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The slack realization space of a matroid

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ABSTRACT We introduce a new model for the realization space of a matroid, which is obtained from a variety defined by a saturated determinantal ideal, called the slack ideal, coming from the vertex-hyperplane incidence matrix of the matroid. This is inspired by a similar model for the slack realization space of a polytope. We show how to use these ideas to certify non-realizability of matroids, and describe an explicit relationship to the standard Grassmann–Plücker realization space model. We also exhibit a way of detecting projectively unique matroids via their slack ideals by introducing a toric ideal that can be associated to any matroid.

1. INTRODUCTION

Realization spaces of matroids are well studied objects [1, 2, 10] which encode not only whether or not the matroid is realizable, but also carry additional information about the structure of the matroid. A realization (or representation) of a rank $d + 1$ matroid M is a set of vectors in \mathbb{k}^{d+1} which captures its independence structure. Roughly speaking, a realization space is the set of all such choices of vectors. Fundamental questions in the study of realization spaces of matroids include discovering whether or not a given matroid is realizable, determining over which field it is realizable, finding the structure of the set of realizations, and characterizing when realizations exist. A celebrated theorem of Mnëv states that every semialgebraic set defined over the integers is stably equivalent to the realization space of some oriented matroid. That is, realization spaces of matroids can become arbitrarily complicated. In light of this, we aim to connect the combinatorics of the matroid to properties of its realization space.

Our realization space model will be based off the *slack matrix* of a matroid. This is a generalization of the slack matrix of a polytope [15], which has been used extensively in the study of extended formulations of polytopes; see for example [7, 13, 15]. In [8] the slack matrix of a polytope was used to construct a realization space for the polytope via its *slack ideal*. This realization space model and its properties were recently explored in detail in [6]. We extend the results in [6] to matroids both defining the slack realization space of a matroid and examining its properties.

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In Section 2 we introduce the main objects of study, as well as preliminary results and notation. In Section 3 we discuss two models for the realization space of a matroid. One of our main theorems, Theorem 3.8, shows how the two realization space models can be described via a single overarching variety. In Section 4 we show how the slack realization model can be used to determine whether a matroid has a realization over a certain field. We also reframe the tools of final polynomials [2] in terms of slack ideals, and show how they can be used to improve computational efficiency of realizability checking. In Section 5 we introduce a toric ideal associated to a matroid and study its relationship to the projective uniqueness of the matroid. In Appendix A we include a table of notation used throughout the paper. The computations in this paper are done in Macaulay2 [9] with the help of the `Matroids` package [3]; code for the computations can be found at <http://sites.math.washington.edu/~awiebe>.

2. THE SLACK MATRIX OF A MATROID

Much of this section is analogous to [6, § 2] to which we refer the reader for further details and excluded proofs. We assume the reader has familiarity with the basic definitions from matroid theory, see [12] or [5]. Throughout the paper, we assume all matroids are simple (having no loops or parallel elements).

Let \mathbb{k} be a field, and $\mathbb{k}^* := \mathbb{k} \setminus \{0\}$. Let $M = (E, \mathcal{B})$ be a matroid of rank $d + 1$ with ground set $E = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, where each $\mathbf{v}_i \in \mathbb{k}^{d+1}$ and \mathcal{B} is its set of bases. If V is the matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$, then the independent sets of M are the linearly independent subsets of columns, and we write $M = M[V]$.

Let $\mathcal{H}(M)$ denote the set of hyperplanes of M , which are the closed subsets (flats) of rank d . In $M[V]$, each hyperplane $H \in \mathcal{H}(M)$ corresponds to a linear subspace of \mathbb{k}^{d+1} , so is determined by some linear equation; that is, $H = \{\mathbf{x} \in E : \alpha_H^\top \mathbf{x} = 0\}$. For $\mathcal{H}(M) = \{H_1, \dots, H_h\}$, let W be the matrix whose columns are the hyperplane defining normals $\alpha_1, \dots, \alpha_h$, or some multiple, $\lambda_j \alpha_j$ for $\lambda_j \in \mathbb{k}^*$, thereof.

DEFINITION 2.1. *A slack matrix of the matroid $M = M[V]$ over \mathbb{k} is an $n \times h$ matrix $S_{M[V]} = V^\top W$ for matrices V and W as above.*

We wish to parametrize the set of realizations of a matroid by its slack matrices. So, we must determine the characteristics which define the set of all possible slack matrices of a given matroid. We begin by considering the rank of a slack matrix. Given a matrix S , its *support* consists of the entries which are non-zero. Two matrices have the same support if they have the same zero pattern.

LEMMA 2.2 (See [6, Lemma 3.1]). *If S is a matrix having the same support as a slack matrix of some rank $d + 1$ matroid $M = M[V]$, then $\text{rank}(S) \geq d + 1$.*

COROLLARY 2.3. *If $M = M[V]$ is a rank $d + 1$ matroid then $\text{rank}(S_M) = d + 1$.*

Proof. The factored form of $S_M \in \mathbb{k}^{n \times (d+1)} \times \mathbb{k}^{(d+1) \times h}$ implies that $\text{rank}(S_M) \leq d + 1$. The result then follows from Lemma 2.2. \square

Now, let $M = (E, \mathcal{B})$ be an abstract matroid of rank $d + 1$. Unless otherwise stated, we take $E = [n] = \{1, \dots, n\}$. A *realization* of M over \mathbb{k} is a collection of vectors $V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathbb{k}^\ell$ such that the independent subsets of V are indexed by the independent sets of the matroid, so $M = M[V]$. A matroid with a realization is called *realizable* (also *representable*, *linear* or *coordinatizable*).

LEMMA 2.4. *The rows of a slack matrix S_M form a realization of the matroid M .*

Proof. It suffices to show that if we label the rows of S_M with $[n]$, the subsets indexing linearly independent rows of S_M are the independent sets of M . Since $S_M = V^\top W$,

if a subset J of E is dependent, then there exists a vector $\beta \in \mathbb{k}^n$ with support indexed by J such that $V\beta = 0$. But now, $\beta^\top S_M = (V\beta)^\top W = 0$, so J also indexes a dependent subset of the rows of S_M .

Conversely, suppose J indexes a dependent subset of the rows of S_M . Then for some $\beta \in \mathbb{k}^n$ with support indexed by J , we have $0 = \beta^\top S_M = (V\beta)^\top W$. Since W has full rank by Corollary 2.3, it must be the case that $V\beta = 0$, so that J also indexes a dependent set of M . \square

From now on we assume that realizations come with a fixed labelling of ground set elements and hyperplanes. Then, two slack matrices of the same matroid (which a priori may have had different labellings) now cannot differ by permutations of rows and columns. This also allows us to identify hyperplanes of a realization by their normal vectors or the indices of those vectors. Now, we characterize the set of matrices which correspond to slack matrices of a matroid M .

THEOREM 2.5. *Let M be a rank $d + 1$ matroid with n elements and hyperplanes $\mathcal{H}(M) = \{H_1, \dots, H_h\}$. A matrix $S \in \mathbb{k}^{n \times h}$ is a slack matrix of some realization of M if and only if both of the following hold:*

- (i) $\text{supp}(S) = \text{supp}(S_M)$
- (ii) $\text{rank}(S) = d + 1$.

Proof. Suppose S is a slack matrix of some realization of M . Then (i) holds trivially, and (ii) holds by Corollary 2.3.

Conversely, suppose S is a matrix satisfying (i) and (ii). By (ii), S has some rank factorization $S = AB$, where $A \in \mathbb{k}^{n \times (d+1)}$ and $B \in \mathbb{k}^{(d+1) \times h}$. Let $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{k}^{d+1}$ be the rows of A and $\mathbf{b}_1, \dots, \mathbf{b}_h \in \mathbb{k}^{d+1}$ be the columns of B . Then we claim that the rows of A give a realization of M ; that is $M = M[A^\top]$. To see this, we show that the hyperplanes of M are also hyperplanes of $M[A^\top]$, and that $M[A^\top]$ can not contain more hyperplanes.

By (i), for each hyperplane H_j of M , there is a column of S with zeros in positions indexed by elements of H_j . Since $S = AB$, we have $\mathbf{b}_j^\top \mathbf{a}_i = 0$ if and only if $i \in H_j$. Thus

$$\{\mathbf{x} \in \mathbb{k}^{d+1} : \mathbf{b}_j^\top \mathbf{x} = 0\} \cap \{\text{rows}(A)\} = \{\mathbf{a}_i\}_{i \in H_j}$$

so that H_j is also a hyperplane of the matroid $M[A^\top]$.

Now suppose $M[A^\top]$ has an extra hyperplane $H \notin \mathcal{H}(M)$. Let $\{i_1, \dots, i_d\} \subset H$ be any d distinct independent elements of H . Since i_1, \dots, i_d are also elements of matroid M , the flat $\overline{\{i_1, \dots, i_d\}}$ is a hyperplane H' of M , and thus also a hyperplane of $M[A^\top]$, but $H' \neq H$. However, this means that $\{i_1, \dots, i_d\}$ are contained in two distinct hyperplanes of $M[A^\top]$, which is not possible, so we arrive at a contradiction. \square

Let $\text{GL}(d, \mathbb{k})$ denote the general linear group of degree d over \mathbb{k} , that is, the set of $d \times d$ invertible matrices over \mathbb{k} . We now recall two equivalence relations on the set of realizations of a matroid M , and illustrate how these equivalences are reflected in slack matrices. For $A \in \text{GL}(d + 1, \mathbb{k})$, it is easy to check that V and AV define the same matroid. We call these realizations *linearly equivalent*. If $P \in \mathbb{k}^{n \times n}$ is a permutation matrix which sends $i \mapsto \sigma(i)$, then V and VP define the same matroid up to relabelling the ground set $E = [n]$ with $\sigma(1), \dots, \sigma(n)$. Thus if $A \in \text{GL}(d + 1, \mathbb{k})$ and B is a permutation matrix with any element of \mathbb{k}^* in the 1's positions, then V, AVB define the same matroid. We call the realizations V, AVB *projectively equivalent*. Call a matroid M *projectively unique* (over \mathbb{k}) if all realizations are projectively equivalent.

LEMMA 2.6. *Two realizations of a matroid M are projectively equivalent if and only if their slack matrices are the same up to row and column scaling.*

Proof. Suppose we have projectively equivalent representations U, V of M . Then $U = AVB$, where $A \in \text{GL}(d + 1, \mathbb{k})$ and without loss of generality B is an invertible $n \times n$ diagonal matrix (since we have assumed a fixed labelling of our matroid).

If $H = \{\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_k}\}$ is a hyperplane of $M[V]$, then $H' = \{\mathbf{u}_{i_1}, \dots, \mathbf{u}_{i_k}\}$ is a hyperplane of $M[U]$. Furthermore, if $\alpha_H \in \mathbb{k}^{d+1}$ is normal to H , then since $\mathbf{u}_i = A\mathbf{v}_i \cdot b_i$, $A^{-\top}\alpha_H$ is normal to H' , so that a slack matrix of $M[U]$ is

$$S_{M[U]} = U^\top [A^{-\top}\alpha_H]_{H \in \mathcal{H}} = B^\top V^\top A^\top [A^{-\top}\alpha_H]_{H \in \mathcal{H}} = B^\top V^\top W = B^\top S_{M[V]}.$$

Since we can always scale columns of a slack matrix, this completes the proof.

Conversely, suppose we have realizations U and V of the matroid M such that $S_{M[U]} = D_n S_{M[V]} D_h$ for invertible diagonal matrices $D_n \in \mathbb{k}^{n \times n}, D_h \in \mathbb{k}^{h \times h}$. By definition, $S_{M[U]} = U^\top W$ and $S_{M[V]} = V^\top W'$. Multiplying both sides of the above equation on the right by the right inverse Y of matrix W , we find

$$U^\top = D_n V^\top W' D_h Y.$$

We see that $W' D_h Y$ is a $(d + 1) \times (d + 1)$ invertible matrix, which makes U and V projectively equivalent, as desired. \square

LEMMA 2.7. *Two realizations of a matroid M are linearly equivalent if and only if their slack matrices are the same up to column scaling.*

Proof. By taking B, D_n each to be the $n \times n$ identity matrix in the proof of Lemma 2.6, we obtain the desired result. \square

We now define an analog of the slack matrix which can be constructed for any abstract matroid, even ones which are not realizable, as follows.

DEFINITION 2.8. *Define the symbolic slack matrix of matroid M to be the matrix $S_M(\mathbf{x})$ with rows indexed by elements $i \in E$, columns indexed by hyperplanes $H_j \in \mathcal{H}(M)$ and (i, j) -entry*

$$S_M(\mathbf{x})_{ij} = \begin{cases} x_{ij} & \text{if } i \notin H_j \\ 0 & \text{if } i \in H_j. \end{cases}$$

The slack ideal of M is the saturation of the ideal generated by the $(d + 2)$ -minors of $S_M(\mathbf{x})$, namely

$$I_M := \langle (d + 2) - \text{minors of } S_M(\mathbf{x}) \rangle : \left(\prod_{i=1}^n \prod_{j: i \notin H_j} x_{ij} \right)^\infty \subset \mathbb{k}[\mathbf{x}].$$

Suppose there are t variables in $S_M(\mathbf{x})$. The slack variety is the variety $\mathcal{V}(I_M) \subset \mathbb{k}^t$. The saturation of I_M by the product of all the variables guarantees that there are no components of $\mathcal{V}(I_M)$ that live entirely in coordinate hyperplanes. If $\mathbf{s} \in \mathbb{k}^t$ is a zero of I_M , then we identify it with the matrix $S_M(\mathbf{s})$.

EXAMPLE 2.9. Consider the rank 3 matroid $M_4 = M[V]$ for V whose columns are $\mathbf{v}_1 = (-2, -2, 1)^\top, \mathbf{v}_2 = (-1, 1, 1)^\top, \mathbf{v}_3 = (0, 4, 1)^\top, \mathbf{v}_4 = (2, -2, 1)^\top, \mathbf{v}_5 = (1, 1, 1)^\top, \mathbf{v}_6 = (0, 0, 1)^\top$. Projecting onto the plane $z = 1$, this can be visualized as the points of intersection of four lines in the plane, as in Figure 1.

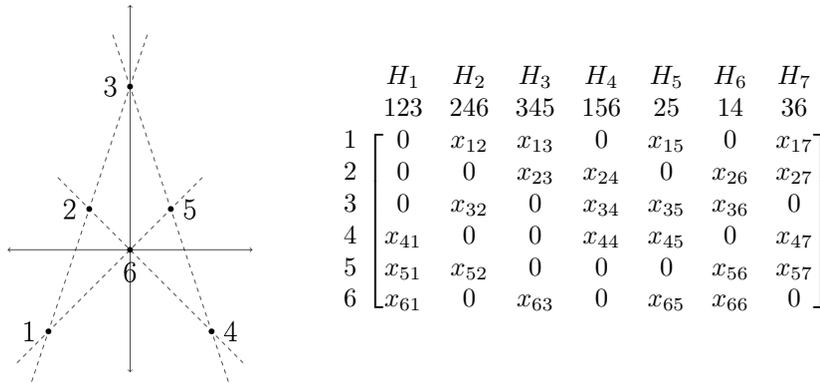


FIGURE 1. The point-line configuration M_4 of Example 2.9, and its symbolic slack matrix $S_{M_4}(\mathbf{x})$.

A slack matrix for this realization is then

$$S_{M_4} = \begin{bmatrix} -2 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 4 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{cccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ 123 & 246 & 345 & 156 & 25 & 14 & 36 \end{array} \begin{bmatrix} -3 & 3 & 6 & -3 & 0 & 0 & 4 \\ 1 & 3 & 2 & 3 & 2 & 4 & 0 \\ -4 & 0 & -8 & 0 & -2 & 8 & 0 \end{bmatrix}$$

$$= \begin{array}{cccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ 123 & 246 & 345 & 156 & 25 & 14 & 36 \end{array} \begin{bmatrix} 1 & \begin{bmatrix} 0 & -12 & -24 & 0 & -6 & 0 & -8 \end{bmatrix} \\ 2 & \begin{bmatrix} 0 & 0 & -12 & 6 & 0 & 12 & -4 \end{bmatrix} \\ 3 & \begin{bmatrix} 0 & 12 & 0 & 12 & 6 & 24 & 0 \end{bmatrix} \\ 4 & \begin{bmatrix} -12 & 0 & 0 & -12 & -6 & 0 & 8 \end{bmatrix} \\ 5 & \begin{bmatrix} -6 & 6 & 0 & 0 & 0 & 12 & 4 \end{bmatrix} \\ 6 & \begin{bmatrix} -4 & 0 & -8 & 0 & -2 & 8 & 0 \end{bmatrix} \end{bmatrix},$$

where using $\{\mathbf{v}_{j_1}, \dots, \mathbf{v}_{j_d}\} \subset H$ independent, we calculate each α_H as

$$(1) \quad \det \begin{bmatrix} \widehat{e}_1 & | & \cdots & | \\ \vdots & | & \mathbf{v}_{j_1} & \cdots & \mathbf{v}_{j_d} & | \\ \widehat{e}_{d+1} & | & \cdots & | \end{bmatrix}.$$

The symbolic slack matrix of M_4 is in Figure 1. We take the ideal of 4-minors of this matrix, and saturate with respect to the product of all of the variables to get the slack ideal I_{M_4} . This has codimension 12, degree 293 and is generated by the 72 binomial generators in Table 1. In Section 5 we will see these correspond to the 72 cycles in the bipartite non-incidence graph of this configuration (Figure 6).

REMARK 2.10. In [6], given a set of n vertices $V \subset \mathbb{k}^d$ defining a d -polytope $P = \text{conv}(V)$, they include only the facet defining hyperplanes in the slack matrix. We can also form a matroid associated to this polytope by considering all the hyperplanes; that is, we define the matroid $M = M[V']$ where $V' \subset \mathbb{k}^{(d+1) \times n}$ is the matrix obtained from V by appending a 1 to each vector. Then the symbolic slack matrix of P defined in [6] is the restriction of the symbolic slack matrix of matroid M to the subset

$x_{36}x_{65} + x_{35}x_{66}, x_{26}x_{63} - x_{23}x_{66}, x_{15}x_{63} - x_{13}x_{65}, x_{56}x_{61} - x_{51}x_{66}, x_{45}x_{61} - x_{41}x_{65},$ $x_{27}x_{56} + x_{26}x_{57}, x_{36}x_{52} - x_{32}x_{56}, x_{17}x_{52} - x_{12}x_{57}, x_{47}x_{51} - x_{41}x_{57}, x_{17}x_{45} + x_{15}x_{47},$ $x_{35}x_{44} - x_{34}x_{45}, x_{27}x_{44} - x_{24}x_{47}, x_{26}x_{34} - x_{24}x_{36}, x_{15}x_{32} - x_{12}x_{35}, x_{17}x_{23} - x_{13}x_{27}$
$x_{47}x_{56}x_{65} - x_{45}x_{57}x_{66}, x_{17}x_{56}x_{65} + x_{15}x_{57}x_{66}, x_{12}x_{56}x_{65} + x_{15}x_{52}x_{66}, x_{26}x_{47}x_{65} + x_{27}x_{45}x_{66},$ $x_{26}x_{44}x_{65} + x_{24}x_{45}x_{66}, x_{17}x_{26}x_{65} - x_{15}x_{27}x_{66}, x_{17}x_{56}x_{63} + x_{13}x_{57}x_{66}, x_{12}x_{56}x_{63} + x_{13}x_{52}x_{66},$ $x_{27}x_{45}x_{63} + x_{23}x_{47}x_{65}, x_{24}x_{45}x_{63} + x_{23}x_{44}x_{65}, x_{12}x_{36}x_{63} + x_{13}x_{32}x_{66}, x_{24}x_{35}x_{63} + x_{23}x_{34}x_{65},$ $x_{23}x_{57}x_{61} + x_{27}x_{51}x_{63}, x_{15}x_{57}x_{61} + x_{17}x_{51}x_{65}, x_{13}x_{57}x_{61} + x_{17}x_{51}x_{63}, x_{35}x_{52}x_{61} + x_{32}x_{51}x_{65},$ $x_{15}x_{52}x_{61} + x_{12}x_{51}x_{65}, x_{13}x_{52}x_{61} + x_{12}x_{51}x_{63}, x_{26}x_{47}x_{61} + x_{27}x_{41}x_{66}, x_{23}x_{47}x_{61} + x_{27}x_{41}x_{63},$ $x_{13}x_{47}x_{61} + x_{17}x_{41}x_{63}, x_{36}x_{44}x_{61} + x_{34}x_{41}x_{66}, x_{26}x_{44}x_{61} + x_{24}x_{41}x_{66}, x_{23}x_{44}x_{61} + x_{24}x_{41}x_{63},$ $x_{35}x_{47}x_{56} + x_{36}x_{45}x_{57}, x_{34}x_{47}x_{56} + x_{36}x_{44}x_{57}, x_{17}x_{35}x_{56} - x_{15}x_{36}x_{57}, x_{35}x_{47}x_{52} + x_{32}x_{45}x_{57},$ $x_{34}x_{47}x_{52} + x_{32}x_{44}x_{57}, x_{27}x_{34}x_{52} + x_{24}x_{32}x_{57}, x_{13}x_{26}x_{52} + x_{12}x_{23}x_{56}, x_{36}x_{45}x_{51} + x_{35}x_{41}x_{56},$ $x_{32}x_{45}x_{51} + x_{35}x_{41}x_{52}, x_{12}x_{45}x_{51} + x_{15}x_{41}x_{52}, x_{36}x_{44}x_{51} + x_{34}x_{41}x_{56}, x_{32}x_{44}x_{51} + x_{34}x_{41}x_{52},$ $x_{26}x_{44}x_{51} + x_{24}x_{41}x_{56}, x_{27}x_{36}x_{45} - x_{26}x_{35}x_{47}, x_{17}x_{32}x_{44} + x_{12}x_{34}x_{47}, x_{15}x_{23}x_{44} + x_{13}x_{24}x_{45},$ $x_{17}x_{26}x_{35} + x_{15}x_{27}x_{36}, x_{13}x_{26}x_{35} + x_{15}x_{23}x_{36}, x_{15}x_{27}x_{34} + x_{17}x_{24}x_{35}, x_{15}x_{23}x_{34} + x_{13}x_{24}x_{35},$ $x_{17}x_{26}x_{32} + x_{12}x_{27}x_{36}, x_{13}x_{26}x_{32} + x_{12}x_{23}x_{36}, x_{17}x_{24}x_{32} + x_{12}x_{27}x_{34}, x_{13}x_{24}x_{32} + x_{12}x_{23}x_{34}$
$x_{27}x_{35}x_{52}x_{63} - x_{23}x_{32}x_{57}x_{65}, x_{17}x_{36}x_{44}x_{63} - x_{13}x_{34}x_{47}x_{66}, x_{24}x_{35}x_{57}x_{61} - x_{27}x_{34}x_{51}x_{65},$ $x_{23}x_{34}x_{52}x_{61} - x_{24}x_{32}x_{51}x_{63}, x_{12}x_{36}x_{47}x_{61} - x_{17}x_{32}x_{41}x_{66}, x_{13}x_{32}x_{44}x_{61} - x_{12}x_{34}x_{41}x_{63},$ $x_{15}x_{26}x_{44}x_{52} - x_{12}x_{24}x_{45}x_{56}, x_{13}x_{26}x_{45}x_{51} - x_{15}x_{23}x_{41}x_{56}, x_{12}x_{23}x_{44}x_{51} - x_{13}x_{24}x_{41}x_{52}$

TABLE 1. The 72 generators of I_{M_4} .

of columns corresponding to facet-defining hyperplanes. Thus the slack ideal of the polytope is always contained in the slack ideal of the matroid, $I_P \subseteq I_M$. We illustrate with the following example.

EXAMPLE 2.11. We consider the triangular prism P labelled as in Figure 2. As a 3-polytope, its facets are given by the hyperplanes 1234, 1256, 3456, 135, 246 and the symbolic slack matrix is in Figure 2. Its slack ideal I_P is generated by three binomials.

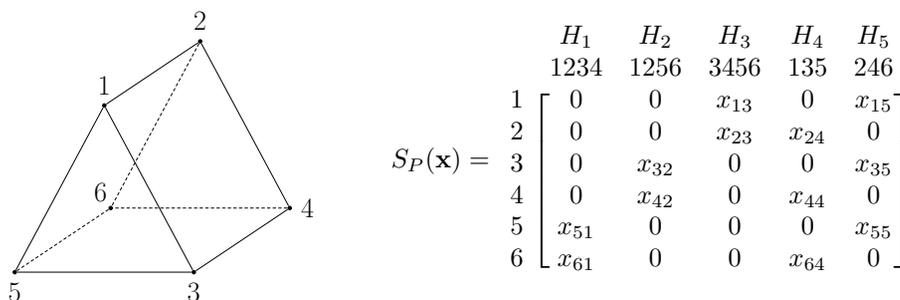


FIGURE 2. The triangular prism P of Example 2.11 and its symbolic slack matrix as a polytope.

Considering P as a rank 4 matroid which has the three facets 1234, 1256, 3456 of P as its non-bases, we obtain following symbolic slack matrix.

$$S_M(\mathbf{x}) = \begin{matrix} & \begin{matrix} H_1 & H_2 & H_3 & H_4 & H_5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1234 & 1256 & 3456 & 135 & 246 \\ 0 & 0 & x_{13} & 0 & x_{15} \\ 0 & 0 & x_{23} & x_{24} & 0 \\ 0 & x_{32} & 0 & 0 & x_{35} \\ 0 & x_{42} & 0 & x_{44} & 0 \\ x_{51} & 0 & 0 & 0 & x_{55} \\ x_{61} & 0 & 0 & x_{64} & 0 \end{bmatrix} \end{matrix} \left| \begin{matrix} H_6 & H_7 & H_8 & H_9 & H_{10} & H_{11} \\ 136 & 146 & 145 & 245 & 235 & 236 \\ 0 & 0 & 0 & x_{19} & x_{1,10} & x_{1,11} \\ x_{26} & x_{27} & x_{28} & 0 & 0 & 0 \\ 0 & x_{37} & x_{38} & x_{39} & 0 & 0 \\ x_{46} & 0 & 0 & 0 & x_{4,10} & x_{4,11} \\ x_{56} & x_{57} & 0 & 0 & 0 & x_{5,11} \\ 0 & 0 & x_{68} & x_{69} & x_{6,10} & 0 \end{matrix} \right.$$

Not only is $I_P \subseteq I_M$ but in this case I_P is the elimination ideal given by eliminating the variables in the columns indexed by the additional hyperplanes H_6, \dots, H_{11} .

3. REALIZATION SPACE MODELS

A realization space for a rank $d + 1$ matroid M with n elements is, roughly speaking, a space whose points are in correspondence with (equivalence classes of) collections of vectors $V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathbb{k}^{d+1}$ whose matroid $M[V]$ is M . In this section we show that the slack variety defined in Section 2 provides such a realization space, and we relate it to another realization space called the Grassmannian of the matroid.

Theorem 2.5 characterizes the slack matrices of realizations of a matroid. The next theorem shows that the slack variety captures exactly these matrices.

THEOREM 3.1. *Let M be a rank $d + 1$ matroid. Then V is a realization of M if and only if $S_{M[V]} = S_M(\mathbf{s})$ where $\mathbf{s} \in \mathcal{V}(I_M) \cap (\mathbb{k}^*)^t$.*

Proof. Let V be a realization of M . Then $S_{M[V]} = S_M(\mathbf{s})$ for some $\mathbf{s} \in (\mathbb{k}^*)^t$ by Theorem 2.5 (i). Furthermore, $\text{rank}(S_{M[V]}) = d + 1$ by Corollary 2.3, so that its $(d + 2)$ -minors vanish and thus $\mathbf{s} \in \mathcal{V}(I_M)$, as desired.

Let $V \in \mathbb{k}^{(d+1) \times n}$ be such that $S_{M[V]} = S_M(\mathbf{s})$ for some $\mathbf{s} \in \mathcal{V}(I_M) \cap (\mathbb{k}^*)^t$. Then $\text{supp}(S_{M[V]}) = \text{supp}(S_M)$ and $\text{rank}(S_{M[V]}) \leq d + 1$. But now by Lemma 2.2, $\text{rank}(S_{M[V]}) \geq d + 1$, making V a realization of M by Theorem 2.5. \square

Since we know that the set of realizations of a matroid is closed under row and column scalings, Theorem 3.1 implies the following corollary. We denote the torus of row and column scalings, $(\mathbb{k}^*)^n \times (\mathbb{k}^*)^h$, by $T_{n,h}$.

COROLLARY 3.2. *The slack variety is closed under the action of the group $T_{n,h}$, where $(\mathbb{k}^*)^n$ acts by row scaling (left multiplication by diagonal matrices) and $(\mathbb{k}^*)^h$ acts by column scaling (right multiplication by diagonal matrices).*

Theorem 3.1 and Corollary 3.2 tell us that the slack variety is a realization space for matroid M and the slack variety modulo the action of $T_{n,h}$ is a realization space for the projective equivalence classes of realizations of M .

DEFINITION 3.3. *The slack realization space of a rank $d + 1$ matroid M on n elements with h hyperplanes is the image of the slack variety inside a product of projective spaces $\mathcal{V}(I_M) \cap (\mathbb{k}^*)^t \hookrightarrow (\mathbb{P}^{n-1})^h$, where \mathbf{s} is sent to the columns of $S_M(\mathbf{s})$.*

PROPOSITION 3.4. *Let M be a rank $d + 1$ matroid on n elements with h hyperplanes. Then the points of its slack realization space are in one-to-one correspondence with linear equivalence classes of realizations of M .*

Proof. Under this embedding, two slack matrices which differ by column scaling are the same point in $(\mathbb{P}^{n-1})^h$. So, the result follows by Lemma 2.7. \square

Next we describe a known model for the realization space of a matroid arising from a subvariety of a Grassmannian. The *Grassmannian* $\text{Gr}(d + 1, n)$ is a variety whose points correspond to $(d + 1)$ -dimensional linear subspaces of a fixed n -dimensional vector space Λ . It embeds into $\mathbb{P}^{\binom{n}{d+1}-1}$ as follows. Any $(d + 1)$ -dimensional linear subspace of Λ can be described as the row space of a $(d + 1) \times n$ matrix of rank $d + 1$. However, two matrices A and B have the same row space when there is a matrix $G \in \text{GL}(d + 1, \mathbb{k})$ such that $A = GB$. Thus, to ensure that we have a one-to-one correspondence between subspaces and points in the Grassmannian, we record a $(d + 1) \times n$ matrix by its vector of $(d + 1)$ -minors. We call this the *Plücker vector*, and it has coordinates indexed by subsets σ of $[n]$ of size $d + 1$. The *Plücker ideal*

$P_{d+1,n} \subseteq \mathbb{k}[\mathbf{p}] = \mathbb{k}[p_\sigma \mid \sigma \subset [n], |\sigma| = d+1]$ is the set of all polynomials which vanish on every vector of $(d+1)$ -minors coming from some $(d+1) \times n$ matrix. It is generated by the homogeneous quadratic Plücker relations and cuts out the Grassmannian as a variety inside $\mathbb{P}^{\binom{n}{d+1}-1}$.

If we have a rank $d+1$ matroid $M = (E, \mathcal{B})$ with realization $V \in \mathbb{k}^{(d+1) \times n}$, the Plücker coordinate p_σ of V is zero if and only if σ is a dependent set of M . Thus, realizations of M correspond to the subvariety of $\text{Gr}(d+1, n)$ defined by setting the Plücker coordinates of non-bases to 0. This variety is also cut out by an ideal, namely, the *Grassmannian ideal* $P_M \subset \mathbb{k}[\mathbf{p}_\mathcal{B}]$ of M ,

$$P_M := (P_{d+1,n} + \langle p_\sigma : \sigma \notin \mathcal{B} \rangle) \cap \mathbb{k}[\mathbf{p}_\mathcal{B}],$$

where $\mathbb{k}[\mathbf{p}_\mathcal{B}] := \mathbb{k}[p_\sigma \mid \sigma \text{ is a basis of } M]$ is the polynomial ring with one variable for each basis $\sigma \in \mathcal{B}$. This is the ideal obtained from $P_{d+1,n} \subset \mathbb{k}[\mathbf{p}]$ by setting the variables indexed by non-bases of M to 0.

DEFINITION 3.5. *The Grassmannian of M , denoted $\text{Gr}(M)$, is $\mathcal{V}(P_M) \cap (\mathbb{k}^*)^{|\mathcal{B}|}$. The points in $\text{Gr}(M)$ correspond to $\text{GL}(d+1, \mathbb{k})$ equivalence classes of $(d+1) \times n$ matrices which realize the matroid M .*

3.1. UNIVERSAL REALIZATION IDEAL. Given a matroid M , we now define an ideal whose variety contains pairs (\mathbf{s}, \mathbf{q}) , where \mathbf{q} is a Plücker vector and \mathbf{s} the nonzero entries of a slack matrix, and both come from the same realization of M .

If V is a realization of a rank $d+1$ matroid $M = (E, \mathcal{B})$, then a slack matrix $S_{M[V]}$ can be filled with the Plücker coordinates of V , which can be seen from (1). Given a hyperplane $H_j \in \mathcal{H}(M)$, we record all possible substitutions of Plücker variables for slack variables using a matrix M_{H_j} whose rows are indexed by $i \in E \setminus H_j$, and whose columns are indexed by subsets $J = \{j_1, \dots, j_d\} \subset E$ with $\overline{J} = H_j$; that is,

$$M_{H_j} = \begin{bmatrix} x_{i_1 j} & \text{sgn}(i_1, J_1) \cdot p_{i_1 \cup J_1} & \cdots & \text{sgn}(i_1, J_k) \cdot p_{i_1 \cup J_k} \\ \vdots & \vdots & \cdots & \vdots \\ x_{i_m j} & \text{sgn}(i_m, J_1) \cdot p_{i_m \cup J_1} & \cdots & \text{sgn}(i_m, J_k) \cdot p_{i_m \cup J_k} \end{bmatrix},$$

where $\text{sgn}(i, J)$ is the sign of the permutation putting (i, j_1, \dots, j_d) in increasing order.

EXAMPLE 3.6. Recall the matroid M_4 of Example 2.9 pictured in Figure 1. Consider the hyperplane $H_2 = 246$. It corresponds to slack variables x_{i2} for $i = 1, 3, 5$ and its independent subsets are 24, 26, and 46. So the matrix M_{246} has the form

$$M_{246} = \begin{bmatrix} x_{12} & p_{124} & p_{126} & p_{146} \\ x_{32} & -p_{234} & -p_{236} & p_{346} \\ x_{52} & p_{245} & -p_{256} & -p_{456} \end{bmatrix}.$$

DEFINITION 3.7. *Let $M = (E, \mathcal{B})$ be a matroid, P_M be the Grassmannian ideal of M and $I_2(M_{H_j})$ be the ideal of 2-minors of the matrix M_{H_j} . The universal realization ideal of M is*

$$U_M := P_M + \sum_{H_j \in \mathcal{H}(M)} I_2(M_{H_j}) \subseteq \mathbb{k}[\mathbf{x}, \mathbf{p}].$$

Intuitively, insisting that the matrices M_{H_j} have rank 1 corresponds to ensuring the columns of the slack matrix are simply scaled versions of the appropriate Plücker coordinates. We now state the main result of this section.

THEOREM 3.8. *Let $M = ([n], \mathcal{B})$ be a rank $d+1$ matroid with universal realization ideal $U_M \subseteq \mathbb{k}[\mathbf{x}, \mathbf{p}]$. Then $\mathcal{V}(U_M) \in \mathbb{k}^t \times \mathbb{k}^{|\mathcal{B}|}$ satisfies*

- (i) the projection of $\mathcal{V}(U_M)$ onto the Plücker coordinates, $\pi_{\mathbf{p}} : \mathcal{V}(U_M) \rightarrow \mathbb{k}^{|\mathcal{B}|}$, is the Grassmannian of the matroid,

$$\overline{\pi_{\mathbf{p}}(\mathcal{V}(U_M))} = \overline{\text{Gr}(M)};$$

- (ii) the projection of $\mathcal{V}(U_M) \cap ((\mathbb{k}^*)^t \times (\mathbb{k}^*)^{|\mathcal{B}|})$ onto the slack coordinates, $\pi_{\mathbf{x}} : \mathcal{V}(U_M) \rightarrow \mathbb{k}^t$, is the set of slack matrices of realizations of M ,

$$\pi_{\mathbf{x}} \left(\mathcal{V}(U_M) \cap \left((\mathbb{k}^*)^t \times (\mathbb{k}^*)^{|\mathcal{B}|} \right) \right) = \mathcal{V}(I_M) \cap (\mathbb{k}^*)^t.$$

The proof of this theorem requires several preliminary lemmas. We first have the following result on Gröbner bases. (For notation and further details see [4].)

LEMMA 3.9. Fix an elimination order on $\mathbb{k}[\mathbf{x}, \mathbf{p}]$. Given two \mathbf{x} -homogeneous polynomials f, g and an \mathbf{x} -homogeneous set $\mathcal{G} \subset \mathbb{k}[\mathbf{x}, \mathbf{p}]$, if $h := \overline{S(f, g)}^{\mathcal{G}}$ with $h \neq 0$, then $\deg_{\mathbf{x}}(h) \geq \max\{\deg_{\mathbf{x}}(f), \deg_{\mathbf{x}}(g)\}$.

LEMMA 3.10. The Grassmannian ideal of a matroid can be obtained by eliminating the slack variables from its universal realization ideal. That is, $P_M = U_M \cap \mathbb{k}[\mathbf{p}]$.

Proof. We obtain one containment by the definition of U_M , since

$$P_M = P_M \cap \mathbb{k}[\mathbf{p}] \subseteq \left(P_M + \sum_{H \in \mathcal{H}(M)} I_2(M_H) \right) \cap \mathbb{k}[\mathbf{p}] = U_M \cap \mathbb{k}[\mathbf{p}].$$

It remains to show the reverse containment. Fix an elimination order on $\mathbb{k}[\mathbf{x}, \mathbf{p}]$ eliminating the \mathbf{x} variables. Let \mathcal{G} be a Gröbner basis for U_M with respect to this ordering. Then, it suffices to show that

$$(2) \quad \mathcal{G} \cap \mathbb{k}[\mathbf{p}] \subset P_M.$$

If we start with a generating set satisfying (2), then by Lemma 3.9, any terms which are added to \mathcal{G} after applying each step of Buchberger’s algorithm with \mathbf{x} -degree 0 must be the reduction of an S -pair of elements which also have \mathbf{x} -degree 0, and are therefore also contained in P_M .

It remains to show that an initial generating set of U_M satisfies (2). Taking the generating set of the definition, it is enough to show that any minor in $I_2(M_H)$ not containing a slack variable is already in P_M . It is not hard to check that any such minor already arises in P_M as some 3-term Plücker relation having a term $p_{\sigma}p_{\tau}$ for some $\sigma \notin \mathcal{B}$ which gets set to zero. \square

Proof of Theorem 3.8.

- (i) This follows from the definition of $\text{Gr}(M)$ and Lemma 3.10.
- (ii, c) Let $(\mathbf{s}, \mathbf{q}) \in \mathcal{V}(U_M) \cap ((\mathbb{k}^*)^t \times (\mathbb{k}^*)^{|\mathcal{B}|})$. From \mathbf{q} we obtain a $(d+1) \times n$ matrix V with Plücker vector whose nonzero coordinates come from \mathbf{q} . We claim that V is a realization of M , so that $S_{M[V]} \in \mathcal{V}(I_M) \cap (\mathbb{k}^*)^t$ by Theorem 3.1. Furthermore, we claim that $S_{M[V]}$ and $S_M(\mathbf{s})$ are the same up to column scaling, so that $\mathbf{s} \in \mathcal{V}(I_M) \cap (\mathbb{k}^*)^t$ by Corollary 3.2.

A subset of $d+1$ columns of V is independent if and only if the corresponding Plücker coordinate is nonzero. Since $\mathbf{q} \in (\mathbb{k}^*)^{|\mathcal{B}|}$, the nonzero Plücker coordinates correspond to bases $B \in \mathcal{B}$ of M . Hence V is a realization of M .

Fix a hyperplane H and an independent subset $J = \{j_1, \dots, j_d\} \subset H$. Then the first column of M_H is $(s_{iH})_{i \notin H}^{\top}$, column J is $(\text{sgn}(i, J) \cdot q_{i \cup J})_{i \notin H}^{\top}$, and by definition of U_M , there exists $\lambda_H \in \mathbb{k}^*$ such that $s_{iH} = \lambda_H \text{sgn}(i, J) \cdot q_{i \cup J}$ for all i . Since \mathbf{q} is the Plücker vector of V , this gives

$$s_{iH} = \lambda_H \text{sgn}(i, J) \cdot q_{i \cup J} = \lambda_H \det(\mathbf{v}_i, \mathbf{v}_{j_1}, \dots, \mathbf{v}_{j_d}), \quad \forall i$$

so that using subset J to calculate α_H , and hence the entries of $S_{M[V]}$, as in Example 2.9, we see that $S_{M[V]}D_h = S_M(\mathbf{s})$, for D_h the diagonal matrix with entries λ_H .

(ii, \supset) Let $\mathbf{s} \in \mathcal{V}(I_M) \cap (\mathbb{k}^*)^t$. By Theorem 3.1, $S_M(\mathbf{s})$ is a slack matrix of some realization V of M . Let \mathbf{q} be the vector of Plücker coordinates of V . We claim $(\mathbf{s}, \mathbf{q}) \in \mathcal{V}(U_M) \cap ((\mathbb{k}^*)^t \times (\mathbb{k}^*)^{|\mathcal{B}|})$. To see this it suffices to show that the columns of M_H , with entries defined using (\mathbf{s}, \mathbf{q}) , are scalar multiples of each other for each $H \in \mathcal{H}(M)$; that is, the $I_2(M_H)$ ideals are satisfied, since the Plücker coordinates necessarily satisfy the Plücker ideal equations, and hence the equations of P_M .

The (i, H) slack entry s_{iH} is of the form $\det(\mathbf{v}_i, \mathbf{w}_1, \dots, \mathbf{w}_d)$ for some choice of $\mathbf{w}_1, \dots, \mathbf{w}_d$ which span the hyperplane H . Each subsequent column of M_H has entries $\det(\mathbf{v}_i, \mathbf{v}_{j_1}, \dots, \mathbf{v}_{j_d})$, obtained from \mathbf{q} as above, for $j_1 < \dots < j_d$ spanning H . Since each \mathbf{v}_{j_k} lies on the hyperplane H , there is a sequence of elementary column operations that takes the matrix with columns $\mathbf{v}_i, \mathbf{w}_1, \dots, \mathbf{w}_d$ to the one with columns $\mathbf{v}_i, \mathbf{v}_{j_1}, \dots, \mathbf{v}_{j_d}$ for each i . These column operations change the determinant by some scale factor $\lambda \in \mathbb{k}^*$ for all i , so that each column of M_H is a scalar multiple of the first column of slack entries as required. □

COROLLARY 3.11. *Let $M = (E, \mathcal{B})$ be a rank $d + 1$ matroid. Then*

$$\sqrt{I_M} = \sqrt{U_M : \left(\prod_{i \in E, H \in \mathcal{H}} x_{iH} \prod_{\sigma \in \mathcal{B}} p_\sigma \right)^\infty} \cap \mathbb{k}[\mathbf{x}].$$

By universality [10], we do not expect that I_M is radical for every matroid. Thus, Corollary 3.11 may be the strongest relationship between I_M and U_M .

EXAMPLE 3.12. We now continue Example 2.9, for which we can verify Theorem 3.8 at the level of ideals. Let P_{M_4} be the Grassmannian ideal of this matroid, which is generated by the Plücker relations with variables $p_{123}, p_{246}, p_{345}, p_{156}$ set to 0. Let I be the ideal which guarantees each of the seven matrices $M_{123}, M_{246}, M_{345}, M_{156}, M_{25}, M_{14}, M_{36}$ have rank 1; that is, $I = \sum_{H \in \mathcal{H}(M_4)} I_2(M_H)$. Then, the universal realization ideal is $U_M = I + P_{M_4}$. In this case, we can compute that $P_M = U_M \cap \mathbb{k}[\mathbf{p}]$ and $I_M = U_M : (\prod x_{iH} \prod p_\sigma)^\infty \cap \mathbb{k}[\mathbf{x}]$.

4. NON-REALIZABILITY

In this section we illustrate how the slack ideal can be used to determine matroid realizability over a given field. This is a well-studied problem [1, 2, 10] for which a complete characterization is only known in a very limited number of cases. Observe that Theorem 3.1 gives us the following criterion for realizability.

COROLLARY 4.1. *A matroid M is realizable over \mathbb{k} if and only if $\mathcal{V}(I_M) \cap (\mathbb{k}^*)^t \neq \emptyset$.*

We now recast this into a test for realizability in terms of the slack ideal.

PROPOSITION 4.2. *Let M be an abstract matroid and \mathbb{k} be a field. If the slack ideal $I_M = \langle 1 \rangle$ over \mathbb{k} , then M is not realizable over \mathbb{k} . If \mathbb{k} is algebraically closed and M is not realizable over \mathbb{k} , then $I_M = \langle 1 \rangle$.*

Proof. If M is realizable over \mathbb{k} , then there exists a slack matrix S_M which is an element of the variety $\mathcal{V}(I_M)$ by Theorem 3.1. Then we cannot have $I_M = \langle 1 \rangle$. On

the other hand, if $I_M \neq \langle 1 \rangle$, then $\mathcal{V}(I_M)$ is not empty by the Nullstellensatz, and since I_M is saturated with respect to the product of the variables, $\mathcal{V}(I_M)$ is not contained entirely in the coordinate hyperplanes. Therefore, by Theorem 3.1 there is a slack matrix S_M , and the rows of S_M give a realization of M by Lemma 2.4. \square

EXAMPLE 4.3. Consider the Fano plane M_F . It is a rank 3 matroid on 7 elements $E = \{0, 1, 2, 3, 4, 5, 6\}$, depicted in Figure 3 with its symbolic slack matrix. It is known that M_F is only realizable in characteristic 2. Using `Macaulay2` [9], we find $I_{M_F} = \langle 1 \rangle$ over \mathbb{Q} . So, the Fano plane is not realizable over \mathbb{Q} by Proposition 4.2.

Over \mathbb{F}_2 , the slack ideal is generated by 126 binomials of degrees 2, 3, and 4. So, setting all variables to 1 in the slack matrix gives the realization of M_F in \mathbb{F}_2^7 .

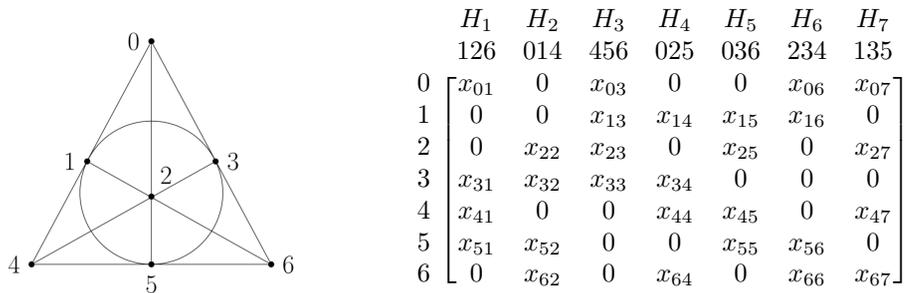


FIGURE 3. The Fano plane M_F of Example 4.3 with its non-bases drawn as lines and a circle, and its symbolic slack matrix $S_{M_F}(\mathbf{x})$.

EXAMPLE 4.4. Consider the complex matroid M_8 from [2, p. 33]. It is a rank 3 matroid on 8 elements with non-bases 124, 235, 346, 457, 568, 167, 278, and 138. It is depicted with its symbolic slack matrix in Figure 4.

To simplify the computation, we use Corollary 3.2 and note that we can select a representative of each projective equivalence class by fixing certain variables in the slack matrix to be 1 (see Section 5.1 for more details). Fixing the variables $x_{14}, x_{27}, x_{28}, x_{3,12}, x_{41}, x_{46}, x_{47}, x_{4,10}, x_{57}, x_{67}, x_{72}, x_{73}, x_{74}, x_{75}, x_{77}, x_{79}, x_{7,11}, x_{7,12}, x_{84}$ to 1 and computing the slack ideal I_{M_8} in `Macaulay2` [9], we find that it is not the unit ideal. However, it contains the polynomial $x_{8,12}^2 + x_{8,12} + 1$. Since this polynomial has only the complex roots $\frac{-1 \pm i\sqrt{3}}{2}$, we get by Corollary 4.1 that M_8 is not realizable over \mathbb{R} , but it is realizable over \mathbb{C} by Proposition 4.2.

4.1. FINAL POLYNOMIALS. The method of final polynomials introduced in [2, § 4.2] certifies when a matroid has no realization. We define an analogous polynomial for the slack ideal, and show how it can be used to improve computational efficiency of checking non-realizability.

DEFINITION 4.5. Let M be a matroid of rank $d+1$. Let \mathcal{S} be the multiplicatively closed set generated by taking finite products of the variables \mathbf{x} in the symbolic slack matrix. A polynomial $f \in \mathbb{k}[\mathbf{x}]$ is a slack final polynomial if

$$f \in I_{d+2}(S_M(\mathbf{x})) \cap (\mathcal{S} + I_M)$$

where $I_{d+2}(S_M(\mathbf{x}))$ is the ideal of $(d+2)$ -minors of the symbolic slack matrix of M .

Intuitively, a slack final polynomial is a $(d+2)$ -minor of the symbolic slack matrix that can be written as the sum of a monomial and an element of the slack ideal. We now have the following result, which demonstrates that the existence of slack final polynomials gives a certificate for non-realizability.

contained in $I_{d+2}(S_M(\mathbf{x}))$. So, if one simply wants to certify non-realizability, a faster method is to compute $I_{d+2}(S_M(\mathbf{x}))$ and check, for example, if $\prod x_{ij} \in I_{d+2}(S_M(\mathbf{x}))$. In the following example we exhibit how this method can be useful for certifying non-realizability.

EXAMPLE 4.10. Consider the Fano matroid M_F of Example 4.3. If we compute $I_4(S_{M_F}(\mathbf{x}))$, then we can verify that the product of all of the variables is contained in this ideal. In fact, even the monomial $x_{07}x_{16}x_{25}x_{33}x_{41}x_{52}x_{64}$ is contained in $I_4(S_{M_F}(\mathbf{x}))$. Verifying this containment (using the laptop of one of the authors) in `Macaulay2` took 0.000067 seconds, while testing $1 \in I_{M_F}$ took 3.40765 seconds, indicating a speed up by a factor of 50,000.

EXAMPLE 4.11. Consider the Vámos matroid pictured in Figure 5. It is a rank 4 matroid M_v on 8 elements whose non-bases are given by the sets 1234, 1256, 3456, 3478, and 5678. It is one of the smallest matroids known to be non-realizable over every field. However, the Vámos matroid has 41 hyperplanes, so that its slack matrix is an 8×41 matrix containing 200 distinct variables. Even computing the full set of minors of this matrix is computationally impractical.

We note though, that it always suffices to show that Proposition 4.6 (ii) holds for some subideal of the ideal of $(d+2)$ -minors. In particular, we can look at the minors of a submatrix of $S_{M_v}(\mathbf{x})$. Consider the submatrix of the Vámos symbolic slack matrix in Figure 5. One can easily check with `Macaulay2` that the monomial given by the product of all the variables in this submatrix is already in the minor ideal of this submatrix (over \mathbb{Q} and various finite fields), making M_v non-realizable over these fields by Propositions 4.2 and 4.6.

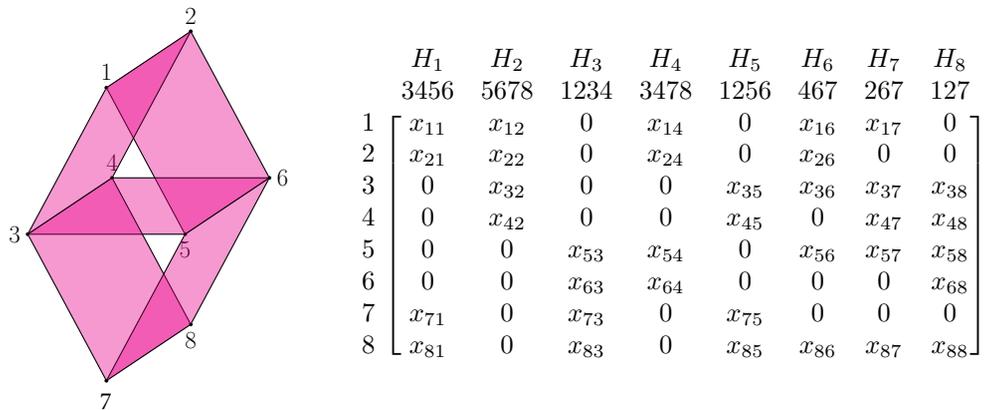


FIGURE 5. The Vámos matroid M_v of Example 4.11 with non-bases pictured as planes, and a submatrix of its symbolic slack matrix.

5. PROJECTIVE UNIQUENESS OF MATROIDS

The simplest slack realization spaces are those belonging to projectively unique matroids. In this case, we know that there is a single realization up to projective transformations; in other words, $\mathcal{V}(I_M)$ is the toric variety which is the closure of the orbit of some realization under the action of $T_{n,h}$. This implies $\sqrt{I_M} = \mathcal{I}(\mathcal{V}(I_M))$ is a toric ideal; however, universality suggests that I_M need not be radical. A natural question which arises is whether projectively unique matroids correspond exactly to matroids with toric slack ideals. To study this question we introduce an intermediate toric ideal associated to a matroid.

DEFINITION 5.1. Define the non-incidence graph of matroid M as the bipartite graph G_M with one node for each element of the ground set of M , one node for each hyperplane, and an edge between element i and hyperplane H_j if and only if $i \notin H_j$. Notice that G_M records the support of the slack matrix S_M , and so we can think of its edges as being labelled by the corresponding entry of $S_M(\mathbf{x})$. (See Figure 6 for an example of the graph G_{M_4} for the matroid M_4 of Example 2.9, and Figure 7 for the non-incidence graph of the non-Fano matroid.)

Let \mathcal{A}_G be the set of vectors forming the columns of the vertex-edge incidence matrix of the graph G , and let T_G be the toric ideal of the vector configuration \mathcal{A}_G . The toric ideal of a bipartite graph is a well-studied object [11, 14]. If a matroid M has a slack matrix S_M which is a 0-1 matrix, then the toric ideal T_{G_M} associated to the graph G_M is the ideal of the orbit of S_M under the action of the torus $T_{n,h}$. So, T_{G_M} describes one projective equivalence class of slack matrices of M . We now define an analogous toric ideal for any projective equivalence class.

DEFINITION 5.2. Let M be an abstract matroid with realization V . Let $\mathbf{s} \in (\mathbb{k}^*)^t$ be such that $S_{M[V]} = S_M(\mathbf{s})$, where t is the number of variables in $S_M(\mathbf{x})$, the symbolic slack matrix of M . We define the cycle ideal C_V of $M[V]$ to be the ideal

$$(3) \quad C_V = \left\langle \mathbf{x}^{c+} - \alpha_c \mathbf{x}^{c-} : c \text{ is a cycle in } G_M \text{ and } \alpha_c = \frac{\mathbf{s}^{c+}}{\mathbf{s}^{c-}} \right\rangle \subseteq \mathbb{k}[\mathbf{x}]$$

where $c+$ and $c-$ are alternating edges from the cycle c .

THEOREM 5.3. Let M be a matroid of rank $d + 1$ on n elements with h hyperplanes. Let V be a realization of M with slack matrix $S_{M[V]} = [s_{i,j}]_{i=1,j=1}^{n,h}$. Then the ideal C_V is the (scaled) toric ideal which is the kernel of the \mathbb{k} -algebra homomorphism $\phi : \mathbb{k}[\mathbf{x}] \rightarrow \mathbb{k}[\mathbf{r}, \mathbf{t}, \mathbf{r}^{-1}, \mathbf{t}^{-1}]$, which sends $x_{ij} \mapsto s_{i,j} r_i t_j$.

We note that the cycle ideal of a realization provides a way to distinguish projective equivalence classes of realizations of M , as well as detect projective uniqueness.

LEMMA 5.4. Let M be a matroid with realizations U and V . Then U, V are projectively equivalent if and only if $C_V = C_U$.

Proof. First suppose that U and V are projectively equivalent. Let $S_{M[V]}$ have entries $s_{i,j}$ for elements i of the ground set of M and hyperplanes H_j of M . By Lemma 2.6 we know that $S_{M[U]}$ is obtained from $S_{M[V]}$ by scaling the rows by $r_1, \dots, r_n \in \mathbb{k}^*$ and scaling the columns by $t_1, \dots, t_h \in \mathbb{k}^*$. Then the entries of $S_{M[U]}$ are $r_i t_j s_{i,j}$. Since $S_{M[U]}$ and $S_{M[V]}$ have the same support, they have the same cycles. One may then show via elementary calculation that the coefficients α_c are the same when calculated from $S_{M[U]}$ and from $S_{M[V]}$ for every cycle $c \in G_M$.

Conversely, suppose $C_V = C_U$. Then $\mathcal{V}(C_V) = \mathcal{V}(C_U)$, and in particular, $S_{M[V]} = S_M(\mathbf{s})$ and $S_{M[U]} = S_M(\mathbf{u})$ for $\mathbf{s}, \mathbf{u} \in \mathcal{V}(C_V) \cap (\mathbb{k}^*)^t$. We now argue that $S_{M[U]}$ can be row and column scaled to be equal to $S_{M[V]}$, which proves the results by Lemma 2.6. Fix a spanning forest T of G_M . By Lemma 5.11 we may scale the entries in $S_M(\mathbf{u})$ corresponding to edges in T to be equal to the corresponding entries of $S_{M[V]}$. Any remaining entry a of $S_M(\mathbf{u})$ will correspond to an edge e in G_M such that $T \cup \{e\}$ contains a unique cycle c , where $e \in c$. The equation $\mathbf{x}^{c+} - \alpha_c \mathbf{x}^{c-} \in C_V$ corresponding to this cycle must be satisfied by the scaling of $S_M(\mathbf{u})$, since $C_V = C_U$. Since all variables in the cycle except the one labelled by edge e have been fixed, we find that there is only one possible value for a . Furthermore, this value equals the corresponding entry in $S_{M[V]}$, since $S_{M[V]}$ satisfies the equations of C_V by definition. \square

COROLLARY 5.5. Let M be a matroid with realization V . Then, $\mathcal{V}(C_V) \cap (\mathbb{k}^*)^t$ consists of exactly the slack matrices of realizations projectively equivalent to V .

LEMMA 5.6. Let \mathbb{k} be algebraically closed and M be a matroid with realization V . Then the slack ideal I_M is contained in the cycle ideal C_V .

Proof. By Corollary 5.5 we know that $\mathcal{V}(C_V) \subset \mathcal{V}(I_M)$. Then, $\mathcal{I}(\mathcal{V}(C_V)) \supset \mathcal{I}(\mathcal{V}(I_M))$. Since C_V is radical, and since $I_M \subset \sqrt{I_M}$, this gives $C_V \supset I_M$. \square

PROPOSITION 5.7. Let M be a matroid with projectively unique realization V . Then $\mathcal{V}(I_M) = \mathcal{V}(C_V)$.

Proof. By Corollary 5.5 and Theorem 3.1, we get $\mathcal{V}(I_M) \cap (\mathbb{k}^*)^t = \mathcal{V}(C_V) \cap (\mathbb{k}^*)^t$. Then since both varieties are irreducible, the result follows. \square

In fact, we can have $I_M = C_V$ for a realization V of M . In this case, call I_M cyclic.

THEOREM 5.8. If the slack ideal of a matroid is cyclic then M is projectively unique and I_M is radical. The converse also holds when \mathbb{k} is algebraically closed.

Proof. Suppose that I_M is cyclic. Then M is projectively unique by Corollary 5.5 and I_M is radical, since it is prime by Theorem 5.3. Conversely, suppose that M is projectively unique and I_M is radical. By Proposition 5.7, $\mathcal{I}(\mathcal{V}(I_M)) = \mathcal{I}(\mathcal{V}(C_V))$, so that $I_M = C_V$, as both ideals are radical. \square

EXAMPLE 5.9. Recall the matroid M_4 from Example 2.9, whose non-incidence graph G_{M_4} is displayed in Figure 6. The matroid M_4 has a cyclic slack ideal. Hence I_{M_4} is a radical slack ideal, and M_4 is projectively unique.

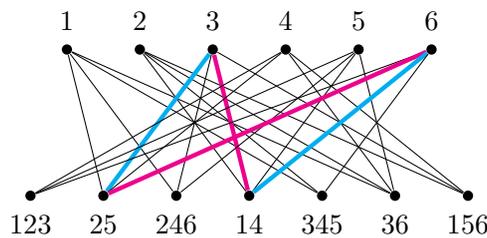


FIGURE 6. The graph G_{M_4} for matroid M_4 with the highlighted cycle corresponding to binomial $x_{36}x_{65} + x_{35}x_{66}$ of Table 1.

EXAMPLE 5.10. Recall the Fano plane discussed in Example 4.3. Over \mathbb{F}_2 , the 126 binomial generators of I_{M_F} found in Example 4.3 correspond to each of the cycles in the graph G_{M_F} . Over \mathbb{F}_2 this is the projectively unique representation of M_F , and this ideal is equal to the cycle ideal of the representation.

5.1. SCALED SLACK MATRICES. From Corollary 3.2 we know that quotienting by the action of $T_{n,h}$ on $\mathcal{V}(I_M) \cap (\mathbb{k}^*)^t$ gives us a realization space for projective equivalence classes of representations of M . We now give an explicit way of computing the variety of these equivalence classes. As in [6, Lemma 5.5], we scale rows and columns of a slack matrix via the following lemma, to fix one representative of each projective equivalence class.

LEMMA 5.11. Given a realization of a matroid M , we may scale the rows and columns of its slack matrix S_M so that it has ones in the entries indexed by the edges in a maximal spanning forest F of the graph G_M ; the resulting realization of M is projectively equivalent to the original realization of M .

DEFINITION 5.12. *Given a matroid M we can take a symbolic slack matrix and set variables corresponding to edges in a maximal spanning forest F to 1 as in Lemma 5.11 to obtain a scaled symbolic slack matrix. Then, the scaled slack ideal is obtained by taking the $(d + 2)$ -minors of this matrix and saturating with respect to the product of all the variables.*

Using the scaled symbolic slack matrix not only allows us to study the projective realization space of M , but also proves to be a useful tool for computations because this matrix will have considerably fewer variables.

EXAMPLE 5.13. Let M_{NF} be the non-Fano matroid. It is a rank 3 matroid on 7 elements depicted in Figure 7 with its symbolic slack matrix. It differs from the Fano plane by the inclusion of 135 as a basis.

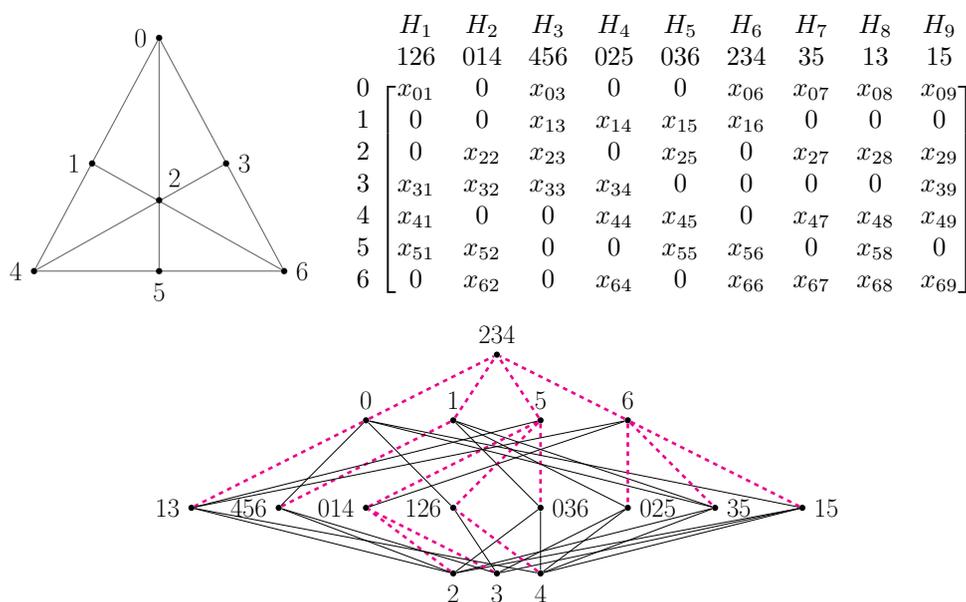
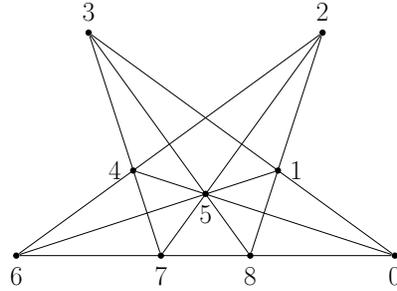


FIGURE 7. The non-Fano matroid M_{NF} , with its non-bases depicted as lines, together with its symbolic slack matrix $S_{M_{NF}}(\mathbf{x})$ and the spanning tree F selected in Example 5.13.

We now show that the non-Fano matroid is projectively unique, and write down a realization from a slack matrix. Let F be the spanning tree of $G_{M_{NF}}$ depicted with dotted lines in Figure 7. We set the corresponding variables $x_{41}, x_{51}, x_{22}, x_{32}, x_{52}, x_{13}, x_{64}, x_{55}, x_{06}, x_{16}, x_{56}, x_{66}, x_{67}, x_{08}, x_{69}$ to 1 in the symbolic slack matrix in Figure 7. Taking the ideal of 4-minors and saturating, we find that the ideal consists of equations of the form $x_{ij} - \alpha_{ij}$, for $\alpha_{ij} \in \mathbb{Q}$, so the configuration is projectively unique over \mathbb{Q} . The slack matrix corresponding to the single point in $\mathcal{V}(I_M) \cap (\mathbb{k}^*)^{39}/T_{7,9}$ is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

EXAMPLE 5.14. Consider the *Perles configuration* M_\star of Figure 8. It is a rank 3 matroid on 9 elements with hyperplanes given by 0678, 347, 156, 128, 045, 358, 013, 246, 257, 48, 17, 36, 14, 23, 02. Its symbolic slack matrix is shown in Figure 8 and has the matrix $S(\mathbf{x})$ studied in [6, § 4.3] as a submatrix.



	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}	H_{15}
	0678	347	156	128	045	358	013	246	257	48	17	36	14	23	02
0	0	x_{02}	x_{03}	x_{04}	0	x_{06}	0	x_{08}	x_{09}	$x_{0,10}$	$x_{0,11}$	$x_{0,12}$	$x_{0,13}$	$x_{0,14}$	0
1	x_{11}	x_{12}	0	0	x_{15}	x_{16}	0	x_{18}	x_{19}	$x_{1,10}$	0	$x_{1,12}$	0	$x_{1,14}$	$x_{1,15}$
2	x_{21}	x_{22}	x_{23}	0	x_{25}	x_{26}	x_{27}	0	0	$x_{2,10}$	$x_{2,11}$	$x_{2,12}$	$x_{2,13}$	0	0
3	x_{31}	0	x_{33}	x_{34}	x_{35}	0	0	x_{38}	x_{39}	$x_{3,10}$	$x_{3,11}$	0	$x_{3,13}$	0	$x_{3,15}$
4	x_{41}	0	x_{43}	x_{44}	0	x_{46}	x_{47}	0	x_{49}	0	$x_{4,11}$	$x_{4,12}$	0	$x_{4,14}$	$x_{4,15}$
5	x_{51}	x_{52}	0	x_{54}	0	0	x_{57}	x_{58}	0	$x_{5,10}$	$x_{5,11}$	$x_{5,12}$	$x_{5,13}$	$x_{5,14}$	$x_{5,15}$
6	0	x_{62}	0	x_{64}	x_{65}	x_{66}	x_{67}	0	x_{69}	$x_{6,10}$	$x_{6,11}$	0	$x_{6,13}$	$x_{6,14}$	$x_{6,15}$
7	0	0	x_{73}	x_{74}	x_{75}	x_{76}	x_{77}	x_{78}	0	$x_{7,10}$	0	$x_{7,12}$	$x_{7,13}$	$x_{7,14}$	$x_{7,15}$
8	0	x_{82}	x_{83}	0	x_{85}	0	x_{87}	x_{88}	x_{89}	0	$x_{8,11}$	$x_{8,12}$	$x_{8,13}$	$x_{8,14}$	$x_{8,15}$

FIGURE 8. The Perles configuration matroid M_\star of Example 5.14 with non-bases shown as lines, and its symbolic slack matrix $S_{M_\star}(\mathbf{x})$.

Since the ideal of $S(\mathbf{x})$ will be contained in the ideal of the whole matrix $S_{M_\star}(\mathbf{x})$, it follows from computation in [6] that M_\star is not realizable over \mathbb{Q} . However, it is realizable over \mathbb{R} , and computing its scaled slack ideal we find that the slack variety consists of the following matrices, where α is a root of the polynomial $\alpha^2 - 3\alpha + 1$:

$$\begin{bmatrix} 0 & 1 & \alpha-3 & 3-\alpha & 0 & 3-\alpha & 0 & -1 & 1 & 3-\alpha & 1 & \alpha-2 & 2-\alpha & 2-\alpha & 0 \\ 2\alpha-5 & 1 & 0 & 0 & 3-\alpha & 3-\alpha & 0 & 2-\alpha & 3-\alpha & \alpha-2 & 0 & \alpha-2 & 0 & 2-\alpha & 5-2\alpha \\ 1 & \alpha & 1 & 0 & \alpha & \alpha-1 & 1 & 0 & 0 & \alpha & 1-\alpha & 1 & \alpha & 0 & 0 \\ 1 & 0 & \alpha-1 & 1-\alpha & \alpha-1 & 0 & 0 & \alpha-1 & -\alpha & 1 & 1-2\alpha & 0 & \alpha & 0 & -1 \\ 2-\alpha & 0 & 2-\alpha & \alpha-1 & 0 & \alpha-2 & 1 & 0 & \alpha-1 & 0 & \alpha & 2-\alpha & 0 & \alpha-1 & \alpha-1 \\ 2-\alpha & 1-\alpha & 0 & 1 & 0 & 0 & 1 & \alpha-1 & 0 & 2-\alpha & 1 & 1-\alpha & 1 & \alpha & \alpha-1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 3-\alpha & \alpha-2 & 1 & \alpha-2 & 1 & 1 & 0 & \alpha-2 & 0 & 2-\alpha & \alpha-1 & \alpha-1 & 1 \\ 0 & \alpha+1 & 1 & 0 & 1 & 0 & 1 & \alpha & 1-\alpha & 0 & 1-\alpha & 1-\alpha & \alpha & \alpha & 1 \end{bmatrix}.$$

Over any field where 5 has a square root, the variety has degree 2 and dimension 0, so it consists of two points obtained by setting $\alpha = \frac{3 \pm \sqrt{5}}{2}$.

APPENDIX A. NOTATION

Name	Description
α_H	a hyperplane normal to the hyperplane H
C_V	cycle ideal of realization V
G_M	non-incidence graph of M
I_M	slack ideal of M , see Definition 2.8
$M[V]$	matroid given by realization V
M_H	Plücker substitution matrices, see Definition 3.7
S_M	slack matrix of M , see Definition 2.1
$S_M(\mathbf{x})$	symbolic slack matrix, see Definition 2.8
t	number of slack variables
$T_{n,h}$	torus of row and column scalings $(\mathbb{k}^*)^n \times (\mathbb{k}^*)^h$
U_M	the universal realization ideal of M , see Definition 3.7
$\mathcal{V}(I_M)$	slack variety of M , see Definition 2.8
x_{ij}	slack variables, see Definition 2.8

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