The second variation we’ll consider is “framing.” This is also a fundamentally graphical operation, accomplished by including red lines, which then interplay with those representing our original Hecke algebra. This case is closely related to the extension from Hecke algebras to cyclotomic Hecke algebras and parabolic category \mathcal{O} of type A.

These algebras lead to categorifications of tensor products of simple representations. In the KLR family, these are precisely the tensor product algebras introduced in [23, Def. 4.7]; in the Hecke family, these algebras do not seem to have appeared in precisely this form before, though they appear naturally as endomorphisms of modules over cyclotomic Hecke algebras.

In particular, we show that our isomorphism and deformation are also compatible with deformations of cyclotomic quotients. For a fixed multiset $\{Q_1, \dots, Q_\ell\}$ of elements of U , there are cyclotomic quotients of both $\widetilde{\mathcal{H}}(q)$ and $R(q)$ (the specializations at $h = 0$), which Brundan and Kleshchev construct an isomorphism between. We can deform this cyclotomic quotient with respect to variables $\mathbf{z} = \{z_j\}$.

For $\widetilde{\mathcal{H}}(\mathbf{q})$, consider the deformed cyclotomic quotient attached to the polynomial $C(A) = \prod_{i=1}^\ell (A - Q_i e^{-z_i})$.

DEFINITION 1.4. *The deformed cyclotomic quotient $\mathcal{H}(\mathbf{q}, \mathbf{Q}_\bullet)$ is the quotient of the base extension $\widetilde{\mathcal{H}}(\mathbf{q}) \otimes_{\mathbb{k}} \mathbb{k}[\mathbf{z}]$ by the 2-sided ideal generated by $C(X_1)$.*

This is precisely the Ariki–Koike algebra of [2, Def. 3.1] for $G(\ell, 1, n)$ with the parameters $u_i = Q_i e^{-z_i}$ (where we use u_i as in the reference of [2]).

For $R(h)$, the corresponding quotient is given by an additive deformation of the roots. For each $u \in U$, we have a polynomial $c_u(a) = \prod_{Q_j=u} (a - z_j)$.

DEFINITION 1.5. *The deformed cyclotomic quotient $R^{Q_\bullet}(h, \mathbf{z})$ is a quotient of the base extension $R(h) \otimes_{\mathbb{k}} \mathbb{k}[\mathbf{z}]$ by the ideal generated by $c_{u_1}(y_1)e_{\mathbf{u}}$ for every length n sequence $\mathbf{u} \in U^n$.*

For the usual indexing of cyclotomic quotients by dominant weights, this is a deformation of the cyclotomic KLR algebra R^λ attached in [11] to a dominant weight λ of \mathfrak{G}_U satisfying

$$\alpha_u^\vee(\lambda) = \#\{i \in [1, \ell] \mid Q_i = u\}.$$

THEOREM 1.6. *The isomorphism $\widehat{\mathcal{H}}(\mathbf{q}) \cong \widehat{R}(h)$ induces an isomorphism of $\mathbb{k}[h, \mathbf{z}]$ -algebras $\mathcal{H}(\mathbf{q}, \mathbf{Q}_\bullet) \cong R^\lambda(h, \mathbf{z})$.*

In Maksimau and Stroppel’s framework [16], these are the cases which are “higher level, not Schur.”

Type WF. Our final goal, the algebras incorporating both these modifications, is the least likely to be familiar to readers. The category of representations over these algebras is equivalent to the category \mathcal{O} for a rational Cherednik algebra for $\mathbb{Z}/\ell\mathbb{Z} \wr S_n$, as we show in [24]. In certain cases, these algebras are also Morita equivalent to cyclotomic q -Schur algebras.

The isomorphism between the two families in this case will prove key in the results of [24], proving the conjecture of Rouquier identifying decomposition numbers in this category \mathcal{O} with parabolic Kazhdan–Lusztig polynomials. This construction is also of some independent interest as a categorification of Uglov’s higher level Fock space, introduced in [20]. In [24], we will show that several natural, but hard-to-motivate structures on the Fock space arise from these algebras.

In Maksimau and Stroppel’s framework [16], these include the cases which are “higher level, Schur” (as before, up to Morita equivalence). We should however, note that the algebras we consider are more general, since they depend on the ratios of parameters corresponding to the weightedness and the framing; the higher level Schur case only captures situations where this ratio is small. This more general context is used in [24, 21] to compare with category \mathcal{O} over Cherednik algebras [7].

2. POLYNOMIAL-STYLE REPRESENTATIONS

First, we will discuss some generalities about completions of algebras and their representations. There are a few facts about these completions we will want to use many times, so it is more convenient to have a general framework from which they follow.

Let A be a \mathbb{K} -algebra for \mathbb{K} a commutative ring. Let B a Noetherian commutative \mathbb{K} -algebra such that $\text{Spec } B$ is a smooth curve over $\text{Spec } \mathbb{K}$. We’ll primarily be interested in the case where $B = \mathbb{K}[X, X^{-1}]$ or $B = \mathbb{K}[y]$, that is the affine line or punctured affine line. The n -fold tensor power $B^{\otimes n} = B \otimes_{\mathbb{K}} B \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} B$ is thus the functions on the n -fold fiber product of $\text{Spec } B$ with its usual induced action of S_n , and the algebra $Z = (B^{\otimes n})^{S_n}$ has smooth spectrum $\text{Spec } Z = \text{Sym}_{\text{Spec } \mathbb{K}}^n(\text{Spec } B)$. As usual, $B^{\otimes n}$ is projective of rank $n!$ over Z , and free if $\text{Spec } B$ is the punctured or unpunctured affine line.

DEFINITION 2.1. *Consider a \mathbb{K} -algebra homomorphism $\psi: B^{\otimes n} \rightarrow A$ and an A -module P . We say that the data (A, B, ψ, P) is a polynomial-style representation of rank p if*

- (1) A is finite rank and free over $B^{\otimes n}$.
- (2) $Z = (B^{\otimes n})^{S_n}$ is central in A .
- (3) P is faithful and free over $B^{\otimes n}$ of some rank p .

We call this a a graded polynomial-style representation if in addition A, B are graded \mathbb{K} -algebras (for some grading on \mathbb{K}), with B graded local with unique graded maximal ideal given by $B_{>0}$, P is a graded module, and ψ a graded homomorphism.

We’ll want to consider representations of such algebras where some fixed ideal $I \subset B$ acts nilpotently under every inclusion $\psi(B \otimes \cdots \otimes B \otimes I \otimes B \otimes \cdots \otimes B)$. We can express this as a topological condition.

Consider B as a topological ring with the I -adic topology, and the obvious induced topologies on $B^{\otimes n}$ and Z . Let $\widehat{B}^{\otimes n}$ and \widehat{Z} be the corresponding completions of these algebras. The former topology is just the $I^{(n)}$ -adic topology for $I^{(n)}$ the sum of all ideals of the form $B \otimes \cdots \otimes I \otimes \cdots \otimes B$.

LEMMA 2.2. *The subspace topology on Z agrees with the $I' = Z \cap I^{(n)}$ -adic topology on this ring. Alternatively, the $I^{(n)}$ -adic topology on $B^{\otimes n}$ is the coarsest topological ring structure such that the inclusion of Z , with the I' -adic topology, is continuous.*

Proof. Obviously $(I')^m \subset (I^{(n)})^k$, so the I' -adic topology is finer than the subspace topology. In order to show the opposite, we need only show that for any fixed m , we have $(I^{(n)})^k \cap Z \subset (I')^m$ for all $k \gg 0$. This will follow if $(I^{(n)})^k \subset B^{\otimes n} \cdot I'$ for some k since $Z \cap (B^{\otimes n} \cdot (I')^m) = (I')^m$ as a simple calculation with projection to invariants (i.e. the Reynolds operator) shows. This will follow if these ideals have the same radical.

Since $B^{\otimes n}$ is integral over Z , every generator of $I^{(n)}$ has a minimal polynomial over Z , whose coefficients, of course, lie in I' . Thus, a power of this generator lies in I' , which establishes the desired equality of radicals. \square

Now we wish to endow A with the coarsest topology compatible with this topology on $B^{\otimes n}$, or equivalently on Z . This is induced by the bases $J_m = A(I^{(n)})^m A$ or $J'_m = A(I')^m A$, which give equivalent topologies by the equality $\sqrt{I^{(n)}} = \sqrt{B^{\otimes n} \cdot I'} \subset B^{\otimes n}$.

If (A, B, ψ, P) is graded, and $I \subset B$ is the unique graded maximal ideal, then there is another description of this topology:

LEMMA 2.3. *If (A, B, ψ, P) is graded, and $I = B_{>0} \subset B$ is the unique graded maximal ideal, then the topology on A is equivalent to usual topology induced by the grading, i.e. the span G_k of the elements of degree $\geq k$ is a neighborhood of 0, and these form a basis of such neighborhoods.*

Proof. The algebra A is finitely generated as a Z -module and thus there is some integer $M \geq 0$ such that the generators of A as a Z -module have degrees in the interval $[-M, M]$. Since the unique graded maximal ideal of Z is $Z_{>0}$, this shows that $G_k G_m \subset G_{k+m-M}$. In particular, since $(I')^m \subset G_m$, we have $J'_m \subset G_{m-2M}$ for all m .

Since Z is Noetherian, $Z_{>0}/Z_{>0}^2$ is a finite dimensional graded vector space over the field $Z/Z_{>0}$, and we can also assume that all the degrees appearing are $\leq M$ (by increasing M if necessary). Note that this means that all elements of degree $> kM$ lie in $Z_{>0}^k$.

We know that elements of degree $\geq (k+1)M$ elements of A are spanned by the products of generators with elements of Z of degree $\geq kM$. As have observed, these elements of Z must lie in $Z_{>0}^k$. Thus we have that $G_{(k+1)M} \subset J'_k$. Thus, these topologies are equivalent. \square

DEFINITION 2.4. *Let \widehat{A} be the completion of A with respect to this topology, and $\widehat{P} = \widehat{A} \otimes_A P$.*

LEMMA 2.5. *The completion \widehat{P} is a faithful representation of \widehat{A} , and is free over $\widehat{B^{\otimes n}}$ of the same rank as P over $B^{\otimes n}$.*

Proof. Note that we have an injective map $A \rightarrow \text{End}_Z(P)$. The projectivity of P over Z implies that $\text{End}_Z(P) \cong \text{Hom}_Z(P, Z) \otimes_Z P$ is also projective over Z . Thus, the induced map

$$\widehat{A} \rightarrow \text{End}_{\widehat{Z}}(\widehat{P}) \cong \widehat{Z} \otimes_Z \text{End}_Z(P)$$

giving the action of \widehat{A} agrees with the base change by \widehat{Z} of the original action map. This remains injective by the flatness of A over Z . \square

3. TYPE O

3.1. HECKE ALGEBRAS. We will follow the conventions of [3] concerning Hecke algebras. Our basic object is $\widetilde{\mathcal{H}}(\mathfrak{q})$, the *affine Hecke algebra*. Let us fix our assumptions on base fields and parameters:

- (*) Let \mathbb{k} be a field of any characteristic. Fix a element $q \in \mathbb{k} \setminus \{0, 1\}$; let e the multiplicative order of q (which may be ∞). Let $d(\mathbf{q}) = 1 + d_1 h + \dots$ be a formal power series in $\mathbb{k}[[z]]$, which satisfies $d(h_1 + h_2) = d(h_1)d(h_2)$, and let $\mathbf{q} = qd(\mathbf{q})$.

Differentiating, we see that this is only possible if $d(h) = e^{d_1 h}$; in particular, if \mathbb{k} has positive characteristic, we must have $d_1 = 0$, whereas if K has characteristic 0, this makes sense for any d_1 .

The algebra $\tilde{\mathcal{H}}(\mathbf{q})$ is generated by $\{X_1^{\pm 1}, \dots, X_n^{\pm 1}\} \cup \{T_1, \dots, T_{n-1}\}$ with the relations:

$$\begin{aligned} X_r^{\pm 1} X_s^{\pm 1} &= X_s^{\pm 1} X_r^{\pm 1} & T_r^2 &= (\mathbf{q} - 1)T_r + \mathbf{q} \\ T_r X_r T_r &= \mathbf{q} X_{r+1} & T_r T_{r+1} T_r &= T_{r+1} T_r T_{r+1} \\ T_r X_s &= X_s T_r \quad (r \neq s, s + 1) & T_r T_s &= T_s T_r \quad (r \neq s \pm 1). \end{aligned}$$

The subalgebra generated by the T_i 's alone is a copy of the (finite) Hecke algebra $\mathcal{H}(\mathbf{q})$, and the subalgebra generated by the $X_i^{\pm 1}$ is a copy of the Laurent polynomial ring $\mathcal{C} = \mathbb{k}[[h]][X_1^{\pm 1}, \dots, X_n^{\pm 1}]$.

In this paper, we'll rely heavily on a diagrammatic visualization of this algebra.

DEFINITION 3.1. Let a rank n type O diagram be a collection of n curves in $\mathbb{R} \times [0, 1]$ with each curve mapping diffeomorphically to $[0, 1]$ via the projection to the y -axis. Each curve is allowed to carry any number of squares or the formal inverse of a square. We assume that these curves have no triple points or tangencies, no squares lie on crossings and consider these up to isotopies that preserve these conditions.

An example of such a rank 5 diagram is given below:

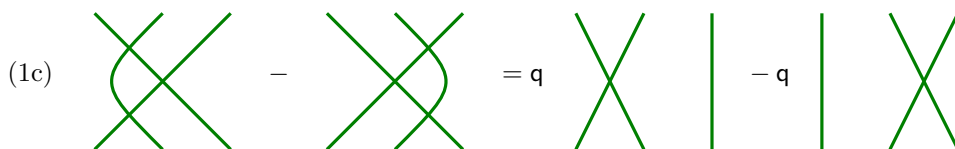


As usual, we can compose these by taking ab to be the diagram where we place a on top of b and attempt to match up the bottom of a and top of b . If the number of strands is the same, the result is unique up to isotopy, and if it is different, we formally declare the result to be 0.

The rank n type O affine Hecke algebra is the quotient of the span of these diagrams over $\mathbb{k}[[h]]$ by the local relations:

(1a)

(1b)

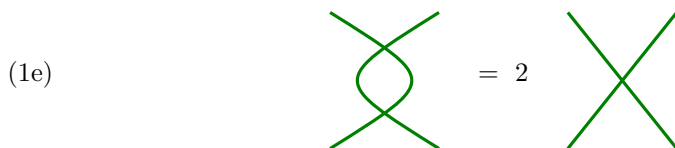
(1c) 

REMARK 3.2. We want to make sure that the reader notices the distinction here between “relations” and “local relations.” Here “relations” has the usual algebraic meaning; generators of the kernel of the homomorphism to an algebra of the free associative algebra on the generators. However, “local relations” means something a bit more subtle: whenever we have two diagrams which are identical outside a small region, and match the two sides of the equation, then they are set equal. Of course, the effect this has depends on what is allowed in the rest of the diagram.

So a relation like 1a can be applied in type O diagrams (as in this section) or in type W diagrams, which we’ll introduce later. While the pictures on the page are the same, the induced relations are different, since we have different rules for how the rest of the diagram is constructed. Thus in later sections, we will refer back to these local relations, but apply them in a different diagrammatic framework.

REMARK 3.3. In the degenerate case, we can write a similar set of local relations, replacing (1a–1c) with the local relations:

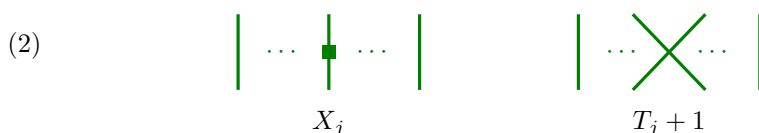
(1d) 

(1e) 

(1f) 

It may not be immediately clear what the additional value of this graphical presentations is. However, this perspective will lead us to generalizations of the affine Hecke algebra which we call types W, F and WF.

THEOREM 3.4. The algebra $\tilde{\mathcal{H}}(\mathfrak{q})$ is isomorphic to the rank n type O Hecke algebra via the map sending $T_r + 1$ to the crossing of the r th and $r + 1$ st strands, and X_r to the square on the r th strand, as shown below:

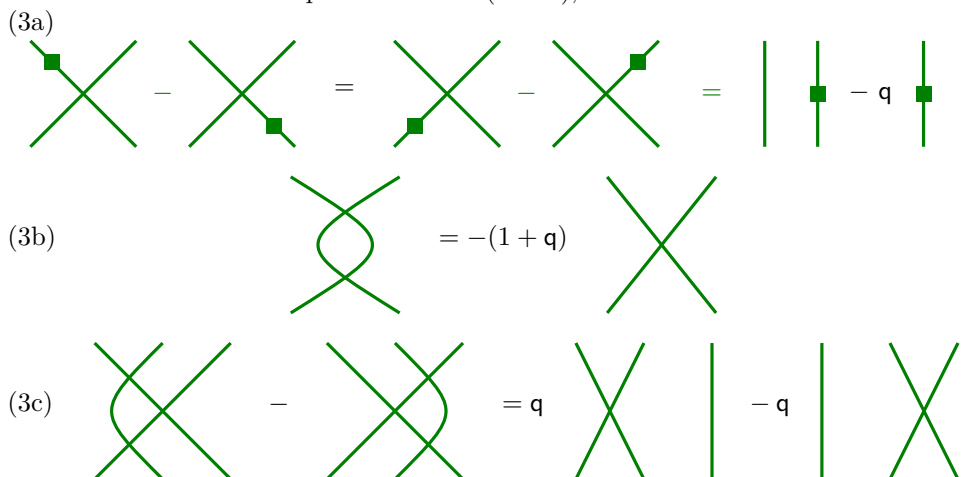
(2) 

Proof. We'll use the relations given in [3, § 4] without additional citation. The equations (1a–1c) become the relations:

$$\begin{aligned}
 X_r(T_r + 1) - (T_r + 1)X_{r+1} &= T_r X_{r+1} + (1 - q)X_{r+1} + X_r - T_r X_{r+1} - X_{r+1} \\
 &= X_r - qX_{r+1} \\
 X_{r+1}(T_r + 1) - (T_r + 1)X_r &= T_r X_r + (q - 1)X_{r+1} + X_{r+1} - T_r X_r - X_r \\
 &= qX_{r+1} - X_r \\
 (T_r + 1)^2 &= T_r^2 + 2T_r + 1 \\
 &= (q - 1)T_r + q + 2T_r + 1 \\
 &= (1 + q)(T_r + 1) \\
 (T_r + 1)(T_{r+1} + 1)(T_r + 1) - (T_{r+1} + 1)(T_r + 1)(T_{r+1} + 1) &= T_r^2 + T_r - T_{r+1}^2 - T_{r+1} \\
 &= q(T_r - T_{r+1}).
 \end{aligned}$$

Similarly, one can easily derive the relations of the affine Hecke from the diagrammatic ones given above. This shows that we have an isomorphism. \square

Note that if we instead sent the element $T_i - q$ to the crossing, we would obtain local relations which are quite similar to (1a–1c), but have a few subtle differences:



Our first task is to describe the completions that are of interest to us.

Consider a finite subset $U \subset \mathbb{k} \setminus \{0\}$; as before, we endow this with a graph structure by adding an edge from u to u' if $u' = qu$. Note that for U chosen generically there will simply be no edges, and that under this graph structure U will always be a union of segments and cycles with e nodes (if $e < \infty$).

We will apply the results of Section 2 in this context with

$$(4) \quad \mathbb{K} = \mathbb{k}[[h]] \quad A = \tilde{\mathcal{H}}(q) \quad B = \mathbb{k}[[h]][X^\pm] \quad I = Bh + B \prod_{u \in U} (X - u).$$

One natural construction of modules over $\tilde{\mathcal{H}}(q)$ is given by induction from $\mathcal{H}(q)$, as discussed in [15, § 4.3]; as discussed there, the result is free as a \mathcal{C} -module if the original module is free over $\mathbb{k}[[h]]$, with ranks matching. In particular, applying this to the two 1-dimensional representations of $\mathcal{H}(q)$, where this algebra acts by the characters χ^\pm where

$$\chi^+(T_i) = q \quad \chi^-(T_i) = -1$$

gives natural (signed) polynomial representation

$$\mathcal{P}^\pm = \tilde{\mathcal{H}}(q) \otimes_{\mathcal{H}(q)} \mathbb{k}[[h]];$$

where $\mathcal{H}(\mathfrak{q})$ acts on $\mathbb{k}[[h]]$ via the homomorphism χ^\pm .

LEMMA 3.5. *The data of (4) defines a polynomial-style representation on $P = \mathcal{P}^\pm$.*

Proof.

- (1) This freeness is clear from the basis in [14, 4.3]; in fact, as $B^{\otimes n} = \mathcal{C}$ -module, we have $\widetilde{\mathcal{H}}(\mathfrak{q}) \cong \mathcal{C} \otimes_{\mathbb{k}[[h]]} \mathcal{H}(\mathfrak{q})$.
- (2) The center of the affine Hecke algebra $\widetilde{\mathcal{H}}(\mathfrak{q})$ is precisely the symmetric Laurent polynomials $Z = \mathcal{C}^{S_n}$ by [14, 4.5].
- (3) The module \mathcal{P}^\pm is faithful by [15, (4.3.10)], and its freeness over \mathcal{C} has already been discussed. \square

Thus, as in Section 2, we have an induced topology on $\widetilde{\mathcal{H}}(\mathfrak{q})$ with completion $\widehat{\mathcal{H}}(\mathfrak{q})$. We could also define $\widehat{\mathcal{H}}(\mathfrak{q})$ as the completion of $\widetilde{\mathcal{H}}(\mathfrak{q})$ in the directed system of all quotients where the spectrum of each X_i lies in U .

We can identify $\text{Spec}(\mathcal{C})$ with $(\mathbb{A}^1 \setminus \{0\})^n \times \widehat{\mathbb{A}}^1$ where the last factor has coordinate h and is completed at 0. Let $\mathcal{U} = U^n \times \{0\} \subset \text{Spec}(\mathcal{C})$. This is the vanishing set of $I^{(n)}$ as defined in Section 2. Thus, the closure of \mathcal{C} in $\widehat{\mathcal{H}}(\mathfrak{q})$ is the completion of \mathcal{C} at this subscheme. In particular, the identity in \mathcal{C} , and thus in $\widehat{\mathcal{H}}(\mathfrak{q})$ decomposes as a sum of idempotents $1 = \sum_{\mathbf{u} \in U^n} e_{\mathbf{u}}$. These have the property that on any topological $\widehat{\mathcal{H}}(\mathfrak{q})$ -module M , we have that

$$e_{\mathbf{u}}M = \{g \in M \mid \lim_{N \rightarrow \infty} (X_j - u_j)^N g = 0\},$$

and for any module, we have $M = \bigoplus_{\mathbf{u}} e_{\mathbf{u}}M$.

In particular, we have that $\widehat{\mathcal{H}}(\mathfrak{q}) = \bigoplus_{\mathbf{u} \in U^n} e_{\mathbf{u}} \widehat{\mathcal{H}}(\mathfrak{q}) = \bigoplus_{\mathbf{u}, \mathbf{u}' \in U^n} e_{\mathbf{u}} \widehat{\mathcal{H}}(\mathfrak{q}) e_{\mathbf{u}'}$.

3.1.1. *Formulas for the polynomial representation.* Now, let us study the action of $\widetilde{\mathcal{H}}(\mathfrak{q})$ on its polynomial representation \mathcal{P}^\pm . Denote the action of S_n on U^n by $\mathbf{u} \mapsto \mathbf{u}^s$ for $s \in S_n$; as usual, we let $s_i = (i, i + 1)$. For any Laurent polynomial F , we let $F^{s_i}(X_1, \dots, X_n) = F(X_1, \dots, X_{i+1}, X_i, \dots, X_n)$.

For notational clarity, we denote $\mathbf{I} = 1 \otimes 1 \in \mathcal{P}^-$, so this representation is generated by this vector, subject to the relation $T_i \mathbf{I} = -\mathbf{I}$. As in [15, (4.3.3)], one can calculate the action of T_i on $F\mathbf{I}$ for any Laurent polynomial F ; this is easiest to see if we expand $F = F_0 + (X_i - X_{i+1})F_1$ where F_0 and F_1 are s_i -invariant Laurent polynomials. Thus, we have that

$$\begin{aligned} T_i F \mathbf{I} &= T_i F_0 \mathbf{I} + T_i (X_i - X_{i+1}) F_1 \mathbf{I} \\ &= -F_0 \mathbf{I} + (X_i - X_{i+1}) F_1 \mathbf{I} + 2(1 - \mathfrak{q}) X_{i+1} F_1 \mathbf{I} \\ &= -F^{s_i} \mathbf{I} + (1 - \mathfrak{q}) X_{i+1} \frac{F^{s_i} - F}{X_{i+1} - X_i} \mathbf{I}. \end{aligned}$$

Thus, we have that

$$(T_i + 1)F \mathbf{I} = \left(F - F^{s_i} + (1 - \mathfrak{q}) X_{i+1} \frac{F^{s_i} - F}{X_{i+1} - X_i} \right) \mathbf{I} + \frac{X_i - \mathfrak{q} X_{i+1}}{X_{i+1} - X_i} (F^{s_i} - F) \mathbf{I}.$$

The Hecke algebra acts faithfully on this representation by [15, (4.3.10)], so we can identify the affine Hecke algebra with a subalgebra of operators on \mathcal{P}^\pm .

Similarly, the representation \mathcal{P}^+ is generated by an element \mathbf{I}^+ satisfying $T_i \mathbf{I}^+ = \mathfrak{q} \mathbf{I}^+$. The action of $\widetilde{\mathcal{H}}(\mathfrak{q})$ in this case is given by the formula

$$T_i F \mathbf{I}^+ = \mathfrak{q} F^{s_i} \mathbf{I}^+ + (1 - \mathfrak{q}) X_{i+1} \frac{F^{s_i} - F}{X_{i+1} - X_i} \mathbf{I}^+$$

so we have that

$$(T_i - \mathfrak{q})\mathbf{I}^+ = \frac{X_{i+1} - \mathfrak{q}X_i}{X_{i+1} - X_i}(F^{s_i} - F).$$

Consider the $\widehat{\mathcal{H}}(\mathfrak{q})$ -module $\widehat{\mathcal{P}}^\pm := \widehat{\mathcal{H}}(\mathfrak{q}) \otimes_{\widehat{\mathcal{H}}(\mathfrak{q})} \mathcal{P}^\pm$. It follows from Lemma 2.5 that:

LEMMA 3.6. *The module $\widehat{\mathcal{P}}^\pm$ is a rank 1 free module over the completion of \mathcal{C} at the set \mathcal{U} , and this representation remains faithful. The space $e_{\mathbf{u}}\widehat{\mathcal{P}}^\pm$ is isomorphic to $\mathbb{k}[(X_1 - u_1), \dots, (X_n - u_n), h]$ via the action map on $e_{\mathbf{u}}\mathbf{I}$.*

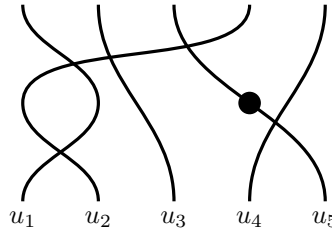
3.2. KLR ALGEBRAS. We wish to define a similar completion of the KLR algebra $R(h)$ for the graph U . We use the conventions of Brundan and Kleshchev, but we record the relations we need here for the sake of completeness and to match our slightly more general context. The rank⁽²⁾ n KLR algebra $R(h)$ attached to the Dynkin diagram U is generated over $\mathbb{k}[h]$ by elements $\{e(\mathbf{u})\}_{\mathbf{u} \in U^n} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$ subject to the relations:

$$\begin{aligned} e(\mathbf{u})e(\mathbf{v}) &= \delta_{\mathbf{u},\mathbf{v}}e(\mathbf{u}); & \sum_{\mathbf{u} \in I^\alpha} e(\mathbf{u}) &= 1; \\ y_r e(\mathbf{u}) &= e(\mathbf{u})y_r; & \psi_r e(\mathbf{u}) &= e(\mathbf{u}^{s_r})\psi_r; \\ y_r y_s &= y_s y_r; & & \\ \psi_r y_s &= y_s \psi_r & & \text{if } s \neq r, r + 1; \\ \psi_r \psi_s &= \psi_s \psi_r & & \text{if } s \neq r \pm 1; \\ \psi_r y_{r+1} e(\mathbf{u}) &= \begin{cases} (y_r \psi_r + 1)e(\mathbf{u}) & \text{if } u_r = u_{r+1}, \\ y_r \psi_r e(\mathbf{u}) & \text{if } u_r \neq u_{r+1}; \end{cases} \\ y_{r+1} \psi_r e(\mathbf{u}) &= \begin{cases} (\psi_r y_r + 1)e(\mathbf{u}) & \text{if } u_r = u_{r+1}, \\ \psi_r y_r e(\mathbf{u}) & \text{if } u_r \neq u_{r+1}; \end{cases} \\ \psi_r^2 e(\mathbf{u}) &= \begin{cases} 0 & \text{if } u_r = u_{r+1}, \\ e(\mathbf{u}) & \text{if } u_r \neq q^{\pm 1}u_{r+1}, u_{r+1}, \\ (y_{r+1} - y_r + d_1 h)e(\mathbf{u}) & \text{if } u_r = q^{-1}u_{r+1}, q \neq -1, \\ (y_r - y_{r+1} + d_1 h)e(\mathbf{u}) & \text{if } u_r = qu_{r+1}, q \neq -1, \\ (y_r - y_{r+1} + d_1 h)(y_{r+1} - y_r + d_1 h)e(\mathbf{u}) & \text{if } u_r = -u_{r+1}, q = -1; \end{cases} \\ \psi_r \psi_{r+1} \psi_r e(\mathbf{u}) &= \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1)e(\mathbf{u}) & \text{if } u_r = u_{r+2} = q^{-1}u_{r+1}, q \neq -1, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1)e(\mathbf{u}) & \text{if } u_r = u_{r+2} = qu_{r+1}, q \neq -1, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 2y_{r+1} + y_r + y_{r+2})e(\mathbf{u}) & \text{if } u_r = u_{r+2} = -u_{r+1}, q = -1, \\ \psi_{r+1} \psi_r \psi_{r+1} e(\mathbf{u}) & \text{otherwise.} \end{cases} \end{aligned}$$



Just as in the Hecke case, there is a graphical presentation for the KLR algebra. Since this is covered in [11] and numerous other sources, we'll just record an example

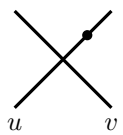
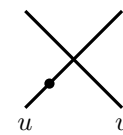
⁽²⁾Note here that the “rank” n has no relationship to the size of the set U (typically called the rank of the corresponding Kac–Moody algebra); for any fixed U , we get a different algebra for each positive integer n .

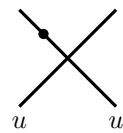
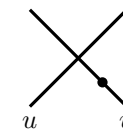
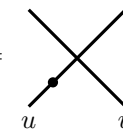
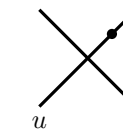

of an appropriate KLR diagram here for comparison purposes:

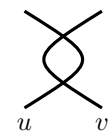


and write out the local relations here for convenience:

(5a)  =  unless $u = v$

(5b)  =  unless $u = v$

(5c)  -  =  -  = 

(5d) 

$$= \begin{cases} 0 & u = v \\ \begin{array}{c} \left| \right| \\ u \quad v \end{array} & u \notin \{v, qv, q^{-1}v\} \\ \begin{array}{c} \left| \right| \bullet - \bullet \left| \right| + d_1 h \left| \right| \left| \right| \\ u \quad v \quad u \quad v \quad u \quad v \end{array} & u = q^{-1}v, q \neq -1 \\ \begin{array}{c} \bullet \left| \right| - \bullet \left| \right| + d_1 h \left| \right| \left| \right| \\ u \quad v \quad u \quad v \quad u \quad v \end{array} & u = qv, q \neq -1 \\ -\left(\begin{array}{c} \left| \right| \left| \right| \\ 2 \quad \left| \right| \\ u \quad v \end{array} \right) + 2\left(\begin{array}{c} \bullet \left| \right| \\ \bullet \left| \right| \\ u \quad v \end{array} \right) - \left(\begin{array}{c} \left| \right| \bullet \\ \left| \right| \left| \right| \\ u \quad v \end{array} \right) + d_1^2 h^2 \left(\begin{array}{c} \left| \right| \left| \right| \\ \left| \right| \left| \right| \\ u \quad v \end{array} \right) & u = -v, q = -1 \end{cases}$$

(5e)

We will again apply the results of Section 2, now with

(6) $\mathbb{K} = \mathbb{k}[h] \quad A = R(h) \quad B = \mathbb{k}[h, y] \quad I = Bh + By.$

We let $C = B^{\otimes n} = \mathbb{k}[y_1, \dots, y_n, h] \subset R(h)$. The algebra $R(h)$ also has a natural polynomial representation P , defined by Rouquier in [18, § 3.2] and Khovanov and Lauda in [11, § 2.3]. This representation P is generated by a single element $\mathbb{1}$, with the relations

$$\psi_k e_{\mathbf{u}} \mathbb{1} = \begin{cases} 0 & u_k = u_{k+1} \\ (y_{k+1} - y_k + d_1 h) e_{\mathbf{u}^{s_k}} \mathbb{1} & u_k = qu_{k+1} \\ e_{\mathbf{u}^{s_k}} \mathbb{1} & u_k \neq u_{k+1}, qu_{k+1}. \end{cases}$$

This can be written as a sum of the images of $e_{\mathbf{u}}$, and we always have that $e_{\mathbf{u}} P$ is a rank 1 free module over C .

Just as in the Hecke algebra, the action of ψ_k on arbitrary polynomials can be written in terms of Demazure operators. For a polynomial $f \in \mathbb{k}[[h]][y_1, \dots, y_n]$, we can describe the action as

(7)
$$\psi_k f e_{\mathbf{u}} \mathbb{1} = \begin{cases} \frac{f^{s_k} - f}{y_{k+1} - y_k} e_{\mathbf{u}} \mathbb{1} & u_k = u_{k+1} \\ (y_{k+1} - y_k + d_1 h) f^{s_k} e_{\mathbf{u}^{s_k}} \mathbb{1} & u_k = qu_{k+1} \\ f^{s_k} e_{\mathbf{u}^{s_k}} \mathbb{1} & u_k \neq u_{k+1}, qu_{k+1}. \end{cases}$$

LEMMA 3.7. *The data of (6) defines a graded polynomial-style representation on P .*

Proof.

- (1) The algebra $\widehat{R}(h)$ is finitely generated free as a C -module by [11, Cor. 2.10].
- (2) By [11, Thm. 2.9], $Z = C^{S_n}$ is central.
- (3) The representation P is faithful by [11, Cor. 2.6], and free over C of rank $(\#U)^n$.

The graded property is clear from the definitions; $\mathbb{k}[h, y]$ is graded local as required because h and y have positive degree. □

Let $\widehat{R}(h)$ be the completion of $R(h)$ respect to the induced topology and

$$\widehat{P} \cong \widehat{R}(h) \otimes_{R(h)} P \cong \widehat{Z} \otimes_Z P$$

be the completion of this polynomial representation. By Lemma 2.3, this is the same as completing these graded abelian groups with respect to their grading. We can easily deduce from Lemma 2.5 that:

LEMMA 3.8. *The module \widehat{P} over $\widehat{R}(h)$ is faithful, and the action of C induces an isomorphism $e_{\mathbf{u}} \widehat{P} \cong \widehat{C}$, the completion of this ring with respect to its grading topology.*

3.3. ISOMORPHISMS. Let $b(h) \in 1 + h + h^2\mathbb{k}[[h]]$ be a formal power series; if $d_1 \neq 0$, we assume that $b(h) = e^h$. Thus, we must have that

$$(8) \quad b(h_1)d(h_2) = b(h_1 + d_1h_2).$$

Our approach will match Brundan and Kleshchev's if we choose $b(h) = 1 + h$.

LEMMA 3.9. *There is a unique vector space isomorphism $\gamma_p: \widehat{\mathcal{P}}^- \rightarrow \widehat{P}$ defined by the formula*

$$(9) \quad \gamma_p((u_1^{-1}X_1)^{a_1} \cdots (u_n^{-1}X_n)^{a_n} e_{\mathbf{u}}) = \prod_{i=1}^n b(y_i)^{a_i} \cdots b(y_n)^{a_n} e_{\mathbf{u}}.$$

In particular, under this map, the operator of multiplication by X_i on $e_{\mathbf{u}}\widehat{\mathcal{P}}^-$ is sent to multiplication by $u_i b(y_i)$.

Here the subscript p is not a parameter, but distinguishes this map from an isomorphism of algebras we'll define later.

Proof. By Lemma 3.6, the elements $(u_1^{-1}X_1)^{a_1} \cdots (u_n^{-1}X_n)^{a_n} e_{\mathbf{u}}$ are a basis of $\widehat{\mathcal{P}}^-$, so this map is well-defined. We will check that it is an isomorphism on the image of each idempotent $e_{\mathbf{u}}$. On this image, this map is induced by the ring homomorphism $\mathbb{k}[[h, (X_1 - u_1), \dots, (X_n - u_n)]] \rightarrow \mathbb{k}[[h, y_1, \dots, y_n]]$ sending $X_i - u_i \mapsto u_i(b(y_i) - 1)$. The induced map modulo the square of the maximal ideal sends $X_i - u_i \mapsto u_i y_i + \dots$, and so defines an isomorphism of these completed polynomial rings. By Lemma 3.8, this shows that the map is an isomorphism. \square

Just as in Brundan and Kleshchev, it will be convenient for us to use different generators for $\widehat{\mathcal{H}}(\mathbf{q})$. Let

$$\Phi_r := T_r + \sum_{\mathbf{u} \text{ s.t. } u_r \neq u_{r+1}} \frac{1 - \mathbf{q}}{1 - X_r X_{r+1}^{-1}} e_{\mathbf{u}} + \sum_{\mathbf{u} \text{ s.t. } u_r = u_{r+1}} e_{\mathbf{u}}.$$

We will freely use the relations involving these given in [3, Lem. 4.1], the most important of which is

$$(10) \quad \Phi_r e_{\mathbf{u}} = e_{\mathbf{u}^{s_r}} \Phi_r$$

Let

$$\varphi_r(y_r, y_{r+1}) = \frac{u_r b(y_r) - \mathbf{q} u_{r+1} b(y_{r+1})}{u_{r+1} b(y_{r+1}) - u_r b(y_r)} = \frac{u_r b(y_r) - \mathbf{q} u_{r+1} b(y_{r+1} + d_1 h)}{u_{r+1} b(y_{r+1}) - u_r b(y_r)}$$

where the second equality holds by (8). Also, let $\beta(w, z) = \frac{b(w) - b(z)}{w - z}$; note that this is an invertible element of $\mathbb{k}[[w, z]]$. Thus, we have that

- $\varphi_r(y_r, y_{r+1})$ is an invertible element of $\mathbb{k}[[h, y_r, y_{r+1}]]$ if and only if $u_r \neq \mathbf{q} u_{r+1}, u_{r+1}$.
- If $u_r = \mathbf{q} u_{r+1}$, then we have

$$\varphi_r(y_r, y_{r+1}) = (y_r - y_{r+1}) \frac{\beta(y_r, y_{r+1} + d_1 h)}{q^{-1} - 1 + q^{-1} b(y_{r+1}) - b(y_r)}.$$

This fraction is an invertible power series, since both the numerator and denominator have non-zero constant terms.

- if $u_r = u_{r+1}$, then

$$\varphi_r(y_r, y_{r+1}) = \frac{b(y_r) - \mathbf{q} b(y_r)}{b(y_{r+1}) - b(y_r)} = \frac{1}{y_{r+1} - y_r} \frac{1 - \mathbf{q} d(h) + b(y_r) - \mathbf{q} d(h) b(y_r)}{\beta(y_{r+1}, y_r)}$$

which is also invertible.

Thus we can define an invertible power series by

$$A_r^{\mathbf{u}} = \begin{cases} \varphi_r(y_r, y_{r+1})(y_{r+1} - y_r) & u_r = u_{r+1} \\ \frac{\varphi_r(y_r, y_{r+1})}{y_{r+1} - y_r + d_1 h} & u_r = qu_{r+1} \\ \varphi_r(y_r, y_{r+1}) & u_r \neq u_{r+1}, qu_{r+1}. \end{cases}$$

THEOREM 3.10. *The isomorphism γ_p induces an isomorphism $\gamma: \widehat{\mathcal{H}}(\mathbf{q}) \cong \widehat{R}(h)$ such that*

$$\gamma(X_r) = \sum_{\mathbf{u}} u_r b(y_r) e_{\mathbf{u}} \quad \gamma(\Phi_r) = \sum_{\mathbf{u}} A_r^{\mathbf{u}} \psi_r e_{\mathbf{u}}$$

which intertwines these two representations, if either $d(h) = 1$ (and $b(h)$ is arbitrary) or $d(h) = b(h) = e^h$.

Proof. The match $\gamma(X_r) = \sum_{\mathbf{u}} u_r b(y_r) e_{\mathbf{u}}$ is clear from the definition of the map (9). Thus, we turn to considering $\gamma(\Phi_r)$. Using (10) and the definition, one can easily calculate that

$$\begin{aligned} \Phi_r e_{\mathbf{u}} \mathbf{I} &= \begin{cases} \frac{X_r - \mathbf{q}X_{r+1}}{X_{r+1} - X_r} e_{\mathbf{u}^{s_r}} \mathbf{I} & u_r \neq u_{r+1} \\ 0 & u_r = u_{r+1} \end{cases} \\ \Phi_r (X_{r+1} - X_r) e_{\mathbf{u}} \mathbf{I} &= \begin{cases} (\mathbf{q}X_{r+1} - X_r) e_{\mathbf{u}^{s_r}} \mathbf{I} & u_r \neq u_{r+1} \\ 2(\mathbf{q}X_{r+1} - X_r) e_{\mathbf{u}^{s_r}} \mathbf{I} & u_r = u_{r+1}. \end{cases} \end{aligned}$$

Using the commutation of Φ_r with symmetric Laurent polynomials in the $X_i^{\pm 1}$'s, we obtain a general form of action of this operator on an arbitrary Laurent polynomial $F \in \mathbb{k}[h, X_1^{\pm 1}, \dots, X_n^{\pm 1}]$.

$$(11) \quad \Phi_r F(X_1, \dots, X_n) e_{\mathbf{u}} \mathbf{I} = \begin{cases} \frac{X_r - \mathbf{q}X_{r+1}}{X_{r+1} - X_r} F^{s_r} e_{\mathbf{u}^{s_r}} \mathbf{I} & u_r \neq u_{r+1} \\ \frac{X_r - \mathbf{q}X_{r+1}}{X_{r+1} - X_r} (F^{s_r} - F) e_{\mathbf{u}} \mathbf{I} & u_r = u_{r+1}. \end{cases}$$

Now, consider how this operator acts if we intertwine with the isomorphism γ_p ; substituting into the formulas (11), we obtain that for a power series $f \in \mathbb{k}[[h, y_1, \dots, y_n]]$,

$$\gamma(\Phi_r) f(y_1, \dots, y_n) e_{\mathbf{u}} \mathbb{1} = \begin{cases} \varphi_r(y_r, y_{r+1}) f^{s_r} e_{\mathbf{u}^{s_r}} \mathbb{1} & u_r \neq u_{r+1} \\ \varphi_r(y_r, y_{r+1}) (f^{s_r} - f) e_{\mathbf{u}} \mathbb{1} & u_r = u_{r+1}. \end{cases}$$

Thus from (7), we immediately obtain that $A_r^{\mathbf{u}} \psi_r e(\mathbf{u}) = \Phi_r e(\mathbf{u})$. Since $A_r^{\mathbf{u}}$ is invertible, this immediately shows that the image of $\widehat{R}(h)$ lies in that of $\widehat{\mathcal{H}}(\mathbf{q})$ and *vice versa*. Thus, we obtain an induced isomorphism between these algebras. \square

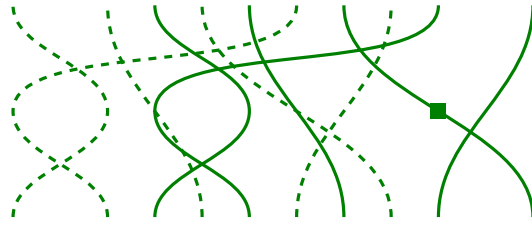
4. TYPE W

4.1. TYPE W HECKE ALGEBRAS. The isomorphism of Theorem 3.10 can be generalized a bit further to include not just KLR algebras but also weighted KLR algebras, a generalization introduced by the author in [22].

Fix a real number $g \neq 0$.

DEFINITION 4.1. *A rank n type W diagram consists of strands $\mathbb{R} \times [0, 1]$ like in a type O diagram defined above, with addition that we draw a dashed line g units to the right of each strand, (which we interpret as $-g$ units left if $g < 0$). We call this a ghost, and require that there are no triple points or tangencies involving any combination of strands or ghosts. We also only consider these diagrams equivalent if they are related by an isotopy that avoids these tangencies and double points.*

An example of a rank 5 type W diagram with $g < 0$ is given below:



DEFINITION 4.2. The rank n type W affine Hecke algebra (WAHA) $\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})$ for some collection \mathcal{B} of finite subsets $B_i \subset \mathbb{R}$ is the $\mathbb{k}[[h]]$ -span of all rank n type W Hecke diagrams such that endpoints of the strands on the lines $y = 0$ and $y = 1$ form a subset in \mathcal{B} , modulo the local relations:

(12a)

(12b)

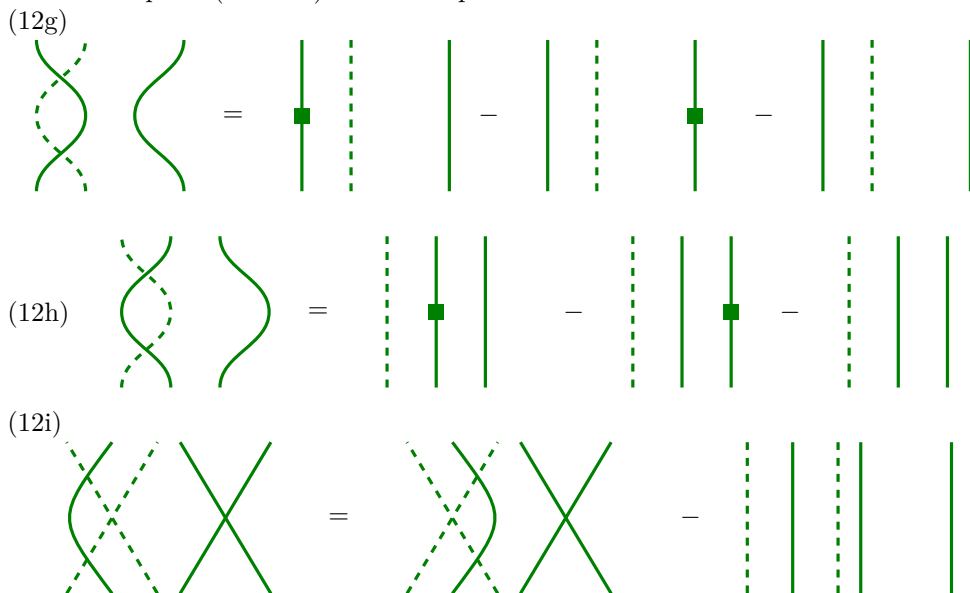
(12c)

(12d)

(12e)

(12f)

REMARK 4.3. As in the type O case, this algebra also has a degenerate analogue, where we replace (12c–12e) with the equations



By convention, we'll let e_B be the diagram with vertical lines at $x = b$ for $b \in B$, and use X_i to represent the square on the i th strand from left.

PROPOSITION 4.4. *The WAHA $W(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})$ for a set \mathcal{B} has a polynomial representation*

$$P_{\mathcal{B}} := \oplus_{B \in \mathcal{B}} \mathbb{k}[[h]][Y_1^{\pm 1}, \dots, Y_{|B|}^{\pm 1}]$$

defined by the rule that

- Each crossing of the r and $r + 1$ st strands acts by the Demazure operator

$$\partial_r(F) = \frac{F^{s_r} - F}{Y_{r+1} - Y_r}.$$

- A crossing between the r th strand and a ghost of s th strand acts by
 - the identity if $g < 0$ and the strand is NE/SW or $g > 0$ and the strand is NW/SE,
 - the multiplication operator of $Y_r - qY_s$ if $g < 0$ and the strand is NW/SE or $g > 0$ and the strand is NE/SW
- A square on the r th strand acts by the multiplication operator Y_r .

Proof. The equations (12a–12b) are the usual relations satisfied by multiplication and Demazure operators. The equations (12c–12d) are clear from the definition of the operators for ghost/strand crossings. Finally, the relations (12e–12f) are calculation with Demazure operators similar to that which is standard for triple points in various KLR calculi. For example, assuming $g < 0$ for (12e), the LHS is

$$\partial_s \circ (Y_r - qY_s) = (Y_r - qY_{s+1}) \circ \partial_s - q$$

using the usual twisted Leibnitz rule for Demazure operators; this is the RHS, so we are done. On the other hand, (12f) follows in a similar way from the equation

$$(Y_r - qY_s) \circ \partial_r = \partial_r \circ (Y_{r+1} - qY_s) + 1.$$

This completes the proof. □

PROPOSITION 4.5. *The rank n type W Hecke algebra $\mathcal{W}(\mathbf{q})_{\mathcal{B}}(\mathbf{q})$ has a basis over $\mathbb{k}[[h]]$ given by the products $e_B D_w X_1^{a_1} \cdots X_n^{a_n} e_{B'}$ for $w \in S_n$ and $(a_1, \dots, a_n) \in \mathbb{Z}^n$; here D_w is a arbitrarily chosen diagram which induces the permutation w on the endpoints at $y = 0$ when they are associated to the endpoint at the top of same strand, and no pair of strands or ghosts cross twice.*

The action of $\mathcal{W}(\mathbf{q})_{\mathcal{B}}(\mathbf{q})$ on its polynomial representation is faithful.

Proof. This proof follows many similar ones in KLR theory. These elements are linearly independent because the elements D_w span the action of $\mathbb{k}[S_n]$ after extending scalars to the fraction field of rational functions, since $D_w = f_w w + \sum_{v < w} f_v v$ for some rational functions f_v with $f_w \neq 0$. Thus our proposed basis is linearly independent over \mathbb{k} in this scalar extension, so must have been linearly independent before.

Note that this shows that the action of these elements on the polynomial representation is linearly independent. Thus, if we show that they span, it will show that the representation is faithful.

Now we need only show that they span. Using relation (12a), we can assume that all squares are at the bottom of the diagram.

Furthermore, any two choices of the diagram D_w differ via a series of isotopies and triple points, so relations (12b,12e,12f) show that these diagrams differ by diagrams with fewer crossings between strands and ghosts. Thus, we need only show that any diagram with a bigon can be written as a sum of diagrams with fewer crossings.

Now, assume we have such a bigon. We should assume that it has no smaller bigons inside it. In this case, we can shrink the bigon, using the relations (12b,12e,12f) whenever we need to move a strand through the top and bottom of the bigon or a crossing out through its side. Thus, we can ultimately assume that the bigon is empty, and apply the relations (12b–12d). \square

We now have the results we need to apply the results of Section 2, in the case of

$$(13) \quad \mathbb{K} = \mathbb{k}[[h]] \quad A = \mathcal{W}(\mathbf{q})_{\mathcal{B}}(\mathbf{q}) \quad B = \mathbb{k}[[h]][X^{\pm}] \quad I = Bh + B \prod_{u \in U} (X - u) \quad P = \mathcal{P}_{\mathcal{B}}.$$

The requisite freeness and the faithfulness of the polynomial representation follow from Proposition 4.5, so this defines a polynomial style representation. Thus, we have an induced completion $\widehat{\mathcal{W}}_{\mathcal{B}}$ with faithful completed polynomial representation by Lemma 2.5.

4.1.1. *Comparison with Hecke and Schur algebras.* Choose

$$\mathcal{V} = \{B_s = \{s, 2s, 3s, \dots, ns\}\}$$

for s some real number with $s \gg |g|$. For every type O diagram on n strands, we can choose an isotopy representative such that the endpoints of the diagram are precisely B_s at both $y = 0$ and $y = 1$. Furthermore, we can choose this representative so that if we think of it as a type W diagram and add ghosts, no strand is between a crossing of strands and the corresponding ghost crossing. Obviously we can do this for individual crossings, and any diagram can be factored into these.

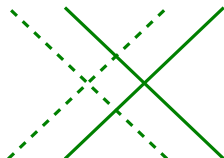
THEOREM 4.6. *This embedding induces an isomorphism between the WAHA $\mathcal{W}(\mathbf{q})_{\mathcal{V}}(\mathbf{q})$ and the honest affine Hecke algebra $\widetilde{\mathcal{H}}(\mathbf{q})$:*

- *If $g < 0$, this isomorphism sends a single crossing to $T_i + 1$. That is, the diagrams satisfy the local relations (1a–1c).*
- *If $g > 0$, this isomorphism sends a single crossing to $T_i - q$. That is, the diagrams satisfy the local relations (3a–3c).*

The polynomial representation defined above is intertwined by this map with the polynomial representation of $\tilde{\mathcal{H}}(\mathbf{q})$ if $g < 0$ and the signed polynomial representation if $g > 0$.

This theorem shows that if we view type O diagrams as type W diagrams where $|g|$ is sufficiently small that we cannot distinguish between a strand and its ghost⁽³⁾, then the local relations (1a–1c) will be consequences of (12a–12f).

Proof. We'll consider the case where $g < 0$. We have that $T_i + 1$ is sent to the diagram



which sent by the polynomial representation of the type W affine Hecke algebra representation to $(Y_r - \mathbf{q}Y_{r+1}) \circ \partial_r$. That is, we have $T_i F = -F^{s_r} + (1 - \mathbf{q})Y_{r+1} \partial_r$. Since $\mathcal{W}(\mathbf{q})_{\mathcal{B}(\mathbf{q})}$ acts faithfully on its polynomial representation, this shows that we have a map of the Hecke algebra to the WAHA; the faithfulness of \mathcal{P}^- implies that this map is injective. Since the diagram D_w and the polynomials in the squares are in the image of this map, the map is surjective.

The case $g > 0$ follows similarly. □

Thus, the WAHA for any set containing \mathcal{V} is a “larger” algebra than the affine Hecke algebra. The category of representations of affine Hecke algebras are a quotient category of its representations via the functor $M \mapsto e_{\mathcal{V}} M$, though in some cases, this quotient will be an equivalence.

For any composition $\mathbf{k} = (k_1, \dots, k_n)$ of m , we have an associated quasi-idempotent $\epsilon_{\mathbf{k}} = \sum_{w \in S_{\mathbf{k}}} T_w$ symmetrizing for the associated Young subgroup. If $\mathbf{k} = (1, \dots, 1)$, then $\epsilon_{\mathbf{k}} = 1$.

DEFINITION 4.7. The affine q -Schur algebra $\mathcal{S}(\mathbf{q}, n, m)$, as defined in [8, Def. 2.1.4], is the algebra defined by

$$\mathcal{S}(\mathbf{q}, n, m) := \text{End}_{\tilde{\mathcal{H}}(\mathbf{q})} \left(\bigoplus_{|\mathbf{k}|=m} \epsilon_{\mathbf{k}} \tilde{\mathcal{H}}(\mathbf{q}) \right)$$

where the sum is over n -part compositions of m .

Following [8], we let $E(n, m)$ denote the $\mathcal{S}(\mathbf{q}, n, m)$ - $\tilde{\mathcal{H}}(\mathbf{q})$ bimodule $\bigoplus_{|\mathbf{k}|=m} \epsilon_{\mathbf{k}} \tilde{\mathcal{H}}(\mathbf{q})$.

By a result of Jimbo [10], the affine Hecke algebra acts naturally on $M \otimes V^{\otimes n}$ for any finite dimensional $U_{\mathbf{q}}(\mathfrak{gl}_n)$ -module M and V the defining representation using universal R-matrices and Casimir operators; analogously, the algebra $\mathcal{S}(\mathbf{q}, n, m)$ naturally acts on

$$\bigoplus_{|\mathbf{k}|=m} M \otimes \text{Sym}^{k_1} V \otimes \dots \otimes \text{Sym}^{k_n} V \cong E(n, m) \otimes_{\tilde{\mathcal{H}}(\mathbf{q})} M \otimes V^{\otimes n}.$$

Furthermore, the algebra $\mathcal{S}(\mathbf{q}, n, m)$ has a natural polynomial representation given by

$$\mathcal{P}_{\mathcal{S}} := \bigoplus_{|\mathbf{k}|=m} \mathcal{P}^{S_{\mathbf{k}}} \cong E(n, m) \otimes_{\tilde{\mathcal{H}}(\mathbf{q})} \mathcal{P}^-.$$

There is a more detailed exposition of this representation in [17, § 4].

LEMMA 4.8. This representation is faithful.

⁽³⁾Perhaps this will be easier if you take off your glasses.

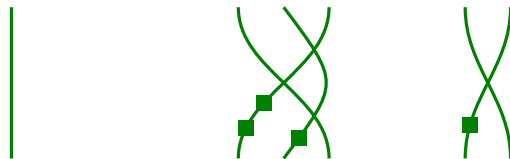
Proof. The algebra $\mathcal{S}(\mathbf{q}, n, m)$ have a basis $\phi_{\mathbf{k}, \mathbf{k}'}^d$ defined in [8, Def. 2.2.3]. This element is defined as a linear combination of left multiplications of elements of $\tilde{\mathcal{H}}(\mathbf{q})$, restricted to $\epsilon_{\mathbf{k}}\tilde{\mathcal{H}}(\mathbf{q})$. Thus, any non-trivial linear combination of these elements has the same property. By the faithfulness of \mathcal{P}^- , this implies that no non-trivial linear combination of $\phi_{\mathbf{k}, \mathbf{k}'}^d$ acts trivially. That is, the action is faithful. \square

If we replace $\epsilon_{\mathbf{k}}$ by the anti-symmetrizing quasi-idempotent $\epsilon_{\mathbf{k}}^- = \sum_{w \in S_{\mathbf{k}}} (-q)^{\ell(w)} T_w$, then we obtain the signed q -Schur algebra $\mathcal{S}_h^-(n, m)$, which instead acts on

$$\bigoplus_{|\mathbf{k}|=m} M \otimes \bigwedge^{k_1} V \otimes \cdots \otimes \bigwedge^{k_n} V.$$

The affine q -Schur algebra has a diagrammatic realization much like the affine Hecke algebra. For each composition $\mu = (\mu_1, \dots, \mu_n)$ of m , we let $C_{\mu} = \{i\epsilon + js \mid 0 \leq i < \mu_j\}$ for some fixed $0 < \epsilon \ll g \ll s$, and let \mathcal{C} be the collection of these sets. That is, we have groups of dots corresponding to the parts of the composition, with sizes given by μ_i .

In the type W affine Hecke algebra $\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})$, we have an idempotent e'_{μ} which on each group in $[js, js + \mu_j\epsilon]$ traces out the primitive idempotent in the nilHecke algebra which acts as $\partial_{w_0} y_1^{\mu_j-1} \cdots y_{\mu_j-1}$ in the polynomial representation. For example, for $\mu = (1, 3, 2)$, this idempotent is given by:



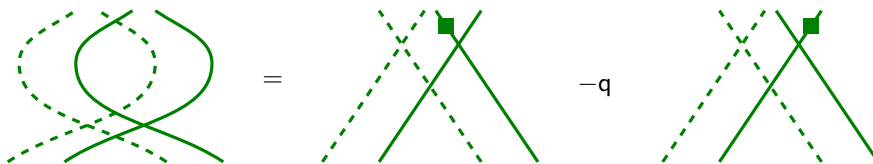
Let $e' = \sum_{\mu} e'_{\mu}$ be the sum of these idempotents over m -part compositions of n .

THEOREM 4.9. *If $g < 0$, we have an isomorphism of algebras $e'\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e' \cong \mathcal{S}(\mathbf{q}, n, m)$ which induces an isomorphism of representations $e'P_{\mathcal{C}} \cong P_{\mathcal{S}(\mathbf{q}, n, m)}$. Similarly, if $g > 0$, we have an isomorphism of algebras $e'\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e' \cong \mathcal{S}_h^-(n, m)$.*

Setting $h = 0$, we obtain an isomorphism between the WAHA $e'\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e'$ (at $h = 0$) with the usual affine Schur algebra for any field \mathbb{k} and any $q \notin \{0, 1\}$. Since this isomorphism requires passing through a Morita equivalence, it is quite difficult to make it explicit. A closely related isomorphism is shown in much greater detail by Miemietz and Stroppel in [17], relating the affine Schur algebra and the quiver Schur algebra from [19]; presumably these results can ultimately be matched by tracing through the Morita equivalence of [22, Th. 3.8], but we will not trace through the details of doing so.

Proof. First, consider the case $g < 0$. Consider the idempotent e_{B_s} in $e'\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e'$. This satisfies $e_{B_s}\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e_{B_s} \cong \tilde{\mathcal{H}}(\mathbf{q})$ by Theorem 4.6.

Thus, $e'e_{C_{\mu}}\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e_{B_s}$ is naturally a right module over $\tilde{\mathcal{H}}(\mathbf{q})$. We wish to show that it is isomorphic to $\epsilon_{\mu}\tilde{\mathcal{H}}(\mathbf{q})$. Consider the diagram $e_{C_{\mu}}D_1e_{B_s}$. Acting on the right by $T_i + 1$ with $(i, i + 1) \in S_{\mu_1} \times \cdots \times S_{\mu_p}$ gives $e_{C_{\mu}}D_1e_{B_s}(T_i + 1) = (q + 1)e_{C_{\mu}}D_1e_{B_s}$, since



Applying (12a), the RHS is equal to $1 + q$ times the identity, plus diagrams with a crossing at top, which are killed by e' . This shows that $e'e_{C_\mu}D_1e_{B_s}$ is invariant. Thus, we have a map of $\epsilon_\mu \tilde{\mathcal{H}}(\mathbf{q}) \rightarrow e'e_{C_\mu} \mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e_{B_s}$ sending $\epsilon_\mu \mapsto e'e_{C_\mu}D_1e_{B_s}$. This map must be surjective, since every $e'e_{C_\mu}D_1e_{B_s}$ is in its image, and comparing ranks over the fraction field $K = \mathbb{k}(X_1, \dots, X_n)$, we see that it must be injective as well. Thus, the action of $e'\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e'$ on $e'\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e_{B_s}$ defines a map $e'\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e' \rightarrow \mathcal{S}_h$.

Assume $a \neq 0$ is in the kernel of this map $e'\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e'$; that is, a acts trivially on $e'\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})e_{B_s}$. Note that $\mathcal{W}(\mathbf{q})_{\mathcal{C}}(\mathbf{q})$ acts faithfully on the rational representation $P_{\mathcal{C}}^K = P_{\mathcal{C}} \otimes_{\mathbb{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]} K$, and the element D_1 induces an isomorphism $e_{C_\mu} P_{\mathcal{C}}^K \otimes F \rightarrow e_{B_s} P_{\mathcal{C}}^K \otimes F$. Thus, we must have that aD_1 then acts non-trivially in $e_{B_s} P_{\mathcal{C}}$, and so $aD_1e_{B_s} \neq 0$, contradicting our assumption that a is in the kernel. Thus, we can only have $a = 0$, and the map to the Schur algebra is injective.


On the other hand, note that the element $e'e_{C_\mu}D_w e_{C_{\mu'}}e$ for w any shortest double coset representative is sent to the element $\phi_w = \sum_{w' \in S_{\mu} w S_{\mu'}} T_{w'}$ plus elements in $\mathbb{k}[[h]][X_1^{\pm 1}, \dots, X_n^{\pm 1}]\phi_v$ for v shorter in Bruhat order. Since $\phi_w X_1^{a_1}, \dots, X_n^{a_n}$ give a basis of \mathcal{S}_h by [8, 2.2.2], the fact that these are in the image shows that this map is surjective.


When $g > 0$, the argument is quite similar, but with $T_i - q$ replacing $T_i + 1$ and using the $g > 0$ version of Theorem 4.6. □

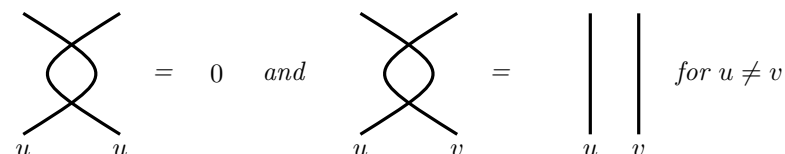
We'll prove in Corollary 6.7 that the idempotent e' induces a Morita equivalence between $\mathcal{W}(\mathbf{q})_{\mathcal{C}}$ and $\mathcal{S}_h(n, m)$. Thus, from the perspective of the Hecke side, introducing the type W relations is an alternate way of understanding the affine Schur algebra.

4.2. WEIGHTED KLR ALGEBRAS. On the other hand, the author has incorporated similar ideas into the theory of KLR algebras, by introducing *weighted KLR algebras* [22].

DEFINITION 4.10. Let $W(\mathbf{q})$ be the rank n weighted KLR algebra attached to the graph U ; that is, $W(\mathbf{q})$ is the quotient of $\mathbb{k}[[h]]$ -span of weighted KLR diagrams with n strands (as defined in [22, Def. 2.3]) by the local relations (note that these relations are drawn with $g < 0$):

(14a) 

(14b) 

(14c) 

(14d)
$$= \left\{ \begin{array}{l} \text{for } u \neq qv \\ \text{for } u = qv \end{array} \right.$$

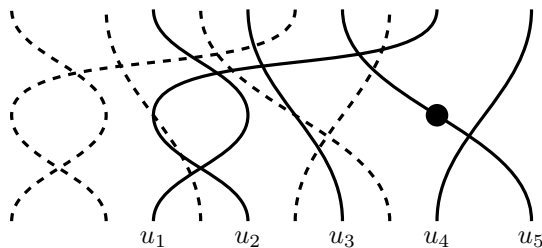
(14e)
$$= \left\{ \begin{array}{l} \text{for } u \neq qv \\ \text{for } u = qv \end{array} \right.$$

(14f)

(14g)
$$= \left\{ \begin{array}{l} \text{if } v = w = qu \\ \text{unless } v = w = qu \end{array} \right.$$

(14h)
$$= \left\{ \begin{array}{l} \text{if } w = qu = qv \\ \text{unless } w = qu = qv \end{array} \right.$$

For the sake of completeness, here is an example of a weighted KLR diagram:



We can define a *degree* function on KL diagrams, a special case of the degree function in [22]. The degrees are given on elementary diagrams by (15)

$$\deg \begin{array}{c} \diagup \\ u \quad v \\ \diagdown \end{array} = -2 \quad \deg \begin{array}{c} \bullet \\ | \\ u \end{array} = 2 \quad \deg \begin{array}{c} \diagup \\ u \quad v \\ \diagdown \end{array} = \deg \begin{array}{c} \diagdown \\ u \quad v \\ \diagup \end{array} = \begin{cases} 2 & u = qv = q^{-1}v \\ 1 & u = q^{\pm 1}v \neq q^{\mp 1}v \\ 0 & u \neq q^{\pm 1}v \end{cases}$$

and h is given grading 2. Note that the relations (14a–14h) are all homogeneous with wKLR diagrams given the grading of (15).

PROPOSITION 4.11 ([22, Prop. 2.7]). *The wKLR algebra $W_{\mathcal{D}}(\mathbf{q})$ for a collection \mathcal{D} has a faithful polynomial representation*

$$P_{\mathcal{D}} := \oplus_{D \in \mathcal{D}} \mathbb{k}[h, y_1, \dots, y_{|D|}]$$

defined by the rule that

- Each crossing of the r and $r + 1$ st strands acts by the Demazure operator

$$\partial_r(f) = \frac{f^{s_r} - f}{y_{r+1} - y_r}.$$

- A crossing between the r th strand and a ghost of s th strand acts by
 - the identity if $g < 0$ and the strand is NE/SW or $g > 0$ and the strand is NW/SE,
 - the multiplication operator of $y_s - y_r + h$ if $g < 0$ and the strand is NW/SE or $g > 0$ and the strand is NE/SW
- A square on the r th strand acts by the multiplication operator Y_r .

Thus, we can again apply the results of Section 2, with

$$(16) \quad \mathbb{K} = \mathbb{k}[h] \quad A = W_{\mathcal{D}}(\mathbf{q}) \quad B = \mathbb{k}[h, y] \quad I = B(h, y) \quad P = P_{\mathcal{D}}.$$

LEMMA 4.12. *The polynomial representation $P_{\mathcal{D}}$ is graded polynomial-style with the data of (16).*

Proof. The algebra $W_{\mathcal{D}}(\mathbf{q})$ is free over $B^{\otimes n} = \mathbb{k}[h, y_1, \dots, y_n]$ by [22, Thm. 2.8], the centrality of Z is clear from the relations and the faithfulness of P follows from Proposition 4.11. The compatibility with grading is also clear from the definition (15). \square

DEFINITION 4.13. *We let $\widehat{W}(h)$ be the completion of the weighted KLR algebra W for U with respect to the grading; since h has degree 2, this completion is naturally a complete $\mathbb{k}[[h]]$ -module. For any collection \mathcal{D} , we let $W_{\mathcal{D}}(\mathbf{q}), \widehat{W}_{\mathcal{D}}(\mathbf{q})$ be the sum of images of the idempotents corresponding to loadings on a set of points in \mathcal{D} .*

Let \mathbf{i} be a loading in the sense of [22], that is, a finite subset $D = \{d_1, \dots, d_n\}$ with $d_1 < \dots < d_n$ of \mathbb{R} together with a map $\mathbf{i}: D \rightarrow U$. In the algebra $\widehat{W}(\mathbf{q})_{\mathcal{D}}(\mathbf{q})$, we have an idempotent $\epsilon_{\mathbf{i}}$ projecting to the stable kernel of $X_j - \mathbf{i}(d_j)$ (that is, the kernel of

a sufficiently large power). We represent $\epsilon_{\mathbf{i}}$ as a type W diagram, with the strands labeled by the elements $u_i = \mathbf{i}(d_j)$.

THEOREM 4.14. *There is an isomorphism $\gamma: \widehat{W}(\mathfrak{q})_{\mathcal{D}}(\mathfrak{q}) \rightarrow \widehat{W}_{\mathcal{D}}(\mathfrak{q})$ such that $\gamma(X_r) = \sum_{\mathbf{u}} u_r b(y_r) e_{\mathbf{u}}$,*

$$(17a) \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \epsilon_{\mathbf{u}} \mapsto \begin{cases} \frac{1}{u_{r+1}b(y_{r+1}) - u_r b(y_r)} (\psi_r - 1) e_{\mathbf{u}} & u_r \neq u_{r+1} \\ \frac{y_{r+1} - y_r}{u_{r+1}(b(y_{r+1}) - b(y_r))} \psi_r e_{\mathbf{u}} & u_r = u_{r+1} \end{cases}$$

$$(17b) \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \epsilon_{\mathbf{u}} \mapsto \begin{cases} \begin{array}{c} \diagup \\ \diagdown \end{array} e_{\mathbf{u}} & u_r \neq qu_s \\ \begin{array}{c} \diagdown \\ \diagup \end{array} e_{\mathbf{u}} & u_r = qu_s \end{cases} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \mapsto \begin{array}{c} \diagdown \\ \diagup \end{array}$$

Proof. This follows from comparing the polynomial representations. Exactly as argued in Lemma 3.9, the map is an isomorphism of vector spaces between the polynomial representations: the polynomial representation $\mathcal{P}_{\mathcal{B}}$ has one copy of \mathcal{C} for each subset in \mathcal{B} . In $\widehat{\mathcal{P}}_{\mathcal{B}}$, each of these copies is completed at U^n , and becomes the direct sum of the images of $e_{\mathbf{u}}$, which is a copy of the completed polynomial ring. We can think of the choice of subset and of \mathbf{u} as giving a loading, which has a corresponding copy of $\widehat{\mathcal{C}}$ in $\mathcal{P}_{\mathcal{B}}$. The map γ_p induces an isomorphism between these completed polynomial rings.

Now, we should consider how identifying completed polynomial representations via γ_p affects how the basic diagrams of the WAHA act on the polynomial representation.

We have that

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \cdot f \epsilon_{\mathbf{u}} = \frac{f^{s_r} - f}{y_{r+1} - y_r} \epsilon_{\mathbf{u}}.$$

If $u_r \neq u_{r+1}$, then $\psi_r \cdot f \epsilon_{\mathbf{u}} = f^{s_r} \epsilon_{\mathbf{u}}$ and $u_{r+1}b(y_{r+1}) - u_r b(y_r)$ is invertible, so the appropriate case of (17a) holds. If $u_r = u_{r+1}$, then $\frac{u_r(b(y_{r+1}) - b(y_r))}{y_{r+1} - y_r}$ is invertible, so the formula is clear.

Now, we turn to (17b). We find that $\begin{array}{c} \diagup \\ \diagdown \end{array} \cdot f \epsilon_{\mathbf{u}} = (u_r b(y_r) - qu_s b(y_s)) f \epsilon_{\mathbf{u}}$. The first case of the isomorphism (17b) thus follows directly from the polynomial representation of the wKLR algebra given in Proposition 4.11. The second case of (17b) is clear. \square

The reader will note that the image of the idempotent e' under this isomorphism is not homogeneous. On abstract grounds, there must exist a homogeneous idempotent e'' with isomorphic image. Let us give a description of one such, which is philosophically quite close to the approach of [19].

Choose an arbitrary order on the elements of U . The idempotent e'_{μ} for a composition μ is replaced by the sum of contributions from a list of multi-subsets Z_i of U such that $|Z_i| = \mu_i$. There's a loading corresponding to these subsets, which we'll denote \mathbf{i}_{Z_*} . The underlying subset is C_{μ} as defined before; the points associated to the j th part at $x = js + \epsilon, \dots, js + \mu_j \epsilon$ are labeled with the elements of Z_j in our fixed order. Finally, e''_{Z_*} is the idempotent on this loading that acts on each group of strands with the same label in U and attached to the same part of μ with a fixed homogeneous primitive idempotent in the nilHecke algebra, for example, that acts as $y_1^{k-1} \cdots y_{k-1} \partial_{w_0}$ in the polynomial representation. Consider the sum e'' of the idempotents e''_{Z_*} over all p -tuples of multi-subsets.

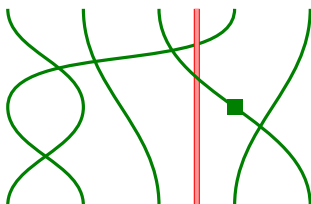
The idempotent e'' has isomorphic image to e' , since We'' is a sum of projectives for each composition μ whose $(\mu_1! \cdots \mu_p!)$ -fold direct sum is $We_{C_{\mu}}$. Thus, the algebra $e''We''$ is graded and isomorphic to the Schur algebra. It would be interesting to make this isomorphism a bit more explicit, but we will leave that to other work.

5. TYPE F

5.1. TYPE F HECKE ALGEBRAS. Now let us turn to our other complication, analogous to that which appeared in [23]:

DEFINITION 5.1. A rank n type F_1 Hecke diagram is a rank n affine Hecke diagram with a vertical red line inserted at $x = 0$. The diagram must avoid tangencies and triple points with this strand as well, and only allow isotopies that preserve these conditions.

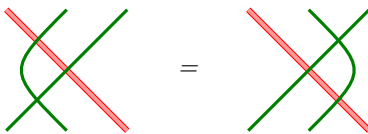
We give an example of such a diagram below:



We decorate this red strand with a multisubset $Q_\bullet = \{Q_1, \dots, Q_\ell\} \subset U$ and let $Q_i = Q_i e^{-z_i}$. To distinguish from other uses of the letter, we let $e_k(\mathbf{z})$ be the degree k elementary symmetric function in an alphabet \mathbf{z} .

DEFINITION 5.2. Let the type F_1 affine Hecke algebra $\tilde{\mathcal{F}}(\mathbf{q}, Q_\bullet)$ be the algebra generated over $\mathbb{k}[[h, \mathbf{z}]]$ by type F_1 Hecke diagrams with m strands modulo the local relations (3a–3c) and the local relations:

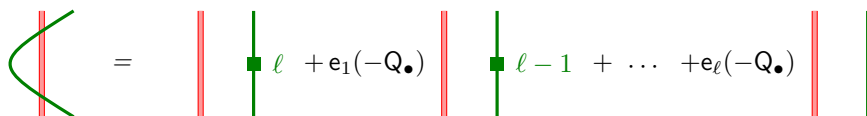
(18a)



(18b)



(18c)



That is, on the RHS, we have the product $p_{\mathbf{Q}} = (X_j - Q_1) \cdots (X_j - Q_\ell)$, where the green strand shown is the j th, and

$$(18d) \quad \text{Diagram} = \sum_{i=1}^{\ell} \sum_{a+b=i-1} e_{\ell-i}(-Q_\bullet) \cdot \left(a+1 \text{ square} \mid b - \mathbf{q} \text{ square} \mid a \text{ square} \mid b+1 \text{ square} \right).$$

The RHS can alternately be written as $(X_i - \mathbf{q}X_{i+1}) \frac{p_{\mathbf{Q}}(X_i) - p_{\mathbf{Q}}(X_{i+1})}{X_i - X_{i+1}}$.

REMARK 5.3. As in the earlier cases, there is a degenerate version of this algebra, where we use the local relations (1d–1f), leave (18a–18c) unchanged, and replace the RHS of (18d) with $(X_i - X_{i+1} + 1) \frac{p_{\mathbf{Q}}(X_i) - p_{\mathbf{Q}}(X_{i+1})}{X_i - X_{i+1}}$.

We'll continue to use our convention of letting X_r denote the sum of all straight-line diagrams with a square on the r th green strand from the left (ignoring red strands).

Given \mathcal{D} a collection of subsets of \mathbb{R} , we'll let $\tilde{\mathcal{F}}_{\mathcal{D}}(\mathfrak{q}, \mathbf{Q}_{\bullet}), \mathcal{F}_{\mathcal{D}}(\mathfrak{q}, \mathbf{Q}_{\bullet})$ denote the subalgebras of $\tilde{\mathcal{F}}(\mathfrak{q}, \mathbf{Q}_{\bullet}), \mathcal{F}(\mathfrak{q}, \mathbf{Q}_{\bullet})$ spanned by diagrams whose tops and bottoms lie in the set \mathcal{D} .

Let e_i be an arbitrarily fixed idempotent in $\tilde{\mathcal{F}}(\mathfrak{q}, \mathbf{Q}_{\bullet})$ given by i strands left of the red strand and $m-i$ right of it; let \mathcal{D}° be the collection of the corresponding sets. Since any idempotent is isomorphic to one of these by a straight-line diagram, enlarging \mathcal{D}° will give a Morita equivalent algebra. Let \tilde{P}_n be the free $S[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ -module generated by elements f_p for $p = 0, \dots, m$.

PROPOSITION 5.4. *The algebra $\tilde{\mathcal{F}}_{\mathcal{D}^{\circ}}(\mathfrak{q}, \mathbf{Q}_{\bullet})$ has a polynomial representation that sends*

- e_i to the identity on the submodule generated by f_i .
- X_i to the multiplication operator and

$$(T_i + 1) \cdot F(X_1, \dots, X_n) f_p \mapsto (X_i - \mathfrak{q} X_{i+1}) \frac{F^{s_i} - F}{X_{i+1} - X_i} f_p.$$

- the action of positive to negative crossing to the identity

$$F(X_1, \dots, X_n) f_i \mapsto F(X_1, \dots, X_n) f_{i+1},$$

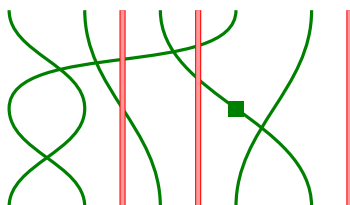
and the opposite crossing to

$$F(X_1, \dots, X_n) f_i \mapsto p_{\mathbf{Q}}(X_i) F(X_1, \dots, X_n) f_{i-1}.$$

Proof. This is a standard computation with Demazure operators. □

Now, we can allow several red lines at various values of x , each of which carries a multiset of values in U . For the sake of notation, we'll still denote the multiset given by all such labels as $\{Q_1, \dots, Q_{\ell}\}$, with a strand with the label Q_i at x -value ϑ_i . So, the situation we had previously considered was $\vartheta_i = 0$ for all i .

DEFINITION 5.5. *A rank n type F Hecke diagram is a rank n affine Hecke diagram with a vertical red lines inserted at $x = \vartheta_i$. The diagram must avoid tangencies and triple points with these strands as well, and only allow isotopies that preserve these conditions. We give an example of such a diagram below:*



Let the rank n type F affine Hecke algebra $\tilde{\mathcal{F}}^{\vartheta}(\mathfrak{q}, \mathbf{Q}_{\bullet})$ be the algebra generated over $\mathbb{k}[[h, \mathbf{z}]]$ by rank n type F Hecke diagrams for ϑ with n strands modulo the local relations (3a–3c) and (18a–18d).

These algebras have a polynomial representation \mathcal{P}^{ϑ} using the same maps attached to basic diagrams as Proposition 5.4, but now with idempotents, and thus copies of Laurent polynomials, indexed by weakly increasing functions $\nu: [1, \ell] \rightarrow [0, m]$ with $\nu(i)$ giving the number of green strands to the left of the i th red strand. This was carried out in more detail in [16, Prop. 1.10]. As before, any two idempotents corresponding to ν are isomorphic by straight-line diagrams.

These affine type F algebras have “finite-type” quotients. In other contexts, these have been called “steadied” or “cyclotomic” quotients.

DEFINITION 5.6. *The rank n type F Hecke algebra $\mathcal{F}^{\vartheta}(\mathfrak{q}, \mathbf{Q}_{\bullet})$ is the quotient of $\tilde{\mathcal{F}}^{\vartheta}(\mathfrak{q}, \mathbf{Q}_{\bullet})$ by the 2-sided ideal generated by e_B for every set B possessing an element $b \in B$ with $b < \vartheta_i$ for all i .*

Pictorially, the idempotents e_B we kill possess a green strand which is left of all the red strands. In [23], the corresponding ideal for KLR algebras is called the *violating ideal* and we will use the same terminology here. Given \mathcal{D} a collection of subsets of \mathbb{R} , we'll let $\tilde{\mathcal{F}}_{\mathcal{D}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet}), \mathcal{F}_{\mathcal{D}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$ denote the subalgebras of $\tilde{\mathcal{F}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet}), \mathcal{F}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$ spanned by diagrams whose tops and bottoms lie in the set \mathcal{D} .

PROPOSITION 5.7. *The rank n cyclotomic affine Hecke algebra $\mathcal{H}(\mathbf{q}, \mathbf{Q}_{\bullet})$ for the parameters $\{Q_1, \dots, Q_{\ell}\}$ is isomorphic to the rank n type F_1 Hecke algebra $\mathcal{F}_{\mathcal{D}^{\circ}}(\mathbf{q}, \mathbf{Q}_{\bullet})$.*

Proof. If we let e be the idempotent given by green lines at $x = 1, \dots, n$, then we see by Theorem 4.6, there is a map from the affine Hecke algebra sending X_i and $T_i + 1$ to diagrams as in (2) which induces a map $\iota: \tilde{\mathcal{H}}(\mathbf{q}) \rightarrow \tilde{\mathcal{F}}_{\mathcal{D}^{\circ}}(\mathbf{q}, \mathbf{Q}_{\bullet})$. Pulling back the polynomial representation of $\tilde{\mathcal{F}}_{\mathcal{D}^{\circ}}(\mathbf{q}, \mathbf{Q}_{\bullet})$ gives the polynomial representation of $\tilde{\mathcal{H}}(\mathbf{q})$, which is faithful, so this map is injective.

Applying (18c) at the leftmost strand shows that $p_{\mathbf{Q}}(X_1)$ lies in the violating ideal, which is the kernel of the map to $\mathcal{F}(\mathbf{q}, \mathbf{Q}_{\bullet})$. Thus, ι induces a map $\mathcal{H}(\mathbf{q}, \mathbf{Q}_{\bullet}) \rightarrow \mathcal{F}_{\mathcal{D}^{\circ}}(\mathbf{q}, \mathbf{Q}_{\bullet})$. This map is clearly surjective, since any F_1 Hecke diagram with no violating strand is a composition of the images.

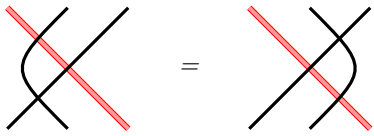
Thus, we need only show that the preimage of the violating ideal under ι lies in the cyclotomic ideal. As in the proof of [23, 3.16], the relations (18c,18d) allow us to reduce to the case where only a single green strand passes into the left half of the plane. In this case, we gain a factor of $p_{\mathbf{Q}}(X_1)$, showing that this is in the cyclotomic ideal. \square

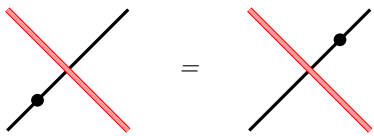
5.2. STENDHAL ALGEBRAS. The type F algebras in the KLR family have been introduced in [23]. Let $o_1 = \min(\vartheta_i)$, and $o_j = \min_{\vartheta_i > o_{j-1}}(\vartheta_i)$; so these are the real numbers that occur as ϑ_i in increasing order. Consider the sequence $\lambda_j = \sum_{\vartheta_i = o_j} \omega_{Q_i}$ of dominant weights for \mathfrak{g}_U , and let $S_{u,j} = \{s \in [1, \ell] \mid \vartheta_s = o_j, u = Q_s\}$.

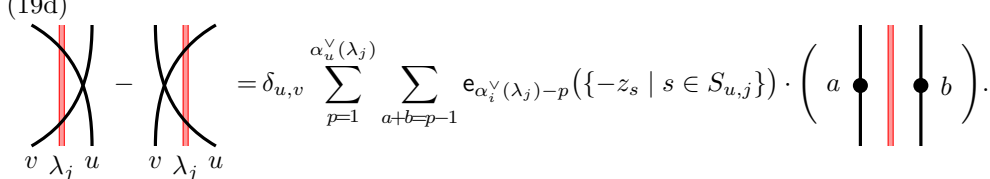
In [23, Def. 4.7], we defined algebras $T^{\lambda}, \tilde{T}^{\lambda}$ attached to this list of weights. These cannot match $\tilde{\mathcal{F}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet}), \mathcal{F}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$ since they are not naturally modules over $\mathbb{k}\llbracket h, \mathbf{z} \rrbracket$; however, we will recover them when we set $h = z_1 = \dots = z_{\ell} = 0$. Instead, we should consider deformed versions of these algebras $\tilde{T}^{\lambda}(h, \mathbf{z}), T^{\lambda}(h, \mathbf{z})$ introduced in [22, § 3.2] based on the canonical deformation of weighted KLR algebras. As usual, we'll let y_r denote the sum of all straight line Stendhal diagrams with a dot on the r th strand.

DEFINITION 5.8. *We let the rank n affine Stendhal algebra $\tilde{T}^{\lambda}(h, \mathbf{z})$ be the quotient of the algebra freely spanned over $\mathbb{k}\llbracket h, \mathbf{z} \rrbracket$ by Stendhal diagrams (as defined in [23, § 3.2]) with n black strands, with the local relations (5a–5e) and the local relations*

$$\begin{aligned}
 (19a) \quad & \begin{array}{c} \text{Diagram 1: A red strand crosses over a black strand. The red strand starts at the top left and ends at the bottom right. The black strand starts at the top right and ends at the bottom left. Labels: } \lambda_j \text{ (top left), } u \text{ (top right).} \\ \text{Diagram 2: A red strand crosses under a black strand. The red strand starts at the top left and ends at the bottom right. The black strand starts at the top right and ends at the bottom left. Labels: } \lambda_j \text{ (top left), } u \text{ (top right).} \end{array} \\
 & = p_{u,j} \left(\begin{array}{c} \text{Diagram 3: Two parallel vertical red strands. Labels: } \lambda_j \text{ (left), } u \text{ (right).} \\ \text{Diagram 4: Two parallel vertical black strands. The left strand has a dot. Labels: } u \text{ (left), } \lambda_j \text{ (right).} \end{array} \right) \\
 & \quad p_{u,j}(y) = \prod_{s \in S_{u,j}} (y - z_s)
 \end{aligned}$$

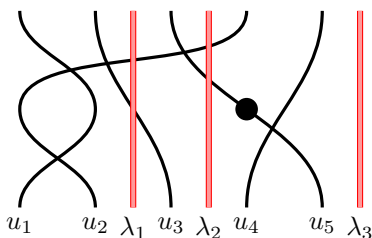
(19b) 

(19c) 

(19d) 

The rank n Stendhal algebra $T^\lambda(h, \mathbf{z})$ is the quotient of $\tilde{T}^\lambda(h, \mathbf{z})$ by violating diagrams as defined in [23, Def. 4.3].

Again for the sake of comparison, here is an example of a Stendhal diagram of rank 5:



This algebra is graded with Stendhal diagrams given their usual grading, summing local contributions given by

$$\deg \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = -\langle \alpha_u, \alpha_v \rangle \quad \deg \begin{array}{c} \bullet \\ | \\ u \end{array} = 2 \quad \deg \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \deg \begin{array}{c} \diagdown \\ \times \\ \diagup \end{array} = \alpha_u^\vee(\lambda),$$

and the variables h and z_i each have degree 2.

The algebra $\tilde{T}^\lambda(h, \mathbf{z})$ has a polynomial representation P_λ , given in [23, Lem. 4.12]. In order to match the Hecke side, we will use the version of this representation that has

$$P_{uv}(a, b) = \begin{cases} b - a + h & u = qv \\ 1 & u \neq qv. \end{cases}$$

For every loading, we have an associated function κ , with $\kappa(k)$ equal to the number of black strands to the left of o_k , and a sequence (u_1, \dots, u_n) given by the eigenvalues we've attached to each black strand. We let $e_{\mathbf{u}, \kappa}$ be the idempotent associated to this data in $\tilde{T}^\lambda(h, \mathbf{z})$ and by extension in $\widehat{\tilde{T}^\lambda}(h, \mathbf{z})$ and $T^\lambda(h, \mathbf{z})$.

5.3. ISOMORPHISMS. As in types O and W, these algebras have polynomial-style representations (graded in the case of $\tilde{T}^\lambda(h, \mathbf{z})$, with the data

$$\mathbb{K} = \mathbb{k}[[h]] \quad A = \tilde{\mathcal{F}}^\vartheta(\mathbf{q}, \mathbf{Q}, \bullet) \quad B = \mathbb{k}[[h]][X^\pm] \quad I = Bh + B \prod_{u \in U} (X - u) \quad P = \mathcal{P}_\theta$$

$$\mathbb{K} = \mathbb{k}[h] \quad A = \tilde{T}^\lambda(h, \mathbf{z}) \quad B = \mathbb{k}[h, y] \quad I = Bh + By \quad P = \underline{P}_\lambda.$$

and the polynomial representations we have defined, with the latter being graded. This is proven exactly as in the earlier cases:

- (1) An explicit basis indexed by permutations, constructed for $\tilde{\mathcal{F}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet)$ in [16, Prop. 1.15] and $\tilde{T}^\lambda(h, \mathbf{z})$ in [23, Prop. 4.16], shows the required freeness.
- (2) The centrality of Z is immediate from the relations.
- (3) The faithfulness of the representations is checked in [16, Prop. 1.11] and implicit in the proof of [23, Prop. 4.16], respectively.

We let $\widehat{\mathcal{F}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet)$ and $\widehat{T}^\lambda(h, \mathbf{z})$ be the completions of these rings with respect to the induced topology. Since h and \mathbf{z} both have positive degree, $\widehat{T}^\lambda(h, \mathbf{z})$ is a complete module over $\mathbb{k}[[h, \mathbf{z}]]$.

THEOREM 5.9. *We have an isomorphism $\widehat{\mathcal{F}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) \cong \widehat{T}^\lambda(h, \mathbf{z})$ which induces an isomorphism $\mathcal{F}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) \cong T^\lambda(h, \mathbf{z})$, given by*

$$\begin{aligned} \epsilon_{\mathbf{u}, \kappa} &\mapsto e_{\mathbf{u}, \kappa} & X_r &\mapsto \sum_{\mathbf{u}, \kappa} u_r b(y_r) e_{\mathbf{u}, \kappa} & \begin{array}{c} \text{green} \\ \text{red} \end{array} \times &\mapsto \begin{array}{c} \text{red} \\ \text{green} \end{array} \times \\ \\ \begin{array}{c} \text{green} \\ \text{red} \end{array} \times \epsilon_{\mathbf{u}, \kappa} &\mapsto \frac{\prod_{\vartheta_s = o_\kappa} (u_r b(y_r - z_s) - Q_s)}{\prod_{s \in S_{u_r, j}} (y_r - z_s)} \begin{array}{c} \text{red} \\ \text{green} \end{array} \times e_{\mathbf{u}, \kappa} & \begin{array}{c} \text{green} \\ \text{red} \end{array} \times e_{\mathbf{u}, \kappa} &\mapsto A_r^{\mathbf{u}} \begin{array}{c} \text{red} \\ \text{green} \end{array} \times e_{\mathbf{u}, \kappa} \end{aligned}$$

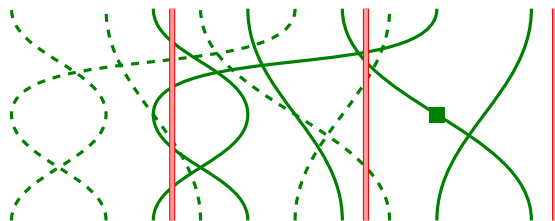
where the leftmost green/black strand shown is the r th from the left, and the red strand shown is the j th from the left.

Proof. Since all generators and relations involve at most one red line, we can assume that $\ell = 1$, and use the representation of Proposition 5.4 for the Hecke side. That diagrams with only green strands have actions that match is just Theorem 3.10. Thus, we only need to check the crossing of green and red strands is intertwined with a crossing of red and black strands. Since we have only one red strand, we have that $\prod_{\vartheta_s = o_\kappa} (u_r e^{y_r} - Q_s) = p_{\mathbf{Q}_\bullet}(u_r e^{y_r})$. Thus, comparing the representation of Proposition 5.4 with the obvious $\mathbb{k}[[h, \mathbf{z}]]$ -deformation of the action in [23, Lem. 4.12] yields the result. \square

6. TYPE WF

6.1. TYPE WF HECKE ALGEBRAS. Finally, we consider these two complications jointly. As mentioned before, these are unlikely to be familiar algebras for the reader, but these results will ultimately be useful in understanding category \mathcal{O} of rational Cherednik algebras in [24].

DEFINITION 6.1. *A rank n type WF Hecke diagram is a type W Hecke diagram with vertical red lines inserted at $x = \vartheta_i$. The diagram must avoid tangencies and triple points between any combination of these strands, green strands and ghosts, and only allow isotopies that preserve these conditions. An example of a rank 5 type WF diagram with $g < 0$ is given below:*



Let the rank n type WF affine Hecke algebra $\widetilde{\mathcal{W}\mathcal{F}^\vartheta}(\mathfrak{q}, \mathbf{Q}_\bullet)$ be the $\mathbb{k}[[h, \mathbf{z}]]$ -algebra generated by type WF Hecke diagrams modulo the local relations (12a–12f, 18a–18c) and

$$(20a) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} - \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \sum_{i=1}^{\ell} \sum_{a+b=i-1} e_{\ell-i}(-\mathbf{Q}_\bullet) \cdot \left(\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right).$$

$$(20b) \quad \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} = \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} = \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} = \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array}$$

REMARK 6.2. The degenerate version of this algebra is defined by the local relations (12a–12b, 12f–12i, 18a–18c, 20a–20b)

Note that relation (18d) is *not* true in this algebra. As before, we should think of type F diagrams as type WF diagrams with g so small that we cannot see that the ghost and strand are separate. Using this approach, we can see that relation (18d) for a strand and a ghost together is a consequence of (14d) and (20a), much as in Theorem 4.6.

This algebra has a polynomial representation P^ϑ , defined using the same formulae as those of Propositions 4.4 and 5.4. We leave the routine computations that these are compatible with (20a) and (20b) to the reader.

We call an idempotent *unsteady* if the strands can be divided into two groups with a gap $> |g|$ between them and all red strands in the right hand group, and *steady* otherwise.

Thus, the idempotents shown in (21a) are steady, and those in (21b) are unsteady.

$$(21a) \quad \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \\ \text{Diagram 17} \\ \text{Diagram 18} \end{array}$$

$$(21b) \quad \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \\ \text{Diagram 21} \\ \text{Diagram 22} \end{array}$$

DEFINITION 6.3. Let the rank n type WF Hecke algebra $\mathcal{W}\mathcal{F}^\vartheta(\mathfrak{q}, \mathbf{Q}_\bullet) \cong \mathfrak{C}^\vartheta$ be the quotient of $\widetilde{\mathcal{W}\mathcal{F}^\vartheta}(\mathfrak{q}, \mathbf{Q}_\bullet)$ by the ideal generated by all unsteady idempotents.

We can also call this a “pictorial Cherednik algebra,” referring to the fact that the representation category of this algebra when $\mathbb{k} = \mathbb{C}$ and we set $h = z_i = 0$ is equivalent to the category \mathcal{O} over a Cherednik algebra for the group $\mathbb{Z}/\ell\mathbb{Z} \wr S_n$ for certain parameters. More precisely, we consider the category \mathcal{O} over the rational Cherednik algebra H for the group $\mathbb{Z}/\ell\mathbb{Z} \wr S_n$ with arbitrary \mathbb{C} -valued parameters k, s_1, \dots, s_ℓ , using the conventions of [24, § 2.1] and consider the algebra $\mathcal{W}\mathcal{F}^\vartheta(\mathfrak{q}, \mathbf{Q}_\bullet)$ where fix the number of green strands to be n , and fix the parameters $g = \text{Re}(k), \vartheta_p = \text{Re}(ks_p), q = e^{2\pi i k}$, and $Q_p = e^{2\pi i ks_p}$.

THEOREM 6.4 ([24, Cor. 3.10]). For the parameters discussed above, we have an equivalence of categories $\mathcal{O} \cong \mathcal{W}\mathcal{F}^\vartheta(\mathfrak{q}, \mathbf{Q}_\bullet)\text{-mod}$.

Obviously, any choice of q, \mathbf{Q}_\bullet can be realized this way for some k, s_\bullet , which are not unique; for any g and ϑ , we can adjust the choice of parameters k, s_\bullet to yield an block of the Cherednik category \mathcal{O} that matches the representations of $\mathcal{W}\mathcal{F}^\vartheta(\mathfrak{q}, \mathbf{Q}_\bullet)$, using the process of Uglovation discussed in [24, Def. 2.8]. This is also useful to

consider as a common generalization of all the algebras we have considered. Given a collection \mathcal{D} of subsets of \mathbb{R} , we'll let $\widetilde{\mathcal{WF}}_{\mathcal{D}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet}), \mathcal{WF}_{\mathcal{D}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$ denote the subalgebras of $\widetilde{\mathcal{WF}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet}), \mathcal{WF}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$ spanned by diagrams whose tops and bottoms lie in the set \mathcal{D} .

As in earlier cases, the algebra $\widetilde{\mathcal{WF}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$ is equipped with a polynomial representation P^{ϑ} using the rules of Proposition 4.4 for diagrams only involving green strands and Proposition 5.4 for basic diagrams involving red and green strands.

6.1.1. *Relation to cyclotomic Schur algebras.* We can extend Theorem 4.6 to this setting. As before, let $\mathcal{V} = \{B_s = \{s, 2s, 3s, \dots, ns\}\}$ for s some real number with $s \gg |g|, |\vartheta_i|$.

THEOREM 6.5. *There is an isomorphism of $\mathcal{WF}_{\mathcal{V}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$ to the rank n cyclotomic affine Hecke algebra $\mathcal{H}(\mathbf{q}, \mathbf{Q}_{\bullet})$ for the parameters $\{Q_1, \dots, Q_{\ell}\}$.*

Proof. First, since $s \gg |\vartheta_i|$, all strands start and end to the right of all red strands. Thus, we have that every diagram can be written, using the relations, in terms of diagrams that remain to the right of all red strands. Thus, we have a surjective map from the type W affine Hecke algebra $\mathcal{W}(\mathbf{q})_{\mathcal{O}}$ onto $\mathcal{WF}_{\mathcal{V}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$. By Theorem 4.6, we can identify $\mathcal{W}(\mathbf{q})_{\mathcal{O}}$ with the usual affine Hecke algebra $\widehat{\mathcal{H}}$.

Now consider a diagram where the first strand starts at $(s, 0)$, goes linearly to $(-s, 1/2)$ then back to $(s, 1)$, while all others remain straight. This diagram is unsteadied, since the horizontal slice at $y = 1/2$ is unsteadied by the leftmost strand. By the relation (18c), this diagram is equal to $\prod_{i=1}^{\ell} (X_1 - Q_i)$ which thus lies in the kernel of the map of the affine Hecke algebra to $\mathcal{WF}_{\mathcal{V}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$.

As in the proof of 5.7, we can easily check that the diagram discussed above generates the kernel so $\mathcal{WF}_{\mathcal{V}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet})$ is isomorphic to this cyclotomic quotient. \square

There is also a version of this theorem relating the type WF Hecke algebras to cyclotomic Schur algebras. Assume that the parameters ϑ_i are ordered with $\vartheta_1 < \dots < \vartheta_{\ell}$. Fix a set Λ of ℓ -multicompositions of n which is an upper order ideal in dominance order. We'll be interested in the cyclotomic q -Schur algebra $\mathcal{S}(\Lambda)$ of rank n attached to the data $(\mathbf{q}, \mathbf{Q}_{\bullet})$ defined by Dipper, James and Mathas [4, 6.1]; let $\mathcal{S}^{-}(\Lambda)$ be the signed version of this algebra defined using signed permutation modules.

Let r be the maximum number of parts of one of the components of $\xi \in \Lambda$. Choose constants $\epsilon \ll g$ and s so that

$$|g| + m\epsilon < s < \min_{k \neq n} (|\vartheta_k - \vartheta_n|/r);$$

of course, this is only possible is $r|g| < |\vartheta_k - \vartheta_n|$ for all $k \neq n$. In this case, we associate to every multicomposition $\xi \in \Lambda$ a subset E_{ξ} that consists of the points $\vartheta_p + i\epsilon + js$ for every $1 \leq j \leq \xi_i^{(p)}$.

In order to simplify the proof below, we'll use some results from [24], in particular, a dimension calculation based on the cellular basis constructed in [24, Thm. 2.26]. Since [24] cites some of the results of this paper, the reader might naturally worry that the author has created a loop of citations and thus utilized circular reasoning. However, we only use these in the proof of Proposition 6.6, which is not used in [24].

As in Section 4, there is an idempotent diagram e'_{ξ} on this subset where we act on the strands with x -value in $[\vartheta_p + js, \vartheta_p + js + \epsilon\mu_j^{(p)}]$ the idempotent $y_1^{\mu_1 - 1} \dots y_{\mu_j - 1} \partial_{w_0}$. Let $e_{\Lambda} = \sum_{\xi \in \Lambda} e'_{\xi}$. Let \mathcal{D} be any collection of n -element subsets containing E_{ξ} for all $\xi \in \Lambda$.

PROPOSITION 6.6. *We have an isomorphism $\mathcal{S}(\Lambda) \cong e_\Lambda \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_\Lambda$ if $g < 0$, and $\mathcal{S}^-(\Lambda) \cong e_\Lambda \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_\Lambda$ if $g > 0$. If Λ contains all ℓ -multipartitions of n , this subalgebra is Morita equivalent to $\mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet)$ via the obvious bimodule.*

Proof. For $t \gg 0$ sufficiently large, we have that $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_{D_{t,m}}$ is the cyclotomic Hecke algebra $\mathcal{H}(\mathbf{q}, \mathbf{Q}_\bullet)$ by Theorem 6.5. Thus, we have that $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_\Lambda$ is a bimodule over $\mathcal{H}(\mathbf{q}, \mathbf{Q}_\bullet)$ and the algebra $e_\Lambda \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_\Lambda$.

Let q_ξ be the diagram that linearly interpolates between $D_{t,m}$ and E_ξ , times e'_ξ on the right. We'll concentrate on the case where $\kappa < 0$. The same argument as the proof of Theorem 4.9 shows that $(T_i - q)q_\xi = 0$ if the i th and $i + 1$ st strands lie in one of the segments $[\vartheta_p + js, \vartheta_p + js + \epsilon\mu_j^{(p)}]$ in E_ξ . If $\kappa > 0$, we instead see that $(T_i + 1)q_\xi = 0$. Note that q_ξ generates $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_\Lambda$ as a left module.

If $\xi^{(p)} = \emptyset$ for $p < \ell$, then this shows that sending $m_\xi \mapsto q_\xi$ induces a map of P_ξ to $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e'_\xi$, which is surjective since q_ξ generates.

For an arbitrary ξ , let ξ° be the multicomposition where $(\xi^\circ)^{(p)} = \emptyset$ for $p < \ell$, and $(\xi^\circ)^{(\ell)}$ is the concatenation of $\xi^{(p)}$ for all p . We have a natural map $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e'_\xi \rightarrow e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e'_{\xi^\circ}$ given by the straight-line diagram interpolating between ξ and ξ° . Applying relation (18c) many times, we find that this map sends

$$q_\xi \mapsto \prod_{j \leq |\xi^{(1)}| + \dots + |\xi^{(k-1)}|} (L_j - Q_k) q_{\xi^\circ}.$$

The submodule of P_{ξ° generated by this element is a copy of P_ξ , thus we have a surjective map $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e'_\xi \rightarrow P_\xi$.

Dimension considerations show that this map is an isomorphism. The dimension of $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e'_\xi$ is $1/\xi!$ times the dimension of $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_{E_\xi}$, since e_{E_ξ} is the sum of $\xi!$ orthogonal idempotents isomorphic to e'_ξ . Thus, by [24, Thm. 2.26], it is equal to $1/\xi!$ times the number of pairs of tableaux of the same shape, one standard and of type E_ξ . The entries of an E_ξ -tableau are of the form $\vartheta_p + i\epsilon + js$ for (i, j, p) a box of the diagram of ξ . A filling will be a E_ξ if and only if the replacement $\vartheta_p + i\epsilon + js \mapsto j_p$ is a semi-standard tableau⁽⁴⁾ increasing weakly along columns and strongly along rows if $\kappa > 0$ and *vice versa* if $\kappa < 0$. In fact, this gives a $\xi! := \prod \xi_k^{(p)}!$ -to-1 map from E_ξ -tableau to semi-standard tableau of type ξ .

Thus, the dimension of $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e'_\xi$ is the number of pairs of tableaux of the same shape, one standard and one semi-standard of type ξ . This is the same as the dimension of the permutation module associated to ξ , so the surjective map $e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e'_\xi \rightarrow P_\xi$ must be an isomorphism.

We have from [24, Lem. 3.3] that the map

$$(22) \quad e_\Lambda \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_\Lambda \rightarrow \text{End}_{\mathcal{H}(\mathbf{q}, \mathbf{Q}_\bullet)}(e_{D_{t,m}} \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_\Lambda)$$

is injective. Applying [24, Thm. 2.26] again, the dimension of $e_\Lambda \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_\Lambda$ is equal to the number of pairs of semi-standard tableaux of the same shape and (possibly different) type in Λ . Thus, the dimension coincides with $\dim \mathcal{S}(\Lambda)$. This shows that the injective map (22) must be an isomorphism.

Finally, we wish to show that the bimodules $e_\Lambda \mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet)$ and $\mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet) e_\Lambda$ induce a Morita equivalence. For this, it suffices to show that no simple $\mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet)$ -module is killed by e_Λ . If this were the case, $\mathcal{WF}_{\mathcal{D}}^\vartheta(\mathbf{q}, \mathbf{Q}_\bullet)$ would have strictly more simple modules than the cyclotomic q -Schur algebra. However, in [24, Thm. 2.26], we

⁽⁴⁾This tableau uses ℓ alphabets (denoted using subscripts) with the order $1_1 < 2_1 < 3_1 \dots < 1_2 < 2_2 < 3_2 \dots < 1_3 < \dots$.

show that this algebra is cellular with the number of cells equal to the number of ℓ -multipartitions of n . By [4, 6.16], this is the number of simples over $\mathcal{S}(\Lambda)$ as well. \square

This also allows us to show:

THEOREM 6.7. *The idempotent e' induces a Morita equivalence between the affine Schur algebra $\mathcal{S}_h(n, m)$ and the type W affine Hecke algebra $\mathcal{W}(\mathfrak{q})_{\mathcal{E}}(\mathfrak{q})$.*

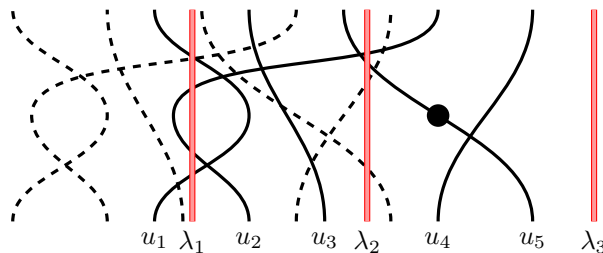
Proof. Since the algebra $\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})$ is Noetherian, if $\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})e'\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q}) \neq \mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})$, then there is at least one simple module L over $\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})/\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})e'\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})$, which must be killed by e' .

This simple module must be finite dimensional since $\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})$ is of finite rank over the center of this module. Thus, X_1 acting on this simple module satisfies some polynomial equation $p(X_1) = 0$, and L factors through the map to a type WF Hecke algebra \mathcal{WF}^ϑ where we choose $\vartheta_i \ll \vartheta_{i+1}$ for all i , and $\vartheta_\ell \ll 0$, with Q_i being the roots of p with multiplicity.

By Proposition 6.6, the identity of \mathcal{WF}^ϑ can be written as a sum of cellular basis vectors factoring through the idempotent e'_ξ at $y = 1/2$. We have some choice in the definition of these vectors, and we can assure that all crossings in them occur to the right of all red line. The relation (18c) allows us to pull all strands to the right. Once all the strands are to the right of all red lines, this slice at $y = 1/2$ will be the idempotent e'_{ξ° , times a polynomial in the dots. Since this idempotent e'_{ξ° lies in $e'\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})e'$, we must have that e' acts non-trivially on L , contradicting our assumption. This shows that $\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})e'\mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q}) = \mathcal{W}(\mathfrak{q})_{\mathcal{B}}(\mathfrak{q})$, proving the Morita equivalence. \square

6.2. WEIGHTED KLR ALGEBRAS. There's also a KLR algebra in type WF. This is also a weighted KLR algebra as defined in [22], but now for the Dynkin diagram U with a Crawley-Boevey vertex added, as discussed in [22, § 3.1].

DEFINITION 6.8. *A rank n WF KLR diagram is a w KLR diagram (as defined in Definition 4.10, with labels in U) with vertical red lines inserted at $x = \vartheta_i$. The diagram must avoid tangencies and triple points between any combination of these strands, green strands and ghosts, and only allow isotopies that preserve these conditions. Here is an example of such a diagram:*



The rank n type WF KLR algebra $\tilde{\mathbb{T}}^\vartheta(h, \mathbf{z})$ is the algebra generated by these diagrams over $\mathbb{k}[h, \mathbf{z}]$ modulo the local relations (14a–14h, 19a–19d) and

$$(23) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} = \dots$$

This is a reduced weighted KLR algebra for the Crawley-Boevey graph of U for the highest weight λ .

The steadied quotient of $\mathbb{T}^\vartheta(h, \mathbf{z})$ is the quotient of $\tilde{\mathbb{T}}^\vartheta(h, \mathbf{z})$ by the 2-sided ideal generated by all unsteady idempotents.

As with the other algebras we've introduced, the algebra $\tilde{\mathbb{T}}^\vartheta(h, \mathbf{z})$ has a natural polynomial representation P^ϑ , defined in [22, Prop. 2.7]. It also has a grading, with the degrees of diagrams given by

$$\begin{aligned} \deg \begin{array}{c} \diagup \\ u \quad v \end{array} &= -2\delta_{u,v} & \deg \begin{array}{c} \bullet \\ | \\ u \end{array} &= 2 & \deg \begin{array}{c} \diagup \\ \cdot \\ u \quad v \end{array} &= \deg \begin{array}{c} \diagdown \\ \cdot \\ u \quad v \end{array} &= \delta_{u,q\pm 1v} \\ \\ \deg \begin{array}{c} \diagup \\ \color{red}{\times} \\ \lambda \quad u \end{array} &= \deg \begin{array}{c} \diagdown \\ \color{red}{\times} \\ u \quad \lambda \end{array} &= \alpha_u^\vee(\lambda), \end{aligned}$$

6.3. ISOMORPHISMS. As in all previous sections, we can show that both of the algebras we have introduced have polynomial style representations (graded in the case of $\tilde{\mathbb{T}}^\vartheta(h, \mathbf{z})$) where

$$\begin{aligned} \mathbb{K} &= \mathbb{k}[h] & A &= \widehat{\mathcal{WF}}^\vartheta(\mathfrak{q}, \mathbf{Q}_\bullet) & B &= \mathbb{k}[h][X^\pm] & I &= Bh + B \prod_{u \in U} (X - u) & P &= \mathcal{P}^\vartheta \\ \mathbb{K} &= \mathbb{k}[h] & A &= \tilde{\mathbb{T}}^\vartheta(h, \mathbf{z}) & B &= \mathbb{k}[h, y] & I &= Bh + By & P &= P^\vartheta. \end{aligned}$$

and thus have suitable completions $\widehat{\mathcal{WF}}^\vartheta(\mathfrak{q}, \mathbf{Q}_\bullet)$ and $\widehat{\mathbb{T}}^\vartheta(h, \mathbf{z})$.

Now, let \mathbf{u} be a loading on a set $D \in \mathcal{D}$, that is, a map $D \rightarrow U$. Let u_1, \dots, u_n be the values of \mathbf{u} read from left to right. Attached to such data, we have an idempotent $e_{\mathbf{u}}$ in $\tilde{\mathbb{T}}^\vartheta(h, \mathbf{z})$ and another $\epsilon_{\mathbf{u}}$ in $\widehat{\mathcal{WF}}^\vartheta(\mathfrak{q}, \mathbf{Q}_\bullet)$ given by projection to the stable kernel of $X_r - u_r$ for all r .

THEOREM 6.9. We have isomorphisms of $\mathbb{k}[h, \mathbf{z}]$ -algebras

$$\mathcal{WF}_\mathcal{D}^\vartheta(\mathfrak{q}, \mathbf{Q}_\bullet) \cong \mathbb{T}_\mathcal{D}^\vartheta(h, \mathbf{z}) \quad \widehat{\mathcal{WF}}^\vartheta(\mathfrak{q}, \mathbf{Q}_\bullet) \cong \widehat{\mathbb{T}}_\mathcal{D}^\vartheta(h, \mathbf{z})$$

which send

$$\begin{aligned} \epsilon_{\mathbf{u}} &\mapsto e_{\mathbf{u}} & X_r &\mapsto \sum_{\mathbf{u}} u_r b(y_r) e_{\mathbf{u}} & \color{red}{\times} &\mapsto \color{red}{\times} \\ \color{red}{\times} \epsilon_{\mathbf{u}} &\mapsto \frac{\prod_{\vartheta_s = \theta_k} (u_r b(y_r - z_s) - Q_s)}{\prod_{s \in S_{u_r, j}} (y_r - z_s)} \color{red}{\times} e_{\mathbf{u}} \end{aligned}$$

$$(23a) \quad \color{green}{\times} \epsilon_{\mathbf{u}} \mapsto \begin{cases} \frac{1}{u_{r+1} b(y_{r+1}) - u_r b(y_r)} (\psi_r - 1) e_{\mathbf{u}} & u_r \neq u_{r+1} \\ \frac{y_{r+1} - y_r}{u_{r+1} (b(y_{r+1}) - b(y_r))} \psi_r e_{\mathbf{u}} & u_r = u_{r+1} \end{cases}$$

$$(23b) \quad \color{green}{\times} \epsilon_{\mathbf{u}} \mapsto \begin{cases} \frac{u_r b(y_r) - q u_s b(y_s)}{u_r b(y_r) - q u_s b(y_s)} \color{red}{\times} e_{\mathbf{u}} & u_r \neq q u_s \\ \frac{y_s - y_r}{u_r b(y_r) - q u_s b(y_s)} \color{red}{\times} e_{\mathbf{u}} & u_r = q u_s, d(h) = 1 \\ \frac{y_s - y_r}{y_s - y_r + h} \color{red}{\times} e_{\mathbf{u}} & u_r = q u_s, d(h) = e^h \end{cases} \quad \color{green}{\times} \mapsto \color{red}{\times}$$

where the solid strand shown is the r th (and $r + 1$ st in the first line), and the ghost is associated to the s th from the left.

Proof. That this map sends unsteady idempotents to unsteady idempotents is clear, so we need only show that we have an isomorphism $\widehat{\mathcal{W}\mathcal{F}}_{\mathcal{D}}^{\vartheta}(\mathbf{q}, \mathbf{Q}_{\bullet}) \cong \widehat{\mathbb{T}}_{\mathcal{D}}^{\vartheta}(h, \mathbf{z})$. As in the proofs of Theorems 3.10, 4.14, and 5.9, we check this by comparing polynomial representations. The comparison for diagrams involving no red strands is covered by the isomorphism of Theorem 4.14 and for crossings with red strands is checked in Theorem 5.9. \square

Just as in Section 4, this isomorphism does not immediately grade the cyclotomic q -Schur algebra, since the idempotent from Theorem 6.6 does not have homogeneous image. One can, however, define a homogenous idempotent e'' with isomorphic image. As before, e'' will be a sum over ℓ -ordered lists of multi-subsets of U whose size gives a multi-composition in Λ . Each of these contributes the idempotent where the points connected to the part $\mu_i^{(s)}$ are labeled with the multi-subset, in increasing order, with a primitive idempotent in the nilHecke algebra acting on the groups with the same label.

Note that in the level one case, a graded version of the q -Schur algebra was defined by Ariki [1]. This grading was uniquely determined by its compatibility with the Brundan–Kleshchev grading on the Hecke algebra, so our algebra must match up to graded Morita equivalence with that of [1, 3.17] (just as we saw with the closely related quiver Schur algebra in [19, Thm. 7.9]).

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GLOSSARY

$R(h)$	The Khovanov–Lauda–Rouquier algebra for the Dynkin diagram U with the h -deformed relations.	3–5, 12, 14
$\widetilde{\mathcal{H}}(\mathbf{q})$	The affine Hecke algebra of S_n with parameter $\mathbf{q} = qe^h$.	3–5, 7–12, 19–22, 28
$\mathcal{S}(\mathbf{q}, n, m)$	The affine q -Schur algebra as defined in Definition 4.7.	3, 20–22, 34
$T^{\lambda}(h, \mathbf{z})$	The Stendhal algebra defined in Definition 5.8	3, 29, 30
$\mathbb{T}^{\vartheta}(h, \mathbf{z})$	The steadied quotient of the WF KLR algebra defined in Definition 6.8.	3, 34, 35
$\widehat{\mathcal{H}}(\mathbf{q})$	The completion of affine Hecke algebra $\widetilde{\mathcal{H}}(\mathbf{q})$ with respect to the topology induced by U .	3, 5, 11, 12, 15, 16, 32
$\widehat{R}(h)$	The Khovanov–Lauda–Rouquier algebra completed with respect to its grading.	3, 5, 14, 16
U	A fixed finite subset of $\mathbb{k} \setminus \{0\}$, which we endow with a quiver structure by connecting $u \rightarrow qu$ whenever $u, qu \in U$.	4, 5, 10–12, 14, 19, 22, 24–29, 34–36
$\mathcal{H}(\mathbf{q}, \mathbf{Q}_{\bullet})$	The cyclotomic quotient of $\widetilde{\mathcal{H}}(\mathbf{q})$ with Definition 1.4.	5, 28, 32, 33
$R^{\mathbf{Q}_{\bullet}}(h, \mathbf{z})$	The cyclotomic quotient of R with Definition 1.5.	5
$\mathcal{H}(\mathbf{q})$	The affine Hecke algebra of S_n with parameter $\mathbf{q} = qe^h$.	8, 10, 11
$\mathcal{W}(\mathbf{q})$	The weighted affine Hecke algebra defined in Definition 4.2.	17–22, 24, 25, 32, 34

$\widehat{W}(\mathfrak{q})$	The completion of the weighted affine Hecke algebra defined in Definition 4.2.	19
\mathcal{E}	The collection of sets C_μ attached to m -part composition of n .	21, 22, 34
e'	The idempotent e' in $\mathcal{W}(\mathfrak{q})(\mathfrak{q})$ which corresponds to all m -part compositions of n .	21, 22, 25, 34
W	The weighted KLR algebra defined in Definition 4.10.	22, 24
\widehat{W}	The completed weighted KLR algebra defined in Definition 4.13.	24, 25
$\tilde{\mathcal{F}}^\theta(\mathfrak{q}, \mathbf{Q}_\bullet)$	The type F affine Hecke algebra defined in Definition 5.5.	27–30
$\mathcal{F}^\theta(\mathfrak{q}, \mathbf{Q}_\bullet)$	The type F Hecke algebra defined in Definition 5.6.	27, 28, 30
$\widehat{\mathcal{F}}^\theta(\mathfrak{q}, \mathbf{Q}_\bullet)$	The completed type F affine Hecke algebra.	28, 30
$\widehat{T}^\lambda(h, \mathbf{z})$	The affine Stendhal algebra defined in Definition 5.8.	28–30
$\widehat{\widetilde{T}}^\lambda(h, \mathbf{z})$	The completion of the affine Stendhal algebra.	29, 30
$\widehat{\mathcal{WF}}(\mathfrak{q}, \mathbf{Q}_\bullet)$	The type WF affine Hecke algebra defined in Definition 6.1.	31, 32, 35, 36
$\mathcal{WF}(\mathfrak{q}, \mathbf{Q}_\bullet)$	The type WF Hecke algebra defined in Definition 6.3.	31–35
$\mathcal{S}^\pm(\Lambda)$	The cyclotomic q -Schur algebra $\mathcal{S}(\Lambda)$ of rank n attached to the data $(\mathfrak{q}, \mathbf{Q}^\bullet)$ defined by Dipper, James and Mathas [4, 6.1].	32–34
$\widehat{\mathbb{T}}^\theta(h, \mathbf{z})$	The WF KLR algebra (reduced weighted KLR algebra) defined in Definition 6.8.	34–36

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