ALGEBRAIC COMBINATORICS

Connor Sawaske

Stanley–Reisner rings of simplicial complexes with a free action by an abelian group

Volume 1, issue 5 (2018), p. 677-695.

<http://alco.centre-mersenne.org/item/ALCO_2018__1_5_677_0>

Algebraic Combinatorics
Volume 1, issue 5 (2018), p. 677–695
https://doi.org/10.5802/alco.29

Stanley–Reisner rings of simplicial complexes with a free action by an abelian group

Connor Sawaske

Abstract We consider simplicial complexes admitting a free action by an abelian group. Specifically, we establish a refinement of the classic result of Hochster describing the local cohomology modules of the associated Stanley–Reisner ring, demonstrating that the topological structure of the free action extends to the algebraic setting. If the complex in question is also Buchsbaum, this new description allows for a specialization of Schenzel's calculation of the Hilbert series of some of the ring's Artinian reductions. In further application, we generalize to the Buchsbaum case the results of Stanley and Adin that provide a lower bound on the h-vector of a Cohen–Macaulay complex admitting a free action by a cyclic group of prime order.

1. Introduction

Since the 1970's, an extensive dictionary translating the topological and combinatorial properties of a simplicial complex Δ into algebraic properties of the associated Stanley–Reisner ring $\Bbbk[\Delta]$ has been constructed. For instance, the local cohomology modules $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ of $\Bbbk[\Delta]$ have a beautiful interpretation due to Hochster (and later Gräbe) as topological invariants of Δ (see [6, Theorem 2], [20], or [24, Section II.4]). The following formulation of their results is the starting point of this paper (here $\operatorname{cost}_\Delta \sigma$ denotes the contrastar of a face σ of Δ, while $H^{i-1}(\Delta, \operatorname{cost}_\Delta \sigma)$ is the relative simplicial cohomology of the pair $(\Delta, \operatorname{cost}_\Delta \sigma)$ computed with coefficients in $\Bbbk$ and $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ is the local cohomology module of $\Bbbk[\Delta]$):

Theorem 1.1. *Let Δ be a simplicial complex on vertex set $\{1,\dots,n\}$ and let $\Bbbk$ be a field. Then*
$$H^i_{\mathfrak{m}}(\Bbbk[\Delta])_{-j} \cong \bigoplus_{U\in T(\Delta)_j} H^{i-1}(\Delta, \operatorname{cost}_\Delta s(U))$$
as vector spaces over $\Bbbk$, where $T(\Delta)_j = \{U \in \mathbb{N}^n : s(U) \in \Delta \text{ and } |U| = j\}$, and for $U = (u_1,\dots,u_n) \in \mathbb{N}^n$, $s(U) = \{k : u_k > 0\}$ and $|U| = \sum_{k=1}^n u_k$.

On the other hand, the effect on the Stanley–Reisner ring of a simplicial complex being endowed with a group action has also been studied in some depth (see [24, Section III.8] and [1]). Our primary goal is to bring these two topics together by studying the additional structure on $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ that appears when Δ admits a group

Manuscript received 16th January 2018, revised 18th May 2018, accepted 20th June 2018.

Keywords. Stanley–Reisner rings, local cohomology, group actions.

Acknowledgements. The author was partially supported by graduate fellowships from NSF grants DMS-1361423 and DMS-1664865.

action. As it turns out, the topological invariants introduced by the group action dictate a more detailed description of $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$, providing our main result.

Our secondary goal is to apply this new decomposition of $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ to extend two separate classes of previously-existing results describing the h-vectors (perhaps the most widely-recognized and studied combinatorial invariants) of certain simplicial complexes. One of these classes of results examines Buchsbaum simplicial complexes: using Theorem 1.1, Schenzel was able to calculate the Hilbert series of the quotient of $\Bbbk[\Delta]$ by a linear system of parameters in the case that Δ is Buchsbaum. The culmination of this line of study was the following theorem appearing in [21], providing an algebraic interpretation of the h-vector of Δ.

Theorem 1.2 (Schenzel). *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex, and let Θ be a linear system of parameters for Δ. Then*

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Theta\Bbbk[\Delta]\right)_i = h_i(\Delta) + \binom{d}{i}\sum_{j=0}^{i-1}(-1)^{i-j-1}\beta_{j-1}(\Delta),$$

where $\beta_{j-1}(\Delta)$ denotes the reduced simplicial Betti number of Δ computed over $\Bbbk$.

Since any dimension must be non-negative, the theorem provides lower bounds for the entries of $h(\Delta)$ in terms of some topological invariants of Δ. More recently, these bounds have been further lowered by the use of socles ([18]) and the sigma module $\Sigma(\Theta;\Bbbk[\Delta])$ (originally introduced by Goto in [5], and whose definition is provided in Section 2).

Theorem 1.3 (Murai–Novik–Yoshida). *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex and let Θ be a linear system of parameters for Δ. Then*

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_i = h_i(\Delta) + \binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta).$$

Furthermore, the symmetry appearing in the h-vector described by Gräbe in [7] was reformulated by Novik and Murai in [15, Proposition 1.1], and using Theorem 1.3 it may be described as follows:

Theorem 1.4 (Murai–Novik–Yoshida). *Let Δ be a $(d-1)$-dimensional orientable $\Bbbk$-homology manifold and let Θ be a linear system of parameters for Δ. Then*

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_i = \dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_{d-i}$$

for $i=0,\dots,d$.

The second class of results that we will enlarge deals with the h-vectors of complexes admitting specific group actions by $\mathbb{Z}/p\mathbb{Z}$ in the Cohen–Macaulay case. In particular, the following two theorems ([22, Theorem 3.2] and [1, Theorem 3.3], respectively) provide impressive bounds on the h-vector of such a complex and are ripe for extensions to more general complexes:

Theorem 1.5 (Stanley). *Let Δ be a $(d-1)$-dimensional Cohen–Macaulay (over $\mathbb{C}$) simplicial complex admitting a very free action by $\mathbb{Z}/p\mathbb{Z}$ with p a prime. Then $h_i(\Delta)\geqslant\binom{d}{i}$ if i is even and $h_i(\Delta)\geqslant(p-1)\binom{d}{i}$ if i is odd.*

Theorem 1.6 (Adin). *Let Δ be a $(d-1)$-dimensional Cohen–Macaulay (over $\mathbb{C}$) simplicial complex with a free action of $\mathbb{Z}/p\mathbb{Z}$ with p a prime such that d is divisible by $p-1$. Then*

$$\sum_{i=0}^{d}h_i(\Delta)\lambda^i \geqslant (1+\lambda+\cdots+\lambda^{p-1})^{d/(p-1)},$$

where the inequality holds coefficient-wise.

In summary, our new results are the following (as much notation must be introduced before explicit statements may be provided, we present here a brief overview):

- *A new version of Hochster and Gräbe's theorem:* We examine how a group action leads to a special decomposition of $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$, and we demonstrate how the isomorphism of Theorem 1.1 respects this decomposition (Theorem 3.1). Later, for the $G = \mathbb{Z}/p\mathbb{Z}$ case, we encounter a piece of the local cohomology of $\Bbbk[\Delta]$ that corresponds to the singular cohomology of the quotient $|\Delta|/G$.
- *Applications to h-vectors:* The dimension calculations of Theorems 1.2 and 1.3 are refined and specialized to the case of a Buchsbaum complex admitting a free action by a cyclic group of prime order (Theorems 4.2 and 4.8). Using this refinement, Theorems 1.5 and 1.6 are generalized to the setting of Buchsbaum complexes; in some cases, the expressions are identical (Section 4.3). Lastly, in Theorems 4.12 and 4.15, we exhibit a symmetry in the dimensions of finely graded pieces of some Artinian reductions of Stanley–Reisner rings of certain orientable homology manifolds without boundary admitting a free group action by a cyclic group of prime order in the flavor of Theorem 1.4.

The paper is organized as follows. In Section 2, we introduce notation, review classical results, and discuss in some depth the theory of graded rings and modules. In Section 3 we study the local cohomology modules $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ and provide our main new result. Section 4 is devoted to calculating a finely-graded Hilbert series of an Artinian reduction of the Stanley–Reisner ring $\Bbbk[\Delta]$, providing many applications of our main theorem. We close with comments and questions in Section 5.

2. Preliminaries

2.1. Combinatorics and topology. A *simplicial complex* Δ on a finite vertex set V is a collection of subsets of V that is closed under inclusion. The elements of Δ are called *faces*, and the maximal faces (with respect to inclusion) are *facets*. The *dimension* of a face σ is defined by $\dim\sigma := |\sigma| - 1$, and the *dimension* of Δ is defined by $\dim\Delta := \max\{\dim\sigma : \sigma \in \Delta\}$. The faces of dimension zero are called *vertices*. If all facets of Δ have the same dimension, then we say that Δ is *pure*.

Given a face σ of Δ, we define the *contrastar* of σ in Δ by

$$\operatorname{cost}_\Delta \sigma := \{\tau \in \Delta : \sigma \not\subseteq \tau\}.$$

Similarly, the *link* of σ in Δ is

$$\operatorname{lk}_\Delta \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \varnothing\}.$$

We define the *f-vector* $f(\Delta)$ of a $(d-1)$-dimensional simplicial complex Δ by

$$f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta)),$$

where $f_i(\Delta)$ is the number of i-dimensional faces of Δ. The *h-vector* $h(\Delta)$ is then defined by $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$, where

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}(\Delta).$$

Let $\Bbbk$ be a field, and let $\widetilde{H}^i(\Delta)$ be the i-th reduced simplicial cohomology group of Δ with coefficients in $\Bbbk$. If Γ is a subcomplex of Δ, then $H^i(\Delta, \Gamma)$ is the i-th relative cohomology group of the pair (Δ, Γ) with coefficients in $\Bbbk$. In the case that $\Gamma = \{\varnothing\}$, this is the same as the reduced cohomology group $\widetilde{H}^i(\Delta)$. Denote the *i-th (reduced) Betti number* of Δ over $\Bbbk$ by

$$\beta_i(\Delta) := \dim_\Bbbk \widetilde{H}^i(\Delta).$$

We call a complex Δ *Cohen–Macaulay* if $\beta_i(\mathrm{lk}_\Delta \sigma) = 0$ for all faces σ and all $i < \dim \mathrm{lk}_\Delta \sigma$. Similarly, we call a complex *Buchsbaum* if Δ is pure and $\beta_i(\mathrm{lk}_\Delta \sigma) = 0$ for all faces $\sigma \neq \varnothing$ and all $i < \dim \mathrm{lk}_\Delta \sigma$. If Δ is $(d-1)$-dimensional and the link of each non-empty face σ of Δ has the homology of a $(d-|\sigma|-1)$-sphere, then we say that Δ is a $\Bbbk$-*homology manifold*; furthermore, if $\beta_{\dim \Delta}(\Delta)$ is equal to the number of connected components of Δ, then we say that Δ is *orientable*.

Now let G be a finite group and suppose G acts on a simplicial complex Δ (in particular, each $g \in G$ acts as a simplicial isomorphism on Δ). We say that G acts *freely* if $g \cdot \sigma \neq \sigma$ for all faces $\sigma \neq \varnothing$ and all non-identity elements $g \in G$ (we will always take our actions to be defined on the left). If G acts freely on Δ and, additionally, $(g \cdot \{v\}) \cup \{v\}$ is not an edge of Δ for every vertex $v \in V$ and every non-identity element $g \in G$, then we say that G acts *very freely*.

In the case that $G = \mathbb{Z}/2\mathbb{Z}$, free and very free actions are equivalent and we call Δ *centrally-symmetric* when such an action by G on Δ exists. When extending a free group action by G to the geometric realization $|\Delta|$ of Δ, the action obtained is a covering space action (see [9, Section 1.3]).

If Δ admits a group action by G, then this extends to an action on the reduced (simplicial) chain complex $\tilde{C}_\bullet(\Delta)$ by permuting its basis elements in accordance with the group action. This extends further to the cochain complex $\tilde{C}^\bullet(\Delta) := \mathrm{Hom}_\Bbbk(\tilde{C}_\bullet(\Delta), \Bbbk)$ in the usual way: if $\hat{\sigma} \in \tilde{C}^i(\Delta)$ is the basis element defined by $\hat{\sigma}(\tau) = 0$ for $\sigma \neq \tau$ and $\hat{\sigma}(\sigma) = 1$, then

$$(g \cdot \hat{\sigma})(\tau) = \hat{\sigma}(g^{-1} \cdot \tau),$$

and hence $g \cdot \hat{\sigma} = \widehat{(g \cdot \sigma)}$.

Lastly, given a vector $U = (u_1, \ldots, u_n) \in \mathbb{N}^n$, let $s(U) = \{i : u_i > 0\}$ be the support of U and let $|U| = \sum_{i=1}^n u_j$ be the L^1-norm of U. If Δ is a simplicial complex on vertex set $[n]$, then denote

$$T(\Delta)_j = \{U \in \mathbb{N}^n : s(U) \in \Delta \text{ and } |U| = j\}.$$

2.2. Modules and group actions. Assume from now on that the field $\Bbbk$ is an extension of $\mathbb{C}$. If G is a finite abelian group and M is a $\Bbbk[G]$-module, then M admits a decomposition into isotypic components according to the action of G. That is,

$$M = \bigoplus_{\chi \in \hat{G}} M^\chi$$

where $\hat{G}$ is the group of irreducible characters χ of G and

$$M^\chi := \{m \in M : g \cdot m = \chi(g)m \text{ for all } g \in G\}.$$

Of vital importance will be the consideration of direct sums of simplicial cohomology modules of the form

$$\bigoplus_{U \in T(\Delta)_j} H^{i-1}(\Delta, \mathrm{cost}_\Delta s(U)),$$

which inherit a G-action from Δ as in the previous section. In particular, Theorem 3.1 will consider the isotypic components of modules of this form. In the $j = 0$ case, we set

$$\beta_i(\Delta)^\chi := \dim_\Bbbk \widetilde{H}^i(\Delta)^\chi. \tag{1}$$

These refined Betti numbers will be one of the main invariants considered in Sections 4 and 5 of this paper.

Now let A be the polynomial ring $\Bbbk[x_1, \ldots, x_n]$, graded by degree, and let M be a finitely-generated $\mathbb{Z}$-graded A-module, written as an abelian group by $M = \bigoplus_i M_i$.

If G acts on M and each M_i is G-invariant (i.e. , $G \cdot M_i \subseteq M_i$), then each M_i can itself be thought of as a $\Bbbk[G]$-module. As above,

$$M_i = \bigoplus_{\chi \in \hat{G}} M_i^\chi \tag{2}$$

where

$$M_i^\chi = \{m \in M_i : g \cdot m = \chi(g)m \text{ for all } g \in G\}.$$

Suppose A is endowed with a degree-preserving action of G. Then the decomposition in (2) makes A into a $(\mathbb{Z} \times \hat{G})$-graded ring, since $a_1 \in A_{i_1}^{\chi_1}$ and $a_2 \in A_{i_2}^{\chi_2}$ implies that $a_1 a_2 \in A_{i_1+i_2}$ and that

$$g \cdot (a_1 a_2) = (g \cdot a_1)(g \cdot a_2) = \chi_1(g) a_1 \chi_2(g) a_2 = (\chi_1\chi_2)(g) a_1 a_2$$

for all $g \in G$. If, additionally,

$$g \cdot (am) = (g \cdot a)(g \cdot m) \tag{3}$$

for all $a \in A$, $g \in G$, and $m \in M$, then the decomposition in (2) allows for a $(\mathbb{Z} \times \hat{G})$-grading for M, where M_i^χ is the component of M consisting of all elements of degree (i, χ).

If N is another $(\mathbb{Z} \times \hat{G})$-graded A-module satisfying the same conditions as M, then G acts on $\mathrm{Hom}_A(M, N)$ by defining

$$(g \cdot f)(m) := g \cdot f(g^{-1} \cdot m)$$

for all $g \in G$, $f \in \mathrm{Hom}_A(M, N)$, and $m \in M$. If $m \in M_j^{\chi_1}$ and $f \in \mathrm{Hom}_A(M, N)_i^{\chi_2}$, then for all $g \in G$,

$$\chi_2(g) f(m) = (g \cdot f)(m) = g \cdot f(g^{-1} \cdot m) = g \cdot f(\chi_1^{-1}(g) m) = \chi_1^{-1}(g) g \cdot f(m),$$

so $g \cdot f(m) = (\chi_1\chi_2)(g) f(m)$ and $f(m) \in N_{i+j}^{\chi_1\chi_2}$. On the other hand, if $f \in \mathrm{Hom}_A(M, N)$ is such that $f(m) \in N_{i+j}^{\chi_1\chi_2}$ for all choices of i, χ_1, and $m \in M_i^{\chi_1}$, then $g \cdot f = \chi_2(g) f$ for all $g \in G$. Hence,

$$\mathrm{Hom}_A(M, N)_i^{\chi_1} = \{f \in \mathrm{Hom}_A(M, N) : f(M_j^{\chi_2}) \subseteq N_{i+j}^{\chi_1\chi_2} \text{ for all } (j, \chi_2) \in \mathbb{Z} \times \hat{G}\}.$$

That is, the standard $(\mathbb{Z} \times \hat{G})$-grading on $\mathrm{Hom}_A(M, N)$ is consistent with the induced one above. In general, the set of graded A-module homomorphisms from M to N is a submodule of $\mathrm{Hom}_A(M, N)$. However, in our case of M being finitely generated, the two submodules are equal (see [2, Section II.11.6]). Unless stated otherwise, a $(\mathbb{Z} \times \hat{G})$-graded map of A-modules will refer to a homomorphism of degree 0.

Let $\mathfrak{m} = (x_1, \ldots, x_n)$ denote the irrelevant ideal of A, let $H^i_{\mathfrak{m}}(M)$ denote the i-th local cohomology module of M with support in $\mathfrak{m}$ (see [10] for some basic properties of these modules), and denote $\mathfrak{m}_j = (x_1^j, \ldots, x_n^j)$ (not to be confused with M_j, the j-th graded piece of M as a $\mathbb{Z}$-graded A-module). We can trace the extra grading of Hom modules through to some standard local cohomology results using the following proposition.

Proposition 2.1. *Let G be a finite abelian group, and let M be a $(\mathbb{Z} \times \hat{G})$-graded A-module with an action of G satisfying* (3). *Then:*

1. *$\mathrm{Ext}^i_A(A/\mathfrak{m}_j, M)$ and $H^i_{\mathfrak{m}}(M)$ are $(\mathbb{Z} \times \hat{G})$-graded A-modules for all i and all $j \geqslant 1$.*
2. *The canonical maps $\psi^M_{i,j} : \mathrm{Ext}^i_A(A/\mathfrak{m}_j, M) \to H^i_{\mathfrak{m}}(M)$ are $(\mathbb{Z} \times \hat{G})$-graded.*
3. *If N satisfies the same conditions as M and $f : M \to N$ is a map of $(\mathbb{Z} \times \hat{G})$-graded A-modules, then the induced map $f^* : H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(N)$ is also $(\mathbb{Z} \times \hat{G})$-graded.*

(4) *If* $0 \to L \to M \to N \to 0$ *is a short exact sequence of* $(\mathbb{Z} \times \hat{G})$*-graded* A*-modules, then there is a long exact sequence*

$$\cdots \to H^{i-1}_{\mathfrak{m}}(N) \xrightarrow{\delta} H^i_{\mathfrak{m}}(L) \to H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(N) \xrightarrow{\delta} H^{i+1}_{\mathfrak{m}}(L) \to \cdots$$

of $(\mathbb{Z} \times \hat{G})$*-graded* A*-modules.*

Remark 2.2. While these statements are to be expected and follow the typical constructions, we could not find any all-encompassing reference for them. Since many of the modules and maps involved are vital to the proofs of Theorems 3.1 and 4.2, the proofs have been included both for the sake of completeness and for a preview of what is to come.

Proof. Let $K^j_\bullet$ denote the Koszul complex of A with respect to the sequence $x_1^j, \dots, x_n^j$. We view each term K^j_t of $K^j_\bullet$ as

$$K^j_t = \bigoplus_{1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant n} A(x^j_{i_1} \wedge x^j_{i_2} \wedge \cdots \wedge x^j_{i_t}),$$

and we can naturally extend the action of G on A to an action of G on $K^j_\bullet$ by defining

$$g \cdot \left(f x^j_{i_1} \wedge \cdots \wedge x^j_{i_t} \right) = (g \cdot f)((g \cdot x_{i_1})^j \wedge \cdots \wedge (g \cdot x_{i_t})^j).$$

Then equation (3) holds for each K^j_t, making them $(\mathbb{Z} \times \hat{G})$-graded A-modules. The differential map on $K^j_\bullet$ preserves the G action, so $\operatorname{Hom}_A(K^j_\bullet, M)$ is a cochain complex of $(\mathbb{Z} \times \hat{G})$-graded A-modules. Furthermore, the cohomology modules obtained from this cochain complex are also $(\mathbb{Z}\times\hat{G})$-graded A-modules (see [2, Proposition II.11.3.3]).

Since $K^j_\bullet$ provides a projective resolution of $A/\mathfrak{m}_j$,

$$H^i(\operatorname{Hom}_A(K^j_\bullet, M)) = \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M).$$

The maps $\varphi_j : K^{j+1}_\bullet \to K^j_\bullet$ defined by $\varphi_j(x^{j+1}_{i_1} \wedge \cdots \wedge x^{j+1}_{i_t}) = (x_{i_1} \cdots x_{i_t}) x^j_{i_1} \wedge \cdots \wedge x^j_{i_t}$ also preserve the action of G, so the pullbacks

$$\varphi_j^* : \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M) \to \operatorname{Ext}^i_A(A/\mathfrak{m}_{j+1}, M)$$

do as well. These maps provide a direct system of $(\mathbb{Z}\times\hat{G})$-graded A-modules in which

$$H^i_{\mathfrak{m}}(M) := \varinjlim \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M).$$

By [2, Remark II.11.3.3], $H^i_{\mathfrak{m}}(M)$ is a $(\mathbb{Z} \times \hat{G})$-graded A-module and the canonical maps $\psi^M_{i,j} : \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M) \to H^i_{\mathfrak{m}}(M)$ also preserve the action of G. This proves (1) and (2), from which (3) follows almost immediately; if $m \in H^i_{\mathfrak{m}}(M)$, choose j and m_j such that $m = \psi^M_{i,j}(m_j)$, and let

$$f_j : \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M) \to \operatorname{Ext}^i_A(A/\mathfrak{m}_j, N)$$

be the map induced in Ext modules. It follows from the definitions that f_j is a $(\mathbb{Z}\times\hat{G})$-graded map, while $f^*(m) = (f \circ \psi^M_{i,j})(m_j) = (\psi^N_{i,j} \circ f_j)(m_j)$. Since all maps here are $(\mathbb{Z} \times \hat{G})$-graded, f^* is as well.

For part (4), all that remains to be checked is that the connecting homomorphism δ is $(\mathbb{Z} \times \hat{G})$-graded. This follows from its standard construction, as it is defined as a composition of certain $(\mathbb{Z} \times \hat{G})$-graded maps. ☐

For any integer a, denote by $M[a]$ the shifted $\mathbb{Z}$-graded A-module defined by $M[a]_i = M_{i+a}$. If we are also given $\chi_1 \in \hat{G}$, then we can define a new G-action $*$ on $M[a]$ by setting

$$g * m = \chi_1^{-1}(g)(g \cdot m).$$

for all $g \in G$. Denote by $M[a, \chi_1]$ the module $M[a]$ endowed with this new action of G. Then for all $g \in G$,

$$\{m \in M_{i+a} : g \cdot m = (\chi_1\chi_2)(g)m\} = \{m \in M[a]_i : g * m = \chi_2(g)m\},$$

hence $M[a, \chi_1]_i^{\chi_2} = M_{i+a}^{\chi_1\chi_2}$, and this makes $M[a, \chi_1]$ into a $(\mathbb{Z} \times \hat{G})$-graded A-module. Indeed, if $m \in M[a, \chi_1]_i^{\chi_2}$ and $f \in A_j^{\chi_3}$, then

$$g * (fm) = \chi_1^{-1}(g)(g \cdot fm) = \chi_1^{-1}(g)(\chi_1\chi_2\chi_3)(g)fm = (\chi_2\chi_3)(g)fm,$$

so that $fm \in M[a, \chi_1]_{i+j}^{\chi_2\chi_3}$. Moreover, if $f \in A_j^{\chi_2}$ then the multiplication map $\cdot f : M[-j, \chi_2^{-1}] \to M$ is a map of $(\mathbb{Z} \times G)$-graded A-modules (of degree 0), since if $m \in M[-j, \chi_2^{-1}]_i^{\chi_1}$, then

$$f(g * m) = f\chi_2(g)(g \cdot m) = (g \cdot f)(g \cdot m) = g \cdot (fm).$$

For the rest of this section, we will only be considering groups G of the form $\mathbb{Z}/p\mathbb{Z}$ for a prime p. In this case, $G \cong \hat{G}$. Once we have fixed a generator g for G and a primitive p-th root of unity ζ, there is a group isomorphism mapping $\chi \in \hat{G}$ to $j \in \mathbb{Z}/p\mathbb{Z}$ such that

$$M^\chi = \{m \in M : g \cdot m = \zeta^j m\}.$$

Hence, we write

$$M = \bigoplus_{j=0}^{p-1} M^j$$

where

$$M^j := \{m \in M : g \cdot m = \zeta^j m\}.$$

With this identification, we view all of the modules constructed above as $(\mathbb{Z} \times G)$-graded modules, where M_i^j is the component of M in degree (i, j). As in our definition (1), we set

$$\beta_i(\Delta)^j := \dim_{\Bbbk} \widetilde{H}^i(\Delta)^j. \tag{4}$$

Thinking now of M as a $(\mathbb{Z} \times G)$-graded vector space over $\Bbbk$, we can define the Hilbert series $\mathrm{Hilb}(M, \lambda, t)$ of M by

$$\mathrm{Hilb}(M, \lambda, t) = \sum_{(i,j)\in(\mathbb{Z}\times G)} (\dim_{\Bbbk} M_i^j)\lambda^i t^j$$

where λ and t are indeterminates with $t^p = 1$.

If M is of Krull dimension $d > 0$, we call a system $\Theta = \theta_1, \theta_2, \ldots, \theta_d$ of homogeneous elements in A a *homogeneous system of parameters* (or an h.s.o.p.) for M if $M/\Theta M$ is a finite-dimensional vector space over $\Bbbk$. If each $\theta_i \in A_1$, then we call Θ a *linear system of parameters* (or an l.s.o.p.) for M. We say that M is *Cohen–Macaulay* if every l.s.o.p. is a regular sequence on M, and M is *Buchsbaum* if every l.s.o.p. satisfies

$$(\theta_1, \ldots, \theta_{i-1})M :_M \theta_i = (\theta_1, \ldots, \theta_{i-1})M :_M \mathfrak{m}$$

for $i = 1, \ldots, d$. Given any h.s.o.p. Θ for M, we also have the notion of the sigma module $\Sigma(\Theta; M)$, defined by

$$\Sigma(\Theta; M) := \Theta M + \sum_{i=1}^{d} \left((\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_d)M :_M \theta_i\right).$$

2.3. Stanley–Reisner rings. Let Δ be a $(d-1)$-dimensional simplicial complex with vertex set $[n] := \{1, \dots, n\}$. Given $\sigma \subset [n]$, write $x_\sigma = \prod_{i \in \sigma} x_i$. The *Stanley–Reisner ideal* of Δ is the ideal I_Δ of A defined by

$$I_\Delta = (x_\sigma : \sigma \subset [n], \sigma \notin \Delta).$$

The *Stanley–Reisner ring* of Δ (over $\Bbbk$) is

$$\Bbbk[\Delta] := A/I_\Delta.$$

We will always consider $\Bbbk[\Delta]$ as a module over A. The geometric notion of Buchsbaumness is tied algebraically to the Stanley–Reisner ring through the following vital result (found in [21]).

Theorem 2.3 (Schenzel). *A pure simplicial complex Δ is Buchsbaum over $\Bbbk$ if and only if $\Bbbk[\Delta]$ is a Buchsbaum A-module.*

If Δ admits an action by the group $G = \mathbb{Z}/p\mathbb{Z}$, then this extends to an action on $\Bbbk[\Delta]$ and induces a $(\mathbb{Z} \times G)$-grading as detailed in the previous section. If this action is free, then for any j and $i \geqslant 1$ we have

$$\dim_\Bbbk \Bbbk[\Delta]_i^j = \frac{1}{p} \dim_\Bbbk \Bbbk[\Delta]_i.$$

Also, $\dim_\Bbbk \Bbbk[\Delta]_0^0 = 1$ and $\dim_\Bbbk \Bbbk[\Delta]_0^j = 0$ for $0 < j < p$. As in [22, Section 3], this implies the following expression for the $(\mathbb{Z} \times G)$-graded Hilbert series of $\Bbbk[\Delta]$.

Theorem 2.4. *Let Δ be a $(d-1)$-dimensional simplicial complex admitting a free action by the group $\mathbb{Z}/p\mathbb{Z}$. Then*

$$\operatorname{Hilb}(\Bbbk[\Delta], \lambda, t) = 1 + \frac{1}{p}\left[\frac{\sum_{i=1}^d h_i(\Delta)\lambda^i}{(1-\lambda)^d} - 1\right](1 + t + \cdots + t^{p-1}).$$

In fact, if Δ is centrally-symmetric (so $p = 2$) and Cohen–Macaulay, then a certain Hilbert series provides a strong inequality bounding the h-vector of Δ; see [24, Theorem III.8.1]. If Θ is an l.s.o.p. for Δ, then we denote by $\Bbbk(\Delta; \Theta)$ the quotient $\Bbbk[\Delta]/\Theta\Bbbk[\Delta]$.

Theorem 2.5 (Stanley). *Let Δ be a $(d-1)$-dimensional centrally-symmetric Cohen–Macaulay simplicial complex, and suppose $\Theta = (\theta_1, \dots, \theta_d)$ is an l.s.o.p. for $\Bbbk[\Delta]$ in which $\theta_i \in \Bbbk[\Delta]_1^1$ for $i = 1, \dots, d$. Then*

$$\operatorname{Hilb}(\Bbbk(\Delta; \Theta), \lambda, t) = \frac{1}{2}\left[(1-t)(1+\lambda)^d + (1+t)\sum_{i=0}^d h_i(\Delta)\lambda^i\right].$$

3. The main theorem: a refinement of Hochster

For the rest of the paper, fix Δ to be a $(d-1)$-dimensional simplicial complex on vertex set $[n]$. Let $\Bbbk$ be an extension of $\mathbb{C}$, and let A be the polynomial ring $\Bbbk[x_1, \dots, x_n]$. Recall that $T(\Delta)_j$ is the set $\{U \in \mathbb{N}^n : s(U) \in \Delta \text{ and } |U| = j\}$.

Theorem 3.1. *Let Δ be a simplicial complex with a (not necessarily free) action by a finite abelian group G. Then the isomorphisms*

$$H^i_\mathfrak{m}(\Bbbk[\Delta])_{-j} \cong \bigoplus_{U \in T(\Delta)_j} H^{i-1}(\Delta, \operatorname{cost}_\Delta s(U))$$

of Theorem 1.1 *induce vector space isomorphisms*

$$H^i_\mathfrak{m}(\Bbbk[\Delta])^\chi_{-j} \cong \left[\bigoplus_{U \in T(\Delta)_j} H^{i-1}(\Delta, \operatorname{cost}_\Delta s(U))\right]^\chi$$

for each irreducible character χ of G.

Remark 3.2. There are a multitude of proofs of Theorem 1.1, such as [6, Theorem 1] or [13, Corollary 4.4]. Our proof of the refinement above will be a modification of that which appears in [12].

Proof. As in Section 2.2, let $\mathfrak{m}_{j+1}$ denote the ideal $(x_1^{j+1},\dots,x_n^{j+1})$ and let $K_\bullet^{j+1}$ denote the Koszul complex of A with respect to the sequence $(x_1^{j+1},\dots,x_n^{j+1})$. If $C^\bullet(\Delta,\Gamma)$ denotes the relative simplicial cochain complex of the pair (Δ,Γ) with coefficients in $\Bbbk$, then we know from the proofs of Proposition 2.1 and [12, Theorem 1] that

(5)
$$H^i(\mathrm{Hom}_A(K_\bullet^{j+1},\Bbbk[\Delta])_{-j}) = \mathrm{Ext}_A^i(A/\mathfrak{m}_{j+1},\Bbbk[\Delta])_{-j} \cong \bigoplus_{U\in T(\Delta)_j} H^{i-1}(C^\bullet(\Delta,\mathrm{cost}_\Delta\, s(U)))$$

as vector spaces over $\Bbbk$. Note that while [12, Theorem 1] is stated with respect to a $\mathbb{Z}^n$-grading, Reisner's proof in [20, Theorem 2] shows that the only non-acyclic components of the cochain complex $\mathrm{Hom}_A(K_\bullet^{j+1},\Bbbk[\Delta])$ of $\mathbb{Z}$-graded A-modules also occur in the $\mathbb{Z}^n$-graded case.

Our first goal is to show that this isomorphism preserves isotypic components in the sense that it induces isomorphisms of the form

$$\mathrm{Ext}_A^i(A/\mathfrak{m}_{j+1},\Bbbk[\Delta])_{-j}^{\chi} \cong \left[\bigoplus_{U\in T(\Delta)_j} H^{i-1}(C^\bullet(\Delta,\mathrm{cost}_\Delta\, s(U)))\right]^{\chi} \tag{6}$$

for each irreducible character χ of G.

Given some set $\sigma=\{i_1,\dots,i_t\}\subset[n]$ with $i_1<i_2<\dots<i_t$, define $\overline{x}_\sigma := x_{i_1}^{j+1}\wedge\dots\wedge x_{i_t}^{j+1}$ in K_t^{j+1} and recall that $x_\sigma := x_{i_1}\cdots x_{i_t}$ in $\Bbbk[\Delta]$. Likewise, if $U=(i_1,\dots,i_n)\in\mathbb{N}^n$, define $x_U := x_1^{i_1}\cdots x_n^{i_n}$ in $\Bbbk[\Delta]$. If $\sigma\subset[n]$ is such that $|\sigma|=t$ and $U\in\mathbb{N}^n$ is such that $s(U)\subset\sigma$ and $|U|=j$, define

$$f_\sigma^U(\overline{x}_\tau) = \begin{cases}(x_\sigma)^{j+1}/x_U & \text{if } \tau=\sigma\\ 0 & \text{if } \tau\neq\sigma.\end{cases}$$

Then $\mathrm{Hom}_A(K_t^{j+1},\Bbbk[\Delta])_{-j}$ has a vector subspace with basis given by

$$D^t=\{f_\sigma^U : |\sigma|=t,\ \sigma\in\Delta,\ |U|=j \text{ and } s(U)\subset\sigma\}.$$

For the right side of (5), given $\sigma\in\Delta$ with $|\sigma|=t$, as in Section 2.1 let $\hat\sigma : C_{t-1}(\Delta)\to\Bbbk$ be the homomorphism defined by $\hat\sigma(\tau)=1$ if $\tau=\sigma$ and $\hat\sigma(\tau)=0$ otherwise and define

$$\langle\sigma,U\rangle := \hat\sigma + C^{t-1}(\mathrm{cost}_\Delta\, s(U)).$$

Then a basis for

$$\bigoplus_{U\in T(\Delta)_j} C^{t-1}(\Delta,\mathrm{cost}_\Delta\, s(U))$$

is given by

$$B=\{\langle\sigma,U\rangle : \sigma\in\Delta,\ |\sigma|=t,\ U\in T(\Delta)_j, \text{ and } s(U)\subseteq\sigma\},$$

and Miyazaki's proof of Theorem 1.1 shows that the direct sum of maps

$$\psi : \bigoplus_{U\in T(\Delta)_j} C^{t-1}(\Delta,\mathrm{cost}_\Delta\, s(U)) \to \mathrm{Hom}_A(K_t^{j+1},\Bbbk[\Delta])_{-j}$$

defined componentwise by

$$\psi(\langle\sigma,U\rangle)=f_\sigma^U$$

is the chain map inducing the isomorphism in cohomology in (5).

Now consider how G acts on each of these cochain complexes. First, $g \cdot f_\sigma^U = f_{g\cdot\sigma}^{g\cdot U}$ from the definition of the action on $\mathrm{Hom}_A(K_\bullet^{j+1}, \Bbbk[\Delta])$. On the other hand,

$$g \cdot \langle \sigma, U\rangle = \langle g\cdot\sigma, g\cdot U\rangle,$$

where G acts on $U \in \mathbb{N}^n$ by permuting indices in accordance with the action of G on the vertex set $[n]$. Hence, ψ is a G-equivariant isomorphism and (6) is established.

Finally, again referring to the proof of [12, Theorem 1], the canonical maps

$$\mathrm{Ext}_A^i(A/\mathfrak{m}_{j+1}, \Bbbk[\Delta]) \to H_{\mathfrak{m}}^i(\Bbbk[\Delta])$$

induce isomorphisms in $\mathbb{Z}$-graded components of degree at least $-j$. Since these are also maps of $(\mathbb{Z}\times\hat{G})$-graded A-modules by Proposition 2.1, we obtain the desired isomorphism in the statement of the theorem. □

4. Applications

4.1. Hilbert series of Artinian reductions and an analog of Schenzel's formula.

From now on, Δ is fixed to be Buchsbaum and $G = \mathbb{Z}/p\mathbb{Z}$ for some prime p with a fixed generator g. We consider A to be $(\mathbb{Z}\times G)$-graded and $\Bbbk[\Delta]$ to be a $(\mathbb{Z}\times G)$-graded A-module as in Section 2. In a way similar to [24, Section 8], if the action of G is very free and $0 \leqslant \delta_i \leqslant p-1$ for $i = 1,\dots,d$, then we can easily construct a l.s.o.p. $\Theta = \theta_1,\dots,\theta_d$ for $\Bbbk[\Delta]$ in which $\theta_i \in A_1^{\delta_i}$ for $1 \leqslant i \leqslant d$ as follows.

First, choose one face from each G-orbit of Δ and collect them into the set Δ_G. Let W be the set of vertices of Δ that are in Δ_G, and choose functions $t_1,\dots,t_d : W \to \Bbbk$ such that their restrictions to any subset of W of size d are linearly independent. Now extend t_i to all of $[n]$ by setting

$$t_i(g^k\cdot v) = \zeta^{-k\delta_i} t_i(v)$$

for all i and k, and let

$$\theta_i = \sum_{v\in[n]} t_i(v) x_v.$$

Then

$$g\cdot\theta_i = \sum_{v\in[n]} t_i(v) x_{g\cdot v} = \sum_{v\in[n]} t_i(g^{-1}\cdot v) x_v = \sum_{v\in[n]} \zeta^{\delta_i} t_i(v) x_v,$$

so $\theta_i \in A_1^{\delta_i}$ for $i = 1,\dots,d$. Furthermore, since no facet of Δ contains $\{g^i\cdot v, g^j\cdot v\}$ for any v with $i \not\equiv j \mod p$, the system $\Theta = (\theta_1,\dots,\theta_d)$ forms an l.s.o.p. that is homogeneous with respect to the $(\mathbb{Z}\times G)$-grading by [24, Lemma III.2.4 (a)].

Remark 4.1. The construction above is included purely for the sake of concreteness in the case of a very free action. In fact, as Adin shows in his thesis ([1]), it is possible to construct an l.s.o.p. with the prescribed properties above even in the case of a free action. The construction involves changing the field $\Bbbk$ to a particular extension of $\mathbb{C}$. However, as we are now turning our focus to certain Hilbert series (which are invariant under field extensions), all of the results that follow hold for any extension of $\mathbb{C}$ and are stated with this understanding in mind.

Now that the existence of l.s.o.p.'s of the form above has been established, we may use them to prove the following theorem (recall the refined Betti number notation of definition (4) and that $\Bbbk(\Delta;\Theta) = \Bbbk[\Delta]/\Theta\Bbbk[\Delta]$ for an l.s.o.p. Θ).

Theorem 4.2. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free group action by $G = \mathbb{Z}/p\mathbb{Z}$, let $0 \leqslant m \leqslant p-1$ be some fixed degree, and let Θ be a*

G-homogeneous l.s.o.p. for $\Bbbk[\Delta]$ such that $\theta_i \in A_1^m$ for all i. Then the $(\mathbb{Z}\times G)$-graded Hilbert series of $\Bbbk(\Delta;\Theta)$ is given by

$$\begin{aligned}\operatorname{Hilb}(\Bbbk(\Delta;\Theta),\lambda,t) = \sum_{i=0}^{d}\left[(-1)^i\binom{d}{i}t^{mi} + \left(\frac{1}{p}\sum_{k=0}^{p-1}t^k\right)\left(h_i(\Delta)+(-1)^{i+1}\binom{d}{i}\right)\right]\lambda^i \\ + \sum_{i=0}^{d}\binom{d}{i}\lambda^i\sum_{j=0}^{i-1}(-1)^{i-j-1}\left(\sum_{k=0}^{p-1}t^k\beta_{j-1}(\Delta)^{k-im}\right).\end{aligned}$$

Remark 4.3. Recall that $t^p = 1$ in this Hilbert series, and note that by setting $t = 1$, we can recover Schenzel's Theorem 1.2. Furthermore, we conclude that $h_i(\Delta) \equiv (-1)^i\binom{d}{i} \pmod p$ for all i.

Proof. Suppose that Θ is an arbitrary l.s.o.p. with $\theta_i \in A_1^{\delta_i}$ for $i = 1,\dots,d$. For $s = 1,\dots,d$, denote

$$\Bbbk_s[\Delta] := \Bbbk[\Delta]/(\theta_1,\dots,\theta_s)\Bbbk[\Delta].$$

Then (recall the shifted module discussion from Section 2.2) we have exact sequences of $(\mathbb{Z}\times G)$-graded A-modules with degree-preserving maps of the form

$$0 \to Q(s)[-1,-\delta_s] \to \Bbbk_{s-1}[\Delta][-1,-\delta_s] \xrightarrow{\cdot\theta_s} \Bbbk_{s-1}[\Delta] \to \Bbbk_s[\Delta] \to 0,$$

where

$$Q(s) = 0 :_{\Bbbk_{s-1}[\Delta]} \theta_s = 0 :_{\Bbbk_{s-1}[\Delta]} \mathfrak{m} = H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]).$$

The second equality above follows from the definition of Buchsbaumness, and the third follows from the proof of [25, Proposition I.2.1]. On the level of Hilbert series, a standard argument yields the following equation:

$$\begin{aligned}\operatorname{Hilb}(\Bbbk(\Delta;\Theta),\lambda,t) &= \operatorname{Hilb}(\Bbbk[\Delta],\lambda,t)\prod_{i=1}^{d}(1-\lambda t^{\delta_i}) \\ &\quad + \sum_{s=1}^{d}\lambda t^{\delta_i}\operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]),\lambda,t)\prod_{j=s+1}^{d}(1-\lambda t^{\delta_j}).\end{aligned}$$

In the case $\theta_s \in A_1^m$ for $s = 1,\dots,d$, the equation above simplifies to

$$\begin{aligned}\operatorname{Hilb}(\Bbbk(\Delta;\Theta),\lambda,t) &= (1-\lambda t^m)^d\operatorname{Hilb}(\Bbbk[\Delta],\lambda,t) \\ &\quad + \sum_{s=1}^{d}\lambda t^m(1-\lambda t^m)^{d-s}\operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]),\lambda,t).\end{aligned}$$

Analyzing the first term yields the following, using Theorem 2.4:

$$\begin{aligned}&(1-\lambda t^m)^d\operatorname{Hilb}(\Bbbk[\Delta],\lambda,t) \\ &\qquad = (1-\lambda t^m)^d\left[1+\frac{1}{p}\left(\frac{\sum_{i=0}^{d}h_i(\Delta)\lambda^i}{(1-\lambda)^d}-1\right)(1+t+\cdots+t^{p-1})\right] \\ &\qquad = (1-\lambda t^m)^d + \frac{(1+t+\cdots+t^{p-1})}{p}\left[\sum_{i=0}^{d}h_i(\Delta)\lambda^i-(1-\lambda t^m)^d\right] \\ &\qquad = \sum_{i=0}^{d}\left[(-1)^i\binom{d}{i}t^{mi}+\frac{1}{p}\sum_{k=0}^{p-1}t^k\left(h_i(\Delta)+(-1)^{i+1}\binom{d}{i}\right)\right]\lambda^i.\end{aligned}$$

For the second term,

$$H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]) \cong \bigoplus_{i=0}^{s-1}\left(\bigoplus_{\binom{s-1}{i}} H^i_{\mathfrak{m}}(\Bbbk[\Delta])[-i,-im]\right)$$

by [25, Proposition II.4.14′] (though the stated result is for $\mathbb{Z}$-graded A-modules, the exact same proof works in the $(\mathbb{Z}\times G)$-graded case using Proposition 2.1). By our Theorem 3.1, $H^i_{\mathfrak{m}}(\Bbbk[\Delta])[-i,-im]$ is concentrated in $\mathbb{Z}$-degree i and has dimension $\beta_{i-1}(\Delta)^{k-im}$ in degree (i,k). Hence,

$$\operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]),\lambda,t) = \sum_{i=0}^{s-1}\binom{s-1}{i}\lambda^i\left(\sum_{k=0}^{p-1} t^k\beta_{i-1}(\Delta)^{k-im}\right),$$

and so the $\lambda^i t^k$ coefficient of $\lambda t^m \operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]),\lambda,t)$ is $\binom{s-1}{i-1}\beta_{i-2}(\Delta)^{k-im}$. Then the $\lambda^i t^k$ coefficient of $(1-\lambda t^m)^{d-s}\lambda t^m \operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]),\lambda,t)$ is given by

$$\sum_{r=0}^{i}(-1)^r\binom{d-s}{r}\binom{s-1}{i-r-1}\beta_{i-r-2}(\Delta)^{k-im}.$$

Now we sum over all values of s, setting $f(s)=\binom{d-s}{r}\binom{s-1}{i-r-1}$:

$$\begin{aligned}\sum_{s=1}^{d}\sum_{r=0}^{i}(-1)^r f(s)\beta_{i-r-2}(\Delta)^{k-im} &= \sum_{r=0}^{i}\left[(-1)^r\beta_{i-r-2}(\Delta)^{k-im}\sum_{s=1}^{d} f(s)\right]\\ &= \binom{d}{i}\sum_{r=0}^{i}(-1)^r\beta_{i-r-2}(\Delta)^{k-im}\\ &= \binom{d}{i}\sum_{j=0}^{i-1}(-1)^{i-j-1}\beta_{j-1}(\Delta)^{k-im}.\end{aligned}$$

Here the second equality follows from the identity $\sum_{s=1}^{d}\binom{d-s}{r}\binom{s-1}{i-r-1}=\binom{d}{i}$ and the third follows from setting $j=i-r-1$. □

Of particular interest is the $G=\mathbb{Z}/2\mathbb{Z}$ case, in which many simplifications can be made to the expression in Theorem 4.2. This results in the following corollary.

Corollary 4.4. *Let Δ be a centrally-symmetric Buchsbaum complex of dimension $d-1$ with $\Theta=(\theta_1,\dots,\theta_d)$ an l.s.o.p. for Δ such that $\theta_i\in A_1^m$ for all i and some fixed m. Then*

$$\dim_{\Bbbk}\Bbbk(\Delta;\Theta)_i^k = \frac{1}{2}\left(h_i(\Delta)+(-1)^{i+k+mi}\binom{d}{i}\right)+\binom{d}{i}\sum_{j=0}^{i-1}(-1)^{i-j-1}\beta_{j-1}(\Delta)^{k-im}.$$

4.2. The sigma module. When only considering a $\mathbb{Z}$-grading on $\Bbbk[\Delta]$, it is possible to mod $\Bbbk(\Delta;\Theta)$ out by an additional submodule in order to get an even tighter bound on the h-vector of Δ. In particular, the sigma module can be used to great effect as follows ([16, Theorem 1.2]).

Theorem 4.5 (Murai–Novik–Yoshida). *Let Δ be a Buchsbaum simplicial complex of dimension $d-1$ and let Θ be an l.s.o.p. for Δ. Then*

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_i = h_i(\Delta)+\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta).$$

Of course, we would like for analogous statements to hold for the $(\mathbb{Z}\times G)$-graded Hilbert series of $\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])$. In order to even consider this series, we must first verify that this module is in fact $(\mathbb{Z}\times G)$-graded by establishing that the sigma module is G-invariant. This is accomplished by the following lemma, whose proof is nearly immediate and has been omitted.

Lemma 4.6. *Let Δ be a Buchsbaum simplicial complex with a free action by G. If $\Theta=(\theta_1,\dots,\theta_d)$ is an l.s.o.p. for Δ and each θ_i is homogeneous with respect to the $(\mathbb{Z}\times G)$-grading of A, then the sigma module $\Sigma(\Theta;\Bbbk[\Delta])$ is G-invariant.*

Fortunately, the proof of our needed aspects of [16, Theorem 2.3] goes through essentially verbatim in the $(\mathbb{Z}\times G)$-graded case using Proposition 2.1. One aspect not covered directly by the proposition (though it is considered in the proof) is that images and kernels of maps of $(\mathbb{Z}\times\hat{G})$-graded A-modules are themselves $(\mathbb{Z}\times\hat{G})$-graded; this follows from [2, Proposition II.11.3.3]. Hence, we obtain the following proposition.

Proposition 4.7. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free action by G and let $\Theta=(\theta_1,\dots,\theta_d)$ be an l.s.o.p. for Δ with $\theta_i\in A_1^m$ for all i. Then*

$$\Sigma(\Theta;\Bbbk[\Delta])/\Theta\Bbbk[\Delta]\cong\bigoplus_{i=0}^{d-1}\left(\bigoplus_{\binom{d}{i}}H^i_{\mathfrak{m}}(\Bbbk[\Delta])[-i,-im]\right)$$

as $(\mathbb{Z}\times G)$-graded A-modules. In particular,

$$\dim_{\Bbbk}\left(\Sigma(\Theta;\Bbbk[\Delta])/\Theta\Bbbk[\Delta]\right)_i^k=\binom{d}{i}\beta_{i-1}(\Delta)^{k-im}.$$

This immediately establishes the following extension of Theorem 4.2:

Theorem 4.8. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free action by G and let Θ be a G-homogeneous l.s.o.p. for $\Bbbk[\Delta]$ such that $\theta_i\in A_1^m$ for all i. Then*

$$\begin{aligned}\operatorname{Hilb}(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta]),\lambda,t)=&\\ &\sum_{i=0}^{d}\left[(-1)^i\binom{d}{i}t^{mi}+\left(\frac{1}{p}\sum_{k=0}^{p-1}t^k\right)\left(h_i(\Delta)+(-1)^{i+1}\binom{d}{i}\right)\right]\lambda^i\\ &+\sum_{i=0}^{d}\binom{d}{i}\lambda^i\sum_{j=0}^{i}(-1)^{i-j-1}\left(\sum_{k=0}^{p-1}t^k\beta_{j-1}(\Delta)^{k-im}\right).\end{aligned}$$

4.3. Inequalities. Theorem 4.5 implies the following inequality bounding the h-vector of any Buchsbaum simplicial complex Δ:

$$h_i(\Delta)\geqslant\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j}\beta_{j-1}(\Delta).\tag{7}$$

For Cohen–Macaulay complexes admitting a very free action by G, bounds on the h-vector may be obtained through Theorem 1.5. In this section we present some similar inequalities by exploiting the $(\mathbb{Z}\times G)$-graded Hilbert series of Buchsbaum complexes admitting free group actions by G. To begin, setting $m=0$ and examining the λ^it^k coefficients in Theorem 4.8 immediately allows us to bound the h-vector of Δ by

$$h_i(\Delta)\geqslant(p-1)(-1)^{i+1}\binom{d}{i}+p\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j}\beta_{j-1}(\Delta)^0\tag{8}$$

and

$$h_i(\Delta) \geqslant (-1)^i \binom{d}{i} + p\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j}\beta_{j-1}(\Delta)^k \tag{9}$$

for $k \not\equiv 0 \mod p$.

The inequalities above provide an immediate "permutable" version of (7) that is worth recognizing.

Corollary 4.9. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free action by the group $\mathbb{Z}/p\mathbb{Z}$. If $\mathcal{M}$ is a multiset of size $p-1$ on $\{1,\dots,p-1\}$, then*

$$h_i(\Delta) \geqslant \binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j}\beta_{j-1}(\Delta)^0 + \binom{d}{i}\sum_{m\in\mathcal{M}}\sum_{j=0}^{i}(-1)^{i-j}\beta_{j-1}(\Delta)^m.$$

Proof. For each $m \in \mathcal{M}$, consider the corresponding inequality (9). Add all of the inequalities together along with (8), then divide by p. □

Note that by taking $\mathcal{M} = [p-1]$, we obtain the original bound (7). On the other hand, by considering the degrees of the group representations produced by G acting on $H^i(\Delta)$, we can also obtain some extensions of Theorem 1.5.

Corollary 4.10. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free action by G. Then:*

(1) *If $\beta_i(\Delta) = 0$ for $i \leqslant j$, then $h_i(\Delta) \geqslant \binom{d}{i}$ for $i \leqslant j+2$.*
(2) *If $\beta_i(\Delta) < (p-1)$ for $i \leqslant j$, then $h_i(\Delta) \geqslant (-1)^i\binom{d}{i}$ for $i \leqslant j+2$.*

Proof. The statement in (1) is an immediate consequence of the inequalities in (8) and (9). For (2), consider the reduced cohomology group $\tilde{H}^i(\Delta;\mathbb{Q})$ with coefficients in $\mathbb{Q}$, which satisfies

$$\beta_i(\Delta) = \dim_{\mathbb{Q}} \tilde{H}^i(\Delta;\mathbb{Q}).$$

Furthermore,

$$\tilde{H}^i(\Delta;\Bbbk) \cong \tilde{H}^i(\Delta;\mathbb{Q}) \otimes_{\mathbb{Q}} \Bbbk$$

as representations of G by taking $\Bbbk$ to have a trivial action.

There are only two irreducible representations of G over $\mathbb{Q}$: the 1-dimensional trivial representation and a $(p-1)$-dimensional non-trivial representation. Hence, if $\dim_{\mathbb{Q}} \tilde{H}^i(\Delta;\mathbb{Q}) < p-1$ then G acts trivially on $\tilde{H}^i(\Delta;\mathbb{Q})$. From the above isomorphism, G acts trivially on $\tilde{H}^i(\Delta;\Bbbk)$ as well. That is, $\beta_i(\Delta)^r = 0$ for $1 \leqslant r \leqslant p-1$ and $i \leqslant j$, and the result follows from inequality (9). □

It is also worth examining the properties of the quotient Δ/G. This quotient has elements corresponding to orbits of the faces of Δ under the action of G, and it can always be made into a poset with zero element $\{\varnothing\}$ (assuming $\Delta \neq \varnothing$) under the ordering defined by setting $x \preceq y$ if $\sigma \subseteq \tau$ for some $\sigma \in x$ and $\tau \in y$. This poset is ranked by setting the rank of an orbit to be the cardinality of any of its members.

In some cases, Δ/G is itself isomorphic to the face poset of a simplicial complex (for instance, when $\mathbb{Z}/2\mathbb{Z}$ acts by rotation on the boundary of a hexagon, the quotient is isomorphic to the face poset of the boundary of a triangle). When this happens, we naturally view Δ/G as the corresponding simplicial complex. Less optimistically, Δ/G may have the property that every interval $[\{\varnothing\}, x]$ is isomorphic to a boolean lattice. In this case, we say that Δ/G is a simplicial poset. Many invariants of simplicial complexes have natural extensions to simplicial posets (see [23] for a nice overview). In particular, the h-vector of a simplicial poset is a well-studied object that our next proposition examines.

Proposition 4.11. *Let Δ be a Buchsbaum simplicial complex admitting a free group action by G. If Δ/G is a simplicial poset, then*

$$h_i(\Delta/G) \geqslant (-1)^i \binom{d}{i} + \binom{d}{i} \sum_{j=0}^{i} (-1)^{i-j} \beta_{j-1}(\Delta)^k$$

for $k \not\equiv 0 \mod p$.

Proof. A straightforward calculation shows that

$$h_i(\Delta) = (p-1)(-1)^{i+1} \binom{d}{i} + p h_i(\Delta/G). \tag{10}$$

Now combine (9) with (10) and divide by p. □

As examples, Δ/G is always a simplicial poset when Δ is the order complex of a poset ([4, Section 6]), or, more generally, when Δ is balanced and the action of G is color-preserving ([19]). Note that this proposition provides a "Buchsbaum-like" bound on the h-vector of Δ/G in the style of equation (9) without any direct consideration of whether or not Δ/G is itself Buchsbaum.

4.4. Dehn–Sommerville relations. Here we present a new results akin to Theorem 1.4, relating the symmetric values of the dimensions $\dim_{\Bbbk} (\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta]))_i^j$. First, let $\tilde{\chi}(\Delta)$ denote the reduced Euler characteristic of a simplicial complex Δ, defined by

$$\tilde{\chi}(\Delta) = \sum_{j=-1}^{d-1} (-1)^j \beta_j(\Delta),$$

where $d-1$ is the dimension of Δ. We will also denote by $\chi(\Delta) := \tilde{\chi}(\Delta) + 1$ the (unreduced) Euler characteristic of Δ. If Δ is an orientable homology manifold without boundary, then Klee's formula

$$h_{d-i}(\Delta) - h_i(\Delta) = (-1)^i \binom{d}{i} ((-1)^{d-1} \tilde{\chi}(\Delta) - 1) \tag{11}$$

(found in [11]) provides a relation between the symmetric entries in the h-vector of Δ in terms of $\tilde{\chi}(\Delta)$, abstracting the Dehn–Sommerville relations for simplicial polytopes. We will first examine the symmetric values of the dimensions of $\dim_{\Bbbk} (\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta]))_i^0$, in the case that $\Theta \subset A_1^0$.

Theorem 4.12. *Let Δ be a $(d-1)$-dimensional triangulation of a connected orientable manifold without boundary admitting a free group action by G. If $\Theta \subset A_1^0$ and $\widetilde{H}^{d-1}(\Delta) = \widetilde{H}^{d-1}(\Delta)^0$, then*

$$\dim_{\Bbbk} (\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta]))_i^0 = \dim_{\Bbbk} (\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta]))_{d-i}^0.$$

Proof. If $\Delta = \{\varnothing\}$, then the result is immediate. Assume that $\Delta \neq \{\varnothing\}$. Applying Theorem 4.8, we can write

$$\dim_{\Bbbk} (\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta]))_i^0 = \frac{h_i(\Delta)}{p} + \frac{(-1)^i(p-1)}{p} \binom{d}{i} + \binom{d}{i} \sum_{j=0}^{i} (-1)^{i-j-1} \beta_{j-1}(\Delta)^0.$$

Using (11),

$$\begin{aligned}\frac{h_i(\Delta)}{p}-\frac{h_{d-i}(\Delta)}{p}&=\frac{(-1)^{i+1}}{p}\binom{d}{i}((-1)^{d-1}\tilde{\chi}(\Delta)-1)\\&=\frac{(-1)^{d-i}}{p}\binom{d}{i}\tilde{\chi}(\Delta)+\frac{(-1)^{i}}{p}\binom{d}{i}\\&=\frac{(-1)^{d-i}}{p}\binom{d}{i}\chi(\Delta)+\frac{(-1)^{d-i-1}}{p}\binom{d}{i}+\frac{(-1)^{i}}{p}\binom{d}{i}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left[\frac{h_i(\Delta)}{p}+\frac{(-1)^i(p-1)}{p}\binom{d}{i}\right]-\left[\frac{h_{d-i}(\Delta)}{p}+\frac{(-1)^{d-i}(p-1)}{p}\binom{d}{i}\right]\\&\qquad=\frac{(-1)^{d-i}}{p}\binom{d}{i}\chi(\Delta)+(-1)^i\binom{d}{i}+(-1)^{d-i-1}\binom{d}{i}.\end{aligned}\tag{12}$$

Identify Δ with its geometric realization $|\Delta|$, and set $\Gamma=|\Delta|/G$. Since G acts freely on Δ, the projection $\pi:|\Delta|\to\Gamma$ is a covering space map. By [9, Proposition 3G.1],

$$H^i(|\Delta|)^0=H^i(|\Delta|)^G\cong H^i(\Gamma).$$

Since $\widetilde{H}^{d-1}(\Delta)=\widetilde{H}^{d-1}(\Delta)^0$ by assumption, Γ is itself an orientable manifold. Hence, Poincaré duality applies and $\beta_{j-1}(\Delta)^0=\beta_{d-j}(\Delta)^0$ for $j>1$ while $\beta_0(\Delta)^0=\beta_{d-1}(\Delta)^0-1$. Furthermore, $\beta_{-1}(\Delta)^0=0$ since $\Delta\neq\{\varnothing\}$. Then

$$\begin{aligned}\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta)^0&=(-1)^{i+1}\binom{d}{i}+\binom{d}{i}\sum_{j=1}^{i}(-1)^{i-j-1}\beta_{d-j}(\Delta)^0\\&=(-1)^{i+1}\binom{d}{i}+\binom{d}{i}\sum_{\ell=i+1}^{d}(-1)^{d-i-\ell}\beta_{\ell-1}(\Delta)^0.\end{aligned}$$

Thus,

$$\begin{aligned}&\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta)^0-\binom{d}{i}\sum_{j=0}^{d-i}(-1)^{d-i-j-1}\beta_{j-1}(\Delta)^0\\&\qquad=(-1)^{i+1}\binom{d}{i}+\binom{d}{i}\sum_{j=0}^{d}(-1)^{d-i-j}\beta_{j-1}(\Delta)^0\\&\qquad=(-1)^{i+1}\binom{d}{i}+(-1)^{d-i-1}\binom{d}{i}\tilde{\chi}(\Gamma).\end{aligned}$$

Now note that Δ is a p-sheeted covering space for Γ. This implies that $\chi(\Delta)=p\chi(\Gamma)$ (see [9, Section 2.2]). Hence, we can re-write the difference as

$$\begin{aligned}&(-1)^{i+1}\binom{d}{i}+(-1)^{d-i-1}\binom{d}{i}\tilde{\chi}(\Gamma)\\&\qquad=(-1)^{i+1}\binom{d}{i}+(-1)^{d-i-1}\binom{d}{i}\chi(\Gamma)+(-1)^{d-i}\binom{d}{i}\\&\qquad=(-1)^{i+1}\binom{d}{i}+\frac{(-1)^{d-i-1}}{p}\binom{d}{i}\chi(\Delta)+(-1)^{d-i}\binom{d}{i}.\end{aligned}$$

Adding this difference to (12) completes the proof. □

Remark 4.13. In the case that $G = \mathbb{Z}/p\mathbb{Z}$ with p odd, the hypothesis $\widetilde{H}^{d-1}(\Delta) = \widetilde{H}^{d-1}(\Delta)^0$ of Theorem 4.12 is always true. Indeed, the fundamental class of Δ generating $\widetilde{H}^{d-1}(\Delta)$ is of the form

$$z = \sum_{\widehat{\sigma} \in C^{d-1}(\Delta)} a_\sigma \widehat{\sigma}$$

with $a_\sigma = \pm 1$ for all σ. If $h \in G$ acts on this class, then it can only flip the sign of some coefficients; it cannot introduce a factor of ζ^i across all coefficients for some non-trivial i.

In fact, the theorem above can be greatly strengthened by appealing to algebraic properties of $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])$. First, we will need the following theorem (see [17, Theorem 1.4] or [16, Remark 3.8]).

Theorem 4.14. *Let Δ be a triangulation of a $(d-1)$-dimensional connected manifold without boundary that is orientable over $\Bbbk$, and let Θ be a linear system of parameters for $\Bbbk[\Delta]$. Then $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])$ is an Artinian Gorenstein $\Bbbk$-algebra.*

With this fact, our next theorem quickly follows.

Theorem 4.15. *Let Δ be a triangulation of a $(d-1)$-dimensional connected manifold without boundary that is orientable over $\Bbbk$ and admits a free group action by G, and let Θ be a $(\mathbb{Z} \times G)$-homogeneous linear system of parameters for $\Bbbk[\Delta]$. If s is such that $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_d$ is concentrated in $(\mathbb{Z} \times G)$-degree (d, s), then*

$$\dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])\right)_i^j = \dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])\right)_{d-i}^{s-j}$$

for $i = 0, \ldots, d$ and any j, where we interpret $s - j$ modulo p.

Before starting the proof of this theorem, note that $\dim_{\Bbbk} \Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_d = 1$ because $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])$ is Gorenstein and hence such an s always exists.

Proof. Since $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])$ is Gorenstein, the product map

$$\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_i \times \Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_{d-i} \to \Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_d$$

is a perfect pairing (see [8, Theorem 2.79]). Furthermore, the finely-graded product maps

$$\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_i^j \times \Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_{d-i}^k \to \Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_d^{j+k}$$

only land in a non-zero component of $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_d$ when $j + k \equiv s \pmod{p}$. That is,

$$\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_i^j \times \Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_{d-i}^{s-j} \to \Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_d^s$$

is a perfect pairing for each i, j and hence

$$\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_i^j \cong \Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_{d-i}^{s-j}$$

as vector spaces over $\Bbbk$. □

By comparing 4.12 with 4.15, we obtain the following corollary.

Corollary 4.16. *Let Δ be a $(d-1)$-dimensional triangulation of a connected orientable manifold without boundary admitting a free group action by G. If $\Theta \subset A_1^0$ and $\widetilde{H}^{d-1}(\Delta) = \widetilde{H}^{d-1}(\Delta)^0$, then $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])_d$ in concentrated in $(\mathbb{Z} \times G)$-degree $(d, 0)$ and*

$$\dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])\right)_i^j = \dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])\right)_{d-i}^{p-j}$$

for $i = 0, \ldots, d$ and $j = 0, \ldots, p-1$.

5. Comments

It was shown by Duval in [3] that a version of Theorem 1.1 also holds when considering the face ring of a simplicial poset. Our main result then begs the following question.

Question 5.1. *Does Theorem* 3.1 *extend to simplicial posets admitting free group actions?*

An answer to this question will likely require a careful examination of a $(\mathbb{Z} \times G)$-grading imposed on the face ring of a simplicial poset, an object that at times is considerably more complicated than the Stanley–Reisner ring.

Stanley was able to examine some implications of equality being attained in his inequalities of Theorem 1.5 for the $G = \mathbb{Z}/2\mathbb{Z}$ case (see [24, Proposition III.8.2]). In light of these results, our next question naturally arises.

Question 5.2. *If equality is attained in either inequality* (8) *or* (9)*, what can be said about the other entries in the h-vector of Δ?*

Lastly, this paper would feel incomplete without some mention of the g-conjecture. To this end, let Δ be a $(d-1)$-dimensional orientable $\Bbbk$-homology manifold and denote

$$h_i''(\Delta) = \dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_i = h_i(\Delta) + \binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta).$$

Kalai's manifold g-conjecture asserts in part that $h_0''(\Delta) \leqslant h_1''(\Delta) \leqslant \cdots \leqslant h_{\lfloor d/2\rfloor}''(\Delta)$ (in the case that Δ is a sphere, this is the same as the corresponding part of the usual g-conjecture). There is much evidence in favor of this statement, and it has been shown to be true in some special cases (see [14, Theorem 5.3], [15, Theorem 1.5], and [17, Theorem 1.6]). The inequality is sometimes demonstrated by exhibiting a Lefschetz element $\omega \in A_1$ such that the multiplication map

$$\cdot\omega : \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_{i-1} \to \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_i$$

is an injection for $1 \leqslant i \leqslant \lfloor d/2\rfloor$. As has been our theme, we would like to show that similar inequalities hold under the finer grading. Indeed, if such an ω can in fact be chosen to be an element of A_1^m for some m, then we immediately have the inequalities

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_{i-1}^{j} \leqslant \dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_{i}^{j+m}$$

for any choice of j. Although the Lefschetz elements that have been found thus far cannot be specified to reside in some fixed homogeneous $(\mathbb{Z} \times G)$-degree of A_1, the strength of the finer grading on $\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])$ prompts the following conjecture.

Conjecture 5.3. *Let Δ be an orientable $\Bbbk$-homology manifold admitting a free group action by $\mathbb{Z}/p\mathbb{Z}$. Then there exists m such that*

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_{i-1}^{j} \leqslant \dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_{i}^{j+m}$$

for $1 \leqslant i \leqslant \lfloor d/2\rfloor$ and $0 \leqslant j \leqslant p-1$.

Acknowledgements. The author would like to thank Martina Juhnke-Kubitzke, Isabella Novik, Travis Scholl, and two anonymous referees for many helpful comments and conversations.

References

[1] Ron Adin, *Combinatorial structure of simplicial complexes with symmetry*, Ph.D. thesis, Hebrew University, Jerusalem, 1991.

[2] Nicolas Bourbaki, *Algebra. I. Chapters 1–3*, Elements of Mathematics, Springer, 1989, Translated from the French, Reprint of the 1974 edition.

[3] Art M. Duval, *Free resolutions of simplicial posets*, J. Algebra **188** (1997), no. 1, 363–399.

[4] Adriano M. Garsia and Dennis Stanton, *Group actions of Stanley-Reisner rings and invariants of permutation groups*, Adv. Math. **51** (1984), no. 2, 107–201.

[5] Shiro Goto, *On the associated graded rings of parameter ideals in Buchsbaum rings*, J. Algebra **85** (1983), no. 2, 490–534.

[6] Hans-Gert Gräbe, *The canonical module of a Stanley-Reisner ring*, J. Algebra **86** (1984), no. 1, 272–281.

[7] ———, *Generalized Dehn-Sommerville equations and an upper bound theorem*, Beitr. Algebra Geom. (1987), no. 25, 47–60.

[8] Tadahito Harima, Toshiaki Maeno, Hideaki Morita, Yasuhide Numata, Akihito Wachi, and Junzo Watanabe, *The Lefschetz properties*, Lecture Notes in Math., vol. 2080, Springer, 2013.

[9] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[10] Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther, *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, vol. 87, American Mathematical Society, 2007.

[11] Victor Klee, *A combinatorial analogue of Poincaré's duality theorem*, Canad. J. Math. **16** (1964), 517–531.

[12] Mitsuhiro Miyazaki, *Characterizations of Buchsbaum complexes*, Manuscr. Math. **63** (1989), no. 2, 245–254.

[13] ———, *On the canonical map to the local cohomology of a Stanley-Reisner ring*, Bull. Kyoto Univ. Educ., Ser. B (1991), no. 79, 1–8.

[14] Satoshi Murai, *Tight combinatorial manifolds and graded Betti numbers*, Collect. Math. **66** (2015), no. 3, 367–386.

[15] Satoshi Murai and Isabella Novik, *Face numbers of manifolds with boundary*, Int. Math. Res. Not. (2017), no. 12, 3603–3646.

[16] Satoshi Murai, Isabella Novik, and Ken-ichi Yoshida, *A duality in Buchsbaum rings and triangulated manifolds*, Algebra Number Theory **11** (2017), no. 3, 635–656.

[17] Isabella Novik and Ed Swartz, *Gorenstein rings through face rings of manifolds*, Compos. Math. **145** (2009), no. 4, 993–1000.

[18] ———, *Socles of Buchsbaum modules, complexes and posets*, Adv. Math. **222** (2009), no. 6, 2059–2084.

[19] Victor Reiner, *Quotients of Coxeter complexes and P-partitions*, Mem. Am. Math. Soc. **95** (1992), no. 460, vi+134.

[20] Gerald Allen Reisner, *Cohen-Macaulay quotients of polynomial rings*, Adv. Math. **21** (1976), no. 1, 30–49.

[21] Peter Schenzel, *On the number of faces of simplicial complexes and the purity of Frobenius*, Math. Z. **178** (1981), no. 1, 125–142.

[22] Richard P. Stanley, *On the number of faces of centrally-symmetric simplicial polytopes*, Graphs Combin. **3** (1987), no. 1, 55–66.

[23] ———, *f-vectors and h-vectors of simplicial posets*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 319–331.

[24] ———, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser, 1996.

[25] Jürgen Stückrad and Wolfgang Vogel, *Buchsbaum rings and applications. An interaction between algebra, geometry, and topology*, Mathematische Monographien, vol. 21, VEB Deutscher Verlag der Wissenschaften, 1986.

Connor Sawaske, Department of Mathematics University of Washington Seattle, WA 98195-4350, USA

E-mail : sawaske@math.washington.edu