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Symmetric group characters of almost square shape

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Abstract We give closed product formulas for the irreducible characters of the symmetric groups related to rectangular *'almost square'* Young diagrams $p \times (p+\delta)$ for a fixed value of an integer δ and an arbitrary integer p.

1. Introduction

1.1. The main result. Let π be a partition of an integer ℓ and let λ be a partition of an integer n. Let $\operatorname{Ch}_\pi(\lambda)$ denote *the normalized character* of the symmetric group $\mathfrak{S}_n$ defined by

$$\operatorname{Ch}_\pi(\lambda) = \begin{cases} n^{\downarrow \ell} \cdot \frac{\chi^\lambda_{\pi \cup (1^{n-\ell})}}{\chi^\lambda_{(1^n)}}, & \text{if } n \geqslant \ell, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where $\chi^\lambda_\mu = \operatorname{Tr} \rho^\lambda(w_\mu)$ is the usual character of the irreducible representation ρ^λ of the symmetric group $\mathfrak{S}_n$ associated with λ, evaluated on an arbitrary permutation w_μ with the cycle decomposition given by μ. Here we use the *falling factorial* defined by

$$a^{\downarrow \ell} = \underbrace{a(a-1)(a-2)\cdots(a-\ell+1)}_{\ell \text{ factors}}$$

for a complex number a and a positive integer ℓ, and by $a^{\downarrow 0} := 1$. This choice of the normalization for the characters is quite natural, in particular in the context of the asymptotic representation theory, see for example [3, 5].

In this note we will concentrate on the case when $\pi = (\ell)$ consists of a single part, i.e., on the characters evaluated on a single cycle (augmented by a necessary number of fixed points). Also, we will concentrate on the special case when

$$\lambda = p \times q = (\underbrace{q, \dots, q}_{p \text{ times}})$$

is a rectangular Young diagram, see Figure 1. We will give closed product formulas for such characters in the *almost square* setting when $q-p$ is a fixed integer and p is arbitrary. The exact form of the formula depends on the parity of the length ℓ of the cycle, as well as on the parity of the difference $q-p$, so altogether there are

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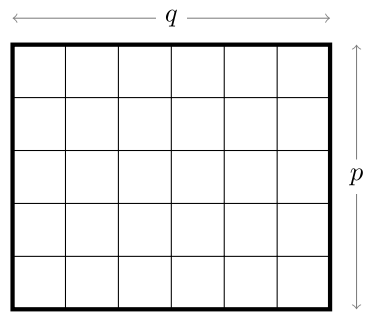

Figure 1. The rectangular Young diagram $p \times q$ for $p = 5$ and $q = 6$.

four distinct formulas for such characters. As a teaser, we start with the case when $\ell = 2j - 1$ is odd while $q - p = 2d$ is even. Let us emphasize that by considering new variables e, d such that $p \times q = (e - d) \times (e + d)$, we get interesting formulas.

Theorem 1.1. *Let j be a positive integer. Then*

$$\begin{aligned} &\operatorname{Ch}_{2j-1}\big((e-d)\times(e+d)\big) \\ &\qquad = (-1)^{j-1}\operatorname{Cat}(j-1)\sum_{k=0}^{j} f_k(j)\left(\prod_{r=0}^{k-1}(d^2-r^2)\right)\left(\prod_{r=k}^{j-1}(e^2-r^2)\right), \end{aligned} \tag{2}$$

where

$$\operatorname{Cat}(j-1) = \frac{(2j-2)!}{(j-1)!\, j!}$$

is the Catalan number; furthermore $f_0(j) = 1$ and

$$f_k(j) = (-1)^k \frac{j^{\downarrow k}\,(2j-1)^{\uparrow\uparrow k}}{k!\,(2k-1)!!} \tag{3}$$

for $1 \leqslant k \leqslant j$.

Above we used the *double rising factorial* $a^{\uparrow\uparrow k}$ (which is somewhat analogous to the *double factorial* $a!!$ in which the factors form an arithmetic progression with the step 2) which is defined by

$$a^{\uparrow\uparrow k} = \prod_{r=0}^{k-1}(a+2r) = 2^k \left(\frac{a}{2}\right)^{\uparrow k}$$

for a complex number a and a positive integer k, and by $a^{\uparrow\uparrow 0} := 1$. Thus, $f_k(j) \in \mathbb{Q}[j]$ is a polynomial in the variable j of degree $2k$.

Remark 1.2. Note that in the above result there is *no* assumption that $e - d$ and $e + d$ are non-negative integers; in fact e and d can be arbitrary complex numbers. The reader may feel uneasy about the case when $(e-d)\times(e+d)$ does not make sense as a Young diagram; later on in Corollary 2.1 we will explain why in this case the left-hand side of (2) still makes sense.

1.2. Convention for products. In the following we will use the following non-standard convention for products:

$$\prod_{r=0}^{l} a_r = \begin{cases} a_0 \cdots a_l & \text{if } l \geqslant 0, \\ 1 & \text{if } l = -1, \\ \frac{1}{a_{l+1}a_{l+2}\cdots a_{-2}a_{-1}} & \text{if } l \leqslant -2. \end{cases}$$

This convention was chosen in such a way that the identity

$$\prod_{r=0}^{l+1} a_r = \left[\prod_{r=0}^{l} a_r\right] \cdot a_{l+1}$$

holds for any (positive or negative) integer l.

1.3. The product formula. In the special case when d is an integer such that its absolute value $|d|$ is small, the formula (2) takes a simpler form because each summand on the right-hand side which corresponds to k such that $k > |d|$ is equal to zero; in this way the sum can be taken over $k \in \{1, \dots, \min(j, |d|)\}$. This observation is especially convenient in the aforementioned *almost square* setting when we consider the character corresponding to a rectangular Young diagram $\lambda = p \times q$ in the setup where $q - p = 2d$ is a fixed even integer and p is arbitrary. In particular, we get the following closed product form for the character.

Corollary 1.3. *Let j, p, q be positive integers; we denote by $n = pq = |p \times q|$ the number of the boxes of the corresponding Young diagram $p \times q$. Suppose that $q - p$ is an even integer which we denote by $2d \coloneqq q - p$. Then*

$$\operatorname{Ch}_{2j-1}\big(p \times q\big) = (-1)^{j-1} \operatorname{Cat}(j-1)\ G_d(j,n) \prod_{r=0}^{j-|d|-1} \big(n - r(r+2|d|)\big), \tag{4}$$

where

$$G_d(j,n) = \sum_{k=0}^{|d|} f_k(j) \left(\prod_{r=0}^{k-1} (d^2 - r^2)\right) \left(\prod_{r=k}^{|d|-1} (n + d^2 - r^2)\right)$$

with $f_k(j)$ given, as before, by (3).

Corollary 1.3 remains valid in the case when $j \leqslant |d|$, however in this case the product on the right hand side of (4) should be understood using the convention from Section 1.2.

Proof of Corollary 1.3. This is obtained by a simple formula transformation from Theorem 1.1. Indeed, recall that

$$d = \frac{q-p}{2}, \qquad e = \frac{q+p}{2}, \qquad \text{and} \qquad n = e^2 - d^2,$$

where d is the integer in our assumption. Then each factor $\prod_{r=0}^{k-1}(d^2 - r^2)$ in (2) vanishes if $k > |d|$. Thus, Theorem 1.1 implies that

$$\operatorname{Ch}_{2j-1}\big(p \times q\big) = (-1)^{j-1} \operatorname{Cat}(j-1) \sum_{k=0}^{|d|} f_k(j) \left(\prod_{r=0}^{k-1} (d^2 - r^2)\right) \left(\prod_{r=k}^{j-1} (e^2 - r^2)\right).$$

Here we split the last product in the above equation as follows. See also the convention in Section 1.2.

$$\begin{aligned}
\prod_{r=k}^{j-1} (e^2 - r^2) &= \prod_{r=k}^{j-1} (n + d^2 - r^2) \\
&= \left(\prod_{r=k}^{|d|-1} (n + d^2 - r^2)\right) \left(\prod_{r=|d|}^{j-1} (n + d^2 - r^2)\right) \\
&= \left(\prod_{r=k}^{|d|-1} (n + d^2 - r^2)\right) \left(\prod_{s=0}^{j-|d|-1} \big(n - s(s+2|d|)\big)\right).
\end{aligned}$$

In the last equality above, we changed the variable by setting $r = s + |d|$. Thus, we obtain the expression in Corollary 1.3. $\square$

We can see that, with d fixed, $G_d(j,n) \in \mathbb{Q}[j,n]$ is a polynomial in the variables j,n of the total degree $2\ |d|$ if we declare that the degrees of the variables j and n are given by $\deg j = 1$ and $\deg n = 2$. In fact, $G_d(j,n) \in \mathbb{Z}[j,n]$ is a polynomial with *integer* coefficients, see Proposition 5.1.

Example 1.4.

$$G_0(j,n) = 1,$$
$$G_1(j,n) = n + 1 - j(2j-1),$$
$$G_2(j,n) = (n+4)(n+3) - 4j(2j-1)(n+3) + 2j(j-1)(2j-1)(2j+1).$$

For an arbitrary integer d, we have

$$G_d(j,n) = G_{-d}(j,n)$$

which is related to the fact that

$$\mathrm{Ch}_{2j-1}\big(p \times q\big) = \mathrm{Ch}_{2j-1}\big(q \times p\big);$$

in this way the values of $G_{-1}(j,n)$ and $G_{-2}(j,n)$ follow immediately.

1.4. Collection of results. Below we present a collection of the results which cover the remaining choices for the parity for the length of the cycle and the difference $q-p$ between the rectangle sides.

1.4.1. *The length of the cycle is odd, the difference of the rectangle sides is odd.* The following result is a counterpart of Theorem 1.1 which is particularly useful for a rectangular Young diagram $\lambda = p \times q$ for which $q - p$ is an odd integer.

Theorem 1.5. *Let j be a positive integer. Then*

$$\mathrm{Ch}_{2j-1}\big((e-d)\times(e+d)\big) = (-1)^{j-1}\,\mathrm{Cat}(j-1) \sum_{k=0}^{j} f_k(j) \left[\prod_{r=0}^{k-1} \big(d^2 - (r+\tfrac{1}{2})^2\big)\right] \left[\prod_{r=k}^{j-1} \big(e^2 - (r+\tfrac{1}{2})^2\big)\right].$$

Corollary 1.6. *Let j,p,q be positive integers and set $n = pq$. Suppose that $q-p$ is an odd integer $2d \coloneqq q-p$, where $d \in \big\{\pm\frac{1}{2}, \pm\frac{3}{2}, \dots\big\}$. Then*

$$\mathrm{Ch}_{2j-1}\big(p\times q\big) = (-1)^{j-1}\,\mathrm{Cat}(j-1)\ H_d(j,n) \prod_{r=0}^{j-|d|-\frac{1}{2}} \big(n - r(r+2|d|)\big),$$

where

$$H_d(j,n) = \sum_{k=0}^{|d|-\frac{1}{2}} (-1)^k \frac{j^{\downarrow k}\,(2j-1)^{\uparrow\uparrow k}}{k!\,(2k-1)!!} \prod_{r=0}^{k-1} \big(d^2 - (r+\tfrac{1}{2})^2\big) \prod_{r=k}^{|d|-\frac{3}{2}} \big(n + d^2 - (r+\tfrac{1}{2})^2\big).$$

Example 1.7.

$$H_{\frac{1}{2}}(j,n) = 1,$$
$$H_{\frac{3}{2}}(j,n) = n + 2 - 2j(2j-1),$$
$$H_{\frac{5}{2}}(j,n) = (n+6)(n+4) - 6j(2j-1)(n+4) + 4j(j-1)(2j-1)(2j+1).$$

Similarly as in Example 1.4 we have

$$H_d(j,n) = H_{-d}(j,n)$$

thus the values of $H_{-\frac{1}{2}}(j,n)$, $H_{-\frac{3}{2}}(j,n)$, $H_{-\frac{5}{2}}(j,n)$ follow immediately.

1.4.2. *The length of the cycle is even, the difference of the rectangle sides is even.* The following result is a direct counterpart of Theorem 1.1 for the even cycle.

THEOREM 1.8. *Let j be a positive integer. Then*

$$\begin{aligned}&\operatorname{Ch}_{2j}\big((e-d)\times(e+d)\big)\\ &\qquad = (-1)^{j-1}\binom{2j-1}{j}\sum_{k=0}^{j} g_k(j)2d\left(\prod_{r=1}^{k}(d^2-r^2)\right)\left(\prod_{r=k+1}^{j}(e^2-r^2)\right),\end{aligned}$$

where $g_0(j)=1$ and

$$g_k(j) = (-1)^k \frac{j^{\downarrow k}\,(2j+1)^{\uparrow\uparrow k}}{k!\,(2k+1)!!}$$

for $1\leqslant k\leqslant j$.

COROLLARY 1.9. *Let j,p,q be positive integers and set $n=pq$. Suppose that $q-p$ is an even integer $2d=q-p$, where $d\in\{0,\pm1,\pm2,\dots\}$. Then*

$$\operatorname{Ch}_{2j}\big(p\times q\big) = (-1)^{j-1}\binom{2j}{j} I_d(j,n)\prod_{r=0}^{j-|d|}\big(n-r(r+2|d|)\big),$$

where $I_0(j,n)=0$ and, for $d\in\{\pm1,\pm2,\dots\}$

$$I_d(j,n) = d\sum_{k=0}^{|d|-1}(-1)^k\frac{j^{\downarrow k}\,(2j+1)^{\uparrow\uparrow k}}{k!\,(2k+1)!!}\left(\prod_{r=1}^{k}(d^2-r^2)\right)\left(\prod_{r=k+1}^{|d|-1}(n+d^2-r^2)\right).$$

With d fixed, $I_d(j,n)\in\mathbb{Q}[j,n]$ is a polynomial in the variables j,n of the total degree $2|d|-1$ if we give $\deg j=1$ and $\deg n=2$.

EXAMPLE 1.10.

$$\begin{aligned}I_0(j,n) &= 0,\\ I_1(j,n) &= 1,\\ I_2(j,n) &= 2(n+3)-2j(2j+1) = 2\big(n-(j-1)(2j+3)\big),\\ I_3(j,n) &= 3(n+8)(n+5)-8j(2j+1)(n+5)+4j(j-1)(2j+1)(2j+3).\end{aligned}$$

For negative integers d, we have $I_d(j,n) = -I_{-d}(j,n)$.

1.4.3. *The length of the cycle is even, the difference of the rectangle sides is odd.* The following result is a direct counterpart of Theorem 1.5 for an even cycle.

THEOREM 1.11. *Let j be a positive integer. Then*

$$\begin{aligned}&\operatorname{Ch}_{2j}\big((e-d)\times(e+d)\big)\\ &= (-1)^{j-1}\binom{2j-1}{j}\sum_{k=0}^{j} g_k(j)2d\left(\prod_{r=1}^{k}\big(d^2-(r-\tfrac12)^2\big)\right)\left(\prod_{r=k+1}^{j}\big(e^2-(r-\tfrac12)^2\big)\right).\end{aligned}$$

COROLLARY 1.12. *Let j,p,q be positive integers and set $n=pq$. Suppose that $q-p$ is an odd integer $2d\coloneqq q-p$, where $d\in\{\pm\frac12,\pm\frac32,\dots\}$. Then*

$$\operatorname{Ch}_{2j}\big(p\times q\big) = (-1)^{j-1}\binom{2j-1}{j} J_d(j,n)\prod_{r=0}^{j-|d|-\frac12}\big(n-r(r+2|d|)\big),$$

where

$$J_d(j,n) = 2d \sum_{k=0}^{|d|-\frac{1}{2}} (-1)^k \frac{j^{\downarrow k}\,(2j+1)^{\uparrow\uparrow k}}{k!\,(2k+1)!!} \left(\prod_{r=1}^{k} \left(d^2 - (r-\tfrac{1}{2})^2\right) \right) \times \left(\prod_{r=k+1}^{|d|-\frac{1}{2}} \left(n + d^2 - (r-\tfrac{1}{2})^2\right) \right).$$

Example 1.13.

$$\begin{aligned} J_{\frac{1}{2}}(j,n) &= 1, \\ J_{\frac{3}{2}}(j,n) &= 3(n+2) - 2j(2j+1) = 3n - 2(j-1)(2j+3), \\ J_{\frac{5}{2}}(j,n) &= 5(n+6)(n+4) - 10j(2j+1)(n+4) + 4j(j-1)(2j+1)(2j+3). \end{aligned}$$

For negative half integers d, we have $J_d(j,n) = -J_{-d}(j,n)$.

1.5. Vanishing of some special characters. The special case of Corollary 1.6 for $d = \frac{3}{2}$ and $p = 2j-2$, $q = 2j+1$ gives rise to the following somewhat surprising corollary for which we failed to find an alternative simple proof.

Corollary 1.14. *For each integer $j \geqslant 2$ the irreducible character related to the rectangular diagram $(2j-2) \times (2j+1)$ vanishes on the cycle of length $2j-1$, i.e.,*

$$\chi^{(2j-2)\times(2j+1)}_{2j-1,\ 1^{(4j^2-4j-1)}} = 0.$$

1.6. The link with spin characters related to the staircase strict partition. One of the motivations for the current paper was the recent progress related to the *spin characters of the symmetric groups* [9]. On one hand, De Stavola [4, Proposition 4.18, page 91] gave an explicit formula for the spin character related to the *staircase strict partition*

$$\Delta_p = (p, p-1, p-2, \dots, 2, 1)$$

which has the property that its *double*

$$D(\Delta_p) = (\underbrace{p+1, p+1, \dots, p+1}_{p \text{ times}}) = p \times (p+1)$$

is a rectangular Young diagram which is almost square. On the other hand, in our recent paper [6] we found an identity which gives a link between the spin characters and their usual (linear) counterparts

$$2\,\mathrm{Ch}^{\mathrm{spin}}_{2j-1}(\xi) = \mathrm{Ch}_{2j-1}(D(\xi))$$

which holds true for any strict partition ξ.

By combining these two results one gets a closed product formula for the linear character $\mathrm{Ch}_{2j-1}\left(p \times (p+1)\right)$ corresponding to a rectangular Young diagram which is almost square; this closed formula coincides with the special case of Corollary 1.6 for $d = \frac{1}{2}$ (in fact the formula in the original paper of De Stavola has an incorrect sign). In his proof, De Stavola employed some computations in Maple; by turning the argument around our Corollary 1.6 gives a purely algebraic proof of his product formula for $\mathrm{Ch}^{\mathrm{spin}}_{2j-1}(\Delta_p)$.

1.7. Sketch of the proof. We will start in Section 2 by collecting some formulas for the irreducible characters related to rectangular shapes. Our strategy towards the proof of Theorem 1.1 is threefold.

- Firstly, we will fix the value of an integer $j \geqslant 1$ and we shall investigate the function
$$(d,e) \mapsto \operatorname{Ch}_{2j-1}\big((e-d)\times(e+d)\big).$$
We will show that it is a polynomial in the variables d, e of the total degree $2j$.
- Secondly, we will show that this polynomial is of the form
$$\operatorname{Ch}_{2j-1}\big((e-d)\times(e+d)\big) = \sum_{k=0}^{j} c_k(j) \left(\prod_{r=0}^{k-1}(d^2-r^2)\right)\left(\prod_{r=k}^{j-1}(e^2-r^2)\right)$$
with certain coefficients $c_k(j)$ independent of d, e.
- Thirdly, by finding explicitly the value of
$$\operatorname{Ch}_{2j-1}\big((-1)\times(2k-1)\big)$$
we will determine the coefficients $c_k(j)$.

The proofs of Theorems 1.5, 1.8, and 1.11 are fully analogous to the proof of Theorem 1.1 and we skip them, see Section 4 for some additional comments.

2. Characters on rectangular diagrams

For any partition π of k, Stanley's character formula for rectangular shapes [10] is given by

$$\operatorname{Ch}_\pi(p\times q) = (-1)^k \sum_{\substack{\sigma_1,\sigma_2\in\mathfrak{S}_k\\ \sigma_1\sigma_2=w_\pi}} (-q)^{\kappa(\sigma_1)} p^{\kappa(\sigma_2)}, \tag{5}$$

where $\kappa(\sigma)$ is the number of cycles in σ and $w_\pi \in \mathfrak{S}_k$ is a fixed permutation of the cycle type π.

Corollary 2.1. *For each partition π the corresponding character*
$$\operatorname{Ch}_\pi(p\times q)\in\mathbb{Z}[p,q]$$
can be identified with a polynomial in the variables p and q. This polynomial is of degree $|\pi|+\ell(\pi)$ and fulfills the equality

$$\operatorname{Ch}_\pi(p\times q) = (-1)^{|\pi|-\ell(\pi)} \operatorname{Ch}_\pi(q\times p). \tag{6}$$

Proof. It is easy to show that if two polynomial functions from $\mathbb{Q}[p,q]$ take equal values on each lattice point (p,q) with $p,q\in\mathbb{N}$ then they are equal as polynomials (see [1, Lemma 2.1] for a stronger result); it follows that the polynomial given by the right-hand side of (5) is unique.

We define the length $\|\sigma\|$ of a permutation $\sigma\in\mathfrak{S}_k$ as the minimal number of factors necessary to write it as a product of transpositions. It is a classical result (see, e.g., [5]) that
$$\|\sigma\| = k-\kappa(\sigma).$$
In this way the exponent on the right-hand side of (5) is bounded from above by
$$\kappa(\sigma_1)+\kappa(\sigma_2) = 2k-\|\sigma_1\|-\|\sigma_2\| \leqslant 2k-\|\sigma_1\sigma_2\| = k+(k-\|\sigma_1\sigma_2\|) = |\pi|+\ell(\pi),$$
as required.

Equation (6) is a consequence of the general formula for the character which corresponds to the transposed Young diagram
$$\operatorname{Ch}_\pi(\lambda^T) = (-1)^{|\pi|-\ell(\pi)}\operatorname{Ch}_\pi(\lambda). \qquad \square$$

We will need the following classical fact (see for instance [11, Proposition 1.3.7]).

Lemma 2.2. *For each integer $k \geqslant 1$*

$$\sum_{\sigma \in \mathfrak{S}_k} p^{\kappa(\sigma)} = p(p+1)\cdots(p+k-1).$$

Corollary 2.3. *For each integer $k \geqslant 1$*

$$\operatorname{Ch}_k\big((-1)\times q\big) = (-1)\ q(q+1)\cdots(q+k-1),$$
$$\operatorname{Ch}_k\big(p\times(-1)\big) = (-1)^k\ p(p+1)\cdots(p+k-1).$$

Proof. From the Stanley formula (5) and Lemma 2.2 it follows that

$$\operatorname{Ch}_k\big(p\times(-1)\big) = (-1)^k \sum_{\sigma\in\mathfrak{S}_k} p^{\kappa(\sigma)} = (-1)^k p(p+1)\cdots(p+k-1),$$

as required.

The other identity follows first one and (6). □

3. Proof of Theorem 1.1

We fix a positive integer j; note that the following notation depends on j implicitly. We denote by

$$\mathcal{P}(d) = \operatorname{Ch}_{2j-1}\big((e-d)\times(e+d)\big)$$

the left-hand side of (2) *viewed as a polynomial in the variable* d with the coefficients in the polynomial ring $\mathbb{Q}[e]$. From the Stanley character formula (5) and Corollary 2.1 it follows that the degree of $\mathcal{P}(d)$ is at most $2j$.

Equation (6) implies that

$$\operatorname{Ch}_{2j-1}\big((e-d)\times(e+d)\big) = \operatorname{Ch}_{2j-1}\big((e+d)\times(e-d)\big);$$

in other words the polynomial $\mathcal{P}(d)$ is even.

The linear space of even polynomials in the variable d has a linear basis

$$1, \quad d^2, \quad d^2(d^2-1^2), \quad d^2(d^2-1^2)(d^2-2^2), \quad \dots .$$

It follows that there exist polynomials $P_0,\dots,P_j \in \mathbb{Q}[e]$ with the property that

$$\begin{aligned}(7)\quad & \operatorname{Ch}_{2j-1}\big((e-d)\times(e+d)\big) \\ &= P_0(e) + P_1(e)\ d^2 + P_2(e)\ d^2(d^2-1^2) + \cdots = \sum_{k=0}^{j} P_k(e) \prod_{r=0}^{k-1}(d^2-r^2).\end{aligned}$$

Additionally, from Corollary 2.1 it follows that the degree of the polynomial $P_k(e)$ is at most $2(j-k)$.

The parity of the total degree of each monomial on the right-hand side of the Stanley formula (5) is the same as the parity of $|\pi| - \ell(\pi)$. In our case $\pi = (2j-1)$, so this parity is even. It follows that

$$\begin{aligned}\operatorname{Ch}_{2j-1}\big((e-d)\times(e+d)\big) &= \operatorname{Ch}_{2j-1}\big((-e+d)\times(-e-d)\big) \\ &= \operatorname{Ch}_{2j-1}\big((-e-d)\times(-e+d)\big),\end{aligned}$$

where the second equality is the consequence of the above observation that the polynomial $\mathcal{P}(d)$ is even. This shows that $\mathcal{P}(d)$ is invariant under the involutive automorphism of the polynomial ring $\mathbb{Q}[e]$ which is given by the substitution $e \mapsto -e$. It follows that each coefficient $P_k(e)$ is an even polynomial in the variable e.

Lemma 3.1. *For each $k \in \{0, \dots, j\}$ there exists some constant c_k with the property that*

$$P_k(e) = c_k \prod_{r=k}^{j-1} (e^2 - r^2). \tag{8}$$

Proof. We will use induction over the variable k. For the induction step let $k_0 \in \{0, \dots, j\}$; we assume that (8) holds true for each integer $k \in \{0, \dots, k_0 - 1\}$.

Our strategy is to evaluate (7) for $d = k_0$ and $e \in \{k_0 - 1, \dots, j-1\}$. Each summand on the right-hand side which corresponds to $k > k_0$ vanishes as it contains the factor $(d^2 - r^2)$ for $r = k_0$. On the other hand, each summand on the right-hand side which corresponds to $k < k_0$ vanishes because either (a) $k_0 = 0$ and there are no such summands, or (b) by the inductive hypothesis $P_k(e)$ contains the factor $(e^2 - r^2)$ for $r = e$. We proved in this way that for $e \in \{k_0 - 1, \dots, j-1\}$

$$\operatorname{Ch}_{2j-1}\big((e - k_0) \times (e + k_0)\big) = P_{k_0}(e) \prod_{r=0}^{k_0 - 1} (k_0^2 - r^2). \tag{9}$$

In fact, in the special case when $k_0 = 0$ the above reasoning shows more, namely that the above equality holds true for an *arbitrary* choice of $e \in \mathbb{C}$ and

$$\operatorname{Ch}_{2j-1}\big(e \times e\big) = P_0(e). \tag{10}$$

In the special case when $e \in \{k_0, \dots, j-1\}$ the rectangular Young diagram $(e - k_0) \times (e + k_0)$ is well-defined and the defining formula (1) can be used. Furthermore, the total number of rows and columns of this Young diagram is at most $(e - k_0) + (e + k_0) = 2e < 2j - 1$ hence it does not contain any rim hooks of length $2j - 1$. From the Murnaghan–Nakayama rule (see, e.g., [8, Corollary 4.10.6]) it follows that the left-hand side of (9) is equal to zero; as a consequence $P_{k_0}(e) = 0$.

We proved in this way that P_{k_0} is an even polynomial which has roots in $k_0, k_0 + 1, \dots, j-1$; it follows that the polynomial $P_{k_0}(e)$ is divisible by the product

$$\prod_{r=k_0}^{j-1} (e^2 - r^2).$$

Since the degree of P_{k_0} is at most $2(j - k_0)$, this determines the polynomial P_{k_0} up to a scalar multiple and shows that (8) holds true for $k \coloneqq k_0$. This completes the proof of the inductive step of Lemma 3.1. □

As an extra bonus, for $k_0 \geqslant 1$ the special case of (9) and (8) for $e = k_0 - 1$ gives (in order to keep the notation lightweight we write $k = k_0$)

$$\operatorname{Ch}_{2j-1}\big((-1) \times (2k - 1)\big) = c_k \prod_{r=k}^{j-1} \big((k-1)^2 - r^2\big) \prod_{r=0}^{k-1} (k^2 - r^2).$$

The left-hand side can be evaluated thanks to Corollary 2.3 which gives an explicit product formula for the constant c_k for $k \geqslant 1$. Note that this argument cannot be applied in the special case when $k = 0$ and $j \geqslant 2$ because the right-hand side contains the factor $\big((k-1)^2 - r^2\big)$ which vanishes for $r = 1$; for this reason in order to evaluate c_0 we will need a different method.

A combination of (8) and (10) gives

$$\operatorname{Ch}_{2j-1}\big(e \times e\big) = c_0 \prod_{r=0}^{j-1} (e^2 - r^2).$$

In order to evaluate the constant c_0 we need some additional piece of information about the polynomial on the left-hand side. One possible approach is to evaluate its value for $e := j$; in this special case the Murnaghan–Nakayama rule has only one summand therefore value of the normalized character is given by a product formula based on the hook-length formula. An alternative approach is based on calculating the leading coefficient $[e^{2j}]\operatorname{Ch}_{2j-1}\big(e\times e\big)$, which stands for the coefficient of e^{2j} in $\operatorname{Ch}_{2j-1}\big(e\times e\big)$, based on the ideas of the asymptotic representation theory, see Remark 3.3.

Thanks to these explicit values of the constants c_k, Theorem 1.1 follows by a straightforward algebra and its proof is now complete.

Theorem 1.1 gives a new proof of the following result.

Corollary 3.2. *For each integer $j \geqslant 1$*

$$\left[e^{2j}\right]\operatorname{Ch}_{2j-1}\big(e\times e\big) = (-1)^{j-1}\operatorname{Cat}(j-1).$$

Remark 3.3. Corollary 3.2 is not new; in the following we only give a rough sketch of an alternative proof based on existing results. The work of Biane ([2, Theorem 1.3] or [3]) implies that

$$\lim_{e\to\infty}\frac{1}{e^{2j}}\operatorname{Ch}_{2j-1}\big(e\times e\big) = R_{2j}(\square),$$

where $R_{2j}(\square)$ denotes the *free cumulant* of the one-box Young diagram $\square = (1)$. More specifically, $R_{2j}(\square)$ is the free cumulant of the *Kerov transition measure* of $\square$ which is equal to the Bernoulli measure

$$\tfrac{1}{2}\left(\delta_{-1}+\delta_1\right).$$

Standard combinatorial tools of free probability [7] give a closed formula for such a free cumulant in terms of Catalan numbers.

4. Comments about the proof of Theorems 1.5, 1.8, and 1.11

As we already mentioned, the proofs of Theorems 1.5, 1.8, and 1.11 are analogous to the proof of Theorem 1.1. Below we revisit only some key places which require an adjustment.

For example, in order to prove Theorem 1.5 we need to write

$$\operatorname{Ch}_{2j-1}\big((e-d)\times(e+d)\big) = \sum_{k=0}^{j} P'_k(e)\prod_{r=0}^{k-1}\left(d^2-\left(r+\tfrac{1}{2}\right)^2\right).$$

and then to show the following analogue of Lemma 3.1: for each $k\in\{0,\dots,j\}$ there exists some constant c'_k with the property that

$$P'_k(e) = c'_k\prod_{r=k}^{j-1}\left(e^2-\left(r+\tfrac{1}{2}\right)^2\right).$$

In order to achieve this goal, the strategy of the induction step is to fix $d = k_0+\frac{1}{2}$ and to consider $e\in\{k_0-\frac{1}{2},\dots,j-\frac{1}{2}\}$.

The calculation of the constants c'_k is particularly easy now because both c'_k as well as the constants c_k from Equations (8) and (7) coincide with the coefficient of a specific monomial in the Stanley polynomial

$$c'_k = \left[d^{2k}e^{2(j-k)}\right]\operatorname{Ch}_{2j-1}\big((e-d)\times(e+d)\big) = c_k$$

hence they are equal.

Theorems 1.8 and 1.11 concern the character Ch_{2j} on an *even* cycle. In this case the corresponding polynomial

$$\mathcal{P}(d) = \mathrm{Ch}_{2j}\big((e-d)\times(e+d)\big)$$

is *odd* and its degree is at most $2j+1$, therefore we may write

$$\begin{aligned}\mathrm{Ch}_{2j}\big((e-d)\times(e+d)\big) &= \sum_{k=0}^{j} P''_k(e)\ d \prod_{r=1}^{k}(d^2-r^2)\\ &= \sum_{k=0}^{j} P'''_k(e)\ d \prod_{r=1}^{k}\Big(d^2-\big(r-\tfrac{1}{2}\big)^2\Big)\end{aligned}$$

for some even polynomials $P''_k(e), P'''_k(e)\in\mathbb{Q}[e]$ which are of order at most $2(j-k)$, where $k\in\{0,\dots,j\}$.

The proof of Theorem 1.8 involves analysis of the polynomials P''_k which is analogous to the one from the proof of Theorem 1.1; in particular an analogue of Lemma 3.1 says that

$$P''_k(e) = c''_k \prod_{r=k+1}^{j}\left(e^2-r^2\right).$$

The proof of its inductive step is based on fixing $d=k_0+1$ and considering $e\in\{k_0,\dots,j\}$. The values $e\in\{k_0+1,\dots,j\}$ are the positive roots of the even polynomial P''_k; the polynomial is therefore determined up to a multiplicative constant. The special case $e=k_0$ allows to find explicitly the value of c''_k; interestingly (opposite to the case in the proof of Theorem 1.1) the case $k=0$ does not require a separate proof.

5. Integrality of the coefficients

In the following we use the notation

$$[\mathit{condition}] = \begin{cases}1 & \text{if } \mathit{condition} \text{ holds true,}\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 5.1. *Let d be an integer. Each coefficient of the polynomial $G_d(j,n)$ (defined in Corollary 1.3) is an integer.*

Proof. We will show a stronger result that for each integer $k\geqslant 1$

$$\frac{1}{2^{k-1}}\frac{\prod_{r=0}^{k-1}(d^2-r^2)}{k!\,(2k-1)!!} = \frac{2d\cdot\prod_{r=-k+1}^{k-1}(d+r)}{(2k)!} \tag{11}$$

is an integer. We will do it by proving that for each prime number p the exponent by which it contributes to the factorization of the numerator is at least its counterpart for the denominator. In the case when $p\neq 2$ these exponents are equal, respectively, to

$$\begin{aligned}&\sum_{c\geqslant 1}\Big(\big[p^c\mid d\big] + \#\Big\{i\in\{d-k+1,\dots,d+k-1\} : p^c\mid i\Big\}\Big)\\ &\quad= \sum_{c\geqslant 1}\Big(\big[p^c\mid d\big] - \big[p^c\mid d+k\big] + \#\Big\{i\in\{d-k+1,\dots,d+k\} : p^c\mid i\Big\}\Big)\end{aligned} \tag{12}$$

and

$$\sum_{c\geqslant 1}\#\Big\{i\in\{1,\dots,2k\} : p^c\mid i\Big\}. \tag{13}$$

We will show that for each $c\geqslant 1$ the corresponding summand on the right-hand side of (12) is greater or equal to its counterpart in (13).

We start with the observation that in any collection of p^c consecutive integers there is exactly one which is divisible by p^c; it follows that a collection of $2k$ consecutive integers contains at least $\left\lfloor \frac{2k}{p^c} \right\rfloor$ such numbers divisible by p^c. As a consequence we get the following lower bound for the summand on (12):

$$\begin{multline}\tag{14} \big[p^c \mid d\big] + \#\Big\{ i \in \{d-k+1, \dots, d+k-1\} : p^c \mid i \Big\} \\ \geqslant [p^c \mid d] - \big[p^c \mid d+k\big] + \left\lfloor \frac{2k}{p^c} \right\rfloor . \end{multline}$$

Since (12) can be alternatively written as

$$\begin{multline}\tag{15} \sum_{c \geqslant 1} \Big(\big[p^c \mid d\big] + \#\Big\{ i \in \{d-k+1, \dots, d+k-1\} : p^c \mid i \Big\} \Big) \\ = \sum_{c \geqslant 1} \Big(\big[p^c \mid d\big] - \big[p^c \mid d-k\big] + \#\Big\{ i \in \{d-k, \dots, d+k-1\} : p^c \mid i \Big\} \Big), \end{multline}$$

an analogous reasoning to the one above gives the following alternative lower bound for the summand on the left-hand side of (12):

$$\begin{multline}\tag{16} \big[p^c \mid d\big] + \#\Big\{ i \in \{d-k+1, \dots, d+k-1\} : p^c \mid i \Big\} \\ \geqslant [p^c \mid d] - \big[p^c \mid d-k\big] + \left\lfloor \frac{2k}{p^c} \right\rfloor . \end{multline}$$

On the other hand, for the corresponding summand in (13) we get the following exact expression

$$\#\Big\{ i \in \{1, \dots, 2k\} : p^c \mid i \Big\} = \left\lfloor \frac{2k}{p^c} \right\rfloor . \tag{17}$$

If one of the following two conditions holds true: (a) the right-hand side of (14) is greater or equal to the right-hand side of (17), or (b) the right-hand side of (16) is greater or equal to the right-hand side of (17), then the desired inequality holds true. The opposite case, i.e., when both $d-k$ and $d+k$ are divisible by p^c and d is not divisible by p^c, is clearly not possible since $(d+k)+(d-k)=2d$.

For the case when $p=2$ let us consider *half* of the expression (11); then (12) and (13) still provide the exponents in the numerator and the denominator. The proof proceeds without any modifications until after Equation (17). It might happen that none of the conditions (a) and (b) holds true; if this is indeed the case then both $d-k$ and $d+k$ are divisible by 2^c (hence d is divisible by 2^{c-1}) and d is not divisible by 2^c. There exists at most one index c with this property. It follows that the sum (12) is at least (13) *less one*. Since we considered *half* of the expression (11), this completes the proof. □

Proposition 5.2. *Let $d \in \{\pm\frac{1}{2}, \pm\frac{3}{2}, \dots\}$. Each coefficient of the polynomial $H_d(j,n)$ (defined in Corollary 1.6) is an integer.*

Proof. By the symmetry $H_d(j,n) = H_{-d}(j,n)$, we may suppose that d is positive and we write it as $d = d' + \frac{1}{2}$, where d' is a non-negative integer. For each integer$k \in \{0,1,2,\dots,d'\}$,

$$\frac{\prod_{r=0}^{k-1}\left(d^2 - (r+\frac{1}{2})^2\right)}{k!\,(2k-1)!!} = \frac{2^k \prod_{r=-k+1}^{k}(d'+r)}{(2k)!} = 2^k \binom{d'+k}{2k}$$

is clearly an integer. Thus, we see that

$$H_{d'+\frac{1}{2}}(j,n) = \sum_{k=0}^{d'} (-1)^k 2^k \binom{d'+k}{2k} j^{\downarrow k} (2j-1)^{\uparrow\uparrow k} \prod_{r=k}^{d'-1} \left(n + d'(d'+1) - r(r+1)\right)$$

has integer coefficient, as required. ☐

Proposition 5.3. *Let d be an integer. Each coefficient of the polynomial $I_d(j,n)$ (defined in Corollary 1.9) is an integer.*

Proof. By the symmetry $I_d(j,n) = -I_{-d}(j,n)$, we may suppose that d is a positive integer. For each integer $k \in \{1, 2, \ldots, d-1\}$,

$$\frac{d \prod_{r=1}^{k} (d^2 - r^2)}{k!\,(2k+1)!!} = \frac{2^k \prod_{r=-k}^{k} (d+r)}{(2k+1)!} = 2^k \binom{d+k}{2k+1}$$

is clearly an integer which completes the proof. ☐

Proposition 5.4. *Let d be a half integer. Each coefficient of the polynomial $J_d(j,n)$ (defined in Corollary 1.12) is an integer.*

Proof. We may write $d = d' - \frac{1}{2}$, where d' is an integer. We will show a stronger result that for each integer $k \geqslant 1$

$$\frac{1}{2^k} \frac{2d \prod_{r=1}^{k} \left(d^2 - (r - \frac{1}{2})^2\right)}{k!\,(2k+1)!!} = \frac{(2d'-1) \prod_{r=-k}^{k-1} (d'+r)}{(2k+1)!}$$

is an integer. The proof is analogous to the proof of Proposition 5.1. ☐

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Sho Matsumoto, Graduate School of Science and Engineering, Kagoshima University, 1-21-35, Korimoto, Kagoshima, Japan
E-mail : shom@sci.kagoshima-u.ac.jp

Piotr Śniady, Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland
E-mail : psniady@impan.pl