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Abstract Let I and J be edge ideals in a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ with $I \subseteq J$. In this paper, we obtain a general upper and lower bound for the Castelnuovo–Mumford regularity of IJ in terms of certain invariants associated with I and J. Using these results, we explicitly compute the regularity of IJ for several classes of edge ideals. In particular, we compute the regularity of IJ when J has a linear resolution. Finally, we compute the precise expression for the regularity of $J_1 J_2 \cdots J_d$, $d \in \{3, 4\}$, where $J_1, \ldots, J_d$ are edge ideals, $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_d$ and J_d is the edge ideal of a complete graph.

1. Introduction

Let M be a finitely generated graded module over $R = \mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field. The Castelnuovo–Mumford regularity (or simply, regularity) of M, denoted by $\operatorname{reg}(M)$, is defined to be the least integer i so that, for every j, the j^{th} syzygy of M is generated in degrees $\leqslant i + j$. Regularity is an important invariant in commutative algebra and algebraic geometry that measures the computational complexity of ideals, modules, and sheaves. In this paper, we study bounds on the regularity of products of ideals in a polynomial ring.

The regularity of products of ideals was studied first by Conca and Herzog [8]. They studied whether for homogeneous ideal I and finitely generated graded module M over R, one has $\operatorname{reg}(IM) \leqslant \operatorname{reg}(I) + \operatorname{reg}(M)$. This question is essentially a generalization of the simple fact that the highest degree of a generator of the product IM is bounded above by the sum of the highest degree of a generator of M and the highest degree of a generator of I. The answer to this question is negative in general. There are several examples already known with $M = I$ such that $\operatorname{reg}(I^2) > 2\operatorname{reg}(I)$, see Sturmfels [23]. They found some special classes of ideals I and modules M such that $\operatorname{reg}(IM) \leqslant \operatorname{reg}(I) + \operatorname{reg}(M)$. In particular, they showed that if I is a homogeneous ideal in a polynomial ring R with $\dim(R/I) \leqslant 1$, then $\operatorname{reg}(IM) \leqslant \operatorname{reg}(I) + \operatorname{reg}(M)$ for any finitely generated module M over R.

In case M is also a homogeneous ideal, the situation becomes particularly interesting. For example, Sidman proved that if $\dim(R/(I + J)) \leqslant 1$, then the regularity of IJ is bounded above by $\operatorname{reg}(I) + \operatorname{reg}(J)$, [22]. Also, she proved that if two ideals of R, say I and J, define schemes whose intersection is a finite set of points, then

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$\operatorname{reg}(IJ) \leqslant \operatorname{reg}(I) + \operatorname{reg}(J)$. Chardin, Minh and Trung [6] proved that if I and J are monomial complete intersections, then $\operatorname{reg}(IJ) \leqslant \operatorname{reg}(I)+\operatorname{reg}(J)$. Cimpoeaş [7] proved that for two monomial ideals of Borel type I, J, we have $\operatorname{reg}(IJ) \leqslant \operatorname{reg}(I) + \operatorname{reg}(J)$. Caviglia [5] and Eisenbud, Huneke and Ulrich [9] studied the more general problem of the regularity of tensor products and various Tor modules of R/I and R/J.

In this paper, we study the same problem for the case of edge ideals and seek better bounds by exploiting the combinatorics of the underlying graph. Let G be a finite simple graph without isolated vertices on the vertex set $\{x_1, \dots, x_n\}$ and $I(G) \coloneqq (\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}) \subset R = \mathbb{K}[x_1, \dots, x_n]$ the *edge ideal* corresponding to G. In general, computing the regularity of $I(G)$ is NP-hard [24, Corollary 23]. Several recent papers have related the $\operatorname{reg}(I(G))$ with various invariants of the graph G (see [2] for a survey in this direction). A primary inspiration for this paper is Katzman's and Woodroofe's theorem from [18] and [24]. They showed that if G is a graph, then

$$\nu(G) + 1 \leqslant \operatorname{reg}(I(G)) \leqslant \text{co-chord}(G) + 1, \tag{1.1}$$

where $\nu(G)$ denotes the induced matching number of G (see Section 2 for the definition) and co-chord(G) denotes the co-chordal cover number of G (see Section 2 for the definition). In this context, a natural question is if I and J are edge ideals in R, then what is the regularity of IJ? This question give rise to two directions of research. One direction is to obtain the precise expression for $\operatorname{reg}(IJ)$ for particular classes of edge ideals. Another direction is to obtain upper and lower bounds for $\operatorname{reg}(IJ)$ using combinatorial invariants associated to graphs. Therefore, one may ask for edge ideals I and J,

- (Q1) can we find lower and upper bounds for the regularity of IJ using combinatorial invariants associated to the graphs?
- (Q2) can we find precise expressions for the regularity of IJ for particular classes of graphs?

This paper revolves around these two questions.

Computing the regularity of products of two edge ideals of graphs seems more challenging compared to the regularity of edge ideals of graphs. Even in the case of simple classes of graphs, a formula for the regularity of products of two edge ideals is not known. So, naturally one restricts the attention to important subclasses. We are therefore interested in families of edge ideals I and J with $I \subseteq J$.

First, we prove a lower bound for the regularity of the product of more than two edge ideals. More precisely, let $J_1 = I(G_1), \dots, J_d = I(G_d)$ be edge ideals of graphs $G_1, \dots, G_d$ with $J_1 \subseteq \cdots \subseteq J_d$. Then we prove $2d + \nu_{G_1 \cdots G_d} - 1 \leqslant \operatorname{reg}(J_1 \cdots J_d)$, where $\nu_{G_1 \cdots G_d}$ denotes the joint induced matching number of G_i (see Section 2 for the definition) for all $1 \leqslant i \leqslant d$ (Theorem 4.1). We prove an upper bound for the regularity of product of two edge ideals in terms of co-chordal cover numbers. We prove that if G is a graph and H is a subgraph of G with $I = I(H)$ and $J = I(G)$, then $\operatorname{reg}(IJ) \leqslant \max\{\text{co-chord}(G)+3,\ \operatorname{reg}(I)\}$. In particular, $\operatorname{reg}(IJ) \leqslant \max\{\text{co-chord}(G)+3,\ \text{co-chord}(H)+1\}$ (Theorem 4.2). The above bound is inspired by the general upper bound for the regularity of powers of edge ideals given in [15, Theorem 3.6] and [16, Theorem 4.4]. Theorem 4.2 has a number of interesting consequences. For example, Corollary 4.4 says that if H is any subgraph of G, then $\operatorname{reg}(IJ) \leqslant \operatorname{m}(G) + 3$ where $\operatorname{m}(G)$ denotes the matching number of G. On the other hand, Corollary 4.6 says that if H is an induced subgraph of G, then $\nu(H) + 3 \leqslant \operatorname{reg}(IJ) \leqslant \text{co-chord}(G) + 3$.

We then move on to compute the precise expression for the regularity of product of edge ideals. As a consequence of the techniques that we have developed, we explicitly compute the regularity of IJ when J has a linear resolution (Theorem 5.1). Next, we study the regularity of products of more than two edge ideals. We compute the precise expression for $\operatorname{reg}(J_1 \cdots J_d)$ when $J_1 \subseteq \cdots \subseteq J_d$, $d \in \{3,4\}$ and J_d is the edge ideal of complete graph (Theorem 5.4). We use Theorem 4.2 and Theorem 5.4 to get an upper bound for the regularity of $J_1 \cdots J_d$ in terms of co-chordal cover numbers (Corollary 5.5). As an immediate consequence of these results, we give sufficient conditions for product of edge ideals to have linear resolutions (Corollary 5.3, Corollary 5.6).

Our paper is organized as follows. In Section 2, we collect the necessary notions, terminologies and some results that are used subsequently. In Section 3 we prove several technical lemmas which are needed for the proof of our main results, which appear in Sections 4 and 5.

2. Preliminaries

In this section, we set up basic definitions and notation needed for the main results. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A subgraph $L \subseteq G$ is called *induced* if $\{u,v\}$ is an edge of L if and only if u and v are vertices of L and $\{u,v\}$ is an edge of G. For $\{u_1,\dots,u_r\} \subseteq V(G)$, let $N_G(u_1,\dots,u_r) = \{v \in V(G) \mid \{u_i,v\} \in E(G) \text{ for some } 1 \leqslant i \leqslant r\}$ and $N_G[u_1,\dots,u_r] = N_G(u_1,\dots,u_r) \cup \{u_1,\dots,u_r\}$. For $U \subseteq V(G)$, we denote by $G \smallsetminus U$ the induced subgraph of G on the vertex set $V(G) \smallsetminus U$. Let C_k denote the cycle on k vertices.

Let G be a graph. We say 2 non-adjacent edges $\{f_1,f_2\}$ form a $2K_2$ in G if G does not have an edge with one endpoint in f_1 and the other in f_2. A graph without $2K_2$ is called $2K_2$-free also called *gap-free* graph.

A *matching* in a graph G is a subgraph consisting of pairwise disjoint edges. The *matching number* of G, denoted by $\mathrm{m}(G)$, is the maximum cardinality among matchings of G. If the subgraph is an induced subgraph, the matching is an *induced matching*. The largest size of an induced matching in G is called its *induced matching number* and denoted by $\nu(G)$. The *complement* of G, denoted by G^c, is the graph on the same vertex set as G, where $\{u,v\}$ is an edge of G^c if and only $\{u,v\} \notin E(G)$. A graph G is *chordal* if every induced cycle in G has length 3, and is co-chordal if G^c is chordal. The *co-chordal cover number*, denoted $\operatorname{co-chord}(G)$, is the minimum number n such that there exist co-chordal subgraphs $H_1,\dots,H_n$ of G with $E(G) = \bigcup_{i=1}^{n} E(H_i)$.

Consider graphs G_i for $1 \leqslant i \leqslant d$ where G_i is a subgraph of G_{i+1} for all $1 \leqslant i \leqslant d-1$. The largest size of an induced matching in G_i for all $1 \leqslant i \leqslant d$ is called the *joint induced matching number* and denoted by $\nu_{G_1 \cdots G_d}$. Note that if G_i is an induced subgraph of G_{i+1} for all $1 \leqslant i \leqslant d-1$, then $\nu_{G_1 \cdots G_d} = \nu(G_1)$.

Example 2.1. Let G be the graph as shown in Figure 1. Then $\{\{x_1,x_2\}, \{x_3,x_4\}, \{x_5,x_6\}, \{x_7,x_8\}\}$ forms a matching of G, but not an induced matching. The set $\{\{x_1,x_2\},\{x_4,x_5\}\}$ forms an induced matching. Then $\nu(G) \geqslant 2$. It is not hard to verify that $\nu(G) = 2$. Let H be a subgraph of G with $E(H) = \{\{x_1,x_2\},\{x_3,x_4\}\}$. Since H is a disjoint union two edges, $\nu(H) = 2$. The set $\{\{x_1,x_2\}\}$ forms an induced matching of G and H. Then $\nu_{HG} \geqslant 1$. Since the set $\{\{x_1,x_2\},\{x_3,x_4\}\}$ forms an induced matching of H but not in G, $\nu_{HG} = 1$.

Let H_1, H_2 and H_3 be subgraphs of G with $E(H_1) = \{\{x_1,x_2\},\{x_2,x_3\},\{x_3,x_4\}\}$, $E(H_2) = \{\{x_4,x_5\},\{x_5,x_6\},\{x_6,x_7\}\}$ and $E(H_3) = \{\{x_7,x_8\},\{x_8,x_1\}\}$ respectively. We can seen that H_1, H_2 and H_3 are co-chordal subgraphs of G and $E(G) = \bigcup_{i=1}^{3} E(H_i)$. Therefore, $\operatorname{co-chord}(G) \leqslant 3$. It is also not hard to verify that $\operatorname{co-chord}(G) = 3$.

Polarization is a process to obtain a squarefree monomial ideal from a given monomial ideal.

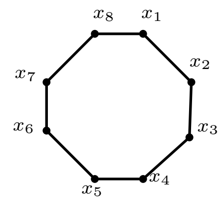

Figure 1. Graph G for Example 2.1

Definition 2.2. *Let $M = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial in $R = \mathbb{K}[x_1, \dots, x_n]$. Then we define the squarefree monomial $P(M)$* (polarization *of M) as*

$$P(M) = x_{11} \cdots x_{1a_1} x_{21} \cdots x_{2a_2} \cdots x_{n1} \cdots x_{na_n}$$

in the polynomial ring $R_1 = \mathbb{K}[x_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant a_i]$. If $I = (M_1, \dots, M_q)$ is an ideal in R, then the polarization of I, denoted by $\widetilde{I}$, is defined as $\widetilde{I} = (P(M_1), \dots, P(M_q))$.

Let M be a graded $R = \mathbb{K}[x_1, \dots, x_n]$ module. For non-negative integers i, j, let $\beta_{i,j}(M)$ denote the (i, j)-th graded Betti number of M. In this paper, we repeatedly use an important property of the polarization, namely:

Corollary 2.3. [14, Corollary 1.6.3(a)] *Let $I \subseteq R = \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal. If $\widetilde{I} \subseteq \widetilde{R}$ is a polarization of I, then for all i, j, we have $\beta_{i,j}(R/I) = \beta_{i,j}(\widetilde{R}/\widetilde{I})$. In particular,* $\operatorname{reg}(R/I) = \operatorname{reg}(\widetilde{R}/\widetilde{I})$.

3. Technical lemmas

In this section we prove several technical results concerning the graph associated with $\widetilde{(IJ : ab)}$, for any $ab \in I$, where I and J are edge ideals and $I \subseteq J$. We first fix the set-up that we consider throughout this paper.

Set-up 1. *Let G be a graph and H be a subgraph of G. Set $I = I(H)$ and $J = I(G)$.*

For a monomial ideal K, let $\mathcal{G}(K)$ denote the minimal generating set of K. For a monomial $m \in R = \mathbb{K}[x_1, \dots, x_n]$, the *support* of m is the set of variables appearing in m and is denoted by $\operatorname{supp}(m)$, i.e. $\operatorname{supp}(m) = \{x_i \mid x_i \text{ divides } m\}$.

The following result is used repeatedly in this paper.

Lemma 3.1. *Let I and J be as in Set-up 1. Then the* colon ideal $(IJ : ab)$ *is a generated by quadratic monomial ideal for any $ab \in I$. More precisely,*

$$(IJ : ab) = J + K_1 + K_2,$$

where $K_1 = (pq \mid p \in N_G(a) \text{ and } q \in N_H(b))$ and $K_2 = (rs \mid r \in N_H(a) \text{ and } s \in N_G(b))$.

Proof. Let $m \in \mathcal{G}((IJ : ab))$. By degree considerations m cannot have degree 1. Suppose $\deg(m) \geqslant 3$. Then there exist $e \in \mathcal{G}(I)$ and $f \in \mathcal{G}(J)$ such that $ef \mid mab$. Since m is a minimal monomial generator of $(IJ : ab)$, there does not exist m', $m' \neq m$ and $m' \mid m$ such that $ef \mid m'ab$. If there exists $g \in \mathcal{G}(J)$ such that $g \mid m$, then the minimality of m and $g \in (IJ : ab)$ imply $g = m$. This is a contradiction to $\deg(m) \geqslant 3$. Therefore, $\deg(m) = 2$. We assume that $g \nmid m$ for any $g \in \mathcal{G}(J)$. Then $e \nmid ab$. Let $e = ax$, where $x \mid m$. Therefore, $xf \mid mb$. If $f = by$, where $y \mid (\frac{m}{x})$, then $xy \mid m$. Hence, by minimality of m, m is a quadratic monomial. Similarly, for $e = bx$ we can prove that m is quadratic in a similar manner.

Clearly, $J + K_1 + K_2 \subseteq (IJ : ab)$. We need to prove the reverse inclusion. Let $uv \in \mathcal{G}(IJ : ab)$. If $uv \in J$, then we are done. Suppose $uv \notin J$. Since $uvab \in IJ$, we have the following cases $ua \in I$ and $vb \in J$ or $ua \in J$ and $vb \in I$ or $ub \in I$ and $va \in J$ or $ub \in J$ and $va \in I$. In all cases, one can show that either $uv \in K_1$ or $uv \in K_2$. Therefore, $(IJ : ab) = J + K_1 + K_2$. $\square$

Let I and J be as in Set-up 1. Then for any $ab \in I$, $\widetilde{(IJ : ab)}$ is a quadratic square-free monomial ideal by Lemma 3.1. There exists a graph $\mathcal{P}$ associated to $\widetilde{(IJ : ab)}$. Suppose xy is a minimal generator of $(IJ : ab)$. If $x \neq y$, then set $\{[x, y]\} = \{x, y\}$ and $\{[x, y]\}$ is an edge of $\mathcal{P}$. If $x = y$, then set $\{[x, y]\} = \{x, z_x\}$, where z_x is a new vertex of $\mathcal{P}$, and $\{[x, y]\}$ is an edge of $\mathcal{P}$. Observe that G is a subgraph of $\mathcal{P}$, i.e. $V(G) \subseteq V(\mathcal{P})$ and $E(G) \subseteq E(\mathcal{P})$. For example, let $I = (x_4x_5, x_5x_6, x_4x_6)$ and $J = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_1x_6, x_4x_6)$. Then $(IJ : x_4x_5) = J + (x_6^2, x_3x_6) \subset \mathbb{K}[x_1, \dots, x_6]$ and $\widetilde{(IJ : x_4x_5)} = J + (x_6z_{x_6}, x_3x_6) \subset \mathbb{K}[x_1, \dots, x_6, z_{x_6}]$. Let $\mathcal{P}$ be the graph associated to $\widetilde{(IJ : x_4x_5)}$. Then $V(\mathcal{P}) = V(G) \cup \{z_{x_6}\}$ and $E(\mathcal{P}) = E(G) \cup \{\{[x_6, x_6]\}, \{x_3, x_6\}\}$.

The following is a useful result on co-chordal graphs that allow us to assume certain order on their edges.

Lemma 3.2. [3, Lemma 1 and Theorem 2] *Let G be a graph and $E(G) = \{e_1 \dots, , e_t\}$. Then G is a co-chordal graph if and only if there is an ordering of edges of G, $e_{i_1} < \cdots < e_{i_t}$, such that for $1 \leqslant r \leqslant t$, $(V(G), \{e_{i_1}, \dots, e_{i_r}\})$ has no induced subgraph isomorphic to $2K_2$.*

One of the key ingredients in the proof of the main results is a new graph $\mathcal{P}$ obtained from the given graphs G and H as in Lemma 3.1. Our main aim in this section is to get an upper bound for the co-chordal cover number of $\mathcal{P}$ which in turn will help us in bounding $\operatorname{reg}(IJ)$. For this purpose, we need to understand the structure of the graph $\mathcal{P}$ in more detail. First we discuss the procedure to get a new graph from the given co-chordal subgraph of G.

Discussion 1. *Let I and J be as in Set-up 1. Let $\mathcal{P}$ be the graph associated to $\widetilde{(IJ : ab)}$ for any $ab \in I$. Suppose* co-chord$(G) = \widetilde{n}$. *Then there exist co-chordal subgraphs $H_1, \dots, H_{\widetilde{n}}$ of G such that $E(G) = \bigcup_{i=1}^{\widetilde{n}} E(H_i)$. Let $N_H(a) \smallsetminus b = \{a_1, \dots, a_{\alpha'}\}$, $N_G(a) \smallsetminus b = \{a_1, \dots, a_{\alpha'}, a_{\alpha'+1}, \dots, a_\alpha\}$, $N_H(b) \smallsetminus a = \{b_1, \dots, b_{\beta'}\}$ and $N_G(b) \smallsetminus a = \{b_1, \dots, b_{\beta'}, b_{\beta'+1}, \dots, b_\beta\}$. Set*

$$\mathcal{N}(G)_a = \{\{a, a_i\} \in E(G) \mid 1 \leqslant i \leqslant \alpha\} \text{ and } \mathcal{N}(G)_b = \{\{b, b_i\} \in E(G) \mid 1 \leqslant i \leqslant \beta\}.$$

Note that if $c \in (N_G(a) \smallsetminus b) \cap (N_G(b) \smallsetminus a)$, then $\{a, c\} \in \mathcal{N}(G)_a$ and $\{b, c\} \in \mathcal{N}(G)_b$. Since H_m is co-chordal for all $1 \leqslant m \leqslant \widetilde{n}$, by Lemma 3.2, there is an ordering of edges of H_m:

$$f_1 < \cdots < f_{t_m}, \tag{3.1}$$

such that for $1 \leqslant r \leqslant t_m$, $(V(H_m), \{f_1, \dots, f_r\})$ has no induced subgraph isomorphic to $2K_2$.

We now define a procedure to add certain edges to H_m to get a new graph H'_m in the following steps:

Step 1 *If $f_k = \{a, b\}$ for some $1 \leqslant k \leqslant t_m$, then we extend the ordered sequence of edges f_is by adding some new edges in the following order:*

$$\cdots < f_k < \{a, a_1\} < \cdots < \{a, a_{\alpha'}\} < \{b, b_1\} < \cdots < \{b, b_{\beta'}\} < \{[a_1, b_1]\} < \cdots$$
$$< \{[a_1, b_{\beta'}]\} < \{[a_2, b_1]\} < \cdots < \{[a_2, b_{\beta'}]\} < \cdots < \{[a_{\alpha'}, b_1]\} < \cdots$$
$$< \{[a_{\alpha'}, b_{\beta'}]\} < f_{k+1} < \cdots$$

Step 2 (i) *If for $1 \leqslant \mu \leqslant \alpha$, $f_{k_1} = \{a, a_\mu\} \in \mathcal{N}(G)_a$ for some $1 \leqslant k_1 \leqslant t_m$, then extend the ordered sequence of edges obtained in* Step 1 *by adding some new edges in the following order:*

$$\cdots < f_{k_1} < \{[a_\mu, b_1]\} < \cdots < \{[a_\mu, b_{\beta'}]\} < f_{k_1+1} < \cdots$$

(ii) *If for $1 \leqslant \mu \leqslant \beta$, $f_{k_2} = \{b, b_\mu\} \in \mathcal{N}(G)_b$ for some $1 \leqslant k_2 \leqslant t_m$, then extend the ordered sequence obtained from* Step 2(i) *by adding new edges in the following order:*

$$\cdots < f_{k_2} < \{[b_\mu, a_1]\} < \cdots < \{[b_\mu, a_{\alpha'}]\} < f_{k_2+1} < \cdots$$

otherwise do not do anything.

Step 3 *After applying* Step 1 *and* Step 2*, we get that the ordered sequence*

$$g_1 < \cdots < g_{t_{m'}} \tag{3.2}$$

of whose elements are edges of H'_m. Note that these steps give us an ordered sequence of edges where some edges may appear more than once, i.e. g_i may be equal to g_j for some $1 \leqslant i, j \leqslant t_{m'}$ in (3.2)*. For each edge we keep the first appearance and delete the subsequent ones in* (3.2) *to get a non repeating ordered sequence*

$$\mathfrak{g}_1 < \cdots < \mathfrak{g}_{t_{m_1}}$$

of edges of H'_m where $t_{m_1} \leqslant t_{m'}$.

First note that $\{g_1, \ldots, g_{t_{m'}}\} = \{\mathfrak{g}_1, \ldots, \mathfrak{g}_{t_{m_1}}\}$. For the convenience of the readers, we give an example in next describing the ordering just defined.

Example 3.3. Let G and H be the graphs as shown in the figure below. Set $I = I(H)$, $J = I(G)$, $a = x_7$ and $b = x_6$. Let $\mathcal{P}$ be the graph associated to $(\widetilde{IJ : ab})$. Note that $N_G(x_6) \smallsetminus \{x_7\} = \{x_5, x_8, x_{10}\}$, $N_G(x_7) \smallsetminus \{x_6\} = \{x_2, x_4, x_8\}$, $N_H(x_6) \smallsetminus \{x_7\} = \{x_5, x_8\}$ and $N_H(x_7) \smallsetminus \{x_6\} = \{x_4, x_8\}$. Let H_1, H_2 and H_3 be co-chordal subgraphs of G such that $E(G) = \bigcup_{i=1}^{3} E(H_i)$; see Figure 3. Therefore co-chord$(G) = 3$.

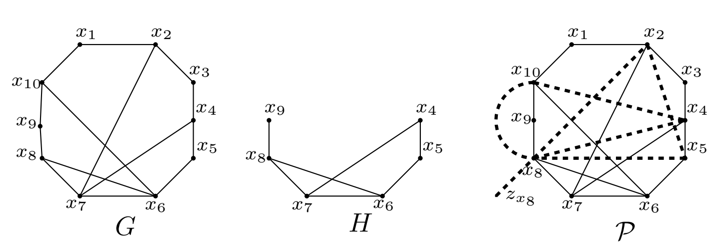

Figure 2. Graphs G and H for Example 3.3.

Let $f_1 = \{x_1, x_2\} < f_2 = \{x_2, x_7\} < f_3 = \{x_2, x_3\} < f_4 = \{x_3, x_4\}$. This is an ordering of the edges of H_1 such that for $1 \leqslant i \leqslant 4$, $(V(H_1), \{f_1, \ldots, f_i\})$ has no induced subgraph isomorphic to $2K_2$. Note that $f_i \neq \{a, b\}$ for all $1 \leqslant i \leqslant 4$.

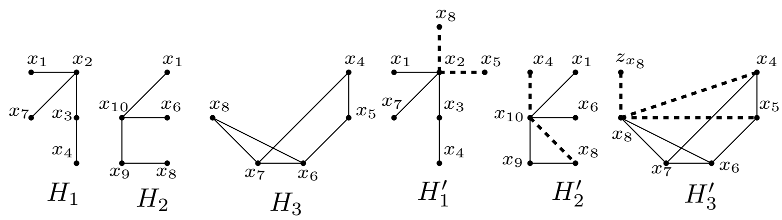

FIGURE 3. Graphs H_1, H_2, and H_3 for Example 3.3

Therefore there is no change in the ordered sequence of edges f_i's. Since $f_2 \in \mathcal{N}(G)_a$, by STEP 2(i), we get

$$f_1 < f_2 < \{x_2, x_5\} < \{x_2, x_8\} < f_3 < f_4.$$

Also note that $f_i \notin \mathcal{N}(G)_b$ for all $1 \leqslant i \leqslant 4$. Since there are no repeated edges in the ordering above, by STEP 3 we have that H'_1 is the graph with the edge set $E(H_1) \cup \{\{x_2, x_5\}, \{x_2, x_8\}\}$ and whose edges appear in the above ordered sequence.

Let $f'_1 = \{x_1, x_{10}\} < f'_2 = \{x_6, x_{10}\} < f'_3 = \{x_9, x_{10}\} < f'_4 = \{x_9, x_8\}$. This is an ordering of the edges of H_2 such that for $1 \leqslant i \leqslant 4$, $(V(H_2), \{f'_1, \dots, f'_i\})$ has no induced subgraph isomorphic to $2K_2$. Note that $f'_i \neq \{a, b\}$ and $f'_i \notin \mathcal{N}(G)_a$ for all $1 \leqslant i \leqslant 4$. Since $f'_2 \in \mathcal{N}(G)_{x_6}$, by STEP 2(ii) we get

$$f'_1 < f'_2 < \{x_{10}, x_4\} < \{x_{10}, x_8\} < f'_3 < f'_4.$$

In this case also there are no repeated edges. By STEP 3, H'_2 is the graph with the edge set $E(H_2) \cup \{\{x_{10}, x_4\}, \{x_{10}, x_8\}\}$ and edges in H'_2 appear in the ordered sequence above.

Let

$$f''_1 = \{x_7, x_6\} < f''_2 = \{x_6, x_5\} < f''_3 = \{x_5, x_4\} < f''_4 = \{x_4, x_7\} < \\ f''_5 = \{x_7, x_8\} < f''_6 = \{x_6, x_8\}.$$

This is an ordering of the edges of H_3 such that for $1 \leqslant i \leqslant 6$, $(V(H_3), \{f''_1, \dots, f''_i\})$ has no induced subgraph isomorphic to $2K_2$. Since $f''_1 = \{a, b\}$, by STEP 1,

$$f''_1 = \{x_7, x_6\} < \{x_7, x_8\} < \{x_7, x_4\} < \{x_6, x_5\} < \{x_6, x_8\} < \{[x_8, x_8]\} < \\ \{x_8, x_5\} < \{x_4, x_5\} < \{x_4, x_8\} < f''_2 < f''_3 < f''_4 < f''_5 < f''_6.$$

Since $f''_4, f''_5 \in \mathcal{N}(G)_a$, by STEP 2(i), we get

$$f''_1 = \{x_7, x_6\} < \{x_7, x_8\} < \{x_7, x_4\} < \{x_6, x_5\} < \{x_6, x_8\} < \{[x_8, x_8]\} < \\ \{x_8, x_5\} < \{x_4, x_5\} < \{x_4, x_8\} < f''_2 < f''_3 < f''_4 < \{x_4, x_5\} < \{x_4, x_8\} < f''_5 \\ < \{[x_8, x_8]\} < \{x_8, x_6\} < f''_6.$$

Since $f''_2, f''_6 \in \mathcal{N}(G)_b$, by STEP 2(ii), we get

$$f''_1 = \{x_7, x_6\} < \{x_7, x_8\} < \{x_7, x_4\} < \{x_6, x_5\} < \{x_6, x_8\} < \{[x_8, x_8]\} < \{x_8, x_5\} \\ < \{x_4, x_5\} < \{x_4, x_8\} < f''_2 = \{x_6, x_5\} < \{x_5, x_4\} < \{x_5, x_8\} < f''_3 = \{x_5, x_4\} \\ < f''_4 = \{x_4, x_7\} < \{x_4, x_5\} < \{x_4, x_8\} < f''_5 = \{x_7, x_8\} < \{[x_8, x_8]\} < \{x_8, x_6\} \\ < f''_6 = \{x_6, x_8\} < \{[x_8, x_8]\} < \{x_8, x_4\}.$$

Since the edges $\{x_6, x_5\}$, $\{x_5, x_4\}$, $\{x_5, x_8\}$, $\{x_4, x_7\}$, $\{x_4, x_5\}$, $\{x_4, x_8\}$, $\{x_7, x_8\}$, $\{x_6, x_8\}$, $\{[x_8, x_8]\}$ are repeated in the above ordering, by Step 3, we get

$$\begin{aligned}&\{x_7, x_6\} < \{x_7, x_8\} < \{x_7, x_4\} < \{x_6, x_5\} < \{x_6, x_8\} < \{[x_8, x_8]\} < \{x_8, x_5\}\\ &< \{x_4, x_5\} < \{x_4, x_8\}.\end{aligned}$$

So, H_3' is the graph with edge set $E(H_3') = E(H_3) \cup \{\{x_8, x_4\}, \{x_8, x_5\}, \{x_8, z_{x_8}\}\}$ and edges in H_3' appear in the ordered sequence above.

The operations used in Step 1 and Step 2 above will also be used subsequently. So we fix notation to refer to them. Instead of separately describing them on each occasion we shall simply refer to the operation number.

Op 1 The operation used in Step 2(i), i.e. if for $1 \leqslant \mu \leqslant \alpha$, $f_{k_1} = \{a, a_\mu\} \in \mathcal{N}(G)_a$ for some $1 \leqslant k_1 \leqslant t_m$, then

$$\cdots < f_{k_1} < \{[a_\mu, b_1]\} < \cdots < \{[a_\mu, b_{\beta'}]\} < f_{k_1+1} < \cdots .$$

Op 2 The operation used in Step 2(ii), i.e. if for $1 \leqslant \mu \leqslant \beta$, $f_{k_2} = \{b, b_\mu\} \in \mathcal{N}(G)_b$ for some $1 \leqslant k_2 \leqslant t_m$, then

$$\cdots < f_{k_2} < \{[b_\mu, a_1]\} < \cdots < \{[b_\mu, a_{\alpha'}]\} < f_{k_2+1} < \cdots .$$

Op 3 The operation used in Step 1, i.e. if $f_k = \{a, b\}$ for some $1 \leqslant k \leqslant t_m$, then

$$\begin{aligned}&\cdots < f_k < \{a, a_1\} < \cdots < \{a, a_{\alpha'}\} < \{b, b_1\} < \cdots < \{b, b_{\beta'}\} < \{[a_1, b_1]\} < \cdots\\ &< \{[a_1, b_{\beta'}]\} < \{[a_2, b_1]\} < \cdots < \{[a_2, b_{\beta'}]\} < \cdots < \{[a_{\alpha'}, b_1]\} < \cdots\\ &< \{[a_{\alpha'}, b_{\beta'}]\} < f_{k+1} < \cdots .\end{aligned}$$

We refer to the added edges in these operations as *new edges*.

We make some observations which follows directly from the preceding discussion.

Observation 1. *We use the same notation as in Discussion* 1.

(1) *Let $\mathcal{P}$ be the graph associated to $(\widetilde{IJ : ab})$ for any $ab \in I$. Let $h = \{[c, d]\}$ be a new edge as in* Op 1–Op 3. *First note that $h \in \mathcal{N}(G)_a \cup \mathcal{N}(G)_b$ or $c \in N_G(a)$, $d \in N_H(b)$ or $c \in N_H(a)$, $d \in N_G(b)$. It follows from Lemma* 3.1 *that $h \in E(\mathcal{P})$. Therefore H_m' is a subgraph of $\mathcal{P}$ for all $1 \leqslant m \leqslant \widetilde{n}$. Hence $\bigcup_{1\leqslant m\leqslant \widetilde{n}} E(H_m') \subseteq E(\mathcal{P})$. It is also not hard to verify that $E(\mathcal{P}) \subseteq \bigcup_{1\leqslant m\leqslant \widetilde{n}} E(H_m')$. Therefore $E(\mathcal{P}) = \bigcup_{1\leqslant m\leqslant \widetilde{n}} E(H_m')$.*

(2) *Let $g_1 < \cdots < g_{t_{m'}}$ be the ordered sequence whose elements are edges of H_m' as in* (3.2). *Suppose g_i is a new edge as in* Op 1 *(* Op 2 *or* Op 3*) where $1 \leqslant i \leqslant t_{m'}$. Then there exists $g_{i'} \in \mathcal{N}(G)_a$ ($g_{i'} \in \mathcal{N}(G)_b$ or $g_{i'} = \{a, b\}$) such that $g_{i'} < g_i$ i.e. $g_1 < \cdots < g_{i'} < \cdots < g_i < \cdots < g_{t_{m_1}}$.*

Now we fix some notation for some of the technical lemmas that are needed for the proof of the main result.

Notation 3.4. We use the same notation as in Discussion 1. Let $g_1 < \cdots < g_{t_{m'}}$ be the ordered sequence whose elements are edges of H_m' as in (3.2). For $1 \leqslant i \leqslant t_{m'}$, let $\mathcal{K}_i$ denote the graph with edge set $\{g_1, \ldots, g_i\}$ and whose edges appearing in the following ordered sequence $g_1 < \cdots < g_i$.

In the next two lemmas, we further reveal the structure of H_m'.

Lemma 3.5. *We use the same notation as in Discussion* 1. *If $\mathcal{K}_i$ has no induced subgraph isomorphic to $2K_2$ for all $1 \leqslant i \leqslant t_{m'}$, then $(V(H_m'), \{\mathfrak{g}_1, \ldots, \mathfrak{g}_j\})$ has no induced subgraph isomorphic to $2K_2$ for all $1 \leqslant j \leqslant t_{m_1}$.*

Proof. Suppose $(V(H'_m), \{\mathfrak{g}_1, \dots, \mathfrak{g}_q\})$ has an induced subgraph isomorphic to $2K_2$, say $\{\mathfrak{g}_p, \mathfrak{g}_q\}$, for some $1 \leqslant p < q \leqslant t_{m_1}$. Set $\mathfrak{g}_q = g_s$ for some $1 \leqslant s \leqslant t_{m'}$. It can also noted that $\mathfrak{g}_p = g_r < g_s$ for some $1 \leqslant r < s$. Since $\{g_r, g_s\}$ cannot form an induced subgraph $2K_2$ in $\mathcal{K}_i$ for all $1 \leqslant i \leqslant t_{m'}$, g_r and g_s have a vertex in common or there exist an edge $f_l \in E(H_m)$ such that $f_l < g_s$ connecting g_r and g_s. Note that $f_l \in \{\mathfrak{g}_1, \dots, \mathfrak{g}_q\}$. In both cases we get a contradiction to the assumption. Therefore $(V(H'_m), \{\mathfrak{g}_1, \dots, \mathfrak{g}_j\})$ has no induced subgraph isomorphic to $2K_2$ for all $1 \leqslant j \leqslant t_{m_1}$. $\square$

Lemma 3.6. *Assume notation as in Discussion* 1. *If $\mathcal{K}_j$ has an induced subgraph isomorphic to $2K_2$, say $\{g_i, g_j\}$, for some $1 \leqslant i < j \leqslant t_{m'}$, then $g_i, g_j \notin E(H_m)$.*

Proof. Let $f_1 < \cdots < f_{t_m}$ be the ordering of edges of H_m as in (3.1). Suppose $g_i, g_j \in E(H_m)$. Set $f_p = g_i$ and $f_q = g_j$ for some $1 \leqslant p < q \leqslant t_m$. Note that $(V(H_m), \{f_1, \dots, f_r\})$ has no induced subgraph isomorphic to $2K_2$ for all $1 \leqslant r \leqslant t_m$. Since $g_i, g_j \in E(H_m)$, by Lemma 3.2, they cannot form an induced $2K_2$-subgraph of H_m. Therefore, either g_i and g_j have a vertex in common or there exist an edge $f_l \in E(H_m)$ such that $f_l < g_j$ connecting g_i and g_j. If g_i and g_j have a vertex in common in H_m, then this contradicts the assumption that $\{g_i, g_j\}$ forms an induced $2K_2$-subgraph in $\mathcal{K}_j$. If f_l is an edge connecting g_i and g_j, then $f_l \in E(\mathcal{K}_j)$. This is a contradiction to $g_i, g_j \in E(H_m)$. Therefore $g_i, g_j \notin E(H_m)$. $\square$

Now we prove that the co-chordal cover number of $\mathcal{P}$ is bounded above by that of G.

Lemma 3.7. *Let I and J be as in Set-up* 1. *Let $\mathcal{P}$ be the graph associated to $(\widetilde{IJ : ab})$ for any $ab \in I$. Then*

$$\text{co-chord}(\mathcal{P}) \leqslant \text{co-chord}(G).$$

Proof. Let $\text{co-chord}(G) = \widetilde{n}$. Then there exist co-chordal subgraphs $H_1, \dots, H_{\widetilde{n}}$ of G such that $E(G) = \bigcup\limits_{i=1}^{\widetilde{n}} E(H_i)$. If $E(G) = E(\mathcal{P})$, then we are done. Suppose $E(G) \neq E(\mathcal{P})$. We use the same notation as in Discussion 1. Since H_m is co-chordal, by Lemma 3.2, there is an ordering of the edges of H_m, $f_1 < \cdots < f_{t_m}$, such that for $1 \leqslant r \leqslant t_m$, $(V(H_m), \{f_1, \dots, f_r\})$ has no induced subgraph isomorphic to $2K_2$. By Observation 1(1), we have $E(\mathcal{P}) = \bigcup\limits_{m=1}^{\widetilde{n}} E(H'_m)$. Let $g_1 < \cdots < g_{t_{m'}}$ be the ordered sequence of edges of H'_m as in (3.2). Now we claim that $\mathcal{K}_r$ has no induced subgraph isomorphic to $2K_2$ for all $1 \leqslant r \leqslant t_{m'}$. Suppose not i.e. there exists a least j such that $\mathcal{K}_j$ has an induced $2K_2$-subgraph, say $\{g_i, g_j\}$ for some $i < j$. First note that both g_i and g_j cannot be new edges as in Op 1–Op 3. By Lemma 3.6, we have $g_i, g_j \notin E(H_m)$. Therefore, we have the following cases:

(1) $g_i \in E(H_m)$, g_j is a new edge as in Op 1 or g_i is a new edge as in Op 1, $g_j \in E(H_m)$;
(2) $g_i \in E(H_m)$, g_j is a new edge as in Op 2 or g_i is a new edge as in Op 2, $g_j \in E(H_m)$;
(3) $g_i \in E(H_m)$, g_j is a new edge as in Op 3 or g_i is a new edge as in Op 3, $g_j \in E(H_m)$;
(4) g_i is a new edge as in Op 1, g_j is a new edge as in Op 2 or g_i is a new edge as in Op 2, g_j is a new edge as in Op 1;
(5) g_i is a new edge as in Op 1, g_j is a new edge as in Op 3 or g_i is a new edge as in Op 3, g_j is a new edge as in Op 1;

(6) g_i is a new edge as in Op 2, g_j is a new edge as in Op 3 or g_i is a new edge as in Op 3, g_j is a new edge as in Op 2.

We consider each case separately.

Case 1: Suppose $g_i \in E(H_m)$ and g_j is a new edge as in Op 1. Let $g_i = \{u, v\} \in E(H_m)$ and $g_j = \{[a_\mu, b_p]\}$ for some $1 \leqslant \mu \leqslant \alpha$, $1 \leqslant p \leqslant \beta'$. By Op 1, we have $g_{j'} = \{a, a_\mu\} < g_j$. Since $g_i, g_{j'} \in E(H_m)$, they cannot form an induced $2K_2$-subgraph of H_m. Therefore, either $g_{j'}$ and g_i have a vertex in common or there exist an edge $g_l \in E(H_m)$ such that $g_l < g_{j'}$ connecting g_i and $g_{j'}$. If g_i and $g_{j'}$ have a vertex in common, then this contradicts the assumption that $\{g_i, g_j\}$ forms an induced $2K_2$-subgraph. Suppose g_l is an edge connecting g_i and $g_{j'}$. Let $g_l = \{u, a\}$ and $u \neq b$. Then $g_l \in \mathcal{N}(G)_a$. By Op 1, $g_l < \{[u, b_p]\}$. We have $g_l < \{[u, b_p]\} < g_{j'} < g_j$. This is a contradiction to $\{g_i, g_j\}$ is an induced $2K_2$-subgraph. If $g_l = \{a, b\}$, then by Op 3, $g_l < \{b, b_p\}$. This also contradicts the assumption that $\{g_i, g_j\}$ is an induced $2K_2$-subgraph. Similarly, if $g_l = \{u, a_\mu\}$ or $g_l = \{v, a\}$ or $g_l = \{v, a_\mu\}$, then one arrives at a contradiction. Therefore $\{g_i, g_j\}$ cannot form an induced $2K_2$-subgraph of H'_m.

If g_i is a new edge as in Op 1 and $g_j \in E(H_m)$, then we get a contradiction in a similar manner.

Case 2: Suppose either $g_i \in E(H_m)$ and g_j is a new edge as in Op 2 or $g_j \in E(H_m)$ and g_i is a new edge as in Op 2. Proceeding as in Case 1, one can show that g_i and g_j cannot form an induced $2K_2$-subgraph.

Case 3: Suppose $g_i \in E(H_m)$ and g_j is a new edge as in Op 3. Let $g_i = \{u, v\} \in E(H_m)$. Then $g_j = \{a, a_\mu\}$ for some $1 \leqslant \mu \leqslant \alpha'$ or $g_j = \{b, b_\mu\}$ for some $1 \leqslant \mu \leqslant \beta'$ or $g_j = \{[a_p, b_q]\}$ for some $1 \leqslant p \leqslant \alpha'$, $1 \leqslant q \leqslant \beta'$. If $g_j = \{a, a_\mu\}$ for some $1 \leqslant \mu \leqslant \alpha'$, then by Op 3, we have $g_{j'} = \{a, b\} < g_j$. Since $g_i, g_{j'} \in E(H_m)$, they cannot form an induced $2K_2$-subgraph of H_m. Therefore, either $g_{j'}$ and g_i have a vertex in common or there exist an edge $g_l \in E(H_m)$ such that $g_l < g_{j'}$ connecting g_i and $g_{j'}$. If g_i and $g_{j'}$ have a vertex in common, then this contradicts the assumption that $\{g_i, g_j\}$ forms an induced $2K_2$-subgraph. Suppose g_l is an edge connecting g_i and $g_{j'}$. If $g_l = \{b, u\} \in \mathcal{N}(G)_b$, then by Op 2, $g_l < \{[u, a_\mu]\}$. This also contradicts the assumption that $\{g_i, g_j\}$ is an induced $2K_2$-subgraph. Similarly, if $g_l = \{v, b\}$ or $g_l = \{u, a\}$ or $g_l = \{u, a\}$, then one arrives at a contradiction. If $g_j = \{b, b_\mu\}$ for some $1 \leqslant \mu \leqslant \beta'$, then we get a contradiction in a similar manner.

Suppose $g_j = \{[a_p, b_q]\}$ for some $1 \leqslant p \leqslant \alpha'$, $1 \leqslant q \leqslant \beta'$. By Op 3, we have $g_{j'} = \{a, b\} < g_j$. Since $g_i, g_{j'} \in E(H_m)$, they cannot form an induced $2K_2$-subgraph of H_m. Therefore, either $g_{j'}$ and g_i have a vertex in common or there exist an edge $g_l \in E(H_m)$ such that $g_l < g_{j'}$ connecting g_i and $g_{j'}$. Suppose g_i and $g_{j'}$ have a vertex in common. If $u = a$, then $g_i \in \mathcal{N}(G)_a$. By Op 1, $g_i < \{[v, b_q]\}$. Therefore, we have $g_i < \{[v, b_q]\} < g_{j'} < g_j$. This is a contradiction to $\{g_i, g_j\}$ forms an induced $2K_2$-subgraph. Similarly, if $u = b$ or $v = a$ or $v = b$, then one arrives at a contradiction. Suppose g_l is an edge connecting g_i and $g_{j'}$. Note that $g_l < g_{j'}$. If $g_l = \{u, a\}$, then $g_l \in \mathcal{N}(G)_a$. By Op 1, $g_l < \{[u, b_q]\}$. This also contradicts the assumption that $\{g_i, g_j\}$ is an induced $2K_2$-subgraph. Similarly, if $g_l = \{v, b\}$ or $g_l = \{v, a\}$ or $g_l = \{u, b\}$, then one arrives at a contradiction.

If g_i is a new edge as in Op 3 and $g_j \in E(H_m)$, then we get a contradiction in a similar manner.

Case 4: Suppose g_i is a new edge as in Op 1 and g_j is a new edge as in Op 2. Let $g_i = \{[a_p, b_q]\}$ and $g_j = \{[a_{p'}, b_{q'}]\}$ for some $1 \leqslant p \leqslant \alpha$, $1 \leqslant q \leqslant \beta'$,

$1 \leqslant p' \leqslant \alpha'$, $1 \leqslant q' \leqslant \beta$. Then by Op 1 and Op 2,

$$g_{i'} = \{a, a_p\} < g_i < g_{j'} = \{b, b_{q'}\} < g_j.$$

Since $g_{i'}, g_{j'} \in E(H_m)$, they cannot form an induced $2K_2$-subgraph of H_m. Therefore, either $g_{i'}$ and $g_{j'}$ have a vertex in common or there exist an edge $g_l \in E(H_m)$ such that $g_l < g_{j'}$ connecting $g_{i'}$ and $g_{j'}$. If $g_{i'}$ and $g_{j'}$ have a vertex in common, then this contradicts the assumption that $\{g_i, g_j\}$ forms an induced $2K_2$-subgraph. Suppose g_l is an edge connecting $g_{i'}$ and $g_{j'}$. If $g_l = \{a_p, b_{q'}\}$, then this contradicts the assumption that $\{g_i, g_j\}$ forms an induced $2K_2$-subgraph. If $g_l = \{a_p, b\}$, then $g_l \in \mathcal{N}(G)_b$. By Op 2, $g_l < \{[a_p, a_{p'}]\}$. This also contradicts the assumption that $\{g_i, g_j\}$ is an induced $2K_2$-subgraph. Similarly, if $g_l = \{a, b_{q'}\}$, then one arrives at a contradiction. If $g_l = \{a, b\}$, then by Op 3, $g_l < \{[a_{p'}, b_{q'}]\}$. This also contradicts the assumption that $\{g_i, g_j\}$ is an induced $2K_2$-subgraph.

If $g_i = \{[a_{p'}, b_{q'}]\}$ is a new edge as in Op 2 and $g_j = \{[a_p, b_q]\}$ is a new edge as in Op 1, then we get a contradiction in a similar manner.

Case 5: Suppose $g_i = \{[a_{p'}, b_{q'}]\}$ is a new edge as in Op 1 for some $1 \leqslant p' \leqslant \alpha$, $1 \leqslant q' \leqslant \beta'$ and g_j is a new edge as in Op 3. Note that $g_j = \{a, a_\mu\}$ for some $1 \leqslant \mu \leqslant \alpha'$ or $g_j = \{b, b_\mu\}$ for some $1 \leqslant \mu \leqslant \beta'$ or $g_j = \{[a_p, b_q]\}$ for some $1 \leqslant p \leqslant \alpha'$, $1 \leqslant q \leqslant \beta'$. Suppose $g_j = \{a, a_\mu\}$ for some $1 \leqslant \mu \leqslant \alpha'$. By Op 1, we have

$$\{a, a_{p'}\} < g_i < g_j = \{a, a_\mu\}.$$

This is a contradiction to the assumption that $\{g_i, g_j\}$ forms an induced $2K_2$-subgraph. Suppose $g_j = \{b, b_\mu\}$ for some $1 \leqslant \mu \leqslant \beta'$. Since g_i is a new edge as in Op 1, we have

$$\{a, a_{p'}\} < \{[a_{p'}, b_1]\} < \cdots < \{[a_{p'}, b_\mu]\} < \cdots < \{[a_{p'}, b_{\beta'}]\}.$$

Therefore $\{[a_{p'}, b_\mu]\} < g_j$. This is a contradiction to the assumption that $\{g_i, g_j\}$ forms an induced $2K_2$-subgraph. Suppose $g_j = \{[a_p, b_q]\}$ for some $1 \leqslant p \leqslant \alpha'$, $1 \leqslant q \leqslant \beta'$. It can also seen that $\{[a_{p'}, b_q]\} < g_j$. This is a contradiction to the assumption that $\{g_i, g_j\}$ forms an induced $2K_2$-subgraph.

If g_i is a new edge as in Op 3 and g_j is a new edge as in Op 1, then we get a contradiction in a similar manner.

Case 6: Suppose either g_i is a new edge as in Op 2 and g_j is a new edge as in Op 3 or g_j is a new edge as in Op 3 and g_i is a new edge as in Op 2. Proceeding as in the Case 5, one can show that g_i and g_j cannot form an induced $2K_2$-subgraph.

In all cases we get a contradiction to the assumption that $\mathcal{K}_j$ has an induced $2K_2$-subgraph for some $1 \leqslant j \leqslant t_{m'}$. Therefore $\mathcal{K}_j$ has no induced $2K_2$-subgraph for all $1 \leqslant j \leqslant t_{m'}$. By Lemma 3.5, $(V(H'_m), \{\mathfrak{g}_1, \ldots, \mathfrak{g}_{r'}\})$ has no induced $2K_2$-subgraph for all $1 \leqslant r' \leqslant t_{m'}$. By Lemma 3.2, H'_m is a co-chordal graph. Therefore, H'_m is a co-chordal graph for all $1 \leqslant m \leqslant \widetilde{n}$. Hence co-chord$(\mathcal{P}) \leqslant \widetilde{n}$. □

As a consequence of Lemma 3.7 one has:

Corollary 3.8. *Let I and J be edge ideals with $I \subseteq J$. If J has a linear minimal free resolution and for any $ab \in I$, then $(IJ : ab)$ also has a linear minimal free resolution*

Proof. Let G and $\mathcal{P}$ be the graphs associated to J and $(\widetilde{IJ : ab})$ respectively. By [12, Theorem 1], G is a co-chordal graph and by Lemma 3.7, $\mathcal{P}$ is also co-chordal. Again by [12, Theorem 1], $\mathcal{P}$ has a linear minimal free resolution. Therefore, $(IJ : ab)$ has a linear minimal free resolution. □

4. Upper and lower bound for the regularity of products of two edge ideals

In this section, we obtain a general upper and lower bounds for the regularity of products of two edge ideals.

We start by recalling the notion of *upper-Koszul simplicial complexes* associated to monomial ideals. Let $I \subseteq R = \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a $\mathbb{N}^n$-graded degree. The *upper-Koszul simplicial complex* associated to I at degree α, denoted by $K^\alpha(I)$, is the simplicial complex over $V = \{x_1, \dots, x_n\}$ whose faces are:

$$\Big\{W \subseteq V \mid \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\prod\limits_{u \in W} u} \in I\Big\}.$$

Given a monomial ideal I, its $\mathbb{N}^n$-graded Betti numbers are given by the following formula of Hochster [19, Theorem 1.34]:

$$\beta_{i,\alpha}(I) = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(K^\alpha(I); \mathbb{K}) \text{ for all } i \geqslant 0 \text{ and } \alpha \in \mathbb{N}^n.$$

We now prove the general lower bound for the regularity of product of edge ideals. One can see that [4, Lemma 4.2] works more generally and we generalize their argument to prove it below:

Theorem 4.1. *Let $J_1 = I(G_i), \dots, J_d = I(G_d)$ be the edge ideals of $G_1, \dots, G_d$ with $J_1 \subseteq \cdots \subseteq J_d$. Then*

$$2d + \nu_{G_1 \cdots G_d} - 1 \leqslant \operatorname{reg}(J_1 \cdots J_d).$$

Proof. Let $f_1, f_2, \dots, f_{\nu_{G_1 \cdots G_d}}$ be the induced matching of G_i for all $1 \leqslant i \leqslant d$. Let Q be an induced subgraph of G_i with $E(Q) = \{f_1, \dots, f_{\nu_{G_1 \cdots G_d}}\}$ for all $1 \leqslant i \leqslant d$. First, we claim that if for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $\operatorname{supp}(\alpha) \subseteq V(Q)$, where $\operatorname{supp}(\alpha) = \{x_i \mid \alpha_i \neq 0\}$, then $K^\alpha(I(Q)^d) = K^\alpha(J_1 \cdots J_d)$. Clearly, $K^\alpha(I(Q)^d) \subseteq K^\alpha(J_1 \cdots J_d)$. Suppose $W \in K^\alpha(J_1 \cdots J_d)$. Since $\operatorname{supp}(\alpha) \subseteq V(Q)$, we have $W \subseteq V(Q)$. Then $m = \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\prod\limits_{u \in W} u} \in J_1 \cdots J_d$, which implies that $g_1 \cdots g_d \mid m$ where $g_i \in J_i$ for all $1 \leqslant i \leqslant d$. Clearly $\operatorname{supp}(g_i) \subseteq \operatorname{supp}(m)$ for all $1 \leqslant i \leqslant d$. Therefore $g_i \in I(Q)$ for all $1 \leqslant i \leqslant d$. Then $m = \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\prod\limits_{u \in W} u} \in I(Q)^d$, which implies that $W \in K^\alpha(I(Q)^d)$, proving the claim. It follows from [19, Theorem 1.34] that

$$\begin{aligned}\beta_{i,\alpha}(I(Q)^d) &= \dim_{\mathbb{K}} \widetilde{H}_{i-1}(K^\alpha(I(Q)^d); \mathbb{K}) \\ &= \dim_{\mathbb{K}} \widetilde{H}_{i-1}(K^\alpha(J_1 \cdots J_d); \mathbb{K}) = \beta_{i,\alpha}(J_1 \cdots J_d).\end{aligned}$$

Therefore,

$$\begin{aligned}\beta_{i,j}(I(Q)^d) &= \sum_{\alpha \in \mathbb{N}^n,\ \operatorname{supp}(\alpha) \subseteq V(Q),\ |\alpha| = j} \beta_{i,\alpha}(I(Q)^d) \\ &= \sum_{\alpha \in \mathbb{N}^n,\ \operatorname{supp}(\alpha) \subseteq V(Q),\ |\alpha| = j} \beta_{i,\alpha}(J_1 \cdots J_d) \\ &\leqslant \sum_{\alpha \in \mathbb{N}^n,\ |\alpha| = j} \beta_{i,\alpha}(J_1 \cdots J_d) = \beta_{i,j}(J_1 \cdots J_d).\end{aligned}$$

Hence $\operatorname{reg}(I(Q)^d) \leqslant \operatorname{reg}(J_1 \cdots J_d)$. By [4, Lemma 4.4], $\operatorname{reg}(I(Q)^d) = 2d + \nu_{G_1 \cdots G_d} - 1$. Hence $2d + \nu_{G_1 \cdots G_d} - 1 \leqslant \operatorname{reg}(J_1 \cdots J_d)$. □

We now prove an upper bound for the regularity of IJ.

Theorem 4.2. *Let I and J be as in Set-up* 1. *Then*

$$\operatorname{reg}(IJ) \leqslant \max\{\text{co-chord}(G)+3,\ \operatorname{reg}(I)\}. \tag{4.1}$$

In particular,

$$\operatorname{reg}(IJ) \leqslant \max\{\text{co-chord}(G)+3,\ \text{co-chord}(H)+1\}.$$

Proof. Set $I=(f_1,\ldots,f_t)$. It follows from the short exact sequences:

$$\begin{aligned} &0 \longrightarrow \frac{R}{(IJ:f_1)}(-2) \xrightarrow{\cdot f_1} \frac{R}{IJ} \longrightarrow \frac{R}{(IJ,f_1)} \longrightarrow 0;\\ &\qquad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &0 \longrightarrow \frac{R}{((IJ,f_1,\ldots,f_{t-1}):f_t)}(-2) \xrightarrow{\cdot f_t} \frac{R}{(IJ,f_1,\ldots,f_{t-1})} \longrightarrow \frac{R}{(IJ,I)} \longrightarrow 0, \end{aligned} \tag{4.2}$$

that

$$\operatorname{reg}\left(\frac{R}{IJ}\right) \leqslant \max\left\{\operatorname{reg}\left(\tfrac{R}{(IJ:f_1)}\right)+2,\ldots,\operatorname{reg}\left(\tfrac{R}{(IJ,f_1,\ldots,f_{t-1}):f_t)}\right)+2,\ \operatorname{reg}\left(\tfrac{R}{I}\right)\right\}.$$

Note that $((IJ,f_1,\ldots,f_{i-1}):f_i)=(IJ:f_i)+(\text{variables})$ for any $1\leqslant i\leqslant t$. By [17, Theorem 1.2] and Corollary 2.3, we have

$$\operatorname{reg}((IJ,f_1,\ldots,f_{i-1}):f_i) \leqslant \operatorname{reg}((IJ:f_i)) = \operatorname{reg}((\widetilde{IJ:f_i})).$$

Let $\mathcal{P}_i$ be the graph associated to $(\widetilde{IJ:f_i})$. Therefore, by [24, Theorem 1] and Lemma 3.7, we get $\operatorname{reg}(\widetilde{IJ:f_i}) \leqslant \text{co-chord}(\mathcal{P}_i)+1 \leqslant \text{co-chord}(G)+1$. Hence $\operatorname{reg}(IJ)\leqslant \max\{\text{co-chord}(G)+3,\ \operatorname{reg}(I)\}$. Now the second assertion follows from [24, Theorem 1]. ☐

Remark 4.3. Let G be a graph and H be a subgraph of G. We would like to note here that the invariant co-chord(G) and co-chord(H) are not comparable in general. For example, if G is the graph with $E(G)=\{\{x_1,x_2\},\ \{x_2,x_3\},\ \{x_3,x_4\},\ \{x_4,x_5\},\ \{x_5,x_1\},\ \{x_1,x_3\}\}$ and H is a subgraph of G with $E(H)=\{\{x_1,x_2\},\ \{x_2,x_3\},\ \{x_3,x_4\},\ \{x_4,x_5\},\ \{x_5,x_1\}\}$, then co-chord$(G)=1$ and co-chord$(H)=2$. If G is a graph with $E(G)=\{\{x_1,x_2\},\ \{x_2,x_3\},\ \{x_3,x_4\},\{x_4,x_5\}\}$ and H is a graph with $E(H)=\{\{x_1,x_2\},\{x_2,x_3\}\}$, then co-chord$(G)=2$ and co-chord$(H)=1$.

As an immediate consequence, we have the following statements.

Corollary 4.4. *Let I and J be as in Set-up* 1. *Then* $\operatorname{reg}(IJ)\leqslant \mathrm{m}(G)+3$.

Proof. Since H is a subgraph of G, $\mathrm{m}(H)\leqslant \mathrm{m}(G)$. Hence the assertion follows from Theorem 4.2. ☐

The following example shows that the inequalities given in Theorem 4.1 and Corollary 4.4 are sharp.

Example 4.5. Let H and G be graphs with $I(H)=(x_2x_3,x_4x_5)$ and $I(G)=(x_1x_2,x_1x_3,x_1x_4,x_1x_5,x_2x_3,x_4x_5)$. It is not hard to verify that $\mathrm{m}(G)=2$ and $\nu_{HG}=2$. Therefore, by Theorem 4.1 and Corollary 4.4, we have $\operatorname{reg}(I(H)I(G))=5$.

Corollary 4.6. *Let I and J be as in Set-up* 1. *If H is an induced subgraph of G, then*

$$\nu(H)+3\leqslant \operatorname{reg}(IJ)\leqslant \text{co-chord}(G)+3.$$

Proof. If H is an induced subgraph of G, then co-chord$(H)\leqslant$ co-chord(G) and $\nu_{HG}=\nu(H)$. Therefore, by Theorems 4.1 and 4.2, we have $\nu(H)+3\leqslant \operatorname{reg}(IJ)\leqslant$ co-chord$(G)+3$. ☐

It follows from Corollary 4.4 that if G_1 is a subgraph of G_2, then

$$\operatorname{reg}(J_1J_2) \leqslant 3 + \operatorname{m}(G_2),$$

where $J_i = I(G_i)$ for all $i = 1, 2$. As a natural extension of this result, one may tend to think that the same expression may hold true for $\operatorname{reg}(J_1 \cdots J_d)$. This question is inspired by previous work [2, 15, 16] of the regularity of powers of edge ideals of graphs. More precisely, we would like to ask:

Question 4.7. *If G_{i-1} is a subgraph of G_i for all $i = 2, \ldots, d$, is it true that*

$$\operatorname{reg}(J_1 \cdots J_d) \leqslant 2d + \operatorname{m}(G_d) - 1,$$

where $J_i = I(G_i)$ for all $1 \leqslant i \leqslant d$? In particular, if G_{i-1} is an induced subgraph of G_i for all $i = 2, \ldots, d$, is it true that

$$\operatorname{reg}(J_1 \cdots J_d) \leqslant 2d + \operatorname{co-chord}(G_d) - 1?$$

The following example shows that the above inequality can be equality.

Example 4.8. Let $J_1 = (\{x_{i-1}x_i \mid 5 \leqslant i \leqslant 6\})$, $J_2 = J_3 = (\{x_{i-1}x_i \mid 3 \leqslant i \leqslant 8\})$ and $J_4 = J_5 = (\{x_{i-1}x_i \mid 2 \leqslant i \leqslant 10\})$ be the edge ideals. Set $J_i = I(G_i)$ for all $1 \leqslant i \leqslant 5$. A computation on Macaulay2 shows that $\operatorname{reg}(J_1 \cdots J_5) = 12$. Note that G_{i-1} is an induced subgraph of G_i for all $2 \leqslant i \leqslant 5$ and $\operatorname{co-chord}(G_5) = 3$. Then $\operatorname{reg}(J_1 \cdots J_5) = 12 \leqslant 2 \cdot 5 + \operatorname{co-chord}(G_5) - 1 = 12$.

Let G_1 and G_2 be graphs with disjoint vertex sets, i.e. $V(G_1) \cap V(G_2) = \varnothing$. The *join* of G_1 and G_2, denoted by $G_1 * G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ whose edge set is $E(G_1 * G_2) = E(G_1) \cup E(G_2) \cup \{\{x, y\} \mid x \in V(G_1) \text{ and } y \in V(G_2)\}$.

Corollary 4.9. *Let G_1, G_2 be graphs with disjoint edges and $G = G_1 * G_2$. If $H = G_1$ or $H = G_2$, then*

$$\nu(H) + 3 \leqslant \operatorname{reg}(I(H)I(G)) \leqslant \max\{\operatorname{co-chord}(G_1), \operatorname{co-chord}(G_2)\} + 3.$$

In particular, if $\operatorname{co-chord}(G_1) \leqslant \operatorname{co-chord}(G_2)$ *and $H = G_2$, then* $\operatorname{reg}(I(H)I(G)) = \nu(G_2) + 3$.

Proof. If H is equal to either G_1 or G_2, then H is an induced subgraph of G. Therefore, by Corollary 4.6, we have

$$\nu(H) + 3 \leqslant \operatorname{reg}(I(H)I(G)) \leqslant \max\{\operatorname{co-chord}(G) + 3, \operatorname{co-chord}(H) + 1\}.$$

By [21, Proposition 4.12], we know that

$$\operatorname{co-chord}(G) = \max\{\operatorname{co-chord}(G_1),\ \operatorname{co-chord}(G_2)\}.$$

Therefore $\operatorname{reg}(I(H)I(G)) \leqslant \max\{\operatorname{co-chord}(G_1), \operatorname{co-chord}(G_2)\} + 3$. □

5. Precise expressions for the regularity of product of edge ideals

In this section, we explicitly compute the regularity of product of edge ideals for certain classes of graphs. First, we compute the regularity of IJ when J has a linear resolution.

Theorem 5.1. *Let I and J be edge ideals with $I \subseteq J$. Suppose J has a linear resolution.*

1. *If $\operatorname{reg}(I) \leqslant 4$, then IJ has a linear resolution.*
2. *If $5 \leqslant \operatorname{reg}(I)$, then $\operatorname{reg}(IJ) = \operatorname{reg}(I)$.*

Proof. Suppose $\operatorname{reg}(I) \leqslant 4$. Since J has a linear resolution, by (4.1), $4 \leqslant \operatorname{reg}(IJ) \leqslant \max\{4,\ \operatorname{reg}(I)\}$. Hence $\operatorname{reg}(IJ) = 4$.

Suppose $\operatorname{reg}(I) \geqslant 5$. By (4.1), we have $\operatorname{reg}(IJ) \leqslant \max\{4,\ \operatorname{reg}(I)\} \leqslant \operatorname{reg}(I)$. Since $4 \leqslant \operatorname{reg}(R/I)$, there exist i, j such that $j - i \geqslant 4$ and $\beta_{i,j}(R/I) \neq 0$. From (4.2), either $\beta_{i,j}\left(\dfrac{R}{(IJ, f_1, \dots, f_{t-1})}\right) \neq 0$ or $\beta_{i-1,j}\left(\dfrac{R}{((IJ, f_1, \dots, f_{t-1}) : f_t)}(-2)\right) \neq 0$. Note that $((IJ, f_1, \dots, f_{t-1}) : f_t) = (IJ : f_t) + (\text{variables})$. Since J has a linear resolution, by Corollary 3.8 we infer that $(IJ : f_t)$ has a linear resolution. Hence $(IJ, f_1, \dots, f_{t-1}) : f_t)$ has a linear resolution, i.e. $\operatorname{reg}((IJ, f_1, \dots, f_{t-1}) : f_t)) = 2$.

If $\beta_{i-1,j-2}\left(\dfrac{R}{((IJ, f_1, \dots, f_{t-1}) : f_t)}\right) \neq 0$, then

$$\operatorname{reg}\left(\frac{R}{((IJ, f_1, \dots, f_{t-1}) : f_t)}\right) \geqslant j - 1 - i \geqslant 4 - 1 = 3.$$

This is a contradiction to $\operatorname{reg}\left(\dfrac{R}{((IJ, f_1, \dots, f_{t-1}) : f_t)}\right) \leqslant 1$. Therefore

$$\beta_{i,j}\left(\frac{R}{(IJ, f_1, \dots, f_{t-1})}\right) \neq 0.$$

Then again either

$$\beta_{i,j}\left(\frac{R}{(IJ, f_1, \dots, f_{t-2})}\right) \neq 0$$

or

$$\beta_{i-1,j}\left(\frac{R}{((IJ, f_1, \dots, f_{t-2}) : f_{t-1})}(-2)\right) \neq 0.$$

As in the previous case, we get $\beta_{i,j}\left(\dfrac{R}{(IJ, f_1, \dots, f_{t-2})}\right) \neq 0$. Then one proceeds in the same manner. At each stage, we get either $\beta_{i,j}\left(\dfrac{R}{(IJ, f_1, \dots, f_{l-1})}\right) \neq 0$ or $\beta_{i-1,j}\left(\dfrac{R}{((IJ, f_1, \dots, f_{l-1}) : f_l)}(-2)\right) \neq 0$ for all l. Therefore, $\beta_{i,j}\left(\dfrac{R}{IJ}\right) \neq 0$. Hence $\operatorname{reg}(R/I) \leqslant \operatorname{reg}(R/IJ)$. $\square$

An immediate consequence of Theorem 5.1 is the following:

Corollary 5.2. *Let I and J be as in Set-up 1. If J has a linear resolution and $\nu(H) \geqslant 4$, then $\operatorname{reg}(IJ) = \operatorname{reg}(I)$. In particular,*

$$\nu(H) + 1 \leqslant \operatorname{reg}(IJ) \leqslant \text{co-chord}(H) + 1.$$

Proof. By (1.1), we have that $5 \leqslant \operatorname{reg}(I)$. Therefore, by Theorem 5.1, $\operatorname{reg}(IJ) = \operatorname{reg}(I)$. The second assertion follows from (1.1). $\square$

A graph which is isomorphic to the graph with vertices a, b, c, d and edges $\{a, b\}$, $\{b, c\}$, $\{a, c\}$, $\{a, d\}$, $\{c, d\}$ is called a *diamond*. A graph which is isomorphic to the graph with vertices w_1, w_2, w_3, w_4, w_5 and edges $\{w_1, w_3\}$, $\{w_2, w_3\}$, $\{w_3, w_4\}$, $\{w_3, w_5\}$, $\{w_4, w_5\}$ is called a *cricket*. A graph without an induced diamond (respectively cricket) is called diamond (respectively cricket)-free.

Corollary 5.3. *Let I and J be as in Set-up 1. Suppose J has a linear resolution. Then IJ has a linear resolution if*

(1) co-chord$(H) \leqslant 3$*;*
(2) *H is (gap,cricket)-free;*
(3) *H is (gap, diamond)-free;*
(4) *H is (gap, C_4)-free or*

(5) *H is a graph such that H^c has no triangle;*

Proof. By (1.1), [1, Theorem 3.4], [10, Theorem 3.5], [11, Proposition 2.11] and [20, Theorem 2.10], we have that $\operatorname{reg}(I) \leqslant 4$. Therefore, by Theorem 5.1, IJ has a linear resolution. □

So far, we have been discussing about the regularity of products of two edge ideals. Now we study the regularity of products of more than two edge ideals.

Theorem 5.4. *Let $J_1, \dots, J_d$ be edge ideals and $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_d$, $d \in \{3,4\}$. Suppose J_d is the edge ideal of a complete graph.*

(1) *If $\operatorname{reg}(J_1 \cdots J_{d-1}) \leqslant 2d$, then $J_1 \cdots J_d$ has a linear resolution.*
(2) *If $\operatorname{reg}(J_1 \cdots J_{d-1}) \geqslant 2d+1$, then $\operatorname{reg}(J_1 \cdots J_d) = \operatorname{reg}(J_1 \cdots J_{d-1})$.*

Proof. Set $\mathcal{J} := J_1 \cdots J_d$ and $J_1 \cdots J_{d-1} = (\mathcal{F}_1, \dots, \mathcal{F}_t)$. Now we claim that, if $(\mathcal{F}_j : \mathcal{F}_i) = (u^s)$ for some $s \geqslant 3$ and $j \neq i$, then $u^2 \in (\mathcal{J} : \mathcal{F}_i)$. Clearly $d > 3$. Set $\mathcal{F}_j = g_1 g_2 g_3$ and $\mathcal{F}_i = f_1 f_2 f_3$, where $g_i, f_i \in J_i$ for all $1 \leqslant i \leqslant 3$. Since $s \geqslant 3$, we have $u \mid g_i$ and $u \nmid f_i$ for all $1 \leqslant i \leqslant 3$. Set $g_1 = ua$, $g_2 = ub$, $g_3 = uc$, $f_1 = x_1x_2$, $f_2 = x_3x_4$ and $f_3 = x_5x_6$ (x_i may be equal to x_j, for some $1 \leqslant i, j \leqslant 5$). Note that $abc \mid f_1f_2f_3$. If $ab \mid f_i$ and $c \mid f_j$, for some $1 \leqslant i, j \leqslant 3$, then $uaubf_jf_k \in \mathcal{J}$, where $k \neq i, j$. If $a \mid f_i$, $b \mid f_j$, $c \mid f_k$ for some $1 \leqslant i, j, k \leqslant 3$, then $uaubf_k(\frac{f_if_j}{ab}) \in \mathcal{J}$. Therefore $u^2 \in (\mathcal{J} : \mathcal{F}_i)$. Hence the claim.

Let $m \in \mathcal{G}(\mathcal{J} : \mathcal{F}_i)$. By degree consideration m cannot have degree 1. We now claim that $\deg(m) = 2$. Suppose $|\operatorname{supp}(m)| \geqslant 2$. Since J_d is an edge ideal of a complete graph, $\deg(m) = 2$. Suppose $|\operatorname{supp}(m)| = 1$. Assume that $\deg(m) \geqslant 3$. Set $m = u^s$ for some $s \geqslant 3$. Clearly $n_1 \cdots n_d \mid u^s\mathcal{F}_i$, where $n_l \in \mathcal{G}(J_l)$ for all $1 \leqslant l \leqslant d$. Then $n_1 \cdots n_{d-1} \mid u^s \mathcal{F}_i$. Also, $u^s \in (n_1 \cdots n_{d-1} : \mathcal{F}_i)$. By the above claim, $u^2 \in (\mathcal{J} : \mathcal{F}_i)$. This is a contradiction to $\deg(m) \geqslant 3$. Therefore $\deg(m) = 2$.

By the above arguments, one can see that the ideal $((\mathcal{J}, \mathcal{F}_1, \dots, \mathcal{F}_{i-1}) : \mathcal{F}_i)$ is generated by quadratic monomial ideals. Note that $J_d \subseteq (\mathcal{J} : \mathcal{F}_i)$. Let K_i be the graph associated to $((\mathcal{J}, \mathcal{F}_1, \dots, \mathcal{F}_{i-1}) : \mathcal{F}_i)$. Since J_d is the edge ideal of complete graph, K_i is the graph obtained from complete graph by attaching pendant to some vertices. Hence K_i is a co-chordal graph. By [12, Theorem 1], $\operatorname{reg}((\mathcal{J}, \mathcal{F}_1, \dots, \mathcal{F}_{i-1}) : \mathcal{F}_i)) = 2$ for all $1 \leqslant i \leqslant t$.

Considering similar exact sequences as in (4.2), we get that the inequality

$$\operatorname{reg}\left(\frac{R}{\mathcal{J}}\right) \leqslant \max \left\{ \begin{array}{l} \operatorname{reg}\left(\frac{R}{(\mathcal{J}:\mathcal{F}_1)}\right) + 2(d-1), \dots, \operatorname{reg}\left(\frac{R}{(\mathcal{J},\mathcal{F}_1,\dots,\mathcal{F}_{t-1}):\mathcal{F}_t)}\right) + 2(d-1), \\ \operatorname{reg}\left(\frac{R}{J_1 \cdots J_{d-1}}\right) \end{array} \right\}$$

holds. Therefore $\operatorname{reg}\left(\frac{R}{\mathcal{J}}\right) \leqslant \max\left\{2d,\ \operatorname{reg}\left(\frac{R}{J_1 \cdots J_{d-1}}\right)\right\}$. Proceeding as in the proof of Theorem 5.1 we get the desired conclusion. □

As an immediate consequence of Theorem 4.2, Theorem 5.4, we obtain an upper bound for the regularity of products of edge ideals in terms of co-chordal cover numbers.

Corollary 5.5. *Let $J_1 = I(G_1), \dots, J_d = I(G_d)$ be the edge ideal of $G_1, \dots, G_d$ with $J_1 \subseteq \cdots \subseteq J_d$.*

(1) *If G_3 is a complete graph, then*
$$\operatorname{reg}(J_1J_2J_3) \leqslant \max\{6,\ \text{co-chord}(G_2) + 3,\ \text{co-chord}(G_1) + 1\}.$$
(2) *If G_i is a complete graph for all $i = 3, 4$, then*
$$\operatorname{reg}(J_1J_2J_3J_4) \leqslant \max\{8,\ \text{co-chord}(G_2) + 3,\ \text{co-chord}(G_1) + 1\}.$$

As a consequence of Theorem 5.4, we give sufficient conditions for products of edge ideals to have linear resolutions.

Corollary 5.6. *Let $J_i = I(G_i)$ be the edge ideal of G_i for all $1 \leqslant i \leqslant d$ and $J_1 \subseteq \cdots \subseteq J_d$.*

(1) *If G_3 is a complete graph and* $\max\{\text{co-chord}(G_2)+3, \text{co-chord}(G_1)+1\} \leqslant 6$*, then $J_1J_2J_3$ has linear resolution.*
(2) *If G_i is a complete graph for all $i = 3, 4$ and* $\max\{\text{co-chord}(G_2) + 3,\ \text{co-chord}(G_1)+1\} \leqslant 8$*, then $J_1J_2J_3J_4$ has linear resolution.*
(3) *If G_4 is a complete graph and G_i is an induced subgraph of G_{i+1} for all $1 \leqslant i \leqslant 3$, then $J_1J_2J_3J_4$ has linear resolution.*
(4) *If G_i is a complete graph for all $i = 3, 4$ and J_1J_2 has a linear resolution, then $J_1J_2J_3J_4$ has a linear resolution.*

Proof. For (1) and (2), the assertions follow from Theorem 4.2 and Theorem 5.4. Consider (3). Since G_4 is a complete graph and G_i is an induced subgraph of G_{i+1} for all $1 \leqslant i \leqslant 3$, G_i is a complete graph for all $1 \leqslant i \leqslant 3$. Therefore, by Corollary 5.5(1), $J_1J_2J_3$ has a linear resolution. Hence, by Theorem 5.4, $J_1J_2J_3J_4$ has a linear resolution. Finally, consider (4). If J_1J_2 has a linear resolution, then by Theorem 5.4, $J_1J_2J_3$ has linear resolution. Therefore, by Theorem 5.4, $J_1J_2J_3J_4$ has a linear resolution. □

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