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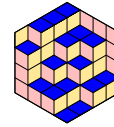


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ABSTRACT A relative simplicial complex is a collection of sets of the form $\Delta \setminus \Gamma$, where $\Gamma \subset \Delta$ are simplicial complexes. Relative complexes have played key roles in recent advances in algebraic, geometric, and topological combinatorics but, in contrast to simplicial complexes, little is known about their general combinatorial structure. In this paper, we address a basic question in this direction and give a characterization of f -vectors of relative (multi)complexes on a ground set of fixed size. On the algebraic side, this yields a characterization of Hilbert functions of quotients of homogeneous ideals over polynomial rings with a fixed number of indeterminates.

Moreover, we characterize h -vectors of fully Cohen–Macaulay relative complexes as well as h -vectors of Cohen–Macaulay relative complexes with minimal faces of given dimensions. The latter resolves a question of Björner.

1. INTRODUCTION

A **simplicial complex** Δ is a collection of subsets of a finite ground set, say $[n] := \{1, \dots, n\}$, such that $\sigma \in \Delta$ and $\tau \subseteq \sigma$ imply $\tau \in \Delta$. Simplicial complexes are fundamental objects in algebraic, geometric, and topological combinatorics; see, for example, [2, 3, 19]. A basic combinatorial statistic of Δ is the **face vector** (or **f -vector**)

$$f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1}),$$

where $f_k = f_k(\Delta)$ records the number of faces $\sigma \in \Delta$ of dimension k , where $\dim \sigma := |\sigma| - 1$ and $d - 1 = \dim \Delta := \max\{\dim \sigma : \sigma \in \Delta\}$. Notice that we allow $\Delta = \emptyset$, the *void* complex, which is the only complex with $f_k(\Delta) = 0$ for all $k \geq -1$. A **relative simplicial complex** Ψ on the ground set $[n]$ is the collection of sets $\Delta \setminus \Gamma = \{\tau \in \Delta : \tau \not\subseteq \Gamma\}$, where $\Gamma \subset \Delta \subseteq 2^{[n]}$ are simplicial complexes. In general, the pair of simplicial complexes (Δ, Γ) is not uniquely determined by Ψ , and we call $\Psi = (\Delta, \Gamma)$ a **presentation** of Ψ . We set $\dim \Psi := \max\{\dim \sigma : \sigma \in \Delta \setminus \Gamma\}$. Relative complexes were introduced by Stanley [18] and made prominent recent appearances in, for example, [1, 9, 15, 16]. The f -vector of a relative complex is given by

$$f(\Psi) := f(\Delta) - f(\Gamma),$$

where we set $f_k(\Gamma) := 0$ for all $k > \dim \Gamma$. When $\Gamma = \emptyset$, then Ψ is simply a simplicial complex and we write Δ instead of Ψ . We call Ψ a **proper** relative complex if $\Gamma \neq \emptyset$ or, equivalently, if $f_{-1}(\Psi) = 0$.

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In contrast to simplicial complexes, much less is known about the combinatorics of relative simplicial complexes. The first goal of this paper is to address the following basic question:

Which vectors $f = (0, f_0, \dots, f_{d-1}) \in \mathbb{Z}_{\geq 0}^{d+1}$ are f -vectors of proper relative simplicial complexes?

For simplicial complexes, this question is beautifully answered by the Kruskal–Katona theorem [12, 13]. Björner and Kalai [5] characterized the pairs $(f(\Delta), \beta(\Delta))$ where Δ is a simplicial complex and $\beta(\Delta)$ is the sequence of Betti numbers of Δ (over a field $\mathbf{k}$). Duval [8] characterized the pairs $(f(\Delta), f(\Gamma))$ where $\Delta \subseteq \Gamma$ but, as stated before, the presentation $\Psi = \Delta \setminus \Gamma$ is generally not unique. Moreover, the following example shows that a characterization of f -vectors of relative complexes is trivial without further qualifications.

EXAMPLE 1.1. If $\Delta = 2^{[k+1]}$ is a k -dimensional simplex and $\partial\Delta := \Delta \setminus \{[k+1]\}$ denotes its boundary complex, then $f_i(\Delta, \partial\Delta) = 1$ if $i = k$ and is zero otherwise. Hence, by observing that relative simplicial complexes are closed under disjoint unions, any vector $f = (0, f_0, \dots, f_{d-1}) \in \mathbb{Z}_{\geq 0}^{d+1}$ can occur as the f -vector of a proper relative simplicial complex.

The main difference between f -vectors of complexes and relative complexes is that $f_0(\Psi)$ does not reveal the size of the ground set and the construction outlined in Example 1.1 produces relative complexes with given f -vectors on large ground sets. Restricting the size of the ground set is the key to a meaningful treatment of f -vectors of relative complexes. Therefore, we are going to characterize the f -vectors of relative complexes $\Psi = \Delta \setminus \Gamma$ with $\Gamma \subset \Delta \subseteq 2^{[n]}$ for fixed n . To state our characterization, we need to recall the binomial representation of a natural number: For any $r, k \in \mathbb{Z}_{\geq 0}$ with $k > 0$, there are unique integers $r_k > r_{k-1} > \dots > r_1 \geq 0$ such that

$$(1) \quad r = \binom{r_k}{k} + \binom{r_{k-1}}{k-1} + \dots + \binom{r_1}{1}.$$

We refer the reader to Greene–Kleitman’s excellent article [10, Section 8] for details and combinatorial motivations for this and the following definition. For the representation given in (1) we define

$$\partial_k(r) := \binom{r_k}{k-1} + \binom{r_{k-1}}{k-2} + \dots + \binom{r_1}{0}.$$

The Kruskal–Katona theorem characterizes f -vectors of simplicial complexes in terms of these $\partial_k(r)$, see Theorem 2.1. We prove the following characterization of f -vectors of proper relative complexes in Section 2.

THEOREM 1.2. *Let $f = (0, f_0, \dots, f_{d-1}) \in \mathbb{Z}_{\geq 0}^{d+1}$ and $n > 0$ and define two sequences $(a_0, \dots, a_{d-1})$ and $(b_0, \dots, b_{d-1})$ by $a_{d-1} := f_{d-1}$ and $b_{d-1} := 0$ and continue recursively*

$$\begin{aligned} a_{k-1} &:= \max(\partial_{k+1}(a_k), f_{k-1} + \partial_{k+1}(b_k)) \\ b_{k-1} &:= \max(\partial_{k+1}(b_k), \partial_{k+1}(a_k) - f_{k-1}) \end{aligned}$$

for $k \geq 0$. Then there is a proper relative simplicial complex Ψ on the ground set $[n]$ with $f = f(\Psi)$ if and only if $a_0 \leq n$.

The two sequences $(1, a_0, \dots, a_{d-1})$ and $(1, b_0, \dots, b_{d-1})$ are the componentwise-minimal f -vectors of simplicial complexes Δ and Γ such that $\Gamma \subseteq \Delta$ and $f_{k-1} = f_{k-1}(\Delta) - f_{k-1}(\Gamma)$ for all $0 \leq k < d$.

(Relative) simplicial complexes can be generalized to (relative) *multicomplexes* by replacing sets with multisets. The notion of an f -vector of a multicomplex is immediate (by taking into account multiplicities) and the question above carries over to relative multicomplexes on a ground set of fixed size. Multicomplexes are more natural from an algebraic perspective: If $S := \mathbf{k}[x_1, \dots, x_n]$ is the polynomial ring over a field $\mathbf{k}$ with n indeterminates and $I \subseteq S$ is a monomial ideal, then the monomials outside I form a (possibly infinite) multicomplex on ground set $[n]$ and every multicomplex over $[n]$ arises this way. In particular, the f -vector of a multicomplex is the Hilbert function of S/I . By appealing to initial ideals it is easy to see that f -vectors of (infinite) multicomplexes are exactly the Hilbert functions of standard graded algebras, which were characterized by Macaulay [14]. In Section 3 we give precise definitions and Theorem 3.1 is the corresponding analogue of Theorem 1.2 for proper, possibly infinite, relative multicomplexes. The corresponding algebraic statement characterizes Hilbert functions of I/J where $J \subset I \subseteq S$ are pairs of homogeneous ideals; see Corollary 3.3.

The **h -vector** $h(\Psi) = (h_0, \dots, h_d)$ of a $(d-1)$ -dimensional relative complex Ψ is defined through

$$(2) \quad \sum_{k=0}^d f_{k-1}(\Psi)t^{d-k} = \sum_{i=0}^d h_i(\Psi)(t+1)^{d-i}.$$

Note that if $\dim \Delta = \dim \Gamma$, then $h(\Psi) = h(\Delta) - h(\Gamma)$. The h -vector clearly carries the same information as the f -vector but it has been amply demonstrated that h -vectors often times reveal more structure; see [19] for example. In particular, if Δ is a **Cohen–Macaulay** simplicial complex (or CM complex, for short) over some field $\mathbf{k}$, then $h_i(\Delta) \geq 0$ for all $i \geq 0$. Stanley [17] showed that Macaulay’s theorem characterizing Hilbert functions of standard graded algebras yields a characterization of h -vectors of CM complexes akin to the Kruskal–Katona theorem. Stronger even, Björner, Frankl, and Stanley [4] showed that all admissible h -vectors can be realized by shellable simplicial complexes, a proper subset of CM complexes.

In Section 4, we recall the definition of a Cohen–Macaulay relative complex and we give a characterization of h -vectors of *fully* CM relative complexes. We call a relative complex Ψ **fully Cohen–Macaulay** over a ground set $[n]$ if it has a presentation $\Psi = (\Delta, \Gamma)$ with $\Gamma \subset \Delta \subseteq 2^{[n]}$, $\dim \Gamma = \dim \Psi$, and Ψ as well as Δ and Γ are Cohen–Macaulay.

For $r, k \in \mathbb{Z}_{\geq 0}$ with $k > 0$, let $r_k > \dots > r_1 \geq 0$ as defined by (1). We define

$$\tilde{\partial}_k(r) := \binom{r_k - 1}{k - 1} + \binom{r_{k-1} - 1}{k - 2} + \dots + \binom{r_1 - 1}{0}.$$

Note that Ψ is proper if and only if $h_0(\Psi) = 0$. Our characterization of h -vectors of fully CM complexes parallels that of CM complexes in that it suffices to consider *fully shellable* relative complexes; see Section 4 for a definition.

THEOREM 1.3. *Let $h = (0, h_1, \dots, h_d) \in \mathbb{Z}_{\geq 0}^{d+1}$ and $n > 0$. Then the following are equivalent:*

- (a) *There is a fully CM relative complex Ψ on ground set $[n]$ with $h = h(\Psi)$;*
- (b) *There is a fully shellable relative complex Ψ on ground set $[n]$ with $h = h(\Psi)$;*
- (c) *Let $(a_0, \dots, a_{d-1})$ and $(b_0, \dots, b_{d-1})$ be the sequences defined through $a_{d-1} := h_d$ and $b_{d-1} := 0$ and recursively continued*

$$a_{i-1} := \max(\tilde{\partial}_{i+1}(a_i), h_i + \tilde{\partial}_{i+1}(b_i))$$

$$b_{i-1} := \max(\tilde{\partial}_{i+1}(b_i), \tilde{\partial}_{i+1}(a_i) - h_i)$$

for $i \geq 1$. Then $a_0 \leq n - d$.

In Section 5, we discuss the difference between CM and fully CM relative complexes. In particular, we show in Theorem 5.4 that every $(d - 1)$ -dimensional CM relative complex has a presentation as a fully CM relative complex if we allow the ground set to grow by at most d elements. From this, we derive the following necessary condition on h -vectors of proper CM relative complexes.

COROLLARY 1.4. *Let $h = (0, h_1, \dots, h_d) \in \mathbb{Z}_{\geq 0}^{d+1}$ and $n > 0$. Further, let $(a_0, \dots, a_{d-1})$ and $(b_0, \dots, b_{d-1})$ be the sequences defined in Theorem 1.3(c). If there exists a CM relative complex Ψ on ground set $[n]$ with $h = h(\Psi)$, then $a_0 \leq n$.*

We conjecture that it actually suffices to extend the ground set by a single new vertex. This would strengthen the bound of Corollary 1.4 to $n - d + 1$.

Finally, Theorem 5.7 gives a characterization of h -vectors of relative multicomplexes if the dimensions of the minimal faces of $\Psi = \Delta \setminus \Gamma$ are given. This resolves a question of A. Björner stated in [18].

2. f -VECTORS OF RELATIVE SIMPLICIAL COMPLEXES

The proof of Theorem 1.2 follows the same ideas as that of the classical Kruskal–Katona theorem given in [10, Section 8]. A simplicial complex $\Delta \subset 2^{[n]}$ is called **compressed** if its set of k -faces forms an initial segment with respect to the reverse lexicographic order on the $(k + 1)$ -subsets of $[n]$, for each k . Note that if Δ and Γ are both compressed simplicial complexes and $f_k(\Gamma) \leq f_k(\Delta)$ for all k , then $\Gamma \subseteq \Delta$. The Kruskal–Katona theorem now states that f is the f -vector of a simplicial complex if and only if it is the f -vector of a compressed simplicial complex, which can be checked by numerical conditions.

THEOREM 2.1 (Kruskal [13], Katona [12]). *For a vector $f = (1, f_0, \dots, f_{d-1}) \in \mathbb{Z}_{\geq 0}^{d+1}$, the following conditions are equivalent:*

- (a) f is the f -vector of a simplicial complex;
- (b) f is the f -vector of a compressed simplicial complex;
- (c) $\partial_{k+1}(f_k) \leq f_{k-1}$ for all $k \geq 1$.

The shadow of a family of k -sets consists of all $(k - 1)$ -subsets of the k -sets of the family. The Kruskal–Katona theorem tells us that $\partial_{k+1}(r)$ is the minimum size of the shadow of a family k -sets of size r . Actually, this minimum is always achieved if the family is compressed. Note that this implies in particular that the functions ∂_k are monotone.

With these preparations, we can now give the proof of our Theorem 1.2.

Proof of Theorem 1.2. Let us recall the definition of the sequences $(a_0, \dots, a_{d-1})$ and $(b_0, \dots, b_{d-1})$. We have that $a_{d-1} = f_{d-1}$, $b_{d-1} = 0$ and

$$\begin{aligned} a_{k-1} &= \max(\partial_{k+1}(a_k), f_{k-1} + \partial_{k+1}(b_k)) \\ &= \partial_{k+1}(a_k) + \max(0, f_{k-1} - (\partial_{k+1}(a_k) - \partial_{k+1}(b_k))); \\ b_{k-1} &= \max(\partial_{k+1}(b_k), \partial_{k+1}(a_k) - f_{k-1}) \\ &= \partial_{k+1}(b_k) + \max(0, (\partial_{k+1}(a_k) - \partial_{k+1}(b_k)) - f_{k-1}), \end{aligned}$$

for $1 \leq k \leq d - 1$. From the second expression for a_{k-1} and b_{k-1} it is easy to see that $a_{k-1} - b_{k-1} = f_{k-1}$. In particular, we have that $a_k \geq b_k$ for $k \geq 0$.

We now show the sufficiency of the condition, so assume that $a_0 \leq n$. As both sequences $(1, a_0, \dots, a_{d-1})$ and $(1, b_0, \dots, b_{d-1})$ satisfy the condition of the Kruskal–Katona theorem (Theorem 2.1), there exist compressed simplicial complexes

$\Gamma, \Delta \subset 2^{[n]}$ whose respective f -vectors equal the two sequences. In particular, since both complexes are compressed and $f_k(\Gamma) = b_k \leq a_k = f_k(\Delta)$, it holds that $\Gamma \subset \Delta$, and the relative complex $\Psi := (\Delta, \Gamma)$ has f -vector f .

Now we turn to the necessity of our condition. Assume that we are given a relative complex $\Psi = (\Delta, \Gamma)$ on the ground set $[n]$ with $f(\Psi) = f$. We show by induction on k that $a_k \leq f_k(\Delta)$ and $b_k \leq f_k(\Gamma)$ for $k \geq 0$.

The base case $k = d - 1$ is obvious. For the inductive step, it follows from Theorem 2.1 that $f_{k-1}(\Delta) \geq \partial_{k+1}(f_k(\Delta))$, and further $f_k(\Delta) \geq a_k$ implies that $\partial_{k+1}(f_k(\Delta)) \geq \partial_{k+1}(a_k)$. Similarly, it holds that $f_{k-1}(\Delta) = f_{k-1} + f_{k-1}(\Gamma) \geq f_{k-1} + \partial_{k+1}(f_k(\Gamma)) \geq f_{k-1} + \partial_{k+1}(b_k)$. Together, this implies that

$$f_{k-1}(\Delta) \geq \max(\partial_{k+1}(a_k), f_{k-1} + \partial_{k+1}(b_k)) = a_{k-1}.$$

Moreover, the last inequality together with the fact that $f_{k-1}(\Delta) - f_{k-1}(\Gamma) = a_{k-1} - b_{k-1}$ implies that $f_{k-1}(\Gamma) \geq b_{k-1}$. In particular, $a_0 \leq f_0(\Delta) \leq n$. $\square$

3. f -VECTORS OF RELATIVE MULTICOMPLEXES

A **k -multiset** is a set with repetitions allowed. A **multicomplex** $\tilde{\Delta}$ is a collection of multisets closed under taking (multi-)subsets. We denote a k -multisubset of $[n]$ by $F = \{s_1, s_2, \dots, s_k\}_{\leq}$ where $1 \leq s_1 \leq s_2 \leq \dots \leq s_k \leq n$. We say that the dimension of F is $k - 1$ and in the same way as for simplicial complexes, one defines f -vectors of multicomplexes. Note that multicomplexes can be infinite, even if the ground set is finite.

The sequences which arise as f -vectors of multicomplexes are called **M -sequences** and they have a well-known classification due to Macaulay. Namely, a sequence $(1, f_0, f_1, \dots)$ is an M -sequence if and only if $f_{k-1} \geq \tilde{\partial}_{k+1}(f_k)$. Moreover, as in the simplicial case, for each M -sequence f there exists a unique *compressed* multicomplex $\tilde{\Delta}$ with $f = f(\tilde{\Delta})$. Here, being compressed is defined as in the simplicial case. We refer the reader to [10, Section 8] or [19, Section II.2] for details.

Using compressed multicomplexes and the characterization of M -sequences, the same proof as for Theorem 1.2 also yields the following characterization for f -vectors of finite proper relative multicomplexes $\tilde{\Psi} = (\tilde{\Delta}, \tilde{\Gamma})$.

THEOREM 3.1. *Let $f = (0, f_0, \dots, f_{d-1}) \in \mathbb{Z}_{\geq 0}^{d+1}$ and $n > 0$ and define two sequences $(a_0, \dots, a_{d-1})$ and $(b_0, \dots, b_{d-1})$ by $a_{d-1} := f_{d-1}$ and $b_{d-1} := 0$ and continue recursively*

$$\begin{aligned} a_{k-1} &:= \max(\tilde{\partial}_{k+1}(a_k), f_{k-1} + \tilde{\partial}_{k+1}(b_k)) \\ b_{k-1} &:= \max(\tilde{\partial}_{k+1}(b_k), \tilde{\partial}_{k+1}(a_k) - f_{k-1}) \end{aligned}$$

for $k \geq 0$. Then there is a proper (finite) relative multicomplex $\tilde{\Psi}$ on the ground set $[n]$ with $f = f(\tilde{\Psi})$ if and only if $a_0 \leq n$.

Now we turn to the classification of f -vectors of not necessarily finite multicomplexes. In the proof of Theorem 1.2, it was crucial that relative simplicial complexes have bounded dimension, so that we could proceed by induction from the top dimension downwards. For general relative multicomplexes, we will instead proceed from dimension 0 upwards. This requires some new notation. For $r, k \in \mathbb{Z}_{\geq 0}$ with $k > 0$, let $r_k > \dots > r_1 \geq 0$ as defined by (1). We define

$$\tilde{\partial}^k(r) := \binom{r_k + 1}{k + 1} + \binom{r_{k-1} + 1}{k + 2} + \dots + \binom{r_1 + 1}{2}.$$

It is not difficult to see that $\tilde{\partial}_{k+1}(\tilde{\partial}^k(r)) = r$ and $\tilde{\partial}^{k-1}(\tilde{\partial}_k(r)) \geq r$. Therefore, M -sequences can be equivalently characterized as those sequences $(f_{-1}, f_0, \dots)$ which satisfy $f_{k+1} \geq \tilde{\partial}^{k+1}(f_k)$ for all k .

THEOREM 3.2. *Let $f = (0, f_0, f_1, \dots)$ be a sequence of non-negative integers and $n > 0$ and define two sequences $(a_0, a_1, \dots)$ and $(b_0, b_1, \dots)$ by $a_0 := n$, $b_0 := n - f_0$ and continue recursively*

$$\begin{aligned} a_{k+1} &:= \min(\tilde{\partial}^{k+1}(a_k), f_{k+1} + \tilde{\partial}^{k+1}(b_k)) \\ b_{k+1} &:= \min(\tilde{\partial}^{k+1}(b_k), \tilde{\partial}^{k+1}(a_k) - f_{k+1}) \end{aligned}$$

for $k \geq 0$. Then, there is a proper relative multicomplex $\tilde{\Psi}$ on the ground set $[n]$ with $f = f(\tilde{\Psi})$ if and only if $b_k \geq 0$ for all $k \geq 0$.

The proof is almost the same as the proof of Theorem 1.2, using the characterization of M -sequences in terms of $\tilde{\partial}^k$. The only difference is that to prove necessity, one needs to start the induction at $k = 0$ and proceed in increasing order.

The classical theorem by Macaulay characterizes Hilbert functions of standard graded algebras, and Theorem 3.2 has a similar interpretation. We denote the Hilbert function of a finitely generated graded module M over the polynomial ring $\mathbf{k}[x_1, \dots, x_n]$ by $H(M, k) := \dim_{\mathbf{k}} M_k$.

COROLLARY 3.3 (Macaulay for quotients of ideals). *Let $H : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ with $H(0) = 0$ and $n \geq H(1)$. Furthermore, let $(a_0, a_1, \dots)$ and $(b_0, b_1, \dots)$ be the two sequences of Theorem 3.2, where we set $f_k = H(k+1)$. Then, there exist two proper homogeneous ideals $J \subset I \subsetneq \mathbf{k}[x_1, \dots, x_n]$ with $H(k) = H(I/J, k)$ for all k , if and only if $b_k \geq 0$ for all $k \geq 0$.*

Proof. Consider a homogeneous ideal $I \subseteq \mathbf{k}[x_1, \dots, x_n]$. For any fixed term order $\preceq$, the collection of standard monomials, that is, the monomials not contained in the initial ideal of I with respect to $\preceq$, is naturally identified with a multicomplex $\tilde{\Delta}$. Since the standard monomials form a vector space basis of $\mathbf{k}[x_1, \dots, x_n]/I$ that respects the grading, the f -vector of $\tilde{\Delta}$ coincides with the Hilbert function of $\mathbf{k}[x_1, \dots, x_n]/I$. Moreover, if $J \subseteq I \subseteq \mathbf{k}[x_1, \dots, x_n]$ are two homogeneous ideals, then passing to the initial ideals (with respect to $\preceq$) preserves the inclusion. Therefore, any Hilbert function of a quotient of ideals also arises as f -vector of a relative multicomplex.

For the converse we associate to any multicomplex $\tilde{\Delta}$ the monomial ideal corresponding to all multisets not in $\tilde{\Delta}$. $\square$

4. h -VECTORS OF RELATIVE COHEN–MACAULAY COMPLEXES

Let $\Psi = (\Delta, \Gamma)$ be a $(d-1)$ -dimensional relative simplicial complex and let $\sigma_1, \dots, \sigma_m$ be some ordering of the inclusion-maximal faces (i.e., the facets) of Ψ . Define

$$\Psi_j := (2^{\sigma_1} \cup 2^{\sigma_2} \cup \dots \cup 2^{\sigma_j}) \cap (\Delta \setminus \Gamma)$$

for $j \geq 1$ and set $\Psi_0 := \emptyset$. We call the ordering of the facets a **shelling order** if $\Psi_j \setminus \Psi_{j-1}$ has a unique inclusion-minimal element $R(\sigma_j)$ for all $j = 1, \dots, m$. Consequently, Ψ is **shellable** if it has a shelling order. If $\Gamma = \emptyset$ and hence Ψ is a simplicial complex, this recovers the usual notion of shellability. The h -vector $h(\Psi)$ of a shellable relative complex has a particularly nice interpretation:

$$h_i(\Psi) = |\{j : |R(\sigma_j)| = i\}|,$$

for $0 \leq i \leq d$. It is shown in [19, Section III.7] that a shellable relative complex is Cohen–Macaulay but the converse does not need to hold.

We will call a relative complex Ψ **fully shellable** if it has a presentation $\Psi = (\Delta, \Gamma)$ such that $\dim \Psi = \dim \Gamma$ and Ψ as well as Δ and Γ are shellable. By the above remarks, it is clear that fully shellable relative complexes are fully Cohen–Macaulay and, again, the converse does not necessarily hold.

In light of Theorem 3.1, condition (c) of Theorem 1.3 states that h is the f -vector of a proper relative multicomplex. In order to prove the implication (c) $\implies$ (b), we will show that for every relative multicomplex on the ground set $[n-d]$ with given f -vector $h = (0, h_1, \dots, h_d)$, there is a fully shellable relative complex Ψ with $h(\Psi) = h$.

Let $\tilde{\Psi} = (\tilde{\Delta}, \tilde{\Gamma})$ be a proper relative $(d-1)$ -dimensional multicomplex on ground set $[n-d]$ and assume that $\tilde{\Delta}$ and $\tilde{\Gamma}$ are compressed. To turn $\tilde{\Psi}$ into a relative complex, we follow the construction in [4]. Order the collection of multisets of size $\leq d$ on the ground set $[n-d]$ by graded reverse lexicographic order, and the collection of d -sets on $[n]$ by reverse lexicographic order. There is a unique bijection Φ_d between these two collections which preserves the given orders. Explicitly, the map is

$$\Phi_d(\{b_1, b_2, \dots, b_k\}_{\leq}) := \{1, 2, \dots, d-k, b_1 + d-k+1, b_2 + d-k+2, \dots, b_k + d\}.$$

We denote by Δ the simplicial complex with facets $\{\Phi_d(F) : F \in \tilde{\Delta}\}$ and Γ likewise. Since $\tilde{\Gamma}$ is a submulticomplex of $\tilde{\Delta}$, it follows that $\Gamma \subset \Delta$ and $\Psi = (\Delta, \Gamma)$ is a relative complex with $\dim \Psi = \dim \Delta = \dim \Gamma = d-1$.

PROPOSITION 4.1. *Let $\tilde{\Psi} = (\tilde{\Delta}, \tilde{\Gamma})$ be a $(d-1)$ -dimensional relative multicomplex such that $\tilde{\Delta}$ and $\tilde{\Gamma}$ are compressed. Let $\Psi = (\Delta, \Gamma)$ be the corresponding relative simplicial complex constructed above. Given an ordering $\prec$ of the faces of $\tilde{\Delta}$ such that $F \prec F'$ whenever $|F| < |F'|$, the induced ordering on the facets $\Phi_d(F)$ of Δ is a shelling order for Δ , Γ , and Ψ .*

Proof. It was shown in [4] that any such ordering gives a shelling order for Δ with restriction sets

$$R(\sigma) = \sigma \setminus \{1, 2, \dots, d-k\} = \{s_1 + d-k+1, \dots, s_k + d\}$$

if $\sigma = \Phi_d(\{s_1, \dots, s_k\}_{\leq})$. We are left to prove that restricting this order to the facets of $\Delta \setminus \Gamma$ yields a shelling order for Ψ . It suffices to show that if σ is a facet of Ψ , i.e., a facet of Δ not contained in Γ , then $R(\sigma) \notin \Gamma$.

Let $F = \{s_1, \dots, s_k\}_{\leq}$ be the face of $\tilde{\Delta}$ such that $\sigma = \Phi_d(F)$. We will show that any facet σ' of Δ which contains $R := R(\sigma)$ does not belong to Γ . By construction, the facets of Γ are a subset of the facets of Δ , and thus $R \notin \Gamma$.

Let σ' be a facet of Δ which contains R and let F' be the corresponding element of $\tilde{\Delta}$ with $\sigma' = \Phi_d(F')$. Observe that either $\sigma' = \sigma$ or $t = |F'| > |F| = k$. Indeed, if $t < k$, $\{1, 2, \dots, d-k+1\} \subseteq \sigma'$, and since $R \cap \{1, 2, \dots, d-k+1\} = \emptyset$, R cannot be a subset of σ' . If $t = k$, then $\sigma' \supseteq R$ implies $\sigma' = \sigma$.

So, let us assume that $t > k$. Let $G = \{r_1, \dots, r_t\}_{\leq}$ be the smallest t -multiset in $\tilde{\Delta}$ in reverse lexicographic order such that $\tau = \Phi_d(G) \supseteq R$. Now $\tau = \{1, \dots, d-t\} \cup S$, with $S = \{d-t+1+r_1, \dots, d+r_t\}$. As before, observe that $R \cap \{1, \dots, d-t\} = \emptyset$. Since Φ_d preserves the reverse lexicographic order on t -multisets, S is also minimal with respect to reverse lexicographic order. Therefore the elements of R are the largest elements in S and

$$G = \underbrace{\{1, \dots, 1\}}_{t-k}, s_1, \dots, s_k\}_{\leq}.$$

Then $F = \{s_1, \dots, s_k\}_{\leq} \subseteq G$, and since $F \notin \tilde{\Gamma}$ and $\tilde{\Gamma}$ is a multicomplex, it follows that $G \notin \tilde{\Gamma}$. Since $\tilde{\Gamma}$ is compressed and G is smaller than F' , F' also does not belong to $\tilde{\Gamma}$. This implies $\sigma \notin \Gamma$. $\square$

Proof of Theorem 1.3: (c) $\implies$ (b) $\implies$ (a). By Theorem 3.1, condition (c) guarantees the existence of a proper relative multicomplex $\tilde{\Psi}$ with f -vector h . By Proposition 4.1, the construction above yields a fully shellable relative simplicial complex Ψ with $h = h(\Psi)$. This proves (c) $\implies$ (b). Theorem 2.5 for relative complexes in [19] asserts that Ψ is fully Cohen–Macaulay and hence proves (b) $\implies$ (a). $\square$

In order to prove the implication (a) $\implies$ (c), we make use of the powerful machinery of Stanley–Reisner modules. Let $\mathbf{k}$ be an infinite field. For a fixed $n > 0$, let $S := \mathbf{k}[x_1, \dots, x_n]$ be the polynomial ring. For a simplicial complex $\Delta \subseteq 2^{[n]}$, its **Stanley–Reisner ideal** is $I_\Delta := \langle \mathbf{x}^\tau : \tau \notin \Delta \rangle$ and we write $\mathbf{k}[\Delta] := S/I_\Delta$ for its **Stanley–Reisner ring**. If $\Gamma \subset \Delta$ is a pair of simplicial complexes, then $\mathbf{k}[\Delta] \twoheadrightarrow \mathbf{k}[\Gamma]$ and the **Stanley–Reisner module** of $\Psi = (\Delta, \Gamma)$ is

$$M[\Psi] := \ker(\mathbf{k}[\Delta] \twoheadrightarrow \mathbf{k}[\Gamma]) = I_\Gamma/I_\Delta.$$

This is a graded S -module and Ψ is a **Cohen–Macaulay** relative complex if $M[\Psi]$ is a Cohen–Macaulay module over S . In particular, any choice of generic linear forms $\theta_1, \dots, \theta_d \in S$ for $d = \dim \Psi + 1$ is a regular sequence for $M[\Psi]$ and

$$\dim_{\mathbf{k}}(M[\Psi]/\langle \theta_1, \dots, \theta_d \rangle M[\Psi])_i = h_i(\Psi),$$

for all $i \geq 0$.

Proof of Theorem 1.3: (a) $\implies$ (c). Let (Δ, Γ) be a presentation of Ψ such that $\dim \Gamma = \dim \Psi$ and Δ and Γ are CM. Consider the short exact sequence

$$(3) \quad 0 \rightarrow M[\Psi] \rightarrow \mathbf{k}[\Delta] \rightarrow \mathbf{k}[\Gamma] \rightarrow 0$$

of S -modules. Let $\theta \in S$ be a generic linear form. Tensoring (3) with S/θ yields

$$(4) \quad \mathrm{Tor}_1^S(\mathbf{k}[\Gamma], S/\theta) \rightarrow M[\Psi]/\theta M[\Psi] \rightarrow \mathbf{k}[\Delta]/\theta \mathbf{k}[\Delta] \rightarrow \mathbf{k}[\Gamma]/\theta \mathbf{k}[\Gamma] \rightarrow 0.$$

By resolving S/θ , it is easy to see that $\mathrm{Tor}_1^S(\mathbf{k}[\Gamma], S/\theta) = (0 :_{\mathbf{k}[\Gamma]} \theta) = 0$, so (4) is a short exact sequence as well.

By our choice of presentation, $\mathbf{k}[\Gamma]$ is Cohen–Macaulay and we may repeat the process for a full regular sequence $\Theta = (\theta_1, \dots, \theta_d)$ to arrive at

$$(5) \quad 0 \rightarrow M[\Psi]/\Theta M[\Psi] \rightarrow \mathbf{k}[\Delta]/\Theta \mathbf{k}[\Delta] \rightarrow \mathbf{k}[\Gamma]/\Theta \mathbf{k}[\Gamma] \rightarrow 0.$$

Since Ψ is Cohen–Macaulay, the Hilbert function of $M[\Psi]/\Theta M[\Psi]$ is exactly the h -vector of Ψ and, moreover, we can identify $M[\Psi]/\Theta M[\Psi]$ with a graded ideal in $\mathbf{k}[\Delta]/\Theta \mathbf{k}[\Delta]$. By a linear change of coordinates, this yields a pair of homogeneous ideals $J_\Delta \subset J_\Gamma \subset R := \mathbf{k}[y_1, \dots, y_{n-d}]$ with difference of Hilbert functions exactly $h(\Psi)$. For any fixed term order $\preceq$, we denote by $\mathrm{in}_{\preceq}(J_\Delta), \mathrm{in}_{\preceq}(J_\Gamma)$ the corresponding initial ideals. The passage to initial ideals leaves the Hilbert functions invariant and $\mathrm{in}_{\preceq}(J_\Delta) \subseteq \mathrm{in}_{\preceq}(J_\Gamma)$; c.f. [7, Proposition 9.3.9]. The corresponding collections of standard monomials are naturally identified with a pair of multicomplexes $\tilde{\Gamma} \subset \tilde{\Delta}$ with f -vector h and this completes the proof. $\square$

5. COHEN–MACAULAY VERSUS FULLY COHEN–MACAULAY

Theorem 1.3 only addresses the characterization of h -vectors of fully CM relative complexes. By definition, a relative simplicial complex Ψ is the set difference of a pair $\Gamma \subset \Delta \subseteq 2^{[n]}$ of simplicial complexes. This presentation is by no means unique and it is natural to ask if in the case that Ψ is Cohen–Macaulay, there are always CM complexes $\Gamma' \subseteq \Delta' \subseteq 2^{[n]}$ of dimension $\dim \Psi$ such that $\Psi = \Delta' \setminus \Gamma'$. The following example shows that this is not the case.

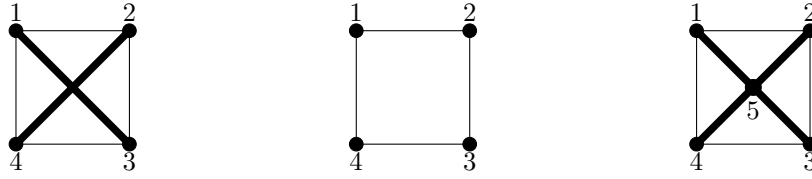


FIGURE 1. The relative complexes of Example 5.1, Example 5.2, and Example 5.3. In each case, Γ is drawn in bold.

EXAMPLE 5.1. Let $\Delta \subset 2^{[4]}$ be the complete graph on 4 vertices, that is, the complex consisting of all subsets of $[4]$ of size at most 2. Let $\Gamma \subset \Delta$ be a perfect matching, see Figure 1. Then $\Delta \setminus \Gamma$ is the relative complex consisting of 4 *open* edges. This is a shellable relative complex. It is easy to check that on the fixed ground set $[4]$, this is the only presentation with $\dim \Delta = \dim \Gamma = 1$ and hence Ψ is not fully Cohen–Macaulay.

There are several possibilities to weaken the requirements on fully Cohen–Macaulay, for example, the requirement that $\dim \Gamma = \dim \Psi$. The next example, however, shows that the characterization of Theorem 1.3 then ceases to hold.

EXAMPLE 5.2. Let $\Delta \subseteq 2^{[4]}$ be the 1-dimensional complex with facets $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}$ and let Γ be the complex composed of the vertices of Δ . Then $\Psi = (\Delta, \Gamma)$ is a relative complex isomorphic to the relative complex of Example 5.1. Both Δ and Γ are Cohen–Macaulay but $\dim \Gamma < \dim \Psi$. In particular, Ψ is shellable with h -vector $h := h(\Psi) = (0, 0, 4)$. However, h is not the f -vector of a relative multicomplex on ground set $[4 - 2]$, as any such (relative) multicomplex can have at most 3 faces of dimension 1.

Nevertheless, it is possible to remedy the problem illustrated in Example 5.1 by allowing more vertices.

EXAMPLE 5.3. Let $\Psi = (\Delta, \Gamma)$ be the relative complex of Example 5.1. Let $\Delta' := \Delta \cup \{\{i, 5\} : i \in [4]\}$ be the graph-theoretic cone over Δ and define Γ' accordingly. Then $\Delta \setminus \Gamma = \Delta' \setminus \Gamma'$ and, since Δ' and Γ' are connected graphs and hence Cohen–Macaulay, this shows that Ψ is a fully Cohen–Macaulay relative complex over the ground set $[5]$.

The following result now shows that every Cohen–Macaulay relative complex is fully Cohen–Macaulay if the ground set is sufficiently enlarged.

THEOREM 5.4. *Let $\Gamma \subset \Delta \subseteq 2^{[n]}$ be simplicial complexes, such that $\Psi = (\Delta, \Gamma)$ is Cohen–Macaulay of dimension $d - 1$. Let e be the depth of $\mathbf{k}[\Gamma]$. Then there exist $\Gamma' \subseteq \Delta' \subseteq 2^{[n+d-e]}$, such that $\Delta' \setminus \Gamma' = \Delta \setminus \Gamma$, and both Δ' and Γ' are Cohen–Macaulay of dimension $d - 1$.*

Proof. Let Γ_1 be the $(d - e)$ -fold cone over Γ and set $\Delta_1 := \Delta \cup \Gamma_1$. Then $\Delta_1 \setminus \Gamma_1 = \Delta \setminus \Gamma$. Further note that $\mathbf{k}[\Gamma_1] = \mathbf{k}[\Gamma][y_1, \dots, y_{d-e}]$, where the y_i are new variables. Thus, the depth of $\mathbf{k}[\Gamma_1]$ is d . Finally, we define Δ' and Γ' to be the $(d - 1)$ -dimensional skeleta of Δ_1 and Γ_1 , respectively. Again, $\Delta' \setminus \Gamma' = \Delta \setminus \Gamma$ and thus $\Psi \cong (\Delta', \Gamma')$. By [11, Corollary 2.6], Γ' is Cohen–Macaulay. By assumption, $\Psi = \Delta' \setminus \Gamma'$ is Cohen–Macaulay, and since $\dim \Psi = \dim \Delta' = \dim \Gamma'$, it follows from [6, Proposition 1.2.9] that Δ' is also Cohen–Macaulay. $\square$

In the construction given in the course of the proof, the complexes Δ and Γ occur as induced subcomplexes. If we are to abandon this requirement, then our computations

suggest that it suffices to add a single new vertex. Based on this evidence, we offer the following conjecture.

CONJECTURE 5.5. *Every Cohen–Macaulay relative complex Ψ on ground set $[n]$ is a fully Cohen–Macaulay relative complex on ground set $[n+1]$. That is, for every $(d-1)$ -dimensional Cohen–Macaulay relative complex $\Psi = (\Delta, \Gamma)$ on ground set $[n]$, there are Cohen–Macaulay simplicial complexes $\Gamma' \subseteq \Delta' \subseteq 2^{[n+1]}$ of dimension $d-1$, such that $\Delta \setminus \Gamma = \Delta' \setminus \Gamma'$.*

We also offer a more precise conjecture on how the complexes $\Gamma' \subset \Delta'$ can be obtained.

CONJECTURE 5.6. *Let $\emptyset \neq \Gamma \subsetneq \Delta \subset 2^{[n]}$ be two simplicial complexes, such that the relative complex (Δ, Γ) is Cohen–Macaulay of dimension $d-1$ over some field $\mathbf{k}$. If Δ and Γ have no common minimal non-faces, then the depth of $\mathbf{k}[\Gamma]$ is at least $d-1$.*

To see that Conjecture 5.6 implies Conjecture 5.5, let $\Psi = (\Delta, \Gamma)$ be a given presentation. We can assume that Δ and Γ have no minimal non-faces in common. Conjecture 5.6 then assures us that $\mathbf{k}[\Gamma]$ has depth $d-1$ and Theorem 5.4 yields Conjecture 5.5.

Instead of fixing the ground set, we may instead consider the dimensions of the minimal faces in $\Psi = (\Delta, \Gamma)$. For a sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ of numbers and $i \geq 0$ we set

$$E^i \alpha := (0, \dots, 0, \alpha_1, \alpha_2, \alpha_3, \dots),$$

$\underbrace{\hspace{1.5cm}}_i$

THEOREM 5.7. *For a vector $h = (h_0, \dots, h_d) \in \mathbb{Z}_{\geq 0}^{d+1}$ and numbers $a_1, \dots, a_r \in \mathbb{Z}_{\geq 0}$, the following are equivalent:*

- (i) $h = h(\Delta, \Gamma)$ for a shellable relative complex (Δ, Γ) , whose minimal faces have cardinalities $a_1, \dots, a_r$;
- (ii) $h = h(\Delta, \Gamma)$ for a Cohen–Macaulay relative complex (Δ, Γ) , whose minimal faces have cardinalities $a_1, \dots, a_r$;
- (iii) h is the h -vector of a graded Cohen–Macaulay module (over some polynomial ring), whose generators have the degrees $a_1, \dots, a_r$;
- (iv) There exist M -sequences $\nu_1, \dots, \nu_r$ such that

$$h = E^{a_1} \nu_1 + E^{a_2} \nu_2 + \dots + E^{a_r} \nu_r.$$

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear, and (iii) $\Rightarrow$ (iv) is Proposition 5.2 of [18]. In loc. cit. Anders Björner asked if the implication (iv) $\Rightarrow$ (iii) also holds.

Proof. We only need to show (iv) $\Rightarrow$ (i). For each i , we can find a shellable simplicial complex Δ_i whose h -vector is ν_i . Further, let $v_{i1}, \dots, v_{ia_i}$ be new vertices and let Ψ_i be the relative complex with faces $\{F \cup \{v_{i1}, \dots, v_{ia_i}\} : F \in \Delta_i\}$. It is clear that any shelling order on Δ_i yields a shelling on Ψ_i , and that $h(\Psi_i) = E^{a_i} \nu_i$. Finally, by taking cones if necessary, we may assume that all the Ψ_i have the same dimension. Then the disjoint union of the Ψ_i is the desired shellable relative complex. $\square$

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