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# Petrie symmetric functions

Darij Grinberg

Abstract For any positive integer $k$ and nonnegative integer $m$, we consider the symmetric function $G(k,m)$ defined as the sum of all monomials of degree $m$ that involve only exponents smaller than $k$. We call $G(k,m)$ a *Petrie symmetric function* in honor of Flinders Petrie, as the coefficients in its expansion in the Schur basis are determinants of Petrie matrices (and thus belong to $\{0,1,-1\}$ by a classical result of Gordon and Wilkinson). More generally, we prove a Pieri-like rule for expanding a product of the form $G(k,m) \cdot s_\mu$ in the Schur basis whenever $\mu$ is a partition; all coefficients in this expansion belong to $\{0,1,-1\}$. We also show that $G(k,1), G(k,2), G(k,3), \ldots$ form an algebraically independent generating set for the symmetric functions when $1-k$ is invertible in the base ring, and we prove a conjecture of Liu and Polo about the expansion of $G(k,2k-1)$ in the Schur basis.

Considered as a ring, the *symmetric functions* (which is short for "formal power series in countably many indeterminates $x_1, x_2, x_3, \ldots$ that are of bounded degree and fixed under permutations of the indeterminates") are hardly a remarkable object: By a classical result essentially known to Gauss, they form a polynomial ring in countably many indeterminates. The true theory of symmetric functions is rather the study of specific *families* of symmetric functions, often defined by combinatorial formulas (e.g. as multivariate generating functions) but interacting deeply with many other fields of mathematics. Classical families are, for example, the *monomial symmetric functions* $m_\lambda$, the *complete homogeneous symmetric functions* $h_n$, the *power-sum symmetric functions* $p_n$, and the *Schur functions* $s_\lambda$. Some of these families – such as the monomial symmetric functions $m_\lambda$ and the Schur functions $s_\lambda$ – form bases of the ring of symmetric functions (as a module over the base ring).

In this paper, we introduce a new family $(G(k,m))_{k \geqslant 1;\ m \geqslant 0}$ of symmetric functions, which we call the *Petrie symmetric functions* in honor of Flinders Petrie. For any integers $k \geqslant 1$ and $m \geqslant 0$, we define $G(k,m)$ as the sum of all monomials of degree $m$ (in $x_1, x_2, x_3, \ldots$) that involve only exponents smaller than $k$. When $G(k,m)$ is expanded in the Schur basis (i.e. as a linear combination of Schur functions $s_\lambda$), all coefficients belong to $\{0,1,-1\}$ by a classical result of Gordon and Wilkinson, as they are determinants of so-called *Petrie matrices* (whence our name for $G(k,m)$). We give an explicit combinatorial description for the coefficients as well. More generally, we prove a Pieri-like rule for expanding a product of the form $G(k,m) \cdot s_\mu$ in the

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Schur basis whenever $\mu$ is a partition; all coefficients in this expansion again belong to $\{0,1,-1\}$ (although we have no explicit combinatorial rule for them). We show some further properties of $G(k,m)$ and prove that if $k$ is a fixed positive integer such that $1-k$ is invertible in the base ring, then $G(k,1), G(k,2), G(k,3), \ldots$ form an algebraically independent generating set for the symmetric functions. We prove a conjecture of Liu and Polo in [19, Remark 1.4.5] about the expansion of $G(k,2k-1)$ in the Schur basis.

This paper begins with Section 1, in which we introduce the notions and notations that the paper will rely on. (Further notations will occasionally be introduced as the need arises.) The rest of the paper consists of two essentially independent parts. The first part comprises Section 2, in which we define the Petrie symmetric functions $G(k,m)$ (and the related power series $G(k)$) and state several of their properties, and Section 3, in which we prove said properties. The second part is Section 4, which is devoted to proving the conjecture of Liu and Polo.(1) A final Section 5 adds comments, formulates two conjectures, and (in its last subsection) explores a more general family of symmetric functions that still shares some of the properties of the Petrie functions $G(k,m)$. (As a byproduct of the latter generalization, a formula for the antipode of $G(k,m)$ – Corollary 5.25 – emerges.)

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This paper was started at the Mathematisches Forschungsinstitut Oberwolfach, where I was staying as a Leibniz fellow in Summer 2019, and finished during a semester program at the Institut Mittag–Leffler in 2020. I thank both institutes for their hospitality. The SageMath computer algebra system [26] has been used in discovering some of the results.

*Remarks.* **1.** A short exposition of the main results of this paper (without proofs), along with an additional question motivated by it, can be found in [14].

**2.** While finishing this work, I have become aware of three independent discoveries of the Petrie symmetric functions $G(k,m)$:

- **(a)** In [8, §3.3], Stephen Doty and Grant Walker define a *modular complete symmetric function* $h'_d$, which is precisely our Petrie symmetric function $G(k,m)$ up to a renaming of variables (namely, their $m$ and $d$ correspond to our $k$ and $m$). Some of our results appear in their work: Our Theorem 2.20 is (a slight generalization of) [8, Corollary 3.9]; our Theorem 2.26 is (part of) [8, Proposition 3.15] restated in the language of Hopf algebras. The $h'_d$ are studied further in Walker's follow-up paper [27], some of whose results mirror ours again (in particular, the maps $\psi^p$ and $\psi_p$ from [27] are our $\mathbf{f}_p$ and $\mathbf{v}_p$).
- **(b)** The preprint [10] by Houshan Fu and Zhousheng Mei introduces the Petrie symmetric functions $G(k,m)$ and refers to them as *truncated homogeneous symmetric functions* $h_m^{[k-1]}$. Some results below are also independently obtained in [10]. In particular, Theorem 2.8 is a formula in [10, §2], and Theorem 2.13 is equivalent to [10, Proposition 2.9]. The particular case of Theorem 2.20 when $\mathbf{k}=\mathbb{Q}$ is part of [10, Theorem 2.7].

(1)This proof is independent of the first part of the paper, except that it uses the very simple Proposition 2.3 **(c)**.

**(c)** The paper [3] by Bazeniar, Ahmia and Belbachir introduces the symmetric functions $G(k,m)$ as well, or rather their evaluations $(G(k,m))(x_1,x_2,\ldots,x_n)$ at finitely many variables $x_1,x_2,\ldots,x_n$; it denotes them by $E_m^{(k-1)}(n) = E_m^{(k-1)}(x_1,x_2,\ldots,x_n)$. Ahmia and Merca continue the study of these $E_m^{(k-1)}(x_1,x_2,\ldots,x_n)$ in [1]. Our Theorem 2.19 is equivalent to the second formula in [1, Theorem 3.3] (although we are using infinitely many variables).

**(d)** The formal power series $G(k)$ also appears in [12, Chapter I, §6], under the guise of *Bott's cannibalistic class* $\theta^j(e)$ (for $j=k$ and rewritten in the language of $\lambda$-ring operations(2)); it is used there to prove an abstract Riemann–Roch theorem. An application to group representations appears in [2].

**3.** The Petrie symmetric functions have been added to Per Alexandersson's collection of symmetric functions at `https://www.math.upenn.edu/~peal/polynomials/petrie.htm` .

Remark on alternative versions. This paper also has an arXiv version [15], which includes some proofs that are here omitted for brevity. It also has a detailed version [16] with many more details.

## 1. Notations

We will use the following notations (most of which are also used in [17, §2.1]):

- We let $\mathbb{N} = \{0,1,2,\ldots\}$.
- The words "positive", "larger", etc. will be used in their Anglophone meaning (so that 0 is neither positive nor larger than itself).
- We fix a commutative ring $\mathbf{k}$; we will use this $\mathbf{k}$ as the base ring in what follows.
- A *weak composition* means an infinite sequence of nonnegative integers that contains only finitely many nonzero entries (i.e. a sequence $(\alpha_1,\alpha_2,\alpha_3,\ldots) \in \mathbb{N}^\infty$ such that all but finitely many $i \in \{1,2,3,\ldots\}$ satisfy $\alpha_i = 0$).
- We let WC denote the set of all weak compositions.
- For any weak composition $\alpha$ and any positive integer $i$, we let $\alpha_i$ denote the $i$-th entry of $\alpha$ (so that $\alpha = (\alpha_1,\alpha_2,\alpha_3,\ldots)$). More generally, we use this notation whenever $\alpha$ is an infinite sequence of any kind of objects.
- The *size* $|\alpha|$ of a weak composition $\alpha$ is defined to be $\alpha_1+\alpha_2+\alpha_3+\cdots \in \mathbb{N}$.
- A *partition* means a weak composition whose entries weakly decrease (i.e. a weak composition $\alpha$ satisfying $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \cdots$).
- If $n \in \mathbb{Z}$, then a *partition of* $n$ means a partition $\alpha$ having size $n$ (that is, satisfying $|\alpha| = n$).
- We let Par denote the set of all partitions. For each $n \in \mathbb{Z}$, we let $\mathrm{Par}_n$ denote the set of all partitions of $n$.
- We will sometimes omit trailing zeroes from partitions: i.e. a partition $\lambda = (\lambda_1,\lambda_2,\lambda_3,\ldots)$ will be identified with the $k$-tuple $(\lambda_1,\lambda_2,\ldots,\lambda_k)$ whenever $k \in \mathbb{N}$ satisfies $\lambda_{k+1} = \lambda_{k+2} = \lambda_{k+3} = \cdots = 0$. For example, $(3,2,1,0,0,0,\ldots) = (3,2,1) = (3,2,1,0)$.

(2)See [18, §16.74] for the connection between symmetric functions (over $\mathbb{Z}$) and universal operations on $\lambda$-rings. To be specific: If $a$ is an element of a $\lambda$-ring $A$, then the canonical $\lambda$-ring morphism $\Lambda_\mathbb{Z} \to A$ (where $\Lambda_\mathbb{Z}$ is the ring of symmetric functions over $\mathbb{Z}$) that sends $e_1 = x_1+x_2+x_3+\cdots \in \Lambda_\mathbb{Z}$ to $a \in A$ will send the Petrie symmetric function $G(k,m)$ to the "$m$-th graded component" of Bott's cannibalistic class $\theta^k(a)$. (Bott's cannibalistic class $\theta^k(a)$ itself is defined only if $a$ is a "positive element" in the sense of [12] (or can only be defined in an appropriate closure of $A$). When it is defined, it is the image of the series $G(k)$. Otherwise, its "graded components" are the right object to consider.)

- The partition $(0, 0, 0, \ldots) = ()$ is called the *empty partition* and denoted by $\varnothing$.
- A *part* of a partition $\lambda$ means a nonzero entry of $\lambda$. For example, the parts of the partition $(3, 1, 1) = (3, 1, 1, 0, 0, 0, \ldots)$ are $3, 1, 1$.
- We will use the notation $1^k$ for "$\underbrace{1, 1, \ldots, 1}_{k \text{ times}}$" in partitions. (For example, $\left(2, 1^4\right) = (2, 1, 1, 1, 1)$. This notation is a particular case of the more general notation $m^k$ for "$\underbrace{m, m, \ldots, m}_{k \text{ times}}$" in partitions, used, e.g. in [17, Definition 2.2.1].)
- We let $\Lambda$ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. This is a subring of the ring $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ of formal power series. To be more specific, $\Lambda$ consists of all power series in $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ that are symmetric (i.e. invariant under permutations of the variables) and of bounded degree (see [17, §2.1] for the precise meaning of this).
- A *monomial* shall mean a formal expression of the form $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ with $\alpha \in \mathrm{WC}$. Formal power series are formal infinite $\mathbf{k}$-linear combinations of such monomials.
- For any weak composition $\alpha$, we let $\mathbf{x}^\alpha$ denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$.
- The *degree* of a monomial $\mathbf{x}^\alpha$ is defined to be $|\alpha|$.
- A formal power series is said to be *homogeneous of degree* $n$ (for some $n \in \mathbb{N}$) if all monomials appearing in it (with nonzero coefficient) have degree $n$. In particular, the power series 0 is homogeneous of any degree.
- If $f \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ is a power series, then there is a unique family $\left(f_i\right)_{i \in \mathbb{N}} = \left(f_0, f_1, f_2, \ldots\right)$ of formal power series $f_i \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ such that each $f_i$ is homogeneous of degree $i$ and such that $f = \sum\limits_{i \in \mathbb{N}} f_i$. This family $\left(f_i\right)_{i \in \mathbb{N}}$ is called the *homogeneous decomposition* of $f$, and its entry $f_i$ (for any given $i \in \mathbb{N}$) is called the *$i$-th degree homogeneous component* of $f$.
- The $\mathbf{k}$-algebra $\Lambda$ is graded: i.e. any symmetric function $f$ can be uniquely written as a sum $\sum\limits_{i \in \mathbb{N}} f_i$, where each $f_i$ is a homogeneous symmetric function of degree $i$, and where all but finitely many $i \in \mathbb{N}$ satisfy $f_i = 0$.

We shall use the symmetric functions $m_\lambda, h_n, e_n, p_n, s_\lambda$ in $\Lambda$ as defined in [17, Sections 2.1 and 2.2]. Let us briefly recall how they are defined:

- For any partition $\lambda$, we define the *monomial symmetric function* $m_\lambda \in \Lambda$ by[(3)]

$$m_\lambda = \sum \mathbf{x}^\alpha,$$

  where the sum ranges over all weak compositions $\alpha \in \mathrm{WC}$ that can be obtained from $\lambda$ by permuting entries[(4)]. For example,

$$m_{(2,2,1)} = \sum_{i<j<k} x_i^2 x_j^2 x_k + \sum_{i<j<k} x_i^2 x_j x_k^2 + \sum_{i<j<k} x_i x_j^2 x_k^2.$$

  The family $\left(m_\lambda\right)_{\lambda \in \mathrm{Par}}$ (that is, the family of the symmetric functions $m_\lambda$ as $\lambda$ ranges over all partitions) is a basis of the $\mathbf{k}$-module $\Lambda$.

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(3)This definition of $m_\lambda$ is not the same as the one given in [17, Definition 2.1.3]; but it is easily seen to be equivalent to the latter (i.e. it defines the same $m_\lambda$). See [16] for the details of the proof.

(4)Here, we understand $\lambda$ to be an infinite sequence, not a finite tuple, so the entries being permuted include infinitely many 0's.

- For each $n \in \mathbb{Z}$, we define the *complete homogeneous symmetric function* $h_n \in \Lambda$ by
$$h_n = \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = n}} \mathbf{x}^\alpha = \sum_{\lambda \in \mathrm{Par}_n} m_\lambda.$$
Thus, $h_0 = 1$ and $h_n = 0$ for all $n < 0$.
We know (e.g. from [17, Proposition 2.4.1]) that the family $(h_n)_{n \geqslant 1} = (h_1, h_2, h_3, \ldots)$ is algebraically independent and generates $\Lambda$ as a $\mathbf{k}$-algebra. In other words, $\Lambda$ is freely generated by $h_1, h_2, h_3, \ldots$ as a commutative $\mathbf{k}$-algebra.
- For each $n \in \mathbb{Z}$, we define the *elementary symmetric function* $e_n \in \Lambda$ by
$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \mathrm{WC} \cap \{0,1\}^\infty; \\ |\alpha| = n}} \mathbf{x}^\alpha.$$
Thus, $e_0 = 1$ and $e_n = 0$ for all $n < 0$. If $n \geqslant 0$, then $e_n = m_{(1^n)}$, where, as we have agreed above, the notation $(1^n)$ stands for the $n$-tuple $(1, 1, \ldots, 1)$.
- For each positive integer $n$, we define the *power-sum symmetric function* $p_n \in \Lambda$ by
$$p_n = x_1^n + x_2^n + x_3^n + \cdots = m_{(n)}.$$
- For each partition $\lambda$, we define the *Schur function* $s_\lambda \in \Lambda$ by
$$s_\lambda = \sum \mathbf{x}_T,$$
where the sum ranges over all semistandard tableaux $T$ of shape $\lambda$, and where $\mathbf{x}_T$ denotes the monomial obtained by multiplying the $x_i$ for all entries $i$ of $T$. We refer the reader to [17, Definition 2.2.1] or to [25, §7.10] for the details of this definition and further descriptions of the Schur functions. One of the most important properties of Schur functions (see, e.g. [17, (2.4.16) for $\mu = \varnothing$] or [21, Theorem 2.32] or [25, Theorem 7.16.1 for $\mu = \varnothing$] or [23, Theorem 7.2.3 (a)]) is the fact that
$$s_\lambda = \det\left( (h_{\lambda_i - i + j})_{1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \ell} \right) \tag{1}$$
for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. This is known as the *(first, straight-shape) Jacobi–Trudi formula.*
The family $(s_\lambda)_{\lambda \in \mathrm{Par}}$ is a basis of the $\mathbf{k}$-module $\Lambda$, and is known as the *Schur basis.* It is easy to see that each $n \in \mathbb{N}$ satisfies $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$. Moreover, for each partition $\lambda$, the Schur function $s_\lambda \in \Lambda$ is homogeneous of degree $|\lambda|$.

Among the many relations between these symmetric functions is an expression for the power-sum symmetric function $p_n$ in terms of the Schur basis:

Proposition 1.1. *Let $n$ be a positive integer. Then,*
$$p_n = \sum_{i=0}^{n-1} (-1)^i \, s_{(n-i, 1^i)}.$$

*Proof.* This is a classical formula, and appears (e.g. ) in [9, Problem 4.21], [17, Exercise 5.4.12(g)] and [21, Exercise 2.2]. Alternatively, this is an easy consequence of the Murnaghan–Nakayama rule (see [21, Theorem 6.3] or [24, Theorem 4.4.2] or [25, Theorem 7.17.3] or [28, (1)]), applied to the product $p_n s_\varnothing$ (since $s_\varnothing = 1$). □

Finally, we will sometimes use the *Hall inner product* $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \to \mathbf{k}$ as defined in [17, Definition 2.5.12].[(5)] This is the $\mathbf{k}$-bilinear form on $\Lambda$ that is defined by the requirement that

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \qquad \text{for any } \lambda, \mu \in \mathrm{Par}$$

(where $\delta_{\lambda,\mu}$ denotes the Kronecker delta). Thus, the Schur basis $(s_\lambda)_{\lambda \in \mathrm{Par}}$ of $\Lambda$ is an orthonormal basis with respect to the Hall inner product. It is easy to see[(6)] that the Hall inner product $(\cdot, \cdot)$ is graded: i.e. we have

$$\langle f, g \rangle = 0 \tag{2}$$

if $f$ and $g$ are two homogeneous symmetric functions of different degrees. We shall also use the following two known evaluations of the Hall inner product:

Proposition 1.2. *Let $n$ be a positive integer. Then, $\langle h_n, p_n \rangle = 1$.*

Proposition 1.3. *Let $n$ be a positive integer. Then, $\langle e_n, p_n \rangle = (-1)^{n-1}$.*

Both of these propositions follow easily from Proposition 1.1 (since $h_n = s_{(n)}$ and $e_n = s_{(1^n)}$). See [15] for details.

## 2. Theorems

2.1. Definitions. The main role in this paper is played by two power series that we will now define:

Definition 2.1.

**(a)** *For any positive integer $k$, we let*[(7)]

$$G(k) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha. \tag{3}$$

*This is a symmetric formal power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (but does not belong to $\Lambda$ in general).*

**(b)** *For any positive integer $k$ and any $m \in \mathbb{N}$, we let*

$$G(k, m) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha \in \Lambda. \tag{4}$$

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(5) However, it is denoted by $(\cdot, \cdot)$ rather than by $\langle \cdot, \cdot \rangle$ in [17]. (That is, what we call $\langle a, b \rangle$ is denoted by $(a, b)$ in [17].)

The Hall inner product also appears (for $\mathbf{k} = \mathbb{Z}$ and $\mathbf{k} = \mathbb{Q}$) in [9, Definition 7.5], in [25, §7.9] and in [20, Section I.4] (note that it is called the "scalar product" in the latter two references). The definitions of the Hall inner product in [25, §7.9] and in [20, Section I.4] are different from ours, but they are equivalent to ours (because of [25, Corollary 7.12.2] and [20, Chapter I, (4.8)]).

(6) See, e.g. [17, Exercise 2.5.13(a)] for a proof.

(7) Here and in all similar situations, "for all $i$" means "for all positive integers $i$".

EXAMPLE 2.2. **(a)** We have

$$\begin{aligned}
G(2) &= \sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<2 \text{ for all } i}} \mathbf{x}^\alpha \\
&= 1 + x_1 + x_2 + x_3 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots \\
&\qquad + x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + \cdots \\
&\qquad + \cdots \\
&= \sum_{m\in\mathbb{N}} \underbrace{\sum_{1\leqslant i_1<i_2<\cdots<i_m} x_{i_1}x_{i_2}\cdots x_{i_m}}_{=e_m} = \sum_{m\in\mathbb{N}} e_m.
\end{aligned}$$

**(b)** For each $m\in\mathbb{N}$, we have

$$G(2,m) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<2 \text{ for all } i}} \mathbf{x}^\alpha = \sum_{1\leqslant i_1<i_2<\cdots<i_m} x_{i_1}x_{i_2}\cdots x_{i_m} = e_m.$$

We suggest the name *$k$-Petrie symmetric series* for $G(k)$ and the name *$(k,m)$-Petrie symmetric function* for $G(k,m)$. The reason for this naming is that the coefficients of these functions in the Schur basis of $\Lambda$ are determinants of Petrie matrices, as we will see in Subsection 3.6.

2.2. BASIC IDENTITIES. We begin our study of the $G(k)$ and $G(k,m)$ with some simple properties:

PROPOSITION 2.3. *Let $k$ be a positive integer.*

**(a)** *The symmetric function $G(k,m)$ is the $m$-th degree homogeneous component of $G(k)$ for each $m\in\mathbb{N}$.*
**(b)** *We have*

$$G(k) = \sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda\in\mathrm{Par};\\ \lambda_i<k \text{ for all } i}} m_\lambda = \prod_{i=1}^{\infty}\left(x_i^0 + x_i^1 + \cdots + x_i^{k-1}\right).$$

**(c)** *We have*

$$G(k,m) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=m;\\ \lambda_i<k \text{ for all } i}} m_\lambda$$

*for each $m\in\mathbb{N}$.*
**(d)** *If $m\in\mathbb{N}$ satisfies $k>m$, then $G(k,m)=h_m$.*
**(e)** *If $m\in\mathbb{N}$ and $k=2$, then $G(k,m)=e_m$.*
**(f)** *If $m=k$, then $G(k,m)=h_m-p_m$.*

Proving Proposition 2.3 makes good practice in understanding the definitions of $m_\lambda$, $h_n$, $e_n$, $p_n$, $G(k)$ and $G(k,n)$. We omit the proof here; it can be found in full (hardly necessary) detail in [16].

Parts **(d)** and **(e)** of Proposition 2.3 suggest to regard the Petrie symmetric functions $G(k,m)$ as an interpolation between the $h_m$ and the $e_m$.

2.3. The Schur expansion. The solution to [25, Exercise 7.3] gives an expansion of $G(3)$ in terms of the elementary symmetric functions (due to I. M. Gessel); this expansion can be rewritten as

$$G(3) = \sum_{n \in \mathbb{N}} e_n^2 + \sum_{m<n} c_{m,n} e_m e_n, \qquad \text{where } c_{m,n} = (-1)^{m-n} \begin{cases} 2, & \text{if } 3 \mid m-n; \\ -1, & \text{if } 3 \nmid m-n. \end{cases}$$

We shall instead expand $G(k)$ in terms of Schur functions. For this, we need to define some notations.

Convention 1. *We shall use the* Iverson bracket notation*: If $\mathcal{A}$ is a logical statement, then $[\mathcal{A}]$ shall denote the truth value of $\mathcal{A}$ (that is, the integer* $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$ *).*

We shall furthermore use the notation $(a_{i,j})_{1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \ell}$ for the $\ell \times \ell$-matrix whose $(i,j)$-th entry is $a_{i,j}$ for each $i, j \in \{1, 2, \ldots, \ell\}$.

Definition 2.4. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathrm{Par}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \mathrm{Par}$, and let $k$ be a positive integer. Then, the $k$-*Petrie number $\operatorname{pet}_k(\lambda, \mu)$ *of $\lambda$ and $\mu$ is the integer defined by*

$$\operatorname{pet}_k(\lambda, \mu) = \det\left( ([0 \leqslant \lambda_i - \mu_j - i + j < k])_{1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \ell} \right).$$

*Note that this integer does not depend on the choice of $\ell$ (in the sense that it does not change if we enlarge $\ell$ by adding trailing zeroes to the representations of $\lambda$ and $\mu$); this follows from Lemma 2.6 below.*

Example 2.5. Let $\lambda$ be the partition $(3, 2, 1) \in \mathrm{Par}$, let $\mu$ be the partition $(1, 1) \in \mathrm{Par}$, let $\ell = 3$, and let $k$ be a positive integer. Then, the definition of $\operatorname{pet}_k(\lambda, \mu)$ yields

$$\begin{aligned}
&\operatorname{pet}_k(\lambda, \mu) \\
&= \det\left( ([0 \leqslant \lambda_i - \mu_j - i + j < k])_{1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \ell} \right) \\
&= \det \begin{pmatrix} [0 \leqslant \lambda_1 - \mu_1 < k] & [0 \leqslant \lambda_1 - \mu_2 + 1 < k] & [0 \leqslant \lambda_1 - \mu_3 + 2 < k] \\ [0 \leqslant \lambda_2 - \mu_1 - 1 < k] & [0 \leqslant \lambda_2 - \mu_2 < k] & [0 \leqslant \lambda_2 - \mu_3 + 1 < k] \\ [0 \leqslant \lambda_3 - \mu_1 - 2 < k] & [0 \leqslant \lambda_3 - \mu_2 - 1 < k] & [0 \leqslant \lambda_3 - \mu_3 < k] \end{pmatrix} \\
&= \det \begin{pmatrix} [0 \leqslant 3 - 1 < k] & [0 \leqslant 3 - 1 + 1 < k] & [0 \leqslant 3 - 0 + 2 < k] \\ [0 \leqslant 2 - 1 - 1 < k] & [0 \leqslant 2 - 1 < k] & [0 \leqslant 2 - 0 + 1 < k] \\ [0 \leqslant 1 - 1 - 2 < k] & [0 \leqslant 1 - 1 - 1 < k] & [0 \leqslant 1 - 0 < k] \end{pmatrix} \\
&\qquad \left( \begin{array}{c} \text{since } \lambda_1 = 3 \text{ and } \lambda_2 = 2 \text{ and } \lambda_3 = 1 \\ \text{and } \mu_1 = 1 \text{ and } \mu_2 = 1 \text{ and } \mu_3 = 0 \end{array} \right) \\
&= \det \begin{pmatrix} [0 \leqslant 2 < k] & [0 \leqslant 3 < k] & [0 \leqslant 5 < k] \\ [0 \leqslant 0 < k] & [0 \leqslant 1 < k] & [0 \leqslant 3 < k] \\ [0 \leqslant -2 < k] & [0 \leqslant -1 < k] & [0 \leqslant 1 < k] \end{pmatrix}.
\end{aligned}$$

Thus, taking $k = 4$, we obtain

$$\operatorname{pet}_4(\lambda, \mu) = \det \begin{pmatrix} [0 \leqslant 2 < 4] & [0 \leqslant 3 < 4] & [0 \leqslant 5 < 4] \\ [0 \leqslant 0 < 4] & [0 \leqslant 1 < 4] & [0 \leqslant 3 < 4] \\ [0 \leqslant -2 < 4] & [0 \leqslant -1 < 4] & [0 \leqslant 1 < 4] \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 0.$$

On the other hand, taking $k = 3$, we obtain

$$\operatorname{pet}_3(\lambda, \mu) = \det \begin{pmatrix} [0 \leqslant 2 < 3] & [0 \leqslant 3 < 3] & [0 \leqslant 5 < 3] \\ [0 \leqslant 0 < 3] & [0 \leqslant 1 < 3] & [0 \leqslant 3 < 3] \\ [0 \leqslant -2 < 3] & [0 \leqslant -1 < 3] & [0 \leqslant 1 < 3] \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

Lemma 2.6. *Let $\lambda \in \operatorname{Par}$ and $\mu \in \operatorname{Par}$, and let $k$ be a positive integer. Let $\ell \in \mathbb{N}$ be such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$. Then, the determinant* $\det\left(\left([0 \leqslant \lambda_i - \mu_j - i + j < k]\right)_{1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \ell}\right)$ *does not depend on the choice of $\ell$.*

See Subsection 3.5 for the simple proof of Lemma 2.6.

Surprisingly, the $k$-Petrie numbers $\operatorname{pet}_k(\lambda, \mu)$ can take only three possible values:

Proposition 2.7. *Let $\lambda \in \operatorname{Par}$ and $\mu \in \operatorname{Par}$, and let $k$ be a positive integer. Then,* $\operatorname{pet}_k(\lambda, \mu) \in \{-1, 0, 1\}$.

Proposition 2.7 will be proved in Subsection 3.6.

We can now expand the Petrie symmetric functions $G(k, m)$ and the power series $G(k)$ in the basis $(s_\lambda)_{\lambda \in \operatorname{Par}}$ of $\Lambda$:

Theorem 2.8. *Let $k$ be a positive integer. Then,*

$$G(k) = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \varnothing)\, s_\lambda.$$

*(Recall that $\varnothing$ denotes the empty partition $() = (0, 0, 0, \ldots)$.)*

We will not prove Theorem 2.8 directly; instead, we will first show a stronger result (Theorem 2.15), and then derive Theorem 2.8 from it in Subsection 3.8.

Corollary 2.9. *Let $k$ be a positive integer. Let $m \in \mathbb{N}$. Then,*

$$G(k, m) = \sum_{\lambda \in \operatorname{Par}_m} \operatorname{pet}_k(\lambda, \varnothing)\, s_\lambda.$$

Corollary 2.9 easily follows from Theorem 2.8 using Proposition 2.3 **(a)**; but again, we shall instead derive it from a stronger result (Corollary 2.16) in Subsection 3.8.

We will see a more explicit description of the $k$-Petrie numbers $\operatorname{pet}_k(\lambda, \varnothing)$ in Subsection 2.4.

Remark 2.10. Corollary 2.9, in combination with Proposition 2.7, shows that each $k$-Petrie function $G(k, m)$ (for any $k > 0$ and $m \in \mathbb{N}$) is a linear combination of Schur functions, with all coefficients belonging to $\{-1, 0, 1\}$. It is natural to expect the more general symmetric functions

$$\widetilde{G}(k, k', m) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ k' \leqslant \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha, \qquad \text{where } 0 < k' \leqslant k,$$

to have the same property. However, this is not the case. For example,

$$\widetilde{G}(4, 2, 5) = m_{(3,2)} = -2s_{(1,1,1,1,1)} + 2s_{(2,1,1,1)} - s_{(2,2,1)} - s_{(3,1,1)} + s_{(3,2)}.$$

2.4. An explicit description of the $k$-Petrie numbers $\operatorname{pet}_k(\lambda, \varnothing)$. Can the $k$-Petrie numbers $\operatorname{pet}_k(\lambda, \varnothing)$ from Definition 2.4 be described more explicitly than as determinants? To be somewhat pedantic, the answer to this question depends on one's notion of "explicit", as determinants are not hard to compute, and another algorithm for calculating $\operatorname{pet}_k(\lambda, \varnothing)$ can be extracted from our proof of Proposition 2.7 (when combined with [13, proof of Theorem 1]). Nevertheless, there is a more explicit description. This description will be stated in Theorem 2.13 further below.

First, let us get a simple case out of the way:

Proposition 2.11. *Let $\lambda \in \operatorname{Par}$, and let $k$ be a positive integer such that $\lambda_1 \geqslant k$. Then,* $\operatorname{pet}_k(\lambda, \varnothing) = 0$.

*Proof of Proposition 2.11.* Write $\lambda$ as $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. Thus, $\ell \geqslant 1$ (since $\lambda_1 \geqslant k > 0$). Moreover, the empty partition $\varnothing$ can be written as $\varnothing = (\varnothing_1, \varnothing_2, \dots, \varnothing_\ell)$ (since $\varnothing_i = 0$ for each integer $i > \ell$).

Thus, we have $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ and $\varnothing = (\varnothing_1, \varnothing_2, \dots, \varnothing_\ell)$. Hence, the definition of $\operatorname{pet}_k(\lambda, \varnothing)$ yields

$$\begin{aligned} \operatorname{pet}_k(\lambda, \varnothing) &= \det\left(\left(\left[0 \leqslant \lambda_i - \underbrace{\varnothing_j}_{=0} - i + j < k\right]\right)_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}\right) \\ &= \det\left(([0 \leqslant \lambda_i - i + j < k])_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}\right). \end{aligned} \tag{5}$$

But each $j \in \{1, 2, \dots, \ell\}$ satisfies $[0 \leqslant \lambda_1 - 1 + j < k] = 0$ (since $\lambda_1 - 1 + \underbrace{j}_{\geqslant 1} \geqslant \lambda_1 - 1 + 1 = \lambda_1 \geqslant k$). In other words, the $\ell \times \ell$-matrix $([0 \leqslant \lambda_i - i + j < k])_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}$ has first row $(0, 0, \dots, 0)$. Therefore, its determinant is 0. In other words, $\operatorname{pet}_k(\lambda, \varnothing) = 0$ (since $\operatorname{pet}_k(\lambda, \varnothing)$ is its determinant[(8)]). This proves Proposition 2.11. ☐

Stating Theorem 2.13 will require some notation:

Definition 2.12. *For any $\lambda \in \operatorname{Par}$, we define the* transpose *of $\lambda$ to be the partition $\lambda^t \in \operatorname{Par}$ determined by*

$$\left(\lambda^t\right)_i = |\{j \in \{1, 2, 3, \dots\} \mid \lambda_j \geqslant i\}| \qquad \text{for each } i \geqslant 1.$$

*This partition $\lambda^t$ is also known as the* conjugate *of $\lambda$, and is perhaps easiest to understand in terms of Young diagrams: To wit, the Young diagram of $\lambda^t$ is obtained from that of $\lambda$ by a flip across the main diagonal.*

One important use of transpose partitions is the following fact (see, e.g. [17, (2.4.17) for $\mu = \varnothing$] or [21, Theorem 2.32] or [25, Theorem 7.16.2 applied to $\lambda^t$ and $\varnothing$ instead of $\lambda$ and $\mu$] for proofs): We have

$$s_{\lambda^t} = \det\left((e_{\lambda_i - i + j})_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}\right) \tag{6}$$

for any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. This is known as the *(second, straight-shape) Jacobi–Trudi formula*.

We will use the following notation for quotients and remainders:

Convention 2. *Let $k$ be a positive integer. Let $n \in \mathbb{Z}$. Then, $n\%k$ shall denote the remainder of $n$ divided by $k$, whereas $n//k$ shall denote the quotient of this division (an integer). Thus, $n//k$ and $n\%k$ are uniquely determined by the three requirements that $n//k \in \mathbb{Z}$ and $n\%k \in \{0, 1, \dots, k-1\}$ and $n = (n//k) \cdot k + (n\%k)$.*

*The "//" and "%" signs bind more strongly than the "+" and "−" signs. That is, for example, the expression "$a + b\%k$" shall be understood to mean "$a + (b\%k)$" rather than "$(a + b)\%k$".*

Now, we can state our "formula" for $k$-Petrie numbers of the form $\operatorname{pet}_k(\lambda, \varnothing)$.

Theorem 2.13. *Let $\lambda \in \operatorname{Par}$, and let $k$ be a positive integer. Let $\mu = \lambda^t$.*

**(a)** *If $\mu_k \neq 0$, then $\operatorname{pet}_k(\lambda, \varnothing) = 0$.*

*From now on, let us assume that $\mu_k = 0$.*

*Define a $(k-1)$-tuple $(\beta_1, \beta_2, \dots, \beta_{k-1}) \in \mathbb{Z}^{k-1}$ by setting*

$$\beta_i = \mu_i - i \qquad \text{for each } i \in \{1, 2, \dots, k-1\}. \tag{7}$$

---

(8) by (5)

*Define a* $(k-1)$*-tuple* $(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}) \in \{1, 2, \ldots, k\}^{k-1}$ *by setting*

$$\gamma_i = 1 + (\beta_i - 1) \% k \qquad \text{for each } i \in \{1, 2, \ldots, k-1\}. \tag{8}$$

**(b)** *If the* $k-1$ *numbers* $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ *are not distinct, then* $\operatorname{pet}_k(\lambda, \varnothing) = 0$.

**(c)** *Assume that the* $k-1$ *numbers* $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ *are distinct. Let*

$$g = \left| \left\{ (i,j) \in \{1, 2, \ldots, k-1\}^2 \ \mid \ i < j \text{ and } \gamma_i < \gamma_j \right\} \right|.$$

*Then,* $\operatorname{pet}_k(\lambda, \varnothing) = (-1)^{(\beta_1 + \beta_2 + \cdots + \beta_{k-1}) + g + (\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1})}$.

The proof of this theorem is technical and will be given in Subsection 3.9.

It is possible to restate part of Theorem 2.13 without using $\lambda^t$:

Proposition 2.14. *Let* $\lambda \in \operatorname{Par}$*, and let* $k$ *be a positive integer. Assume that* $\lambda_1 < k$*. Define a subset* $B$ *of* $\mathbb{Z}$ *by*

$$B = \{\lambda_i - i \ \mid \ i \in \{1, 2, 3, \ldots\}\}.$$

*Let* $\overline{0}, \overline{1}, \ldots, \overline{k-1}$ *be the residue classes of the integers* $0, 1, \ldots, k-1$ *modulo* $k$ *(considered as subsets of* $\mathbb{Z}$*). Let* $W$ *be the set of all integers smaller than* $k-1$*.*

*Then,* $\operatorname{pet}_k(\lambda, \varnothing) \neq 0$ *if and only if each* $i \in \{0, 1, \ldots, k-1\}$ *satisfies* $\left| \left( \overline{i} \cap W \right) \setminus B \right| \leqslant 1$.

In Subsection 3.9, we will outline how this proposition can be derived from Theorem 2.13.

The sets $B$ and $\left( \overline{i} \cap W \right) \setminus B$ in Proposition 2.14 are related to the *k-modular structure* of the partition $\lambda$, such as the *β-set*, the *k-abacus*, the *k-core* and the *k-quotient* (see [22, §§1–3] for some of these concepts). Essentially equivalent concepts include the *Maya diagram* of $\lambda$ (see, e.g. [7, §3.3])(9) and the *first column hook lengths* of $\lambda$ (see [22, Proposition (1.3)]).

### 2.5. A "Pieri" rule.

Now, the following generalization of Theorem 2.8 holds:

Theorem 2.15. *Let* $k$ *be a positive integer. Let* $\mu \in \operatorname{Par}$*. Then,*

$$G(k) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \mu) \, s_\lambda.$$

Theorem 2.8 is the particular case of Theorem 2.15 for $\mu = \varnothing$.

We shall prove Theorem 2.15 in Subsection 3.7 (see [15] for another proof).

We can also generalize Corollary 2.9 to obtain a Pieri-like rule for multiplication by $G(k, m)$:

Corollary 2.16. *Let* $k$ *be a positive integer. Let* $m \in \mathbb{N}$*. Let* $\mu \in \operatorname{Par}$*. Then,*

$$G(k, m) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}_{m + |\mu|}} \operatorname{pet}_k(\lambda, \mu) \, s_\lambda.$$

Corollary 2.16 follows from Theorem 2.15 by projecting onto the $(m + |\mu|)$-th graded component of $\Lambda$. (We shall explain this argument in more detail in Subsection 3.8.)

(9)The *Maya diagram* of $\lambda$ is a coloring of the set $\left\{ z + \frac{1}{2} \ \mid \ z \in \mathbb{Z} \right\}$ with the colors black and white, in which the elements $\lambda_i - i + \frac{1}{2}$ (for all $i \in \{1, 2, 3, \ldots\}$) are colored black while all remaining elements are colored white. Borcherds's proof of the Jacobi triple product identity ([6, §13.3]) also essentially constructs this Maya diagram (wording it in terms of the "*Dirac sea*" model for electrons).

2.6. Coproducts of Petrie functions. In the following, the "$\otimes$" sign will always stand for $\otimes_{\mathbf{k}}$ (that is, tensor product of $\mathbf{k}$-modules or of $\mathbf{k}$-algebras).

The $\mathbf{k}$-algebra $\Lambda$ is a Hopf algebra due to the presence of a comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$. We recall (from [17, §2.1]) one way to define this comultiplication:

Consider the rings

$$\mathbf{k}[[\mathbf{x}]] := \mathbf{k}[[x_1, x_2, x_3, \ldots]], \text{ and}$$
$$\mathbf{k}[[\mathbf{x}, \mathbf{y}]] := \mathbf{k}[[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]]$$

of formal power series. We shall use the notations $\mathbf{x}$ and $\mathbf{y}$ for the sequences $(x_1, x_2, x_3, \ldots)$ and $(y_1, y_2, y_3, \ldots)$ of indeterminates. If $f \in \mathbf{k}[[\mathbf{x}]]$ is any formal power series, then $f(\mathbf{y})$ shall mean the result of substituting $y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ in $f$. (This will be a formal power series in $\mathbf{k}[[y_1, y_2, y_3, \ldots]]$.) For the sake of symmetry, we also use the analogous notation $f(\mathbf{x})$ for the result of substituting $x_1, x_2, x_3, \ldots$ for $x_1, x_2, x_3, \ldots$ in $f$; of course, this $f(\mathbf{x})$ is just $f$. Finally, if the power series $f \in \mathbf{k}[[\mathbf{x}]]$ is symmetric, then we use the notation $f(\mathbf{x}, \mathbf{y})$ for the result of substituting the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ in $f$ (that is, choosing some bijection[(10)] $\phi : \{x_1, x_2, x_3, \ldots\} \to \{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$ and substituting $\phi(x_i)$ for each $x_i$ in $f$). This result does not depend on the order in which the former variables are substituted for the latter (i.e. on the choice of the bijection $\phi$) because $f$ is symmetric.

Now, the comultiplication of $\Lambda$ is the map $\Delta : \Lambda \to \Lambda \otimes \Lambda$ determined as follows: For a symmetric function $f \in \Lambda$, we have

$$\Delta(f) = \sum_{i \in I} f_{1,i} \otimes f_{2,i}, \tag{9}$$

where $f_{1,i}, f_{2,i} \in \Lambda$ are such that

$$f(\mathbf{x}, \mathbf{y}) = \sum_{i \in I} f_{1,i}(\mathbf{x}) f_{2,i}(\mathbf{y}). \tag{10}$$

More precisely, if $f \in \Lambda$, if $I$ is a finite set, and if $(f_{1,i})_{i \in I} \in \Lambda^I$ and $(f_{2,i})_{i \in I} \in \Lambda^I$ are two families satisfying (10), then $\Delta(f)$ is given by (9). [(11)]

For example, for any $n \in \mathbb{N}$, it is easy to see that

$$e_n(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{n} e_i(\mathbf{x}) e_{n-i}(\mathbf{y}),$$

and thus the above definition of $\Delta$ yields

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}.$$

A similar formula exists for the image of a Petrie symmetric function under $\Delta$:

Theorem 2.17. *Let $k$ be a positive integer. Let $m \in \mathbb{N}$. Then,*

$$\Delta(G(k, m)) = \sum_{i=0}^{m} G(k, i) \otimes G(k, m-i).$$

---

[(10)]Such bijections clearly exist, since the sets $\{x_1, x_2, x_3, \ldots\}$ and $\{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$ have the same cardinality (namely, $\aleph_0$). (This is part of the familiar metaphor of "Hilbert's hotel".)

[(11)]In the language of [17, §2.1], this can be restated as $\Delta(f) = f(\mathbf{x}, \mathbf{y})$, because $\Lambda \otimes \Lambda$ is identified with a certain subring of $\mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ in [17, §2.1] (via the injection $\Lambda \otimes \Lambda \to \mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ that sends any $u \otimes v \in \Lambda \otimes \Lambda$ to $u(\mathbf{x}) v(\mathbf{y}) \in \mathbf{k}[[\mathbf{x}, \mathbf{y}]]$).

The proof of Theorem 2.17 is a simple consequence of (9) (see [15] for the details). We omit it here, since we will prove a more general fact (Theorem 5.15) in Subsection 5.7 further below.

It is well-known that $\Delta : \Lambda \to \Lambda \otimes \Lambda$ is a $\mathbf{k}$-algebra homomorphism. Equipping the $\mathbf{k}$-algebra $\Lambda$ with the comultiplication $\Delta$ (as well as a counit $\varepsilon : \Lambda \to \mathbf{k}$, which we won't need here) yields a connected graded Hopf algebra. (See, e.g. [17, §2.1] for proofs.)

2.7. The Frobenius endomorphisms and Petrie functions. We shall next derive another formula for the Petrie symmetric functions $G(k,m)$. For this formula, we need the following definition ([17, Exercise 2.9.9]):

Definition 2.18. *Let $n \in \{1,2,3,\ldots\}$. We define a map $\mathbf{f}_n : \Lambda \to \Lambda$ by*

$$\mathbf{f}_n(a) = a\left(x_1^n, x_2^n, x_3^n, \ldots\right) \qquad \text{for each } a \in \Lambda.$$

*This map $\mathbf{f}_n$ is called the $n$-th* Frobenius endomorphism *of $\Lambda$.*

Clearly, this map $\mathbf{f}_n$ is a $\mathbf{k}$-algebra endomorphism of $\Lambda$ (since it amounts to a substitution of indeterminates). It is known (from [17, Exercise 2.9.9(d)]) that this map $\mathbf{f}_n : \Lambda \to \Lambda$ is a Hopf algebra endomorphism of $\Lambda$.

Using the notion of *plethysm* (see, e.g. [25, Chapter 7, Definition A2.6] or [20, §I.8][(12)]), we can view the map $\mathbf{f}_n$ as a plethysm with the $n$-th power-sum symmetric function $p_n$, in the sense that any $a \in \Lambda$ satisfies $\mathbf{f}_n(a) = a[p_n] = p_n[a]$ as long as $\mathbf{k} = \mathbb{Z}$. (Plethysm becomes somewhat subtle when the base ring $\mathbf{k}$ is complicated; $\mathbf{f}_n(a) = a[p_n]$ holds for any $\mathbf{k}$, while $\mathbf{f}_n(a) = p_n[a]$ relies on good properties of $\mathbf{k}$.) The plethystic viewpoint makes some properties of $\mathbf{f}_n$ clear, but we shall avoid it for reasons of elementarity.

Now, we can express the Petrie symmetric functions $G(k,m)$ using Frobenius endomorphisms as follows:

Theorem 2.19. *Let $k$ be a positive integer. Let $m \in \mathbb{N}$. Then,*

$$G(k,m) = \sum_{i\in\mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k(e_i).$$

*(The sum on the right hand side of this equality is well-defined, since all sufficiently high $i \in \mathbb{N}$ satisfy $m - ki < 0$ and thus $h_{m-ki} = 0$.)*

Theorem 2.19 will be proved in Subsection 3.10 below.

2.8. The Petrie functions as polynomial generators of $\Lambda$. We now claim the following:

Theorem 2.20. *Fix a positive integer $k$. Assume that $1-k$ is invertible in $\mathbf{k}$. Then, the family $(G(k,m))_{m\geqslant 1} = (G(k,1), G(k,2), G(k,3), \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra $\Lambda$. (In other words, the canonical $\mathbf{k}$-algebra homomorphism*

$$\begin{aligned} \mathbf{k}[u_1,u_2,u_3,\ldots] &\to \Lambda, \\ u_m &\mapsto G(k,m) \end{aligned}$$

*is an isomorphism.)*

We shall prove Theorem 2.20 in Subsection 3.11. The proof uses the following two formulas for Hall inner products (which use Convention 1 again):

(12)Note that [25] uses the notation $f[g]$ for the plethysm of $f$ with $g$, whereas [20] uses the notation $f \circ g$ for this. We shall use $f[g]$.

Lemma 2.21. *Let $k$ and $m$ be positive integers. Let $j \in \mathbb{N}$. Then, $\langle p_m, \mathbf{f}_k (e_j) \rangle = (-1)^{j-1} [m = kj] \, k$.*

Proposition 2.22. *Let $k$ and $m$ be positive integers. Then*

$$\langle p_m, G(k,m) \rangle = 1 - [k|m] \, k.$$

Both of these formulas will be proved in Subsection 3.11 as well.

2.9. The Verschiebung endomorphisms. Now we recall another definition ([17, Exercise 2.9.10]):

Definition 2.23. *Let $n \in \{1, 2, 3, \ldots\}$. We define a $\mathbf{k}$-algebra homomorphism $\mathbf{v}_n : \Lambda \to \Lambda$ by*

$$\mathbf{v}_n (h_m) = \begin{cases} h_{m/n}, & \text{if } n \mid m; \\ 0, & \text{if } n \nmid m \end{cases} \qquad \text{for each } m > 0.$$

(*This is well-defined, since the sequence $(h_1, h_2, h_3, \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra $\Lambda$.*)

*This map $\mathbf{v}_n$ is called the $n$-th* Verschiebung endomorphism *of $\Lambda$.*

Again, it is known ([17, Exercise 2.9.10(e)]) that this map $\mathbf{v}_n : \Lambda \to \Lambda$ is a Hopf algebra endomorphism of $\Lambda$. Moreover, the following holds ([17, Exercise 2.9.10(f)]):

Proposition 2.24. *Let $n \in \{1, 2, 3, \ldots\}$. Then, the maps $\mathbf{f}_n : \Lambda \to \Lambda$ and $\mathbf{v}_n : \Lambda \to \Lambda$ are adjoint with respect to the Hall inner product on $\Lambda$. That is, we have*

$$\langle a, \mathbf{f}_n (b) \rangle = \langle \mathbf{v}_n (a), b \rangle \qquad \text{for any } a \in \Lambda \text{ and } b \in \Lambda.$$

Furthermore, it is known (see, e.g. [17, Exercise 2.9.10(a)]) that any positive integers $n$ and $m$ satisfy

$$\mathbf{v}_n (p_m) = \begin{cases} n p_{m/n}, & \text{if } n \mid m; \\ 0, & \text{if } n \nmid m. \end{cases} \tag{11}$$

2.10. The Hopf endomorphisms $U_k$ and $V_k$. In this final subsection, we shall show another way to obtain the Petrie symmetric functions $G(k,m)$ using the machinery of Hopf algebras. We refer, e.g. to [17, Chapters 1 and 2] for everything we will use about Hopf algebras.

Convention 3. *As already mentioned, $\Lambda$ is a connected graded Hopf algebra. We let $S$ denote its antipode.*

Definition 2.25. *If $C$ is a $\mathbf{k}$-coalgebra and $A$ is a $\mathbf{k}$-algebra, and if $f, g : C \to A$ are two $\mathbf{k}$-linear maps, then the* convolution $f \star g$ *of $f$ and $g$ is defined to be the $\mathbf{k}$-linear map $m_A \circ (f \otimes g) \circ \Delta_C : C \to A$, where $\Delta_C : C \to C \otimes C$ is the comultiplication of the $\mathbf{k}$-coalgebra $C$, and where $m_A : A \otimes A \to A$ is the $\mathbf{k}$-linear map sending each pure tensor $a \otimes b \in A \otimes A$ to $ab \in A$.*

We also recall Definition 2.23 and Definition 2.18. We now claim the following.

Theorem 2.26. *Fix a positive integer $k$. Let $U_k$ be the map $\mathbf{f}_k \circ S \circ \mathbf{v}_k : \Lambda \to \Lambda$. Let $V_k$ be the map $\mathrm{id}_\Lambda \star U_k : \Lambda \to \Lambda$. (This is well-defined by Definition 2.25, since $\Lambda$ is both a $\mathbf{k}$-coalgebra and a $\mathbf{k}$-algebra.) Then:*

**(a)** *The map $U_k$ is a $\mathbf{k}$-Hopf algebra homomorphism.*
**(b)** *The map $V_k$ is a $\mathbf{k}$-Hopf algebra homomorphism.*
**(c)** *We have $V_k (h_m) = G(k,m)$ for each $m \in \mathbb{N}$.*
**(d)** *We have $V_k (p_n) = (1 - [k \mid n] \, k) \, p_n$ for each positive integer $n$.*

See Subsection 3.12 for a proof of this theorem.

Note that Theorem 2.26 can be used to give a second proof of Theorem 2.17; see [16] for this.

We also obtain the following corollary from Theorem 2.17:

Corollary 2.27. *Let $k$ and $n$ be two positive integers. Then, there exists a polynomial $f \in \mathbf{k}[x_1, x_2, x_3, \ldots]$ such that*

$$(1 - [k|n]\, k)\, p_n = f\left(G(k,1), G(k,2), G(k,3), \ldots\right). \tag{12}$$

This corollary will be proved in Subsection 3.13.

## 3. Proofs

3.1. The symmetric functions $h_\lambda$. We shall now approach the proofs of the claims made above. First, let us introduce a family of symmetric functions, obtained by multiplying several $h_n$'s:

Definition 3.1. *Let $\lambda$ be a partition. Write $\lambda$ in the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are positive integers. Then, we define a symmetric function $h_\lambda \in \Lambda$ by*

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}.$$

The symmetric function $h_\lambda$ is called the *complete homogeneous symmetric function* corresponding to the partition $\lambda$.

From [17, Corollary 2.5.17(a)], we know that the families $(h_\lambda)_{\lambda \in \mathrm{Par}}$ and $(m_\lambda)_{\lambda \in \mathrm{Par}}$ are dual bases with respect to the Hall inner product. Thus,

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu} \qquad \text{for any } \lambda \in \mathrm{Par} \text{ and } \mu \in \mathrm{Par}. \tag{13}$$

We note that $h_0 = 1$; therefore, Definition 3.1 can be restated as follows: For any partition $\lambda$, we have

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots$$

(this infinite product $h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots$ is well-defined, since every sufficiently high positive integer $i$ satisfies $\lambda_i = 0$ and thus $h_{\lambda_i} = h_0 = 1$). This is how $h_\lambda$ is defined in [20, Section I.2].

3.2. Skew Schur functions. Let us define a classical partial order on Par (see, e.g. [17, Definition 2.3.1]):

Definition 3.2. *Let $\lambda$ and $\mu$ be two partitions.*

*We say that $\mu \subseteq \lambda$ if each $i \in \{1, 2, 3, \ldots\}$ satisfies $\mu_i \leqslant \lambda_i$.*

*We say that $\mu \not\subseteq \lambda$ if we don't have $\mu \subseteq \lambda$.*

For example, $(3,2) \subseteq (4,2,1)$, but $(3,2,1) \not\subseteq (4,2)$ (since $(3,2,1)_3 = 1$ is not $\leqslant$ to $(4,2)_3 = 0$).

For any two partitions $\lambda$ and $\mu$, a symmetric function $s_{\lambda/\mu}$ called a *skew Schur function* is defined in [17, Definition 2.3.1] and in [20, §I.5] (see also [25, Definition 7.10.1] for the case when $\mu \subseteq \lambda$). We shall not recall its standard definition here, but rather state a few properties.

The first property (which can in fact be used as an alternative definition of $s_{\lambda/\mu}$) is the *first Jacobi–Trudi formula* for skew shapes; it states the following:

Theorem 3.3. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be two partitions. Then,*

$$s_{\lambda/\mu} = \det\left(\left(h_{\lambda_i - \mu_j - i + j}\right)_{1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \ell}\right). \tag{14}$$

Theorem 3.3 appears (with proof) in [17, (2.4.16)] and in [20, Chapter I, (5.4)].

The following properties of skew Schur functions are easy to see:

- If $\lambda$ is any partition, then $s_{\lambda/\varnothing} = s_\lambda$. (Recall that $\varnothing$ denotes the empty partition.)
- If $\lambda$ and $\mu$ are two partitions satisfying $\mu \not\subseteq \lambda$, then $s_{\lambda/\mu} = 0$.

3.3. A Cauchy-like identity. We shall use the following identity, which connects the skew Schur functions $s_{\lambda/\mu}$, the symmetric functions $h_\lambda$ from Definition 3.1 and the monomial symmetric functions $m_\lambda$:

Theorem 3.4. *Recall the symmetric functions $h_\lambda$ defined in Definition 3.1. Let $\mu$ be any partition. Then, in the ring $\mathbf{k}[[\mathbf{x}, \mathbf{y}]]$, we have*

$$\sum_{\lambda \in \mathrm{Par}} s_{\lambda/\mu}(\mathbf{x}) s_\lambda(\mathbf{y}) = s_\mu(\mathbf{y}) \cdot \sum_{\lambda \in \mathrm{Par}} h_\lambda(\mathbf{x}) m_\lambda(\mathbf{y}).$$

*Here, we are using the notations introduced in Subsection 2.6.*

Theorem 3.4 appears in [20, fourth display on page 70], so we omit its proof.(13)

3.4. The $\mathbf{k}$-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$. Recall that the family $(h_n)_{n \geqslant 1} = (h_1, h_2, h_3, \ldots)$ is algebraically independent and generates $\Lambda$ as a $\mathbf{k}$-algebra. Thus, $\Lambda$ can be viewed as a polynomial ring in the (infinitely many) indeterminates $h_1, h_2, h_3, \ldots$. The universal property of a polynomial ring thus shows that if $A$ is any commutative $\mathbf{k}$-algebra, and if $(a_1, a_2, a_3, \ldots)$ is any sequence of elements of $A$, then there is a unique $\mathbf{k}$-algebra homomorphism from $\Lambda$ to $A$ that sends $h_i$ to $a_i$ for all positive integers $i$. We shall refer to this as the *h-universal property of* $\Lambda$. It lets us make the following definition:(14)

Definition 3.5. *Let $k$ be a positive integer. The h-universal property of $\Lambda$ shows that there is a unique $\mathbf{k}$-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ that sends $h_i$ to $[i < k]$ for all positive integers $i$. Consider this $\alpha_k$.*

The following elementary properties of $\alpha_k$ will be used many times:

Lemma 3.6. *Let $k$ be a positive integer.*

**(a)** *We have*

$$\alpha_k(h_i) = [i < k] \qquad \textit{for all } i \in \mathbb{N}. \tag{15}$$

**(b)** *We have*

$$\alpha_k(h_i) = [0 \leqslant i < k] \qquad \textit{for all } i \in \mathbb{Z}. \tag{16}$$

**(c)** *Let $\lambda$ be a partition. Define $h_\lambda$ as in Definition 3.1. Then,*

$$\alpha_k(h_\lambda) = [\lambda_i < k \textit{ for all } i]. \tag{17}$$

*(Here, "for all $i$" means "for all positive integers $i$".)*

*Proof of Lemma 3.6.* Since $\alpha_k$ is a $\mathbf{k}$-algebra homomorphism, we have $\alpha_k(1) = 1$ and $\alpha_k(0) = 0$. Parts **(a)** and **(b)** follow by combining this with Definition 3.5. Part **(c)** follows from Definition 3.5 as well, because if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, then $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$ and $[\lambda_1 < k] \cdot [\lambda_2 < k] \cdot \cdots \cdot [\lambda_\ell < k] = [\lambda_i < k \text{ for all } i]$. $\square$

(13) See also [16] for a detailed proof. (The main ingredients are [17, proof of Proposition 2.5.15] and [17, Exercise 2.5.11(a)].)

(14) We are using the Iverson bracket notation (see Convention 1) here.

3.5. Proof of Lemma 2.6. Lemma 2.6 can be proved directly using Laplace expansion of determinants. But the homomorphism $\alpha_k$ from Definition 3.5 allows for a slicker proof:

*Proof of Lemma 2.6.* Recall that $\alpha_k$ is a $\mathbf{k}$-algebra homomorphism. Thus, applying $\alpha_k$ to both sides of (14), we obtain

$$\begin{aligned}\alpha_k\left(s_{\lambda/\mu}\right) &= \det\left(\left(\alpha_k\left(h_{\lambda_i-\mu_j-i+j}\right)\right)_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}\right)\\ &= \det\left(\left([0\leqslant \lambda_i-\mu_j-i+j<k]\right)_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}\right) \qquad (18)\end{aligned}$$

(by (16)). Thus, $\det\left(\left([0\leqslant \lambda_i-\mu_j-i+j<k]\right)_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}\right)$ does not depend on the choice of $\ell$ (since $\alpha_k\left(s_{\lambda/\mu}\right)$ does not depend on the choice of $\ell$). This proves Lemma 2.6. $\square$

We record a restatement of (18) for subsequent use:

Lemma 3.7. *Let $k$ be a positive integer. Let $\lambda$ and $\mu$ be two partitions. Then, the homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ from Definition* 3.5 *satisfies*

$$\alpha_k\left(s_{\lambda/\mu}\right) = \operatorname{pet}_k(\lambda,\mu). \qquad (19)$$

3.6. Proof of Proposition 2.7.

*Proof of Proposition 2.7.* Write the partitions $\lambda$ and $\mu$ in the forms $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ and $\mu = (\mu_1,\mu_2,\ldots,\mu_\ell)$ for some $\ell\in\mathbb{N}$ [15]. The definition of $\operatorname{pet}_k(\lambda,\mu)$ yields

$$\begin{aligned}\operatorname{pet}_k(\lambda,\mu) &= \det\left(\left([0\leqslant \lambda_i-\mu_j-i+j<k]\right)_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}\right)\\ &= \det\left(\left([\mu_j-j\leqslant \lambda_i-i<\mu_j-j+k]\right)_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}\right) \qquad (20)\end{aligned}$$

(since the statement "$0\leqslant \lambda_i-\mu_j-i+j<k$" is equivalent to "$\mu_j-j\leqslant\lambda_i-i<\mu_j-j+k$" for any $i,j\in\{1,2,\ldots,\ell\}$).

Let $B$ be the $\ell\times\ell$-matrix $\left([\mu_j-j\leqslant \lambda_i-i<\mu_j-j+k]\right)_{1\leqslant i\leqslant \ell,\ 1\leqslant j\leqslant \ell}\in\mathbf{k}^{\ell\times\ell}$. Then, (20) rewrites as follows:

$$\operatorname{pet}_k(\lambda,\mu) = \det B. \qquad (21)$$

We will use the concept of Petrie matrices (see [13, Theorem 1]). Namely, a *Petrie matrix* is a matrix whose entries all belong to $\{0,1\}$ and such that the 1's in each column occur consecutively (i.e. as a contiguous block). In other words, a Petrie matrix is a matrix whose each column has the form

$$(\underbrace{0,0,\ldots,0}_{a\text{ zeroes}},\underbrace{1,1,\ldots,1}_{b\text{ ones}},\underbrace{0,0,\ldots,0}_{c\text{ zeroes}})^T \qquad (22)$$

for some nonnegative integers $a,b,c$ (where any of $a,b,c$ can be 0). For example, $\begin{pmatrix}0&0&1\\1&0&1\\0&0&0\end{pmatrix}$ is a Petrie matrix, but $\begin{pmatrix}0&1&0\\1&0&0\\0&1&1\end{pmatrix}$ is not.

A well-known result due to Fulkerson and Gross (first stated in [11, §8][16]) says that if a square matrix $A$ is a Petrie matrix, then

$$\det A\in\{-1,0,1\}. \qquad (23)$$

Now, we shall show that $B$ is a Petrie matrix.

[15]Such an $\ell$ can always be found, since each of $\lambda$ and $\mu$ has only finitely many nonzero entries.

[16]See [13, Theorem 1] for an explicit proof.

Indeed, fix some $j \in \{1, 2, \ldots, \ell\}$. We have $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell$ (since $\lambda$ is a partition) and thus $\lambda_1 - 1 > \lambda_2 - 2 > \cdots > \lambda_\ell - \ell$. Hence, the set of all $i \in \{1, 2, \ldots, \ell\}$ satisfying $\mu_j - j \leqslant \lambda_i - i < \mu_j - j + k$ is a (possibly empty) integer interval(17). Thus, the $j$-th column of the matrix $B$ has a contiguous (but possibly empty) block of 1's (in the rows corresponding to all these $i$'s), while all other entries of this column are 0. In other words, this column has the form (22) for some nonnegative integers $a, b, c$.

Since $j$ was arbitrary, we thus have shown that $B$ is a Petrie matrix. Therefore, (23) (applied to $A = B$) yields $\det B \in \{-1, 0, 1\}$. In view of (21), this proves Proposition 2.7. $\square$

### 3.7. Proof of Theorem 2.15.

We are now ready to prove Theorem 2.15:

*Proof of Theorem 2.15.* We shall use the notations $\mathbf{k}[[\mathbf{x}]]$, $\mathbf{k}[[\mathbf{x}, \mathbf{y}]]$, $\mathbf{x}$, $\mathbf{y}$, $f(\mathbf{x})$ and $f(\mathbf{y})$ introduced in Subsection 2.6. If $R$ is any commutative ring, then $R[[\mathbf{y}]]$ shall denote the ring $R[[y_1, y_2, y_3, \ldots]]$ of formal power series in the indeterminates $y_1, y_2, y_3, \ldots$ over the ring $R$. We will identify the ring $\mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ with the ring $(\mathbf{k}[[\mathbf{x}]])[[\mathbf{y}]] = (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[y_1, y_2, y_3, \ldots]]$. Note that $\Lambda \subseteq \mathbf{k}[[\mathbf{x}]]$ and thus $\Lambda[[\mathbf{y}]] \subseteq (\mathbf{k}[[\mathbf{x}]])[[\mathbf{y}]] = \mathbf{k}[[\mathbf{x}, \mathbf{y}]]$. We equip the rings $\mathbf{k}[[\mathbf{y}]]$, $\Lambda[[\mathbf{y}]]$ and $\mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ with the usual topologies that are defined on rings of power series, where $\Lambda$ itself is equipped with the discrete topology. This has the somewhat confusing consequence that $\Lambda[[\mathbf{y}]] \subseteq \mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ is an inclusion of rings but not of topological spaces; however, this will not cause us any trouble, since all infinite sums in $\Lambda[[\mathbf{y}]]$ we will consider (such as $\sum_{\lambda \in \mathrm{Par}} s_{\lambda/\mu}(\mathbf{x}) s_\lambda(\mathbf{y})$ and $\sum_{\lambda \in \mathrm{Par}} h_\lambda(\mathbf{x}) m_\lambda(\mathbf{y})$) will converge to the same value in either topology.

We consider both $\mathbf{k}[[\mathbf{y}]]$ and $\Lambda$ as subrings of $\Lambda[[\mathbf{y}]]$ (indeed, $\mathbf{k}[[\mathbf{y}]]$ embeds into $\Lambda[[\mathbf{y}]]$ because $\mathbf{k}$ is a subring of $\Lambda$, whereas $\Lambda$ embeds into $\Lambda[[\mathbf{y}]]$ because $\Lambda[[\mathbf{y}]]$ is a ring of power series over $\Lambda$).

In this proof, the word "monomial" may refer to a monomial in any set of variables (not necessarily in $x_1, x_2, x_3, \ldots$).

Recall the $\mathbf{k}$-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ from Definition 3.5. This $\mathbf{k}$-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ induces a $\mathbf{k}[[\mathbf{y}]]$-algebra homomorphism $\alpha_k[[\mathbf{y}]] : \Lambda[[\mathbf{y}]] \to \mathbf{k}[[\mathbf{y}]]$, which is given by the formula

$$(\alpha_k[[\mathbf{y}]]) \left( \sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} f_\mathfrak{n} \mathfrak{n} \right) = \sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} \alpha_k(f_\mathfrak{n}) \mathfrak{n}$$

for any family $(f_\mathfrak{n})_{\mathfrak{n} \text{ is a monomial in } y_1, y_2, y_3, \ldots}$ of elements of $\Lambda$. This induced $\mathbf{k}[[\mathbf{y}]]$-algebra homomorphism $\alpha_k[[\mathbf{y}]]$ is $\mathbf{k}[[\mathbf{y}]]$-linear and continuous (with respect to the usual topologies on the power series rings $\Lambda[[\mathbf{y}]]$ and $\mathbf{k}[[\mathbf{y}]]$), and thus preserves infinite $\mathbf{k}[[\mathbf{y}]]$-linear combinations. Moreover, it extends $\alpha_k$ (that is, for any $f \in \Lambda$, we have $(\alpha_k[[\mathbf{y}]])(f) = \alpha_k(f)$).

Recall the skew Schur functions $s_{\lambda/\mu}$ defined in Subsection 3.2. Also, recall the symmetric functions $h_\lambda$ defined in Definition 3.1. Theorem 3.4 yields

$$\begin{aligned}\sum_{\lambda \in \mathrm{Par}} s_{\lambda/\mu}(\mathbf{x}) s_\lambda(\mathbf{y}) &= s_\mu(\mathbf{y}) \cdot \sum_{\lambda \in \mathrm{Par}} h_\lambda(\mathbf{x}) m_\lambda(\mathbf{y}) = \sum_{\lambda \in \mathrm{Par}} s_\mu(\mathbf{y}) h_\lambda(\mathbf{x}) m_\lambda(\mathbf{y}) \\ &= \sum_{\lambda \in \mathrm{Par}} s_\mu(\mathbf{y}) m_\lambda(\mathbf{y}) \underbrace{h_\lambda(\mathbf{x})}_{= h_\lambda} = \sum_{\lambda \in \mathrm{Par}} s_\mu(\mathbf{y}) m_\lambda(\mathbf{y}) h_\lambda.\end{aligned}$$

(17) An *integer interval* means a subset of $\mathbb{Z}$ that has the form $\{a, a+1, \ldots, b\}$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. (If $a > b$, then this is the empty set.)

Comparing this with

$$\sum_{\lambda\in\operatorname{Par}} s_{\lambda/\mu}(\mathbf{x})\, s_{\lambda}(\mathbf{y}) = \sum_{\lambda\in\operatorname{Par}} s_{\lambda}(\mathbf{y})\, \underbrace{s_{\lambda/\mu}(\mathbf{x})}_{=s_{\lambda/\mu}} = \sum_{\lambda\in\operatorname{Par}} s_{\lambda}(\mathbf{y})\, s_{\lambda/\mu},$$

we obtain

$$\sum_{\lambda\in\operatorname{Par}} s_{\lambda}(\mathbf{y})\, s_{\lambda/\mu} = \sum_{\lambda\in\operatorname{Par}} s_{\mu}(\mathbf{y})\, m_{\lambda}(\mathbf{y})\, h_{\lambda}.$$

Consider this as an equality in the ring $\Lambda[[\mathbf{y}]] = \Lambda[[y_1, y_2, y_3, \ldots]]$. Apply the map $\alpha_k[[\mathbf{y}]] : \Lambda[[\mathbf{y}]] \to \mathbf{k}[[\mathbf{y}]]$ to both sides of this equality. We obtain

$$(\alpha_k[[\mathbf{y}]])\left(\sum_{\lambda\in\operatorname{Par}} s_{\lambda}(\mathbf{y})\, s_{\lambda/\mu}\right) = (\alpha_k[[\mathbf{y}]])\left(\sum_{\lambda\in\operatorname{Par}} s_{\mu}(\mathbf{y})\, m_{\lambda}(\mathbf{y})\, h_{\lambda}\right).$$

Comparing this with

$$\begin{aligned}
&(\alpha_k[[\mathbf{y}]])\left(\sum_{\lambda\in\operatorname{Par}} s_{\lambda}(\mathbf{y})\, s_{\lambda/\mu}\right) \\
&= \sum_{\lambda\in\operatorname{Par}} s_{\lambda}(\mathbf{y}) \cdot \underbrace{(\alpha_k[[\mathbf{y}]])\left(s_{\lambda/\mu}\right)}_{\substack{=\alpha_k\left(s_{\lambda/\mu}\right) \\ (\text{since } \alpha_k[[\mathbf{y}]] \text{ extends } \alpha_k)}} \\
&\qquad (\text{since the map } \alpha_k[[\mathbf{y}]] \text{ preserves infinite } \mathbf{k}[[\mathbf{y}]]\text{-linear combinations}) \\
&= \sum_{\lambda\in\operatorname{Par}} s_{\lambda}(\mathbf{y}) \cdot \underbrace{\alpha_k\left(s_{\lambda/\mu}\right)}_{\substack{=\operatorname{pet}_k(\lambda,\mu) \\ (\text{by } (19))}} = \sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_k(\lambda,\mu) \cdot s_{\lambda}(\mathbf{y}),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_k(\lambda,\mu) \cdot s_{\lambda}(\mathbf{y}) \\
&= (\alpha_k[[\mathbf{y}]])\left(\sum_{\lambda\in\operatorname{Par}} s_{\mu}(\mathbf{y})\, m_{\lambda}(\mathbf{y})\, h_{\lambda}\right) = \sum_{\lambda\in\operatorname{Par}} s_{\mu}(\mathbf{y})\, m_{\lambda}(\mathbf{y}) \underbrace{(\alpha_k[[\mathbf{y}]])(h_{\lambda})}_{\substack{=\alpha_k(h_{\lambda}) \\ (\text{since } \alpha_k[[\mathbf{y}]] \text{ extends } \alpha_k)}} \\
&\qquad (\text{since the map } \alpha_k[[\mathbf{y}]] \text{ preserves infinite } \mathbf{k}[[\mathbf{y}]]\text{-linear combinations}) \\
&= \sum_{\lambda\in\operatorname{Par}} s_{\mu}(\mathbf{y})\, m_{\lambda}(\mathbf{y}) \underbrace{\alpha_k(h_{\lambda})}_{\substack{=[\lambda_i<k \text{ for all } i] \\ (\text{by } (17))}} = \sum_{\lambda\in\operatorname{Par}} [\lambda_i < k \text{ for all } i] \cdot s_{\mu}(\mathbf{y})\, m_{\lambda}(\mathbf{y}) \\
&= \sum_{\substack{\lambda\in\operatorname{Par}; \\ \lambda_i<k \text{ for all } i}} s_{\mu}(\mathbf{y})\, m_{\lambda}(\mathbf{y})
\end{aligned}$$

(since the $[\lambda_i < k \text{ for all } i]$ factor inside the sum causes all addends to vanish except for the addends that satisfy "$\lambda_i < k$ for all $i$"). Renaming the indeterminates $\mathbf{y} =$

$(y_1, y_2, y_3, \ldots)$ as $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ on both sides of this equality, we obtain

$$\sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k (\lambda, \mu) \cdot s_\lambda (\mathbf{x}) = \sum_{\substack{\lambda \in \operatorname{Par}; \\ \lambda_i < k \text{ for all } i}} \underbrace{s_\mu (\mathbf{x})}_{= s_\mu} \underbrace{m_\lambda (\mathbf{x})}_{= m_\lambda} = \sum_{\substack{\lambda \in \operatorname{Par}; \\ \lambda_i < k \text{ for all } i}} s_\mu m_\lambda$$
$$= s_\mu \cdot \underbrace{\sum_{\substack{\lambda \in \operatorname{Par}; \\ \lambda_i < k \text{ for all } i}} m_\lambda}_{\substack{= G(k) \\ \text{(by Proposition 2.3 } \textbf{(b)}\text{)}}} = s_\mu \cdot G(k) = G(k) \cdot s_\mu .$$

This proves Theorem 2.15. □

A second proof of Theorem 2.15 can be found in [15].

3.8. Proofs of Corollary 2.16, Theorem 2.8 and Corollary 2.9. Having proved Theorem 2.15, we can now obtain Corollary 2.16, Theorem 2.8 and Corollary 2.9 as easy consequences:

*Proof of Corollary 2.16.* Proposition 2.3**(a)** yields that the $m$-th degree homogeneous component of $G(k)$ is $G(k, m)$. Hence, the $(m + |\mu|)$-th degree homogeneous component of $G(k) \cdot s_\mu$ is $G(k, m) \cdot s_\mu$ (because $s_\mu$ is homogeneous of degree $|\mu|$).

Theorem 2.15 yields

$$G(k) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k (\lambda, \mu) s_\lambda .$$

Taking the $(m + |\mu|)$-th degree homogeneous components on both sides of this equality, we obtain

$$G(k, m) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}_{m + |\mu|}} \operatorname{pet}_k (\lambda, \mu) s_\lambda$$

(because each Schur function $s_\lambda$ is homogeneous of degree $|\lambda|$, whereas the $(m + |\mu|)$-th degree homogeneous component of $G(k) \cdot s_\mu$ is $G(k, m) \cdot s_\mu$). This proves Corollary 2.16. □

*Proof of Theorem 2.8.* Theorem 2.15 (applied to $\mu = \varnothing$) yields

$$G(k) \cdot s_\varnothing = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k (\lambda, \varnothing) s_\lambda .$$

Comparing this with $G(k) \cdot \underbrace{s_\varnothing}_{=1} = G(k)$, we obtain

$$G(k) = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k (\lambda, \varnothing) s_\lambda .$$

This proves Theorem 2.8. □

*Proof of Corollary 2.9.* Corollary 2.16 (applied to $\mu = \varnothing$) yields

$$G(k, m) \cdot s_\varnothing = \sum_{\lambda \in \operatorname{Par}_{m + |\varnothing|}} \operatorname{pet}_k (\lambda, \varnothing) s_\lambda .$$

In view of $G(k, m) \cdot \underbrace{s_\varnothing}_{=1} = G(k, m)$ and $m + \underbrace{|\varnothing|}_{=0} = m$, we can rewrite this as

$$G(k, m) = \sum_{\lambda \in \operatorname{Par}_m} \operatorname{pet}_k (\lambda, \varnothing) s_\lambda .$$

This proves Corollary 2.9. □

3.9. Proof of Theorem 2.13. Our proof of Theorem 2.13 will depend on two lemmas about determinants:

Lemma 3.8. *Let $m \in \mathbb{N}$. Let $R$ be a commutative ring. Let $(a_{i,j})_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant m} \in R^{m\times m}$ be an $m\times m$-matrix.*

**(a)** *If $\tau$ is any permutation of $\{1,2,\ldots,m\}$, then*
$$\det\left(\left(a_{\tau(i),j}\right)_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant m}\right) = (-1)^{\tau}\cdot\det\left(\left(a_{i,j}\right)_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant m}\right).$$
*Here, $(-1)^{\tau}$ denotes the sign of the permutation $\tau$.*

**(b)** *Let $u_1,u_2,\ldots,u_m$ be $m$ elements of $R$. Let $v_1,v_2,\ldots,v_m$ be $m$ elements of $R$. Then,*
$$\det\left(\left(u_iv_ja_{i,j}\right)_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant m}\right) = \left(\prod_{i=1}^{m}(u_iv_i)\right)\cdot\det\left(\left(a_{i,j}\right)_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant m}\right).$$

*Proof of Lemma 3.8.* **(a)** This is just the well-known fact that if the rows of a square matrix are permuted using a permutation $\tau$, then the determinant of this matrix gets multiplied by $(-1)^{\tau}$.

**(b)** This follows easily from the definition of the determinant. ☐

Lemma 3.9. *Let $k$ be a positive integer. Let $\gamma_1,\gamma_2,\ldots,\gamma_{k-1}$ be $k-1$ elements of the set $\{1,2,\ldots,k\}$. Let $G$ be the $(k-1)\times(k-1)$-matrix*
$$\left((-1)^{(\gamma_i+j)\%k}\left[(\gamma_i+j)\%k\in\{0,1\}\right]\right)_{1\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1}.$$

**(a)** *If the $k-1$ numbers $\gamma_1,\gamma_2,\ldots,\gamma_{k-1}$ are not distinct, then*
$$\det G = 0.$$

**(b)** *If $\gamma_1>\gamma_2>\cdots>\gamma_{k-1}$, then*
$$\det G = (-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.$$

**(c)** *Assume that the $k-1$ numbers $\gamma_1,\gamma_2,\ldots,\gamma_{k-1}$ are distinct. Let*
$$g = \left|\left\{(i,j)\in\{1,2,\ldots,k-1\}^2 \mid i<j \text{ and } \gamma_i<\gamma_j\right\}\right|.$$
*Then,*
$$\det G = (-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.$$

*Proof of Lemma 3.9.* **(a)** Assume that $\gamma_1,\gamma_2,\ldots,\gamma_{k-1}$ are not distinct. Then, the matrix $G$ has two equal rows. Thus, $\det G=0$. This proves Lemma 3.9 **(a)**.

**(b)** Assume that $\gamma_1>\gamma_2>\cdots>\gamma_{k-1}$. Thus, $\{\gamma_1,\gamma_2,\ldots,\gamma_{k-1}\}$ is a $(k-1)$-element subset of $\{1,2,\ldots,k\}$.

Hence, $\{\gamma_1,\gamma_2,\ldots,\gamma_{k-1}\}=\{1,2,\ldots,k\}\setminus\{u\}$ for some $u\in\{1,2,\ldots,k\}$ (since any $(k-1)$-element subset of $\{1,2,\ldots,k\}$ has such a form). Consider this $u$. From $\{\gamma_1,\gamma_2,\ldots,\gamma_{k-1}\}=\{1,2,\ldots,k\}\setminus\{u\}$, we conclude that $\gamma_1,\gamma_2,\ldots,\gamma_{k-1}$ are the $k-1$ elements of the set $\{1,2,\ldots,k\}\setminus\{u\}$, listed in decreasing order (since $\gamma_1>\gamma_2>\cdots>\gamma_{k-1}$). In other words,

$$(\gamma_1,\gamma_2,\ldots,\gamma_{k-1}) = (k,k-1,\ldots,\widehat{u},\ldots,2,1), \tag{24}$$

where the "hat" over the $u$ signifies that $u$ is omitted from the list (i.e. the expression "$(k,k-1,\ldots,\widehat{u},\ldots,2,1)$" is understood to mean the $(k-1)$-element list $(k,k-1,\ldots,u+1,u-1,\ldots,2,1)$, which contains all $k$ integers from 1 to $k$ in decreasing order except for $u$).

Now, we claim that

$$(-1)^{(\gamma_i+j)\%k}\left[(\gamma_i+j)\%k\in\{0,1\}\right] \\ = (-1)^{\gamma_i+j-k}\left[\gamma_i+j\in\{k,k+1\}\right] \tag{25}$$

for any $i\in\{1,2,\ldots,k-1\}$ and $j\in\{1,2,\ldots,k-1\}$.

[*Proof of (*25*):* Let $i\in\{1,2,\ldots,k-1\}$ and $j\in\{1,2,\ldots,k-1\}$. We must prove the equality (25).

From $i\in\{1,2,\ldots,k-1\}$, we obtain $1\leqslant i\leqslant k-1$ and thus $k-1\geqslant 1$. Thus, $k>k-1\geqslant 1$. Hence, $k+1<2k$, so that $(k+1)\%k=1$.

If we don't have $(\gamma_i+j)\%k\in\{0,1\}$, then both truth values $\left[(\gamma_i+j)\%k\in\{0,1\}\right]$ and $\left[\gamma_i+j\in\{k,k+1\}\right]$ are 0 (indeed, the statement "$\gamma_i+j\in\{k,k+1\}$" is false, since otherwise it would imply $(\gamma_i+j)\%k\in\{0,1\}$), and therefore the equality (25) simplifies to $(-1)^{(\gamma_i+j)\%k}0=(-1)^{\gamma_i+j-k}0$ in this case, which is obviously true. Hence, for the rest of this proof, we WLOG assume that we do have $(\gamma_i+j)\%k\in\{0,1\}$.

Adding $\gamma_i\in\{1,2,\ldots,k\}$ with $j\in\{1,2,\ldots,k-1\}$, we find that $\gamma_i+j\in\{2,3,\ldots,2k-1\}$. Thus, $\gamma_i+j$ is an integer in the set $\{2,3,\ldots,2k-1\}$ that leaves the remainder 0 or 1 upon division by $k$ (since $(\gamma_i+j)\%k\in\{0,1\}$). But the only such integers are $k$ and $k+1$. Hence, we obtain $\gamma_i+j\in\{k,k+1\}\subseteq\{k,k+1,\ldots,2k-1\}$ (since $k+1<2k$), and therefore $(\gamma_i+j)\%k=\gamma_i+j-k$. Thus,

$$\underbrace{(-1)^{(\gamma_i+j)\%k}}_{\substack{=(-1)^{\gamma_i+j-k}\\ (\text{since } (\gamma_i+j)\%k=\gamma_i+j-k)}}\underbrace{\left[(\gamma_i+j)\%k\in\{0,1\}\right]}_{\substack{=1\\ (\text{since } (\gamma_i+j)\%k\in\{0,1\})}}=(-1)^{\gamma_i+j-k}.$$

Comparing this with

$$(-1)^{\gamma_i+j-k}\underbrace{\left[\gamma_i+j\in\{k,k+1\}\right]}_{\substack{=1\\ (\text{since } \gamma_i+j\in\{k,k+1\})}}=(-1)^{\gamma_i+j-k},$$

we obtain

$$(-1)^{(\gamma_i+j)\%k}\left[(\gamma_i+j)\%k\in\{0,1\}\right]=(-1)^{\gamma_i+j-k}\left[\gamma_i+j\in\{k,k+1\}\right].$$

This proves (25).]

Now, $G$ is a $(k-1)\times(k-1)$-matrix. For each $i\in\{1,2,\ldots,k-1\}$ and $j\in\{1,2,\ldots,k-1\}$, we have

$$\begin{aligned}
&(\text{the } (i,j)\text{-th entry of } G)\\
&=(-1)^{(\gamma_i+j)\%k}\left[(\gamma_i+j)\%k\in\{0,1\}\right] \qquad (\text{by the definition of } G)\\
&=(-1)^{\gamma_i+j-k}\left[\gamma_i+j\in\{k,k+1\}\right] \qquad (\text{by (25)})\\
&=\begin{cases}1, & \text{if } \gamma_i+j=k;\\ -1, & \text{if } \gamma_i+j=k+1;\\ 0, & \text{otherwise}\end{cases} \quad =\begin{cases}1, & \text{if } j=k-\gamma_i;\\ -1, & \text{if } j=k-\gamma_i+1;\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Thus, we can explicitly describe the matrix $G$ as follows: For each $i\in\{1,2,\ldots,k-1\}$, the $i$-th row of $G$ has an entry equal to 1 in position $k-\gamma_i$ if $k-\gamma_i>0$, and an entry equal to $-1$ in position $k-\gamma_i+1$ if $k-\gamma_i+1<k$; all remaining entries of this row

are 0. Recalling (24), we thus see that $G$ has the following form:[18]

$$G = \left(\begin{array}{ccccc|ccccc}
-1 &&&&&&&&& \\
1 & -1 &&&&&&&& \\
& 1 & -1 &&&&&&& \\
&& \ddots & \ddots &&&&&& \\
&&& 1 & -1 &&&&& \\
\hline
&&&&& 1 & -1 &&& \\
&&&&&& 1 & -1 && \\
&&&&&&& 1 & -1 & \\
&&&&&&&& \ddots & \ddots \\
&&&&&&&&& 1
\end{array}\right),$$

where the horizontal bar separates the $(k-u)$-th row from the $(k-u+1)$-st row, while the vertical bar separates the $(k-u)$-th column from the $(k-u+1)$-st column. Thus, $G$ can be written as a block-diagonal matrix

$$G = \begin{pmatrix} A & 0_{(k-u)\times(u-1)} \\ 0_{(u-1)\times(k-u)} & B \end{pmatrix}, \tag{26}$$

where $A$ is a lower-triangular $(k-u)\times(k-u)$-matrix with all diagonal entries equal to $-1$, and where $B$ is an upper-triangular $(u-1)\times(u-1)$-matrix with all diagonal entries equal to 1. Since the determinant of a block-diagonal matrix equals the product of the determinants of its diagonal blocks, we thus conclude that $\det G = \det A \cdot \det B$.

However, $\det A = (-1)^{k-u}$ (since $A$ is a lower-triangular $(k-u)\times(k-u)$-matrix with all diagonal entries equal to $-1$) and $\det B = 1$ (since $B$ is an upper-triangular $(u-1)\times(u-1)$-matrix with all diagonal entries equal to 1). Hence, this becomes

$$\det G = \underbrace{\det A}_{=(-1)^{k-u}} \cdot \underbrace{\det B}_{=1} = (-1)^{k-u}. \tag{27}$$

But (24) yields

$$\begin{aligned}\gamma_1+\gamma_2+\cdots+\gamma_{k-1} &= k+(k-1)+\cdots+\widehat{u}+\cdots+2+1 \\ &= (1+2+\cdots+k)-u = (1+2+\cdots+(k-1))+k-u.\end{aligned}$$

Solving this for $k-u$, we find

$$k-u = (\gamma_1+\gamma_2+\cdots+\gamma_{k-1}) - (1+2+\cdots+(k-1)).$$

Hence, (27) rewrites as

$$\det G = (-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.$$

This proves Lemma 3.9 **(b)**.

**(c)** Assume that the $k-1$ numbers $\gamma_1,\gamma_2,\ldots,\gamma_{k-1}$ are distinct. Then, there exists a unique permutation $\sigma$ of $\{1,2,\ldots,k-1\}$ such that $\gamma_{\sigma(1)} > \gamma_{\sigma(2)} > \cdots > \gamma_{\sigma(k-1)}$. Consider this $\sigma$.

Let $\tau$ denote the permutation $\sigma^{-1}$.

Let $\delta_1,\delta_2,\ldots,\delta_{k-1}$ denote the $k-1$ elements $\gamma_{\sigma(1)},\gamma_{\sigma(2)},\ldots,\gamma_{\sigma(k-1)}$ of $\{1,2,\ldots,k\}$. Thus, for each $j\in\{1,2,\ldots,k-1\}$, we have

$$\delta_j = \gamma_{\sigma(j)}. \tag{28}$$

Hence, the chain of inequalities $\gamma_{\sigma(1)} > \gamma_{\sigma(2)} > \cdots > \gamma_{\sigma(k-1)}$ (which is true) can be rewritten as $\delta_1 > \delta_2 > \cdots > \delta_{k-1}$.

[18] Empty cells are understood to have entry 0.

Moreover, from (28), we obtain

$$
\begin{aligned}
\delta_1+\delta_2+\cdots+\delta_{k-1} &= \gamma_{\sigma(1)}+\gamma_{\sigma(2)}+\cdots+\gamma_{\sigma(k-1)} \\
&= \gamma_1+\gamma_2+\cdots+\gamma_{k-1}
\end{aligned} \tag{29}
$$

(since $\sigma$ is a permutation of $\{1,2,\ldots,k-1\}$).

Moreover, for each $i\in\{1,2,\ldots,k-1\}$, we have

$$
\begin{aligned}
\delta_{\tau(i)} &= \gamma_{\sigma(\tau(i))} && (\text{by (28), applied to } j=\tau(i)) \\
&= \gamma_i && \left(\text{since } \tau=\sigma^{-1} \text{ entails } \sigma(\tau(i))=i\right).
\end{aligned} \tag{30}
$$

Recall that an *inversion* of the permutation $\tau$ is defined to be a pair $(i,j)$ of elements of $\{1,2,\ldots,k-1\}$ satisfying $i<j$ and $\tau(i)>\tau(j)$. Hence, the inversions of $\tau$ are precisely the pairs $(i,j)$ of elements of $\{1,2,\ldots,k-1\}$ satisfying $i<j$ and $\delta_{\tau(i)}<\delta_{\tau(j)}$ (since $\delta_1>\delta_2>\cdots>\delta_{k-1}$ shows that the statement "$\tau(i)>\tau(j)$" is equivalent to "$\delta_{\tau(i)}<\delta_{\tau(j)}$"). In other words, the inversions of $\tau$ are precisely the pairs $(i,j)$ of elements of $\{1,2,\ldots,k-1\}$ satisfying $i<j$ and $\gamma_i<\gamma_j$ (since (30) shows that any $i,j\in\{1,2,\ldots,k-1\}$ satisfy $\gamma_i=\delta_{\tau(i)}$ and $\gamma_j=\delta_{\tau(j)}$). Hence, the number of inversions of $\tau$ equals

$$\left|\left\{(i,j)\in\{1,2,\ldots,k-1\}^2 \ \mid\ i<j \text{ and } \gamma_i<\gamma_j\right\}\right| = g$$

(by the definition of $g$). Therefore, the sign $(-1)^\tau$ of the permutation $\tau$ is $(-1)^g$ (since the sign of a permutation is defined to be $(-1)^m$, where $m$ is the number of inversions of this permutation). Thus, we have shown that $(-1)^\tau=(-1)^g$.

Let $H$ be the $(k-1)\times(k-1)$-matrix

$$\left((-1)^{(\delta_i+j)\%k}\left[(\delta_i+j)\%k\in\{0,1\}\right]\right)_{1\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1}.$$

Then, we can apply Lemma 3.9 **(b)** to $\delta_i$ and $H$ instead of $\gamma_i$ and $G$ (since $\delta_1,\delta_2,\ldots,\delta_{k-1}$ are $k-1$ elements of $\{1,2,\ldots,k\}$ and satisfy $\delta_1>\delta_2>\cdots>\delta_{k-1}$). We thus obtain

$$\det H = (-1)^{(\delta_1+\delta_2+\cdots+\delta_{k-1})-(1+2+\cdots+(k-1))} = (-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}$$

(by (29)).

But the definition of $G$ yields

$$
\begin{aligned}
G &= \left((-1)^{(\gamma_i+j)\%k}\left[(\gamma_i+j)\%k\in\{0,1\}\right]\right)_{1\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1} \\
&= \left((-1)^{\left(\delta_{\tau(i)}+j\right)\%k}\left[\left(\delta_{\tau(i)}+j\right)\%k\in\{0,1\}\right]\right)_{1\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1}
\end{aligned}
$$

(since (30) yields $\gamma_i = \delta_{\tau(i)}$ for each $i \in \{1, 2, \ldots, k-1\}$). Hence,

$$\begin{aligned}
\det G &= \det\left(\left((-1)^{\left(\delta_{\tau(i)}+j\right)\%k}\left[\left(\delta_{\tau(i)}+j\right)\%k \in \{0,1\}\right]\right)_{1\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1}\right) \\
&= \underbrace{(-1)^{\tau}}_{=(-1)^g} \det\left(\underbrace{\left((-1)^{(\delta_i+j)\%k}\left[(\delta_i+j)\%k \in \{0,1\}\right]\right)_{1\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1}}_{\substack{=H \\ \text{(by the definition of } H\text{)}}}\right) \\
&\qquad \left(\begin{array}{c}\text{by Lemma 3.8 } \textbf{(a)}\text{, applied to } m=k-1 \text{ and } R=\mathbf{k} \\ \text{and } a_{i,j} = (-1)^{(\delta_i+j)\%k}\left[(\delta_i+j)\%k \in \{0,1\}\right]\end{array}\right) \\
&= (-1)^g \underbrace{\det H}_{=(-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}} \\
&= (-1)^g (-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))} \\
&= (-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.
\end{aligned}$$

This proves Lemma 3.9 **(c)**. □

Next, we recall a well-known property of symmetric functions (see, e.g. [25, proof of Theorem 7.6.1] or [17, (2.4.3)]):

LEMMA 3.10. *Consider the ring* $\Lambda[[t]]$ *of formal power series in one indeterminate* $t$ *over* $\Lambda$. *In this ring, we have*

$$1 = \left(\sum_{n\geqslant 0}(-1)^n e_n t^n\right)\left(\sum_{n\geqslant 0} h_n t^n\right). \tag{31}$$

Next, we shall prove yet another evaluation of the homomorphism $\alpha_k$:

LEMMA 3.11. *Let* $k$ *be a positive integer such that* $k > 1$. *Consider the* $\mathbf{k}$*-algebra homomorphism* $\alpha_k : \Lambda \to \mathbf{k}$ *from Definition* 3.5. *Also, recall Convention* 1. *Let* $r$ *be an integer such that* $r > -k+1$. *Then,*

$$\alpha_k(e_r) = (-1)^{r+r\%k}\left[r\%k \in \{0,1\}\right]. \tag{32}$$

*Proof of Lemma 3.11.* Consider the ring $\Lambda[[t]]$ of formal power series in one indeterminate $t$ over $\Lambda$. Consider also the analogous ring $\mathbf{k}[[t]]$ over $\mathbf{k}$.

The map $\alpha_k : \Lambda \to \mathbf{k}$ is a $\mathbf{k}$-algebra homomorphism. Hence, it induces a $\mathbf{k}[[t]]$-algebra homomorphism

$$\alpha_k[[t]] : \Lambda[[t]] \to \mathbf{k}[[t]]$$

that sends each formal power series $\sum_{n\geqslant 0} a_n t^n \in \Lambda[[t]]$ (with $a_n \in \Lambda$) to $\sum_{n\geqslant 0} \alpha_k(a_n) t^n$. Consider this $\alpha_k[[t]]$.

Applying the map $\alpha_k[[t]]$ to both sides of the equality (31), we obtain

$$\begin{aligned}
(\alpha_k[[t]])(1) &= (\alpha_k[[t]])\left(\left(\sum_{n\geqslant 0}(-1)^n e_n t^n\right)\left(\sum_{n\geqslant 0}h_n t^n\right)\right)\\
&= \underbrace{(\alpha_k[[t]])\left(\sum_{n\geqslant 0}(-1)^n e_n t^n\right)}_{\substack{=\sum_{n\geqslant 0}\alpha_k((-1)^n e_n)t^n\\ \text{(by the definition of }\alpha_k[[t]])}}\cdot\underbrace{(\alpha_k[[t]])\left(\sum_{n\geqslant 0}h_n t^n\right)}_{\substack{=\sum_{n\geqslant 0}\alpha_k(h_n)t^n\\ \text{(by the definition of }\alpha_k[[t]])}}\\
&\qquad\qquad(\text{since }\alpha_k[[t]]\text{ is a }\mathbf{k}[[t]]\text{-algebra homomorphism})\\
&= \left(\sum_{n\geqslant 0}\underbrace{\alpha_k((-1)^n e_n)}_{\substack{=(-1)^n\alpha_k(e_n)\\ \text{(since }\alpha_k\text{ is }\mathbf{k}\text{-linear)}}}t^n\right)\cdot\left(\sum_{n\geqslant 0}\underbrace{\alpha_k(h_n)}_{\substack{=[n<k]\\ \text{(by (15))}}}t^n\right)\\
&= \left(\sum_{n\geqslant 0}(-1)^n\alpha_k(e_n)t^n\right)\cdot\underbrace{\left(\sum_{n\geqslant 0}[n<k]\,t^n\right)}_{=t^0+t^1+\cdots+t^{k-1}=\dfrac{1-t^k}{1-t}}\\
&= \left(\sum_{n\geqslant 0}(-1)^n\alpha_k(e_n)t^n\right)\cdot\frac{1-t^k}{1-t}.
\end{aligned}$$

Comparing this with

$$(\alpha_k[[t]])(1)=1\qquad(\text{since }\alpha_k[[t]]\text{ is a }\mathbf{k}[[t]]\text{-algebra homomorphism}),$$

we obtain

$$\left(\sum_{n\geqslant 0}(-1)^n\alpha_k(e_n)t^n\right)\cdot\frac{1-t^k}{1-t}=1.$$

Hence,

$$\begin{aligned}
\sum_{n\geqslant 0}(-1)^n\alpha_k(e_n)t^n &= \frac{1-t}{1-t^k}=(1-t)\cdot\underbrace{\frac{1}{1-t^k}}_{=1+t^k+t^{2k}+t^{3k}+\cdots}\\
&= (1-t)\cdot\left(1+t^k+t^{2k}+t^{3k}+\cdots\right)\\
&= 1-t+t^k-t^{k+1}+t^{2k}-t^{2k+1}+t^{3k}-t^{3k+1}\pm\cdots\\
&= \sum_{n\geqslant 0}(-1)^{n\%k}[n\%k\in\{0,1\}]\,t^n
\end{aligned}$$

(here, we have used that $k>1$, since for $k=1$ there would be cancellations in the sum $1-t+t^k-t^{k+1}+t^{2k}-t^{2k+1}+t^{3k}-t^{3k+1}\pm\cdots$). Comparing coefficients before $t^m$ on both sides of this equality, we obtain

$$(-1)^m\alpha_k(e_m)=(-1)^{m\%k}[m\%k\in\{0,1\}]\tag{33}$$

for each $m\in\mathbb{N}$.

Multiplying both sides of this equality by $(-1)^m$, we obtain

$$\alpha_k(e_m)=(-1)^{m+m\%k}[m\%k\in\{0,1\}].\tag{34}$$

We must prove that

$$\alpha_k(e_r) = (-1)^{r+r\%k}[r\%k \in \{0,1\}].$$

If $r \in \mathbb{N}$, then this follows by applying (34) to $m = r$. Hence, for the rest of this proof, we WLOG assume that $r \notin \mathbb{N}$. Thus, $r$ is negative, so that $r \in \{-k+2, -k+3, \ldots, -1\}$ (since $r > -k+1$). Hence, $r\%k \in \{2, 3, \ldots, k-1\}$, so that $r\%k \notin \{0,1\}$. Consequently, $[r\%k \in \{0,1\}] = 0$. Also, $e_r = 0$ (since $r$ is negative) and thus $\alpha_k(e_r) = \alpha_k(0) = 0$. Comparing this with $(-1)^{r+r\%k}\underbrace{[r\%k \in \{0,1\}]}_{=0} = 0$, we obtain $\alpha_k(e_r) = (-1)^{r+r\%k}[r\%k \in \{0,1\}]$. This concludes the proof of Lemma 3.11. □

*Proof of Theorem 2.13.* **(a)** Assume that $\mu_k \neq 0$. Thus, the partition $\mu$ has at least $k$ entries. But we have $\mu = \lambda^t$, and thus (by a known and easy fact) the partition $\mu$ has precisely $\lambda_1$ entries. Reconciling the previous two sentences, we see that $\lambda_1 \geqslant k$. Thus, Proposition 2.11 yields $\operatorname{pet}_k(\lambda, \varnothing) = 0$. This proves Theorem 2.13 **(a)**.

Now, let us prepare for the proof of parts **(b)** and **(c)**.

Consider the $\mathbf{k}$-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ from Definition 3.5.

We have $s_\lambda = s_{\lambda/\varnothing}$, so that

$$\alpha_k(s_\lambda) = \alpha_k\left(s_{\lambda/\varnothing}\right) = \operatorname{pet}_k(\lambda, \varnothing) \tag{35}$$

(by (19), applied to $\varnothing$ instead of $\mu$).

For each $i \in \{1, 2, \ldots, k-1\}$, we have $\gamma_i \in \{1, 2, \ldots, k\}$ (by (8), since the remainder $(\beta_i - 1)\%k$ clearly belongs to $\{0, 1, \ldots, k-1\}$). In other words, $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are $k-1$ elements of the set $\{1, 2, \ldots, k\}$.

Assume that $\mu_k = 0$. Thus, $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$ (since $\mu \in \operatorname{Par}$).

It is known that $(\lambda^t)^t = \lambda$. Hence, $\lambda = (\lambda^t)^t = \mu^t$ (since $\lambda^t = \mu$). Therefore,

$$s_\lambda = s_{\mu^t} = \det\left(\left(e_{\mu_i - i + j}\right)_{1 \leqslant i \leqslant k-1,\ 1 \leqslant j \leqslant k-1}\right)$$

(by (6), applied to $\mu$ and $k-1$ instead of $\lambda$ and $\ell$), because $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$. This rewrites as

$$s_\lambda = \det\left(\left(e_{\beta_i + j}\right)_{1 \leqslant i \leqslant k-1,\ 1 \leqslant j \leqslant k-1}\right)$$

(since (7) yields $\mu_i - i = \beta_i$ for each $i \in \{1, 2, \ldots, k-1\}$). Applying the map $\alpha_k$ to both sides of this equality, we find

$$\alpha_k(s_\lambda) = \alpha_k\left(\det\left(\left(e_{\beta_i + j}\right)_{1 \leqslant i \leqslant k-1,\ 1 \leqslant j \leqslant k-1}\right)\right) = \det\left(\left(\alpha_k\left(e_{\beta_i + j}\right)\right)_{1 \leqslant i \leqslant k-1,\ 1 \leqslant j \leqslant k-1}\right)$$

(since $\alpha_k$ is a $\mathbf{k}$-algebra homomorphism, and thus commutes with taking determinants of matrices). Comparing this with (35), we obtain

$$\operatorname{pet}_k(\lambda, \varnothing) = \det\left(\left(\alpha_k\left(e_{\beta_i + j}\right)\right)_{1 \leqslant i \leqslant k-1,\ 1 \leqslant j \leqslant k-1}\right). \tag{36}$$

But each $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$ satisfy $k > 1$ [19] and

$$\underbrace{\beta_i}_{\substack{=\mu_i - i \\ \text{(by (7))}}} + j = \underbrace{\mu_i}_{\geqslant 0} - \underbrace{i}_{\leqslant k-1} + \underbrace{j}_{>0} > 0 - (k-1) + 0 = -k+1$$

and thus

$$\alpha_k\left(e_{\beta_i + j}\right) = (-1)^{(\beta_i + j) + (\beta_i + j)\%k}\left[(\beta_i + j)\%k \in \{0,1\}\right] \tag{37}$$

(by (32), applied to $r = \beta_i + j$).

[19] Indeed, if $i \in \{1, 2, \ldots, k-1\}$, then $1 \leqslant i \leqslant k-1$ and thus $k-1 \geqslant 1 > 0$, so that $k > 1$.

Furthermore, each $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$ satisfy

$$\begin{aligned}
&(-1)^{(\beta_i+j)+(\beta_i+j)\%k} \left[(\beta_i + j)\%k \in \{0,1\}\right] \\
&= (-1)^{\beta_i} (-1)^j (-1)^{(\gamma_i+j)\%k} \left[(\gamma_i + j)\%k \in \{0,1\}\right]. \qquad (38)
\end{aligned}$$

[*Proof of (38):* Let $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$. We have $(\beta_i - 1)\%k \equiv \beta_i - 1 \bmod k$ (since $u\%k \equiv u \bmod k$ for any $u \in \mathbb{Z}$), so that $1 + (\beta_i - 1)\%k \equiv \beta_i \bmod k$. But the definition of $\gamma_i$ yields $\gamma_i = 1 + (\beta_i - 1)\%k \equiv \beta_i \bmod k$. Hence, $\gamma_i + j \equiv \beta_i + j \bmod k$, and therefore $(\gamma_i + j)\%k = (\beta_i + j)\%k$ (because if two integers are congruent modulo $k$, then their remainders upon division by $k$ are equal). Hence,

$$\begin{aligned}
&(-1)^{\beta_i} (-1)^j (-1)^{(\gamma_i+j)\%k} \left[(\gamma_i + j)\%k \in \{0,1\}\right] \\
&= (-1)^{\beta_i} (-1)^j (-1)^{(\beta_i+j)\%k} \left[(\beta_i + j)\%k \in \{0,1\}\right] \\
&= (-1)^{(\beta_i+j)+(\beta_i+j)\%k} \left[(\beta_i + j)\%k \in \{0,1\}\right].
\end{aligned}$$

This proves (38).]

Now, for each $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$, we have

$$\begin{aligned}
\alpha_k\left(e_{\beta_i+j}\right) &= (-1)^{(\beta_i+j)+(\beta_i+j)\%k} \left[(\beta_i + j)\%k \in \{0,1\}\right] && (\text{by } (37)) \\
&= (-1)^{\beta_i} (-1)^j (-1)^{(\gamma_i+j)\%k} \left[(\gamma_i + j)\%k \in \{0,1\}\right] && (\text{by } (38)).
\end{aligned}$$

Thus, (36) can be rewritten as follows:

$$\begin{aligned}
&\operatorname{pet}_k(\lambda, \varnothing) \\
&= \det\left(\left((-1)^{\beta_i} (-1)^j (-1)^{(\gamma_i+j)\%k} \left[(\gamma_i + j)\%k \in \{0,1\}\right]\right)_{1\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1}\right) \\
&= \left(\prod_{i=1}^{k-1} \left((-1)^{\beta_i} (-1)^i\right)\right) \\
&\qquad \cdot \det\left(\left((-1)^{(\gamma_i+j)\%k} \left[(\gamma_i + j)\%k \in \{0,1\}\right]\right)_{1\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1}\right)
\end{aligned}$$

(by Lemma 3.8 **(b)**, applied to $m = k-1$ and $R = \mathbf{k}$ and $a_{i,j} = (-1)^{(\gamma_i+j)\%k} \left[(\gamma_i + j)\%k \in \{0,1\}\right]$ and $u_i = (-1)^{\beta_i}$ and $v_j = (-1)^j$).

Define a $(k-1) \times (k-1)$-matrix $G$ as in Lemma 3.9. Then, the matrix whose determinant appears on the right hand side of this equality is precisely $G$; thus, this equality rewrites as

$$\operatorname{pet}_k(\lambda, \varnothing) = \left(\prod_{i=1}^{k-1} \left((-1)^{\beta_i} (-1)^i\right)\right) \cdot \det G. \qquad (39)$$

Now, we can readily prove parts **(b)** and **(c)** of Theorem 2.13:

**(b)** Assume that the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are not distinct. Then, Lemma 3.9 **(a)** yields $\det G = 0$. Hence, (39) simplifies to $\operatorname{pet}_k(\lambda, \varnothing) = 0$. This proves Theorem 2.13 **(b)**.

**(c)** The equality (39) becomes

$$\begin{aligned}
&\operatorname{pet}_k(\lambda,\varnothing)\\
&= \underbrace{\left(\prod_{i=1}^{k-1}\left((-1)^{\beta_i}(-1)^i\right)\right)}_{=(-1)^{(\beta_1+\beta_2+\cdots+\beta_{k-1})+(1+2+\cdots+(k-1))}} \cdot \underbrace{\det G}_{\substack{=(-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}\\ \text{(by Lemma 3.9 (c))}}}\\
&= (-1)^{(\beta_1+\beta_2+\cdots+\beta_{k-1})+(1+2+\cdots+(k-1))}\cdot(-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}\\
&= (-1)^{(\beta_1+\beta_2+\cdots+\beta_{k-1})+g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})}.
\end{aligned}$$

This proves Theorem 2.13 **(c)**. ☐

The proof of Proposition 2.14 relies on the following known fact:

Proposition 3.12. *Let $\lambda \in \operatorname{Par}$. Let $\mu = \lambda^t$. Then:*

- **(a)** *If $i$ and $j$ are two positive integers satisfying $\lambda_i \geqslant j$, then $\mu_j \geqslant i$.*
- **(b)** *If $i$ and $j$ are two positive integers satisfying $\lambda_i < j$, then $\mu_j < i$.*
- **(c)** *Any two positive integers $i$ and $j$ satisfy $\lambda_i + \mu_j - i - j \neq -1$.*

*For each positive integer $i$, set $\alpha_i = \lambda_i - i$. For each positive integer $j$, set $\beta_j = \mu_j - j$ and $\eta_j = -1 - \beta_j$. Then:*

- **(d)** *The two sets $\{\alpha_1,\alpha_2,\alpha_3,\ldots\}$ and $\{\eta_1,\eta_2,\eta_3,\ldots\}$ are disjoint, and their union is $\mathbb{Z}$.*
- **(e)** *Let $p$ be an integer such that $p \geqslant \lambda_1$. Then, the two sets $\{\alpha_1,\alpha_2,\alpha_3,\ldots\}$ and $\{\eta_1,\eta_2,\ldots,\eta_p\}$ are disjoint, and their union is*
$$\{\ldots,p-3,p-2,p-1\} = \{k \in \mathbb{Z} \mid k < p\}.$$
- **(f)** *Let $p$ and $q$ be two integers such that $p \geqslant \lambda_1$ and $q \geqslant \mu_1$. Then, the two sets $\{\alpha_1,\alpha_2,\ldots,\alpha_q\}$ and $\{\eta_1,\eta_2,\ldots,\eta_p\}$ are disjoint, and their union is*
$$\{-q,-q+1,\ldots,p-1\} = \{k \in \mathbb{Z} \mid -q \leqslant k < p\}.$$

Note that Proposition 3.12 **(f)** is a restatement of [20, Chapter I, (1.7)].

*Proof of Proposition 3.12.* Left to the reader (see [16] for a detailed proof). The easiest way to proceed is by proving **(a)** and **(b)** first, then deriving **(c)** as their consequence, then deriving **(f)** from it, then concluding **(d)** and **(e)**. ☐

*Proof of Proposition 2.14.* Let $\mu = \lambda^t$. Then, the number of parts of $\mu$ is $\lambda_1$. Hence, from $\lambda_1 < k$, we conclude that $\mu$ has fewer than $k$ parts. Thus, $\mu_k = 0$.

For each positive integer $i$, set $\alpha_i = \lambda_i - i$. Hence,

$$\begin{aligned}
\{\alpha_1,\alpha_2,\alpha_3,\ldots\} &= \left\{ \underbrace{\alpha_i}_{=\lambda_i - i} \mid i \in \{1,2,3,\ldots\} \right\} = \{\lambda_i - i \mid i \in \{1,2,3,\ldots\}\}\\
&= B \qquad \text{(by the definition of } B\text{)}.
\end{aligned}$$

For each positive integer $j$, set $\beta_j = \mu_j - j$ and $\eta_j = -1 - \beta_j$. Note that $(\beta_1,\beta_2,\ldots,\beta_{k-1}) \in \mathbb{Z}^{k-1}$ is thus the same $(k-1)$-tuple that was called $(\beta_1,\beta_2,\ldots,\beta_{k-1})$ in Theorem 2.13. It is easy to see that $\beta_1 > \beta_2 > \cdots > \beta_{k-1}$ and $\lambda_1 - 1 > \lambda_2 - 2 > \lambda_3 - 3 > \cdots$.

From $\lambda_1 < k$, we obtain $k - 1 \geqslant \lambda_1$. Hence, Proposition 3.12 **(e)** (applied to $p = k-1$) yields that the two sets $\{\alpha_1,\alpha_2,\alpha_3,\ldots\}$ and $\{\eta_1,\eta_2,\ldots,\eta_{k-1}\}$ are disjoint, and their union is

$$\{\ldots,(k-1)-3,(k-1)-2,(k-1)-1\} = \{\text{all integers smaller than } k-1\} = W.$$

Since $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} = B$, we can restate this as follows: The two sets $B$ and $\{\eta_1, \eta_2, \ldots, \eta_{k-1}\}$ are disjoint, and their union is $W$. Hence, $\{\eta_1, \eta_2, \ldots, \eta_{k-1}\} = W \setminus B$.

It is also easy to see that $\beta_1 > \beta_2 > \cdots > \beta_{k-1}$, so that $\eta_1 < \eta_2 < \cdots < \eta_{k-1}$. Hence, $\eta_1, \eta_2, \ldots, \eta_{k-1}$ are the elements of the set $\{\eta_1, \eta_2, \ldots, \eta_{k-1}\}$ listed in increasing order (with no repetition).

Let us define a $(k-1)$-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}) \in \{1, 2, \ldots, k\}^{k-1}$ as in Theorem 2.13. Now, we have the following chain of logical equivalences:

$$\begin{aligned}
&\left(\operatorname{pet}_k(\lambda, \varnothing) \neq 0\right) \\
\Longleftrightarrow\ &\left(\text{the } k-1 \text{ numbers } \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \text{ are distinct}\right) \\
&\qquad\qquad (\text{by parts } \textbf{(b)} \text{ and } \textbf{(c)} \text{ of Theorem 2.13}) \\
\Longleftrightarrow\ &\left(\text{the } k-1 \text{ numbers } (\beta_1 - 1)\,\%k, (\beta_2 - 1)\,\%k, \ldots, (\beta_{k-1} - 1)\,\%k \text{ are distinct}\right) \\
&\qquad\qquad (\text{since } \gamma_i = 1 + (\beta_i - 1)\,\%k \text{ for each } i) \\
\Longleftrightarrow\ &(\text{no two of the } k-1 \text{ numbers } \beta_1 - 1, \beta_2 - 1, \ldots, \beta_{k-1} - 1 \\
&\quad \text{are congruent modulo } k) \\
\Longleftrightarrow\ &\left(\text{no two of the } k-1 \text{ numbers } \beta_1, \beta_2, \ldots, \beta_{k-1} \text{ are congruent modulo } k\right) \\
\Longleftrightarrow\ &(\text{no two of the } k-1 \text{ numbers } -1-\beta_1, -1-\beta_2, \ldots, -1-\beta_{k-1} \\
&\quad \text{are congruent modulo } k) \\
\Longleftrightarrow\ &\left(\text{no two of the } k-1 \text{ numbers } \eta_1, \eta_2, \ldots, \eta_{k-1} \text{ are congruent modulo } k\right) \\
&\qquad\qquad (\text{since } \eta_j = -1 - \beta_j \text{ for each } j) \\
\Longleftrightarrow\ &\left(\text{no two of the } k-1 \text{ elements of } \{\eta_1, \eta_2, \ldots, \eta_{k-1}\} \text{ are congruent modulo } k\right) \\
&\qquad \left(\begin{array}{c}\text{since } \eta_1, \eta_2, \ldots, \eta_{k-1} \text{ are the elements of the set } \{\eta_1, \eta_2, \ldots, \eta_{k-1}\} \\ \text{listed in increasing order (with no repetition)}\end{array}\right) \\
\Longleftrightarrow\ &\left(\text{no two of the } k-1 \text{ elements of } W \setminus B \text{ are congruent modulo } k\right) \\
&\qquad\qquad (\text{since } \{\eta_1, \eta_2, \ldots, \eta_{k-1}\} = W \setminus B) \\
\Longleftrightarrow\ &\left(\text{each congruence class } \bar{i} \text{ has at most 1 element in common with } W \setminus B\right) \\
\Longleftrightarrow\ &\left(\text{each } i \in \{0, 1, \ldots, k-1\} \text{ satisfies } \left|\bar{i} \cap (W \setminus B)\right| \leqslant 1\right) \\
\Longleftrightarrow\ &\left(\text{each } i \in \{0, 1, \ldots, k-1\} \text{ satisfies } \left|\left(\bar{i} \cap W\right) \setminus B\right| \leqslant 1\right)
\end{aligned}$$

(since $\bar{i} \cap (W \setminus B) = \left(\bar{i} \cap W\right) \setminus B$ for each $i$). This proves Proposition 2.14. ☐

### 3.10. Proof of Theorem 2.19.

*Proof of Theorem 2.19.* Consider the ring $(\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]$ of formal power series in one indeterminate $t$ over $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. We equip this ring with the topology that is obtained by identifying it with $\mathbf{k}[[x_1, x_2, x_3, \ldots, t]]$ (or, equivalently, which is obtained by considering $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ itself as equipped with the standard topology on a ring of formal power series, and then adjoining the extra indeterminate $t$).

Now, consider the map

$$\begin{aligned}
\mathbf{F}_k : \mathbf{k}[[x_1, x_2, x_3, \ldots]] &\to \mathbf{k}[[x_1, x_2, x_3, \ldots]], \\
a &\mapsto a\left(x_1^k, x_2^k, x_3^k, \ldots\right).
\end{aligned}$$

This map $\mathbf{F}_k$ is a continuous $\mathbf{k}$-algebra homomorphism (since it is an evaluation homomorphism).[20] Hence, it induces a continuous[21] $\mathbf{k}[[t]]$-algebra homomorphism

$$\mathbf{F}_k[[t]] : (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]] \to (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]$$

that sends each formal power series $\sum_{n\geqslant 0} a_n t^n \in (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]$ (with $a_n \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$) to $\sum_{n\geqslant 0} \mathbf{F}_k(a_n) t^n$. Consider this $\mathbf{k}[[t]]$-algebra homomorphism $\mathbf{F}_k[[t]]$.

The definition of $\mathbf{F}_k$ yields

$$\mathbf{F}_k(x_i) = x_i^k \qquad \text{for each } i \in \{1, 2, 3, \ldots\}. \tag{40}$$

Also, for each $a \in \Lambda$, we have

$$\begin{aligned} \mathbf{F}_k(a) &= a\left(x_1^k, x_2^k, x_3^k, \ldots\right) && (\text{by the definition of } \mathbf{F}_k) \\ &= \mathbf{f}_k(a) && (\text{by the definition of } \mathbf{f}_k). \end{aligned} \tag{41}$$

It is known (e.g. [17, (2.2.18) and (2.2.19)]) that

$$\prod_{i=1}^{\infty} (1 - x_i t)^{-1} = \sum_{n\geqslant 0} h_n t^n \tag{42}$$

and

$$\prod_{i=1}^{\infty} (1 + x_i t) = \sum_{n\geqslant 0} e_n t^n.$$

Applying the map $\mathbf{F}_k[[t]]$ to both sides of the latter equality, we obtain

$$(\mathbf{F}_k[[t]])\left(\prod_{i=1}^{\infty} (1 + x_i t)\right) = (\mathbf{F}_k[[t]])\left(\sum_{n\geqslant 0} e_n t^n\right) = \sum_{n\geqslant 0} \mathbf{F}_k(e_n) t^n$$

(by the definition of $\mathbf{F}_k[[t]]$). Hence,

$$\begin{aligned} \sum_{n\geqslant 0} \mathbf{F}_k(e_n) t^n &= (\mathbf{F}_k[[t]])\left(\prod_{i=1}^{\infty} (1 + x_i t)\right) = \prod_{i=1}^{\infty} \underbrace{(\mathbf{F}_k[[t]])(1 + x_i t)}_{\substack{=1+\mathbf{F}_k(x_i)t \\ (\text{by the definition of } \mathbf{F}_k[[t]])}} \\ &\qquad \left(\begin{array}{c}\text{since } \mathbf{F}_k[[t]] \text{ is a continuous } \mathbf{k}[[t]]\text{-algebra homomorphism,} \\ \text{and thus respects infinite products}\end{array}\right) \\ &= \prod_{i=1}^{\infty} \left(1 + \underbrace{\mathbf{F}_k(x_i)}_{\substack{=x_i^k \\ (\text{by (40)})}} t\right) = \prod_{i=1}^{\infty} \left(1 + x_i^k t\right). \end{aligned}$$

Substituting $-t^k$ for $t$ in this equality, we find

$$\sum_{n\geqslant 0} \mathbf{F}_k(e_n) \left(-t^k\right)^n = \prod_{i=1}^{\infty} \underbrace{\left(1 + x_i^k \left(-t^k\right)\right)}_{=1-(x_i t)^k} = \prod_{i=1}^{\infty} \left(1 - (x_i t)^k\right).$$

[20] It is well-defined, since $k$ is positive.

[21] Continuity is defined with respect to the topology that we defined on $(\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]$.

We can divide both sides of this equality by $\prod_{i=1}^{\infty}(1-x_it)$ (since the formal power series $\prod_{i=1}^{\infty}(1-x_it)$ has constant term 1 and thus is invertible), and thus obtain

$$\begin{aligned}
&\frac{\sum_{n\geqslant 0}\mathbf{F}_k(e_n)\left(-t^k\right)^n}{\prod_{i=1}^{\infty}(1-x_it)} \\
&=\frac{\prod_{i=1}^{\infty}\left(1-(x_it)^k\right)}{\prod_{i=1}^{\infty}(1-x_it)}=\prod_{i=1}^{\infty}\underbrace{\frac{1-(x_it)^k}{1-x_it}}_{\substack{=(x_it)^0+(x_it)^1+\cdots+(x_it)^{k-1}\\ =\sum_{u=0}^{k-1}(x_it)^u}}=\prod_{i=1}^{\infty}\sum_{u=0}^{k-1}(x_it)^u \\
&=\underbrace{\sum_{\substack{\alpha=(\alpha_1,\alpha_2,\alpha_3,\ldots)\in\{0,1,\ldots,k-1\}^{\infty};\\ \alpha_i=0\text{ for all but finitely many }i}}}_{=\sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k\text{ for all }i}}}\underbrace{(x_1t)^{\alpha_1}(x_2t)^{\alpha_2}(x_3t)^{\alpha_3}\cdots}_{=\left(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\cdots\right)t^{\alpha_1+\alpha_2+\alpha_3+\cdots}} \\
&\qquad\qquad\text{(here, we have expanded the product)} \\
&=\sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k\text{ for all }i}}\underbrace{\left(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\cdots\right)}_{\substack{=\mathbf{x}^\alpha\\ \text{(by the definition of }\mathbf{x}^\alpha)}}\underbrace{t^{\alpha_1+\alpha_2+\alpha_3+\cdots}}_{\substack{=t^{|\alpha|}\\ \text{(since }\alpha_1+\alpha_2+\alpha_3+\cdots=|\alpha|)}}=\sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k\text{ for all }i}}\mathbf{x}^\alpha t^{|\alpha|}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k\text{ for all }i}}\mathbf{x}^\alpha t^{|\alpha|} \\
&=\frac{\sum_{n\geqslant 0}\mathbf{F}_k(e_n)\left(-t^k\right)^n}{\prod_{i=1}^{\infty}(1-x_it)}=\left(\sum_{n\geqslant 0}\underbrace{\mathbf{F}_k(e_n)}_{\substack{=\mathbf{f}_k(e_n)\\ \text{(by (41))}}}\underbrace{\left(-t^k\right)^n}_{=(-1)^nt^{kn}}\right)\cdot\underbrace{\prod_{i=1}^{\infty}(1-x_it)^{-1}}_{\substack{=\sum_{n\geqslant 0}h_nt^n\\ \text{(by (42))}}} \\
&=\left(\sum_{n\geqslant 0}\mathbf{f}_k(e_n)(-1)^nt^{kn}\right)\cdot\underbrace{\left(\sum_{n\geqslant 0}h_nt^n\right)}_{=\sum_{j\geqslant 0}h_jt^j}=\left(\sum_{n\geqslant 0}\mathbf{f}_k(e_n)(-1)^nt^{kn}\right)\cdot\left(\sum_{j\geqslant 0}h_jt^j\right) \\
&=\underbrace{\sum_{n\geqslant 0}\sum_{j\geqslant 0}}_{=\sum_{(n,j)\in\mathbb{N}^2}}\mathbf{f}_k(e_n)(-1)^n\underbrace{t^{kn}h_jt^j}_{=h_jt^{kn+j}}=\sum_{(n,j)\in\mathbb{N}^2}\mathbf{f}_k(e_n)(-1)^nh_jt^{kn+j}.
\end{aligned}$$

This is an equality between two power series in $(\mathbf{k}[[x_1,x_2,x_3,\ldots]])[[t]]$. If we compare the coefficients of $t^m$ on both sides of it (where $x_1,x_2,x_3,\ldots$ are considered scalars, not monomials), we obtain

$$\sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{(n,j)\in\mathbb{N}^2;\\ kn+j=m}} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n h_j = \sum_{n\in\mathbb{N}} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n \cdot \sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j.$$

However, the left hand side of this equality is $G(k,m)$ (since $G(k,m)$ was defined this way). Thus, the right hand side is $G(k,m)$ as well. That is, we have

$$G(k,m) = \sum_{n\in\mathbb{N}} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n \cdot \sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j. \tag{43}$$

But the right hand side of this equality can be simplified. Namely, for each $n\in\mathbb{N}$, we have

$$\sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j = h_{m-kn}. \tag{44}$$

[*Proof of (*44*):* Let $n\in\mathbb{N}$. We must prove the equality (44). If $m-kn<0$, then it boils down to $0=0$ (since its left hand side is an empty sum and its right hand side is $h_{m-kn}=0$ by definition). If $m-kn\geqslant 0$, then the sum $\sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j$ has exactly one addend, namely the addend for $j=m-kn$, and thus this sum equals $h_{m-kn}$. Thus, (44) holds in either case.]

Now, using (44), we can rewrite (43) as

$$G(k,m) = \sum_{n\in\mathbb{N}} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n \cdot h_{m-kn} = \sum_{n\in\mathbb{N}} \left(-1\right)^n h_{m-kn} \cdot \mathbf{f}_k\left(e_n\right).$$

Renaming the summation index $n$ as $i$ on the right hand side, we obtain the claim of Theorem 2.19. ☐

Another proof of Theorem 2.19 is sketched in a footnote in Section 4 below.

3.11. Proofs of the results from Section 2.8. We shall now prove the results from Section 2.8. We begin with Lemma 2.21. This will rely on the Verschiebung endomorphisms $\mathbf{v}_n$ introduced in Definition 2.23, and on Proposition 2.24 and the equality (11).

*Proof of Lemma 2.21.* Applying (11) to $n=k$, we obtain

$$\mathbf{v}_k\left(p_m\right) = \begin{cases} kp_{m/k}, & \text{if } k\mid m; \\ 0, & \text{if } k\nmid m. \end{cases} \tag{45}$$

Applying Proposition 2.24 to $n=k$, $a=p_m$ and $b=e_j$, we obtain

$$\left\langle p_m, \mathbf{f}_k\left(e_j\right)\right\rangle = \left\langle \mathbf{v}_k\left(p_m\right), e_j\right\rangle. \tag{46}$$

Now, we are in one of the following three cases:

*Case 1:* We have $m=kj$.

*Case 2:* We have $k\nmid m$.

*Case 3:* We have neither $m=kj$ nor $k\nmid m$.

Let us first consider Case 1. In this case, we have $m=kj$. Thus, $k\mid m$ (since $j\in\mathbb{N}\subseteq\mathbb{Z}$) and $m/k=j$. Hence, $j=m/k$, so that the integer $j$ is positive (since $m$ and $k$ are positive). Recall again that $k\mid m$; thus, (45) simplifies to

$$\mathbf{v}_k\left(p_m\right) = kp_{m/k} = kp_j \qquad \left(\text{since } m/k=j\right).$$

Hence, (46) rewrites as

$$\begin{aligned}\langle p_m, \mathbf{f}_k(e_j)\rangle &= \langle kp_j, e_j\rangle \\ &= \langle e_j, kp_j\rangle \qquad \text{(since the Hall inner product is symmetric)} \\ &= k \underbrace{\langle e_j, p_j\rangle}_{\substack{=(-1)^{j-1} \\ \text{(by Proposition 1.3)}}} = k(-1)^{j-1} = (-1)^{j-1} k.\end{aligned}$$

Comparing this with

$$(-1)^{j-1} \underbrace{[m = kj]}_{\substack{=1 \\ (\text{since } m=kj)}} k = (-1)^{j-1} k,$$

we obtain $\langle p_m, \mathbf{f}_k(e_j)\rangle = (-1)^{j-1}[m = kj]\, k$. Thus, Lemma 2.21 is proven in Case 1.

A similar (but simpler) argument can be used to prove Lemma 2.21 in Case 2. The main "idea" here is that $k \nmid m$ implies $m \neq kj$. The details are left to the reader.

Let us finally consider Case 3. In this case, we have neither $m = kj$ nor $k \nmid m$. In other words, we have $m \neq kj$ and $k \mid m$. From $k \mid m$, we conclude that $m/k$ is a positive integer[22]. From $m \neq kj$, we obtain $m/k \neq j$. Thus, the symmetric functions $p_{m/k}$ and $e_j$ are homogeneous of different degrees (namely, of degrees $m/k$ and $j$), and therefore satisfy $\langle p_{m/k}, e_j\rangle = 0$ (by (2)).

Now, recall that $k \mid m$. Hence, (45) simplifies to

$$\mathbf{v}_k(p_m) = kp_{m/k}.$$

Thus, (46) rewrites as

$$\langle p_m, \mathbf{f}_k(e_j)\rangle = \langle kp_{m/k}, e_j\rangle = k \underbrace{\langle p_{m/k}, e_j\rangle}_{=0} = 0.$$

Comparing this with

$$(-1)^{j-1} \underbrace{[m = kj]}_{\substack{=0 \\ (\text{since } m\neq kj)}} k = 0,$$

we obtain $\langle p_m, \mathbf{f}_k(e_j)\rangle = (-1)^{j-1}[m = kj]\, k$. Thus, Lemma 2.21 is proven in Case 3.

We have thus proven Lemma 2.21 in all three possible cases. ◻

Next, we shall need a simple property of Hall inner products:

Lemma 3.13. *Let $m$, $\alpha$ and $\beta$ be positive integers. Let $a$ be a homogeneous symmetric function of degree $\alpha$. Let $b$ be a homogeneous symmetric function of degree $\beta$. Then, $\langle p_m, ab\rangle = 0$.*

*Proof of Lemma 3.13.* It is known (see, e.g. [17, Proposition 2.4.3(j)]) that the family $(h_\lambda)_{\lambda\in\mathrm{Par}}$ is a graded basis of the graded $\mathbf{k}$-module $\Lambda$. Thus, $a$ is a $\mathbf{k}$-linear combination of symmetric functions $h_\lambda$ with $\lambda \in \mathrm{Par}_\alpha$ (since $a \in \Lambda$ is homogeneous of degree $\alpha$), and a similar fact holds for $b$. Hence, it suffices to prove that $\langle p_m, h_\lambda h_\mu\rangle = 0$ whenever $\lambda \in \mathrm{Par}_\alpha$ and $\mu \in \mathrm{Par}_\beta$ (since the Hall inner product is $\mathbf{k}$-bilinear).

However, proving this is easy: Let $\lambda \in \mathrm{Par}_\alpha$ and $\mu \in \mathrm{Par}_\beta$. Let $\lambda \sqcup \mu$ be the partition obtained by listing all parts of $\lambda$ and of $\mu$ and sorting the resulting list in weakly decreasing order.[23] Using Definition 3.1, we can easily see that $h_{\lambda\sqcup\mu} = h_\lambda h_\mu$. However, each of the partitions $\lambda$ and $\mu$ has a positive size (since $\alpha$ and $\beta$ are positive), and thus has at least one part. Therefore, the partition $\lambda \sqcup \mu$ has at least 2 parts.

[22]Indeed, it is positive since $m$ and $k$ are positive.

[23]For example: If $\lambda = (5, 3, 2)$ and $\mu = (6, 4, 3, 1, 1)$, then $\lambda \sqcup \mu = (6, 5, 4, 3, 3, 2, 1, 1)$.

Consequently, $\lambda \sqcup \mu \neq (m)$. Now, recall that $p_m = m_{(m)}$ (where, of course, the two "$m$"s in "$m_{(m)}$" mean completely unrelated things). Since the Hall inner product is symmetric, we have

$$\begin{aligned} \langle p_m, h_\lambda h_\mu \rangle &= \left\langle \underbrace{h_\lambda h_\mu}_{=h_{\lambda \sqcup \mu}}, \underbrace{p_m}_{=m_{(m)}} \right\rangle = \left\langle h_{\lambda \sqcup \mu}, m_{(m)} \right\rangle = \delta_{\lambda \sqcup \mu, (m)} \qquad \text{(by (13))} \\ &= 0 \qquad \text{(since } \lambda \sqcup \mu \neq (m)) . \end{aligned}$$

As explained above, this proves Lemma 3.13. □

See [16] for a different proof of Lemma 3.13, using the graded dual $\Lambda^o$ of the Hopf algebra $\Lambda$ and the primitivity of the element $p_m \in \Lambda$.

We can now prove Proposition 2.22:

*Proof of Proposition 2.22.* Theorem 2.19 yields $G(k, m) = \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k(e_i)$. Hence,

$$\langle p_m, G(k, m) \rangle = \sum_{i \in \mathbb{N}} (-1)^i \langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i) \rangle \tag{47}$$

(since the Hall inner product is **k**-bilinear).

Now, we claim that every $i \in \mathbb{N} \setminus \{0, m/k\}$ satisfies

$$\langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i) \rangle = 0. \tag{48}$$

[*Proof of (*48*):* Let $i \in \mathbb{N} \setminus \{0, m/k\}$. Thus, $i \in \mathbb{N}$ and $i \neq 0$ and $i \neq m/k$. From $i \neq m/k$, we obtain $ki \neq m$, so that $m - ki \neq 0$.

We must prove the equality (48). This equality clearly holds if $m - ki < 0$ (since $h_{m-ki} = 0$ in this case). Thus, for the rest of this proof, we WLOG assume that $m - ki \geqslant 0$. Combining this with $m - ki \neq 0$, we obtain $m - ki > 0$. Thus, $m - ki$ is a positive integer. Also, $i$ is a positive integer (since $i \in \mathbb{N}$ and $i \neq 0$), and thus $ki$ is a positive integer (since $k$ is a positive integer).

The map $\mathbf{f}_k : \Lambda \to \Lambda$ operates by replacing each $x_i$ by $x_i^k$ in a symmetric function (by the definition of $\mathbf{f}_k$). Thus, if $g \in \Lambda$ is any homogeneous symmetric function of some degree $\gamma$, then $\mathbf{f}_k(g)$ is a homogeneous symmetric function of degree $k\gamma$. Therefore, $\mathbf{f}_k(e_i)$ is a homogeneous symmetric function of degree $ki$ (since $e_i$ is a homogeneous symmetric function of degree $i$). Also, $h_{m-ki}$ is a homogeneous symmetric function of degree $m - ki$.

Hence, Lemma 3.13 (applied to $\alpha = m - ki$, $a = h_{m-ki}$, $\beta = ki$ and $b = \mathbf{f}_k(e_i)$) yields $\langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i) \rangle = 0$. This proves (48).]

Note that $e_0 = 1$ and thus $\mathbf{f}_k(e_0) = \mathbf{f}_k(1) = 1$ (by the definition of $\mathbf{f}_k$).

Note that $m/k > 0$ (since $m$ and $k$ are positive). Hence, $m/k \neq 0$. Now, we are in one of the following two cases:

*Case 1:* We have $k | m$.

*Case 2:* We have $k \nmid m$.

Let us consider Case 1 first. In this case, we have $k | m$. Hence, $m/k$ is a positive integer (since $m/k > 0$). Thus, 0 and $m/k$ are two distinct elements of $\mathbb{N}$.

Now, consider the sum on the right hand side of (47). All addends of this sum are 0, except for the one for $i = 0$ and the one for $i = m/k$ (because (48) shows that the Hall inner products $\langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i) \rangle$ that appear in these addends vanish whenever $i \notin \{0, m/k\}$). Thus, (47) simplifies to[24]

$$\begin{aligned}
&\langle p_m, G(k,m) \rangle \\
&= \underbrace{(-1)^0}_{=1} \left\langle p_m, \underbrace{h_{m-k\cdot 0}}_{=h_m} \cdot \underbrace{\mathbf{f}_k(e_0)}_{=1} \right\rangle + (-1)^{m/k} \left\langle p_m, \underbrace{h_{m-k\cdot m/k}}_{=h_{m-m}=h_0=1} \cdot \mathbf{f}_k\left(e_{m/k}\right) \right\rangle \\
&= \underbrace{\langle p_m, h_m \rangle}_{\substack{=\langle h_m, p_m \rangle \\ \text{(since the Hall inner} \\ \text{product is symmetric)}}} + (-1)^{m/k} \underbrace{\left\langle p_m, \mathbf{f}_k\left(e_{m/k}\right) \right\rangle}_{\substack{=(-1)^{m/k-1}[m=k(m/k)]k \\ \text{(by Lemma 2.21, applied to } j=m/k)}} \\
&= \underbrace{\langle h_m, p_m \rangle}_{\substack{=1 \\ \text{(by Proposition 1.2)}}} + \underbrace{(-1)^{m/k}(-1)^{m/k-1}}_{=-1} \underbrace{[m = k(m/k)]}_{\substack{=1 \\ \text{(since } m=k(m/k))}} k = 1 - k.
\end{aligned}$$

Comparing this with

$$1 - \underbrace{[k|m]}_{\substack{=1 \\ \text{(since } k|m)}} k = 1 - k,$$

we obtain $\langle p_m, G(k,m) \rangle = 1 - [k|m]\, k$. Hence, Proposition 2.22 is proven in Case 1.

Case 2 is similar to Case 1, but simpler because the addend for $i = m/k$ does not exist (since $m/k \notin \mathbb{N}$ in this case). We leave it to the reader.

We have now proven Proposition 2.22 in both possible cases. ☐

Theorem 2.20 will follow from Proposition 2.22 using the following general criterion for generating sets of $\Lambda$:

Proposition 3.14. *For each positive integer $m$, let $v_m \in \Lambda$ be a homogeneous symmetric function of degree $m$.*

*Assume that $\langle p_m, v_m \rangle$ is an invertible element of $\mathbf{k}$ for each positive integer $m$.*

*Then, the family $(v_m)_{m \geqslant 1} = (v_1, v_2, v_3, \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra $\Lambda$.*

*Proof of Proposition 3.14.* Proposition 3.14 is [17, Exercise 2.5.24]. ☐

*Proof of Theorem 2.20.* Let $m$ be a positive integer. Proposition 2.22 yields that

$$\langle p_m, G(k,m) \rangle = 1 - [k|m]\, k = \begin{cases} 1-k, & \text{if } k|m; \\ 1, & \text{if } k \nmid m. \end{cases}$$

Hence, $\langle p_m, G(k,m) \rangle$ is an invertible element of $\mathbf{k}$ (because both $1-k$ and $1$ are invertible elements of $\mathbf{k}$).

Forget that we fixed $m$. We thus have showed that $\langle p_m, G(k,m) \rangle$ is an invertible element of $\mathbf{k}$ for each positive integer $m$. Also, clearly, for each positive integer $m$, the element $G(k,m) \in \Lambda$ is a homogeneous symmetric function of degree $m$. Thus, Proposition 3.14 (applied to $v_m = G(k,m)$) shows that the family $(G(k,m))_{m \geqslant 1} = (G(k,1), G(k,2), G(k,3), \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra $\Lambda$. This proves Theorem 2.20. ☐

[24] We note that the addends for $i = 0$ and for $i = m/k$ really do exist and are two different addends (since $0$ and $m/k$ are two distinct elements of $\mathbb{N}$).

3.12. Proof of Theorem 2.26.

*Proof of Theorem 2.26.* The $\mathbf{k}$-Hopf algebra $\Lambda$ is both commutative and cocommutative (by [17, Exercise 2.3.7(a)]). Its antipode $S$ is a $\mathbf{k}$-Hopf algebra homomorphism (by [17, Proposition 2.4.3(g)]).

Let $\Delta = \Delta_\Lambda : \Lambda \to \Lambda \otimes \Lambda$ be the comultiplication of the $\mathbf{k}$-coalgebra $\Lambda$. Let $m_\Lambda : \Lambda \otimes \Lambda \to \Lambda$ be the $\mathbf{k}$-linear map sending each pure tensor $a \otimes b \in \Lambda \otimes \Lambda$ to $ab \in \Lambda$. Definition 2.25 then yields $\mathrm{id}_\Lambda \star U_k = m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \Delta$. Thus,

$$V_k = \mathrm{id}_\Lambda \star U_k = m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \Delta. \tag{49}$$

**(a)** The map $\mathbf{f}_k$ is a $\mathbf{k}$-Hopf algebra homomorphism (by [17, Exercise 2.9.9(d)], applied to $n = k$). The map $\mathbf{v}_k$ is a $\mathbf{k}$-Hopf algebra homomorphism (by [17, Exercise 2.9.10(e)], applied to $n = k$). Thus, we have shown that all three maps $\mathbf{f}_k$, $S$ and $\mathbf{v}_k$ are $\mathbf{k}$-Hopf algebra homomorphisms. Hence, their composition $\mathbf{f}_k \circ S \circ \mathbf{v}_k$ is a $\mathbf{k}$-Hopf algebra homomorphism as well. In other words, $U_k$ is a $\mathbf{k}$-Hopf algebra homomorphism (since $U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k$). This proves Theorem 2.26 **(a)**.

**(b)** Recall (from [17, Exercise 1.5.11(a)]) the following fact:

> *Claim 1:* If $H$ is a $\mathbf{k}$-bialgebra and $A$ is a commutative $\mathbf{k}$-algebra, then the convolution $f \star g$ of any two $\mathbf{k}$-algebra homomorphisms $f, g : H \to A$ is again a $\mathbf{k}$-algebra homomorphism.

The following fact is dual to Claim 1:

> *Claim 2:* If $H$ is a $\mathbf{k}$-bialgebra and $C$ is a cocommutative $\mathbf{k}$-coalgebra, then the convolution $f \star g$ of any two $\mathbf{k}$-coalgebra homomorphisms $f, g : C \to H$ is again a $\mathbf{k}$-coalgebra homomorphism.

(See [17, solution to Exercise 1.5.11(h)] for why exactly Claim 2 is dual to Claim 1, and how it can be proved.)

Theorem 2.26 **(a)** yields that the map $U_k$ is a $\mathbf{k}$-Hopf algebra homomorphism. Hence, $U_k$ is both a $\mathbf{k}$-algebra homomorphism and a $\mathbf{k}$-coalgebra homomorphism.

Now, recall that $\Lambda$ is commutative, and that $\mathrm{id}_\Lambda$ and $U_k$ are two $\mathbf{k}$-algebra homomorphisms from $\Lambda$ to $\Lambda$. Hence, Claim 1 (applied to $H = \Lambda$, $A = \Lambda$, $f = \mathrm{id}_\Lambda$ and $g = U_k$) shows that the convolution $\mathrm{id}_\Lambda \star U_k$ is a $\mathbf{k}$-algebra homomorphism. In other words, $V_k$ is a $\mathbf{k}$-algebra homomorphism (since $V_k = \mathrm{id}_\Lambda \star U_k$).

Next, recall that $\Lambda$ is cocommutative, and that $\mathrm{id}_\Lambda$ and $U_k$ are two $\mathbf{k}$-coalgebra homomorphisms from $\Lambda$ to $\Lambda$. Hence, Claim 2 (applied to $H = \Lambda$, $C = \Lambda$, $f = \mathrm{id}_\Lambda$ and $g = U_k$) shows that the convolution $\mathrm{id}_\Lambda \star U_k$ is a $\mathbf{k}$-coalgebra homomorphism. In other words, $V_k$ is a $\mathbf{k}$-coalgebra homomorphism (since $V_k = \mathrm{id}_\Lambda \star U_k$).

So we know that the map $V_k$ is both a $\mathbf{k}$-algebra homomorphism and a $\mathbf{k}$-coalgebra homomorphism. Thus, $V_k$ is a $\mathbf{k}$-bialgebra homomorphism, therefore a $\mathbf{k}$-Hopf algebra homomorphism.(25) This proves Theorem 2.26 **(b)**.

**(c)** The map $\mathbf{v}_k$ is a $\mathbf{k}$-algebra homomorphism; thus, $\mathbf{v}_k(1) = 1$. Now, we have

$$\mathbf{v}_k(h_m) = \begin{cases} h_{m/k}, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases} \tag{50}$$

for each $m \in \mathbb{N}$. (Indeed, if $m > 0$, then this follows from the definition of $\mathbf{v}_k$. But if $m = 0$, then this follows from $\mathbf{v}_k(1) = 1$, since $h_0 = 1$.)

We have

$$S(h_n) = (-1)^n e_n \qquad \text{for each } n \in \mathbb{N}. \tag{51}$$

(This follows from [17, Proposition 2.4.1(iii)].)

(25)This follows since any $\mathbf{k}$-bialgebra homomorphism between two $\mathbf{k}$-Hopf algebras is automatically a $\mathbf{k}$-Hopf algebra homomorphism.

Each $i \in \mathbb{N}$ satisfies

$$\begin{aligned}
\mathbf{v}_k \left(h_{ki}\right) &= \begin{cases} h_{ki/k}, & \text{if } k \mid ki; \\ 0, & \text{if } k \nmid ki \end{cases} && \text{(by (50), applied to } m = ki\text{)} \\
&= h_{ki/k} && \text{(since } k \mid ki\text{)} \\
&= h_i \qquad (52)
\end{aligned}$$

and

$$\begin{aligned}
U_k \left(h_{ki}\right) &= \left(\mathbf{f}_k \circ S \circ \mathbf{v}_k\right) \left(h_{ki}\right) && \text{(since } U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k\text{)} \\
&= \mathbf{f}_k \left( S \left( \underbrace{\mathbf{v}_k \left(h_{ki}\right)}_{\substack{=h_i \\ \text{(by (52))}}} \right) \right) = \mathbf{f}_k \left( \underbrace{S \left(h_i\right)}_{\substack{=(-1)^i e_i \\ \text{(by (51))}}} \right) = \mathbf{f}_k \left( (-1)^i e_i \right) \\
&= (-1)^i \mathbf{f}_k \left(e_i\right). \qquad (53)
\end{aligned}$$

On the other hand, if $j \in \mathbb{N}$ satisfies $k \nmid j$, then

$$\mathbf{v}_k \left(h_j\right) = 0 \qquad \text{(by (50), applied to } m = j\text{)} \qquad (54)$$

and

$$\begin{aligned}
U_k \left(h_j\right) &= \left(\mathbf{f}_k \circ S \circ \mathbf{v}_k\right) \left(h_j\right) && \text{(since } U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k\text{)} \\
&= \left(\mathbf{f}_k \circ S\right) \left(\mathbf{v}_k \left(h_j\right)\right) = \left(\mathbf{f}_k \circ S\right)(0) && \text{(by (54))} \\
&= 0. \qquad (55)
\end{aligned}$$

Let $m \in \mathbb{N}$ (not to be mistaken for the map $m_\Lambda$). Then, [17, Proposition 2.3.6(iii)] (applied to $n = m$) yields

$$\begin{aligned}
\Delta \left(h_m\right) &= \sum_{i+j=m} h_i \otimes h_j \\
&\qquad \text{(where the sum ranges over all pairs } (i,j) \in \mathbb{N} \times \mathbb{N} \text{ with } i + j = m\text{)} \\
&= \sum_{j=0}^{m} h_{m-j} \otimes h_j
\end{aligned}$$

(here, we have substituted $(m-j, j)$ for $(i, j)$ in the sum). Applying the map $\mathrm{id}_\Lambda \otimes U_k$ to both sides of this equality, we obtain

$$\left(\mathrm{id}_\Lambda \otimes U_k\right) \left(\Delta \left(h_m\right)\right) = \left(\mathrm{id}_\Lambda \otimes U_k\right) \left( \sum_{j=0}^{m} h_{m-j} \otimes h_j \right) = \sum_{j=0}^{m} h_{m-j} \otimes U_k \left(h_j\right).$$

Applying the map $m_\Lambda$ to both sides of this equality, we find

$$\begin{aligned}
& m_\Lambda\left(\left(\mathrm{id}_\Lambda \otimes U_k\right)\left(\Delta\left(h_m\right)\right)\right) \\
&= m_\Lambda\left(\sum_{j=0}^{m} h_{m-j} \otimes U_k\left(h_j\right)\right) = \sum_{j=0}^{m} h_{m-j} U_k\left(h_j\right) \qquad \text{(by the definition of } m_\Lambda\text{)} \\
&= \underbrace{\sum_{j=0}^{\infty}}_{=\sum_{j\in\mathbb{N}}} h_{m-j} U_k\left(h_j\right) - \sum_{j=m+1}^{\infty} \underbrace{h_{m-j}}_{\substack{=0 \\ \text{(since } m-j<0\text{)}}} U_k\left(h_j\right) \\
&= \sum_{j\in\mathbb{N}} h_{m-j} U_k\left(h_j\right) - \underbrace{\sum_{j=m+1}^{\infty} 0 U_k\left(h_j\right)}_{=0} \\
&= \sum_{j\in\mathbb{N}} h_{m-j} U_k\left(h_j\right) = \sum_{\substack{j\in\mathbb{N}; \\ k\mid j}} h_{m-j} U_k\left(h_j\right) + \sum_{\substack{j\in\mathbb{N}; \\ k\nmid j}} h_{m-j} \underbrace{U_k\left(h_j\right)}_{\substack{=0 \\ \text{(by (55))}}} \\
&= \sum_{\substack{j\in\mathbb{N}; \\ k\mid j}} h_{m-j} U_k\left(h_j\right) + \underbrace{\sum_{\substack{j\in\mathbb{N}; \\ k\nmid j}} h_{m-j} 0}_{=0} = \sum_{\substack{j\in\mathbb{N}; \\ k\mid j}} h_{m-j} U_k\left(h_j\right) = \sum_{i\in\mathbb{N}} h_{m-ki} \underbrace{U_k\left(h_{ki}\right)}_{\substack{=(-1)^i \mathbf{f}_k(e_i) \\ \text{(by (53))}}} \\
&\qquad \text{(here, we have substituted } ki \text{ for } j \text{ in the sum)} \\
&= \sum_{i\in\mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k\left(e_i\right).
\end{aligned}$$

Comparing this with

$$G(k,m) = \sum_{i\in\mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k\left(e_i\right) \qquad \text{(by Theorem 2.19)},$$

we obtain

$$G(k,m) = m_\Lambda\left(\left(\mathrm{id}_\Lambda \otimes U_k\right)\left(\Delta\left(h_m\right)\right)\right) = \underbrace{\left(m_\Lambda \circ \left(\mathrm{id}_\Lambda \otimes U_k\right) \circ \Delta\right)}_{\substack{=V_k \\ \text{(by (49))}}}\left(h_m\right) = V_k\left(h_m\right).$$

This proves Theorem 2.26 **(c)**.

**(d)** From [17, Exercise 2.9.10(a)], we know that every positive integers $n$ and $m$ satisfy

$$\mathbf{v}_n\left(p_m\right) = \begin{cases} n p_{m/n}, & \text{if } n\mid m; \\ 0, & \text{if } n\nmid m. \end{cases} \tag{56}$$

On the other hand, it is easy to see (directly using the definition of $\mathbf{f}_n$) that every positive integers $n$ and $m$ satisfy

$$\mathbf{f}_n\left(p_m\right) = p_{nm}. \tag{57}$$

Finally, [17, Proposition 2.4.1(i)] yields that every positive integer $n$ satisfies

$$S\left(p_n\right) = -p_n. \tag{58}$$

Now, let $n$ be a positive integer. We first claim the following:

*Claim 1:* We have $U_k\left(p_n\right) = -\left[k\mid n\right] k p_n$.

[*Proof of Claim 1:* We are in one of the following two cases:

*Case 1:* We have $k|n$.

*Case 2:* We have $k\nmid n$.

Let us first consider Case 1. In this case, we have $k \mid n$. Hence, $n/k$ is a positive integer. Now, (56) (applied to $k$ and $n$ instead of $n$ and $m$) yields

$$\mathbf{v}_k(p_n) = kp_{n/k} \qquad (\text{since } k|n).$$

Applying the map $S$ to both sides of this equality, we find

$$S(\mathbf{v}_k(p_n)) = S\left(kp_{n/k}\right) = k \underbrace{S\left(p_{n/k}\right)}_{\substack{=-p_{n/k} \\ \text{(by (58),} \\ \text{applied to } n/k \text{ instead of } n)}} = k\left(-p_{n/k}\right) = -kp_{n/k}.$$

Applying the map $\mathbf{f}_k$ to both sides of this equality, we find

$$\mathbf{f}_k(S(\mathbf{v}_k(p_n))) = \mathbf{f}_k\left(-kp_{n/k}\right) = -k \underbrace{\mathbf{f}_k\left(p_{n/k}\right)}_{\substack{=p_{k(n/k)} \\ \text{(by (57),} \\ \text{applied to } k \text{ and } n/k \\ \text{instead of } n \text{ and } m)}} = -kp_{k(n/k)} = -kp_n.$$

Now, the definition of $U_k$ yields $U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k$. Hence,

$$U_k(p_n) = (\mathbf{f}_k \circ S \circ \mathbf{v}_k)(p_n) = \mathbf{f}_k(S(\mathbf{v}_k(p_n))) = -kp_n.$$

Comparing this with

$$-\underbrace{[k|n]}_{\substack{=1 \\ (\text{since } k|n)}} kp_n = -kp_n,$$

we obtain $U_k(p_n) = -[k|n]\,kp_n$. Hence, Claim 1 is proved in Case 1.

Let us now consider Case 2. In this case, we have $k \nmid n$. Therefore, (56) (applied to $k$ and $n$ instead of $n$ and $m$) yields

$$\mathbf{v}_k(p_n) = 0.$$

But the definition of $U_k$ yields $U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k$. Hence,

$$U_k(p_n) = (\mathbf{f}_k \circ S \circ \mathbf{v}_k)(p_n) = (\mathbf{f}_k \circ S)\left(\underbrace{\mathbf{v}_k(p_n)}_{=0}\right) = (\mathbf{f}_k \circ S)(0) = 0.$$

Comparing this with

$$-\underbrace{[k|n]}_{\substack{=0 \\ (\text{since } k\nmid n)}} kp_n = 0,$$

we obtain $U_k(p_n) = -[k|n]\,kp_n$. Hence, Claim 1 is proved in Case 2.

We have now proved Claim 1 in both Cases 1 and 2. Thus, Claim 1 always holds.]

Theorem 2.26 **(a)** shows that the map $U_k$ is a $\mathbf{k}$-Hopf algebra homomorphism. Hence, $U_k(1) = 1$.

But [17, Proposition 2.3.6(i)] yields $\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1$. Now,

$$\begin{aligned}
&\underbrace{V_k}_{\substack{=m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \Delta \\ \text{(by (49))}}} (p_n) \\
&= (m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \Delta)(p_n) = m_\Lambda \left( (\mathrm{id}_\Lambda \otimes U_k) \left( \underbrace{\Delta(p_n)}_{=1 \otimes p_n + p_n \otimes 1} \right) \right) \\
&= m_\Lambda \left( \underbrace{(\mathrm{id}_\Lambda \otimes U_k)(1 \otimes p_n + p_n \otimes 1)}_{=\mathrm{id}_\Lambda(1) \otimes U_k(p_n) + \mathrm{id}_\Lambda(p_n) \otimes U_k(1)} \right) \\
&= m_\Lambda (\mathrm{id}_\Lambda(1) \otimes U_k(p_n) + \mathrm{id}_\Lambda(p_n) \otimes U_k(1)) \\
&= \underbrace{\mathrm{id}_\Lambda(1)}_{=1} \cdot \underbrace{U_k(p_n)}_{\substack{=-[k|n]kp_n \\ \text{(by Claim 1)}}} + \underbrace{\mathrm{id}_\Lambda(p_n)}_{=p_n} \cdot \underbrace{U_k(1)}_{=1} \qquad \text{(by the definition of } m_\Lambda) \\
&= -[k|n]\, kp_n + p_n = (1 - [k|n]\, k)\, p_n.
\end{aligned}$$

This proves Theorem 2.26 **(d)**. ☐

### 3.13. Proof of Corollary 2.27.

*Proof of Corollary 2.27.* Recall that the family $(h_n)_{n \geqslant 1} = (h_1, h_2, h_3, \ldots)$ generates $\Lambda$ as a $\mathbf{k}$-algebra. Hence, each $g \in \Lambda$ can be written as a polynomial in $h_1, h_2, h_3, \ldots$. Applying this to $g = p_n$, we conclude that $p_n$ can be written as a polynomial in $h_1, h_2, h_3, \ldots$. In other words, there exists a polynomial $f \in \mathbf{k}[x_1, x_2, x_3, \ldots]$ such that

$$p_n = f(h_1, h_2, h_3, \ldots). \tag{59}$$

Consider this $f$. We shall show that this $f$ satisfies (12). This will clearly prove Corollary 2.27.

Consider the map $V_k$ defined in Theorem 2.26. Theorem 2.26 **(c)** yields that $V_k(h_m) = G(k, m)$ for each positive integer $m$. In other words,

$$(V_k(h_1), V_k(h_2), V_k(h_3), \ldots) = (G(k,1), G(k,2), G(k,3), \ldots). \tag{60}$$

The map $V_k$ is a $\mathbf{k}$-Hopf algebra homomorphism (by Theorem 2.26 **(b)**, and thus is a $\mathbf{k}$-algebra homomorphism. Hence, it commutes with polynomials over $\mathbf{k}$. Thus,

$$\begin{aligned}
V_k(f(h_1, h_2, h_3, \ldots)) &= f(V_k(h_1), V_k(h_2), V_k(h_3), \ldots) \\
&= f(G(k,1), G(k,2), G(k,3), \ldots) \qquad \text{(by (60))}.
\end{aligned}$$

Now, applying the map $V_k$ to both sides of the equality (59), we obtain

$$V_k(p_n) = V_k(f(h_1, h_2, h_3, \ldots)) = f(G(k,1), G(k,2), G(k,3), \ldots).$$

Comparing this with

$$V_k(p_n) = (1 - [k|n]\, k)\, p_n \qquad \text{(by Theorem 2.26 } \mathbf{(d)}),$$

we obtain

$$(1 - [k|n]\, k)\, p_n = f(G(k,1), G(k,2), G(k,3), \ldots).$$

Thus, we have shown that our $f$ satisfies (12). As we said, this proves Corollary 2.27. ☐

## 4. Proof of the Liu–Polo conjecture

Let us recall a well-known partial order on the set of partitions of a given $n \in \mathbb{N}$:

Definition 4.1. *Let $n \in \mathbb{N}$. We define a binary relation $\triangleright$ on the set $\operatorname{Par}_n$ as follows: Two partitions $\lambda, \mu \in \operatorname{Par}_n$ shall satisfy $\lambda \triangleright \mu$ if and only if we have*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \geqslant \mu_1 + \mu_2 + \cdots + \mu_k \qquad \text{for each } k \in \{1, 2, \ldots, n\}.$$

*This relation $\triangleright$ is the greater-or-equal relation of a partial order on $\operatorname{Par}_n$, which is known as the* dominance order *(or the* majorization order*).*

This definition is precisely [17, Definition 2.2.7]. Note that if we replace "for each $k \in \{1, 2, \ldots, n\}$" by "for each $k \in \{1, 2, 3, \ldots\}$" in this definition, then the relation $\triangleright$ does not change.

Our goal in this section is to prove the conjecture made in [19, Remark 1.4.5]. We state this conjecture as follows:[26]

Theorem 4.2. *Let $n$ be an integer such that $n > 1$. Then:*

**(a)** *We have*

$$\sum_{\substack{\lambda \in \operatorname{Par}_n; \\ (n-1,1) \triangleright \lambda}} m_\lambda = \sum_{i=0}^{n-2} (-1)^i s_{(n-1-i, 1^{i+1})}.$$

**(b)** *We have*

$$\sum_{\substack{\lambda \in \operatorname{Par}_{2n-1}; \\ (n-1,n-1,1) \triangleright \lambda}} m_\lambda = \sum_{i=0}^{n-2} (-1)^i s_{(n-1, n-1-i, 1^{i+1})}.$$

Example 4.3. For this example, let $n = 3$. Then, $n - 1 = 2$ and $2n - 1 = 5$. Hence, the left hand side of the equality in Theorem 4.2 **(b)** is

$$\sum_{\substack{\lambda \in \operatorname{Par}_{2n-1}; \\ (n-1,n-1,1) \triangleright \lambda}} m_\lambda = \sum_{\substack{\lambda \in \operatorname{Par}_5; \\ (2,2,1) \triangleright \lambda}} m_\lambda = m_{(2,2,1)} + m_{(2,1,1,1)} + m_{(1,1,1,1,1)}.$$

Meanwhile, the right hand side of the equality in Theorem 4.2 **(b)** is

$$\sum_{i=0}^{n-2} (-1)^i s_{(n-1, n-1-i, 1^{i+1})} = \sum_{i=0}^{1} (-1)^i s_{(2, 2-i, 1^{i+1})} = s_{(2,2,1)} - s_{(2,1,1,1)}.$$

Thus, Theorem 4.2 **(b)** claims that $m_{(2,2,1)} + m_{(2,1,1,1)} + m_{(1,1,1,1,1)} = s_{(2,2,1)} - s_{(2,1,1,1)}$ in this case.

We will pave our way to the proof of Theorem 4.2 by several lemmas. We begin with a particularly simple one:

Lemma 4.4. *Let $n$ be an integer such that $n > 1$. Let $\lambda \in \operatorname{Par}_{2n-1}$. Then, $(n-1, n-1, 1) \triangleright \lambda$ if and only if all positive integers $i$ satisfy $\lambda_i < n$.*

*Proof.* This simple proof (an exercise in following Definition 4.1) is left to the reader. □

Lemma 4.5. *Let $n$ be an integer such that $n > 1$. Let $\lambda \in \operatorname{Par}_n$. Then, $(n-1, 1) \triangleright \lambda$ if and only if all positive integers $i$ satisfy $\lambda_i < n$.*

*Proof of Lemma 4.5.* This is analogous to the proof of Lemma 4.4. □

[26] Note that $(n-1, n-1, 1)$ is a partition whenever $n > 1$ is an integer.

The next lemma identifies the left hand side of Theorem 4.2 **(a)** as the Petrie symmetric function $G(n,n)$, and the left hand side of Theorem 4.2 **(b)** as the Petrie symmetric function $G(n,2n-1)$:

COROLLARY 4.6. *Let $n$ be an integer such that $n>1$. Then:*

**(a)** *We have*

$$\sum_{\substack{\lambda\in\operatorname{Par}_n;\\ (n-1,1)\rhd\lambda}} m_\lambda = G(n,n).$$

**(b)** *We have*

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\rhd\lambda}} m_\lambda = G(n,2n-1).$$

*Proof.* **(b)** Proposition 2.3 **(c)** (applied to $k=n$ and $m=2n-1$) yields

$$G(n,2n-1) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=2n-1;\\ \alpha_i<n \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=2n-1;\\ \lambda_i<n \text{ for all } i}} m_\lambda. \tag{61}$$

But Lemma 4.4 yields the following equality of summation signs:

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\rhd\lambda}} = \sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ \lambda_i<n \text{ for all } i}} = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=2n-1;\\ \lambda_i<n \text{ for all } i}} .$$

Hence,

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\rhd\lambda}} m_\lambda = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=2n-1;\\ \lambda_i<n \text{ for all } i}} m_\lambda.$$

Comparing this with (61), we obtain

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\rhd\lambda}} m_\lambda = G(n,2n-1).$$

This proves Corollary 4.6 **(b)**.

**(a)** This is analogous to Corollary 4.6 **(b)**, but uses Lemma 4.5 instead of Lemma 4.4. □

It was Corollary 4.6 that led the author to introduce and study the Petrie symmetric functions $G(k,m)$ in general, even if little of their general properties has proven relevant to Theorem 4.2.

The next proposition gives a simple formula for certain kinds of Petrie symmetric functions:

PROPOSITION 4.7. *Let $n$ be a positive integer. Let $k\in\{0,1,\ldots,n-1\}$. Then,*

$$G(n,n+k) = h_{n+k} - h_k p_n.$$

Proposition 4.7 can be viewed as a particular case of Theorem 2.19 (applied to $n$ and $n+k$ instead of $k$ and $m$), after realizing that in the sum on the right hand side of Theorem 2.19, only the first two addends will (potentially) be nonzero in this case. However, let us give an independent proof of the proposition.

*Proof of Proposition 4.7.* From $k\in\{0,1,\ldots,n-1\}$, we obtain $k<n$ and thus $n+k<n+n$. Thus we conclude:

> *Observation 1:* A monomial of degree $n+k$ cannot have more than one variable appear in it with exponent $\geqslant n$ (since this would require it to have degree $\geqslant n+n>n+k$).

Let $\mathfrak{M}_k$ be the set of all monomials of degree $k$. The definition of $h_k$ shows that

$$h_k = \sum_{\mathfrak{m}\in\mathfrak{M}_k} \mathfrak{m}. \tag{62}$$

Let $\mathfrak{M}_{n+k}$ be the set of all monomials of degree $n+k$. The definition of $h_{n+k}$ shows that

$$h_{n+k} = \sum_{\mathfrak{n}\in\mathfrak{M}_{n+k}} \mathfrak{n}. \tag{63}$$

Let $\mathfrak{N}$ be the set of all monomials of degree $n+k$ in which all exponents are $<n$. These monomials are exactly the $\mathbf{x}^\alpha$ for $\alpha\in\mathrm{WC}$ satisfying $|\alpha|=n+k$ and ($\alpha_i<n$ for all $i$). Hence,

$$\sum_{\mathfrak{n}\in\mathfrak{N}} \mathfrak{n} = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n+k;\\ \alpha_i<n \text{ for all } i}} \mathbf{x}^\alpha. \tag{64}$$

But Proposition 2.3 **(c)** (applied to $n$ and $n+k$ instead of $k$ and $m$) yields

$$G(n,n+k) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n+k;\\ \alpha_i<n \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=n+k;\\ \lambda_i<n \text{ for all } i}} m_\lambda.$$

Hence,

$$G(n,n+k) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n+k;\\ \alpha_i<n \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\mathfrak{n}\in\mathfrak{N}} \mathfrak{n} \tag{65}$$

(by (64)).

Clearly, the set $\mathfrak{N}$ is a subset of $\mathfrak{M}_{n+k}$, and furthermore its complement $\mathfrak{M}_{n+k}\setminus\mathfrak{N}$ is the set of all monomials of degree $n+k$ in which at least one exponent is $\geqslant n$. Hence, the map

$$\mathfrak{M}_k \times \{1,2,3,\ldots\} \to \mathfrak{M}_{n+k}\setminus\mathfrak{N},$$
$$(\mathfrak{m},i) \mapsto \mathfrak{m}\cdot x_i^n$$

is well-defined (because if $\mathfrak{m}$ is a monomial of degree $k$, and if $i\in\{1,2,3,\ldots\}$, then $\mathfrak{m}\cdot x_i^n$ is a monomial of degree $k+n=n+k$, and the variable $x_i$ appears in it with exponent $\geqslant n$). This map is furthermore surjective (for simple reasons) and injective (in fact, if $\mathfrak{n}\in\mathfrak{M}_{n+k}\setminus\mathfrak{N}$, then $\mathfrak{n}$ is a monomial of degree $n+k$, and thus Observation 1 yields that there is **at most** one variable $x_i$ that appears in $\mathfrak{n}$ with exponent $\geqslant n$; but this means that the only preimage of $\mathfrak{n}$ under our map is $\left(\dfrac{\mathfrak{n}}{x_i^n}, i\right)$). Hence, this map is a bijection. We can thus use it to substitute $\mathfrak{m}\cdot x_i^n$ for $\mathfrak{n}$ in the sum $\sum\limits_{\mathfrak{n}\in\mathfrak{M}_{n+k}\setminus\mathfrak{N}} \mathfrak{n}$. We thus obtain

$$\sum_{\mathfrak{n}\in\mathfrak{M}_{n+k}\setminus\mathfrak{N}} \mathfrak{n} = \sum_{(\mathfrak{m},i)\in\mathfrak{M}_k\times\{1,2,3,\ldots\}} \mathfrak{m}\cdot x_i^n = \underbrace{\left(\sum_{\mathfrak{m}\in\mathfrak{M}_k}\mathfrak{m}\right)}_{\substack{=h_k\\ \text{(by (62))}}} \cdot \underbrace{\sum_{i\in\{1,2,3,\ldots\}} x_i^n}_{=p_n}$$
$$= h_k p_n. \tag{66}$$

But (63) becomes

$$\begin{aligned} h_{n+k} &= \sum_{\mathfrak{n}\in\mathfrak{M}_{n+k}} \mathfrak{n} = \underbrace{\sum_{\mathfrak{n}\in\mathfrak{N}} \mathfrak{n}}_{\substack{=G(n,n+k)\\ \text{(by (65))}}} + \underbrace{\sum_{\mathfrak{n}\in\mathfrak{M}_{n+k}\setminus\mathfrak{N}} \mathfrak{n}}_{\substack{=h_k p_n\\ \text{(by (66))}}} \qquad (\text{since } \mathfrak{N}\subseteq\mathfrak{M}_{n+k}) \\ &= G(n,n+k) + h_k p_n. \end{aligned}$$

In other words, $G(n,n+k) = h_{n+k} - h_k p_n$. This proves Proposition 4.7. $\square$

We note in passing that the idea used in the above proof of Proposition 4.7 can be generalized to yield a second proof of Theorem 2.19, using an inclusion/exclusion argument.[27]

Corollary 4.8. *Let $n$ be an integer such that $n > 1$. Then:*

**(a)** *We have*

$$\sum_{\substack{\lambda\in\operatorname{Par}_n;\\ (n-1,1)\rhd\lambda}} m_\lambda = h_n - p_n.$$

**(b)** *We have*

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\rhd\lambda}} m_\lambda = h_{2n-1} - h_{n-1} p_n.$$

---

[27] Here is an outline of this second proof: For any positive integer $k$ and any $m\in\mathbb{N}$, we have

$$G(k,m) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{I\subseteq\{1,2,3,\ldots\}} (-1)^{|I|} \underbrace{\sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i\geqslant k \text{ for all } i\in I}} \mathbf{x}^\alpha}_{=\left(\prod_{i\in I} x_i^k\right)\cdot \sum_{\substack{\beta\in\mathrm{WC};\\ |\beta|=m-k|I|}} \mathbf{x}^\beta}$$

(by an infinite-set version of the inclusion-exclusion principle)

$$\begin{aligned} &= \underbrace{\sum_{I\subseteq\{1,2,3,\ldots\}}}_{=\sum_{p\in\mathbb{N}} \sum_{\substack{I\subseteq\{1,2,3,\ldots\};\\ |I|=p}}} (-1)^{|I|} \left(\prod_{i\in I} x_i^k\right)\cdot \underbrace{\sum_{\substack{\beta\in\mathrm{WC};\\ |\beta|=m-k|I|}} \mathbf{x}^\beta}_{=h_{m-k|I|}} \\ &= \sum_{p\in\mathbb{N}} \sum_{\substack{I\subseteq\{1,2,3,\ldots\};\\ |I|=p}} \underbrace{(-1)^{|I|}}_{=(-1)^p} \left(\prod_{i\in I} x_i^k\right)\cdot \underbrace{h_{m-k|I|}}_{\substack{=h_{m-kp}\\ (\text{since } |I|=p)}} \\ &= \sum_{p\in\mathbb{N}} (-1)^p h_{m-kp} \underbrace{\sum_{\substack{I\subseteq\{1,2,3,\ldots\};\\ |I|=p}} \prod_{i\in I} x_i^k}_{\substack{=\mathbf{f}_k(e_p)\\ \text{(this is easy to check)}}} = \sum_{p\in\mathbb{N}} (-1)^p h_{m-kp}\cdot \mathbf{f}_k(e_p) \\ &= \sum_{i\in\mathbb{N}} (-1)^i h_{m-ki}\cdot \mathbf{f}_k(e_i). \end{aligned}$$

*Proof.* **(b)** Corollary 4.6 **(b)** yields

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\rhd\lambda}} m_\lambda = G\left(n,2n-1\right) = h_{2n-1} - h_{n-1}p_n$$

(by Proposition 4.7, applied to $k=n-1$). This proves Corollary 4.8 **(b)**.

**(a)** Corollary 4.6 **(a)** yields

$$\sum_{\substack{\lambda\in\operatorname{Par}_{n};\\ (n-1,1)\rhd\lambda}} m_\lambda = G\left(n,n\right) = h_n - h_0 p_n$$

(by Proposition 4.7, applied to $k=0$). Since $h_0=1$, this proves Corollary 4.8 **(a)**. $\square$

Our next claim is an easy consequence of Proposition 1.1:

Corollary 4.9. *Let $n$ be a positive integer. Then,*

$$h_n - p_n = \sum_{i=0}^{n-2} \left(-1\right)^i s_{\left(n-1-i,1^{i+1}\right)}.$$

*Proof.* Proposition 1.1 yields

$$\begin{aligned} p_n &= \sum_{i=0}^{n-1} \left(-1\right)^i s_{\left(n-i,1^i\right)} = \underbrace{\left(-1\right)^0}_{=1} \underbrace{s_{\left(n-0,1^0\right)}}_{=s_{\left(n-0\right)}=s_{\left(n\right)}=h_n} + \sum_{i=1}^{n-1} \left(-1\right)^i s_{\left(n-i,1^i\right)} \\ &= h_n + \sum_{i=1}^{n-1} \left(-1\right)^i s_{\left(n-i,1^i\right)}, \end{aligned}$$

so that

$$\begin{aligned} h_n - p_n &= -\sum_{i=1}^{n-1} \left(-1\right)^i s_{\left(n-i,1^i\right)} = \sum_{i=1}^{n-1} \underbrace{\left(-\left(-1\right)^i\right)}_{=\left(-1\right)^{i-1}} s_{\left(n-i,1^i\right)} = \sum_{i=1}^{n-1} \left(-1\right)^{i-1} s_{\left(n-i,1^i\right)} \\ &= \sum_{i=0}^{n-2} \left(-1\right)^i s_{\left(n-1-i,1^{i+1}\right)} \end{aligned}$$

(here, we have substituted $i+1$ for $i$ in the sum). $\square$

We can now immediately prove Theorem 4.2 **(a)**:

*Proof of Theorem 4.2 **(a)**.* Corollary 4.8 **(a)** yields

$$\sum_{\substack{\lambda\in\operatorname{Par}_{n};\\ (n-1,1)\rhd\lambda}} m_\lambda = h_n - p_n = \sum_{i=0}^{n-2} \left(-1\right)^i s_{\left(n-1-i,1^{i+1}\right)} \qquad \text{(by Corollary 4.9)}.$$

This proves Theorem 4.2 **(a)**. $\square$

We shall use the *skewing operators* $f^\perp : \Lambda \to \Lambda$ for all $f \in \Lambda$ as defined in [17, §2.8] or in [20, Chapter I, Section 5, Example 3]. The easiest way to define them (following [20, Chapter I, Section 5, Example 3]) is as follows: For each $f \in \Lambda$, we let $f^\perp : \Lambda \to \Lambda$ be the $\mathbf{k}$-linear map adjoint to the map $L_f : \Lambda \to \Lambda,\ g \mapsto fg$ (that is, to the map that multiplies every element of $\Lambda$ by $f$) with respect to the Hall inner product. That is, $f^\perp$ is the $\mathbf{k}$-linear map from $\Lambda$ to $\Lambda$ that satisfies

$$\left\langle g, f^\perp\left(a\right)\right\rangle = \left\langle fg, a\right\rangle \qquad \text{for all } a\in\Lambda \text{ and } g\in\Lambda.$$

It is not hard to show that such an operator $f^{\perp}$ exists.[28] The definition of $f^{\perp}$ in [17, §2.8] is different but equivalent (because of [17, Proposition 2.8.2(i)]). One of the most important properties of skewing operators is the following fact ([17, (2.8.2)]):

Lemma 4.10. *Let $\lambda$ and $\mu$ be any two partitions. Then,*

$$s_{\mu}^{\perp}\left(s_{\lambda}\right)=s_{\lambda/\mu}. \tag{67}$$

(*Here, $s_{\lambda/\mu}$ is a skew Schur function, defined in Subsection* 3.2.)

Using skewing operators, we can define another helpful family of operators on $\Lambda$:

Definition 4.11. *For any $m\in\mathbb{Z}$, we define a map $\mathbf{B}_m:\Lambda\to\Lambda$ by setting*

$$\mathbf{B}_m\left(f\right)=\sum_{i\in\mathbb{N}}\left(-1\right)^i h_{m+i}e_i^{\perp}f \qquad \text{for all } f\in\Lambda.$$

*It is known (*[17, Exercise 2.9.1(a)]*) that this map $\mathbf{B}_m$ is well-defined and $\mathbf{k}$-linear.*

(Actually, the well-definedness of $\mathbf{B}_m$ is easy to check: If $f\in\Lambda$ has degree $d$, then all integers $i>d$ satisfy $e_i^{\perp}f=0$ for degree reasons, and thus the sum $\sum_{i\in\mathbb{N}}\left(-1\right)^i h_{m+i}e_i^{\perp}f$ has only finitely many nonzero addends. The $\mathbf{k}$-linearity of $\mathbf{B}_m$ is even clearer.)

The operators $\mathbf{B}_m$ for $m\in\mathbb{Z}$ have first appeared in Zelevinsky's [29, §4.20] (in the different-looking but secretly equivalent setting of a PSH-algebra), where they are credited to J. N. Bernstein. They have since been dubbed the *Bernstein creation operators* and proved useful in various contexts (e.g. the definition of the "dual immaculate functions" in [4] takes them for inspiration). One of their most fundamental properties is the following fact (which originates in [29, 4.20, (∗∗)] and appears implicitly in [20, Chapter I, Section 5, Example 29]):

Proposition 4.12. *Let $\lambda$ be any partition. Let $m\in\mathbb{Z}$ satisfy $m\geqslant\lambda_1$. Then,*

$$\sum_{i\in\mathbb{N}}\left(-1\right)^i h_{m+i}e_i^{\perp}s_{\lambda}=s_{(m,\lambda_1,\lambda_2,\lambda_3,\ldots)}. \tag{68}$$

See [17, Exercise 2.9.1(b)] for a proof of Proposition 4.12. Thus, if $\lambda$ is any partition, and if $m\in\mathbb{Z}$ satisfies $m\geqslant\lambda_1$, then

$$\begin{aligned}\mathbf{B}_m\left(s_{\lambda}\right)&=\sum_{i\in\mathbb{N}}\left(-1\right)^i h_{m+i}e_i^{\perp}s_{\lambda} && \text{(by the definition of } \mathbf{B}_m\text{)}\\ &=s_{(m,\lambda_1,\lambda_2,\lambda_3,\ldots)} && \text{(by (68))}. \qquad (69)\end{aligned}$$

Lemma 4.13. *Let $n$ be a positive integer. Let $m\in\mathbb{N}$. Then, $\mathbf{B}_m\left(h_n\right)=h_mh_n-h_{m+1}h_{n-1}$.*

*Proof of Lemma 4.13.* We have $e_0=1$ and thus $e_0^{\perp}=1^{\perp}=\mathrm{id}$. Hence, $e_0^{\perp}\left(h_n\right)=h_n$.

We shall use the notion of skew Schur functions $s_{\lambda/\mu}$ (as in Subsection 3.2). Recall that $s_{\lambda/\mu}=0$ when $\mu\not\subseteq\lambda$.

From $e_1=s_{(1)}$ and $h_n=s_{(n)}$, we obtain

$$e_1^{\perp}\left(h_n\right)=s_{(1)}^{\perp}\left(s_{(n)}\right)=s_{(n)/(1)} \qquad \text{(by (67))}.$$

[28] This is not completely automatic: Not every $\mathbf{k}$-linear map from $\Lambda$ to $\Lambda$ has an adjoint with respect to the Hall inner product! (For example, the $\mathbf{k}$-linear map $\Lambda\to\Lambda$ that sends each Schur function $s_{\lambda}$ to 1 has none.) The reason why the map $L_f:\Lambda\to\Lambda,\ g\mapsto fg$ has an adjoint is that when $f$ is homogeneous of degree $k$, this map $L_f$ sends each graded component $\Lambda_m$ of $\Lambda$ to $\Lambda_{m+k}$, and both of these graded components $\Lambda_m$ and $\Lambda_{m+k}$ are $\mathbf{k}$-modules with **finite** bases. (The case when $f$ is not homogeneous can be reduced to the case when $f$ is homogeneous, since each $f\in\Lambda$ is a sum of finitely many homogeneous elements.)

But it is easy to see that $s_{(n)/(1)} = s_{(n-1)}$. (Indeed, this follows from the combinatorial definition of skew Schur functions, since the skew Ferrers diagram of $(n)/(1)$ can be obtained from the Ferrers diagram of $(n-1)$ by parallel shift[29]. Alternatively, this follows easily from Theorem 3.3, because $s_{(n-1)} = h_{n-1}$.)

Thus, we obtain

$$e_1^{\perp}(h_n) = s_{(n)/(1)} = s_{(n-1)} = h_{n-1}.$$

For each integer $i > 1$, we have

$$\begin{aligned} e_i^{\perp}(h_n) &= s_{(1^i)}^{\perp}\left(s_{(n)}\right) && \left(\text{since } e_i = s_{(1^i)} \text{ and } h_n = s_{(n)}\right) \\ &= s_{(n)/(1^i)} && (\text{by (67)}) \\ &= 0 && \left(\text{since } \left(1^i\right) \not\subseteq (n) \ \ (\text{because } i > 1)\right). \end{aligned} \tag{70}$$

Now, the definition of $\mathbf{B}_m$ yields

$$\begin{aligned} \mathbf{B}_m(h_n) &= \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^{\perp}(h_n) \\ &= \underbrace{(-1)^0}_{=1} \underbrace{h_{m+0}}_{=h_m} \underbrace{e_0^{\perp}(h_n)}_{=h_n} + \underbrace{(-1)^1}_{=-1} h_{m+1} \underbrace{e_1^{\perp}(h_n)}_{=h_{n-1}} + \sum_{i \geqslant 2} (-1)^i h_{m+i} \underbrace{e_i^{\perp}(h_n)}_{\substack{=0 \\ (\text{by (70)})}} \\ &= h_m h_n - h_{m+1} h_{n-1}. \end{aligned}$$

□

Corollary 4.14. *Let $n$ be a positive integer. Then, $\mathbf{B}_{n-1}(h_n) = 0$.*

*Proof.* Apply Lemma 4.13 to $m = n-1$ and simplify. □

Lemma 4.15. *Let $m \in \mathbb{N}$. Let $n$ be a positive integer. Then, $\mathbf{B}_m(p_n) = h_m p_n - h_{m+n}$.*

*Proof.* This is [17, Exercise 2.9.1(f)]. But here is a more direct proof: We will use the comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$ of the Hopf algebra $\Lambda$ (see [17, §2.3]). Here and in the following, the "$\otimes$" sign denotes $\otimes_{\mathbf{k}}$. The power-sum symmetric function $p_n$ is primitive[30] (see [17, Proposition 2.3.6(i)]); thus,

$$\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1.$$

Hence, for each $i \in \mathbb{N}$, the definition of $e_i^{\perp}$ given in [17, Definition 2.8.1] (not the equivalent definition we gave above) yields

$$e_i^{\perp}(p_n) = \langle e_i, 1 \rangle p_n + \langle e_i, p_n \rangle 1. \tag{71}$$

[29] See [17, §2.3] for the notions we are using here.

[30] Recall that an element $x$ of a Hopf algebra $H$ is said to be *primitive* if the comultiplication $\Delta_H$ of $H$ satisfies $\Delta_H(x) = 1 \otimes x + x \otimes 1$.

Now, the definition of $\mathbf{B}_m$ yields

$$\begin{aligned}
\mathbf{B}_m(p_n) &= \sum_{i\in\mathbb{N}} (-1)^i h_{m+i} \underbrace{e_i^{\perp}(p_n)}_{\substack{=\langle e_i,1\rangle p_n+\langle e_i,p_n\rangle 1\\ \text{(by (71))}}} \\
&= \sum_{i\in\mathbb{N}} (-1)^i h_{m+i} \left(\langle e_i,1\rangle p_n + \langle e_i,p_n\rangle 1\right) \\
&= \underbrace{\sum_{i\in\mathbb{N}} (-1)^i h_{m+i}\cdot\langle e_i,1\rangle p_n}_{\substack{=(-1)^0 h_{m+0}\cdot\langle e_0,1\rangle p_n\\ \text{(because the Hall inner product } \langle e_i,1\rangle\\ \text{equals 0 whenever } i\neq 0 \text{ (by (2)),}\\ \text{and thus the only nonzero addend of this}\\ \text{sum is the addend for } i=0)}} + \underbrace{\sum_{i\in\mathbb{N}} (-1)^i h_{m+i}\cdot\langle e_i,p_n\rangle 1}_{\substack{=(-1)^n h_{m+n}\cdot\langle e_n,p_n\rangle 1\\ \text{(because the Hall inner product } \langle e_i,p_n\rangle\\ \text{equals 0 whenever } i\neq n \text{ (by (2)),}\\ \text{and thus the only nonzero addend of this}\\ \text{sum is the addend for } i=n)}} \\
&= \underbrace{(-1)^0}_{=1} \underbrace{h_{m+0}}_{=h_m}\cdot \underbrace{\langle e_0,1\rangle}_{=\langle 1,1\rangle=1} p_n + (-1)^n h_{m+n}\cdot \underbrace{\langle e_n,p_n\rangle}_{\substack{=(-1)^{n-1}\\ \text{(by Proposition 1.3)}}} 1 \\
&= h_m p_n + \underbrace{(-1)^n h_{m+n}\cdot(-1)^{n-1} 1}_{=-h_{m+n}} = h_m p_n - h_{m+n}. \qquad \square
\end{aligned}$$

Lemma 4.16. *Let $n$ be a positive integer. Then,*

$$\mathbf{B}_{n-1}(h_n - p_n) = h_{2n-1} - h_{n-1}p_n.$$

*Proof.* The map $\mathbf{B}_{n-1}$ is $\mathbf{k}$-linear. Thus,

$$\begin{aligned}
\mathbf{B}_{n-1}(h_n - p_n) &= \underbrace{\mathbf{B}_{n-1}(h_n)}_{\substack{=0\\ \text{(by Corollary 4.14)}}} - \underbrace{\mathbf{B}_{n-1}(p_n)}_{\substack{=h_{n-1}p_n - h_{(n-1)+n}\\ \text{(by Lemma 4.15,}\\ \text{applied to } m=n-1)}} \\
&= -\left(h_{n-1}p_n - h_{(n-1)+n}\right) = \underbrace{h_{(n-1)+n}}_{=h_{2n-1}} - h_{n-1}p_n = h_{2n-1} - h_{n-1}p_n.
\end{aligned}$$

$\square$

Lemma 4.17. *Let $n$ be a positive integer. Then,*

$$\mathbf{B}_{n-1}(h_n - p_n) = \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})}.$$

*Proof of Lemma 4.17.* We have

$$\begin{aligned}
\mathbf{B}_{n-1}(h_n - p_n) &= \mathbf{B}_{n-1}\left(\sum_{i=0}^{n-2} (-1)^i s_{(n-1-i,1^{i+1})}\right) && \text{(by Corollary 4.9)} \\
&= \sum_{i=0}^{n-2} (-1)^i \underbrace{\mathbf{B}_{n-1}\left(s_{(n-1-i,1^{i+1})}\right)}_{\substack{=s_{(n-1,n-1-i,1^{i+1})}\\ \text{(by (69), applied to } m=n-1\\ \text{and } \lambda=\left(n-1-i,1^{i+1}\right)\\ \text{(since } n-1\geqslant n-1-i))}} && \text{(since } \mathbf{B}_{n-1} \text{ is } \mathbf{k}\text{-linear)} \\
&= \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})}. && \square
\end{aligned}$$

Now the proof of Theorem 4.2 **(b)** is a trivial concatenation of equalities:

*Proof of Theorem 4.2* ***(b)***. Corollary 4.8 **(b)** yields

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\rhd\lambda}} m_\lambda = h_{2n-1} - h_{n-1}p_n = \mathbf{B}_{n-1}\left(h_n - p_n\right) \qquad \text{(by Lemma 4.16)}$$
$$= \sum_{i=0}^{n-2} (-1)^i \, s_{(n-1,n-1-i,1^{i+1})} \qquad \text{(by Lemma 4.17)}. \qquad \square$$

## 5. Final remarks

5.1. SageMath code. The SageMath computer algebra system [26] does not (yet) natively know the Petrie symmetric functions $G(k,m)$; but they can be easily constructed in it. For example, the code that follows computes $G(k,n)$ expanded in the Schur basis:

```
Sym = SymmetricFunctions(QQ) # Replace QQ by your favorite base ring.
m = Sym.m() # monomial symmetric functions
s = Sym.s() # Schur functions

def G(k, n): # a Petrie function
  return s(m.sum(m[lam] for lam in Partitions(n, max_part=k-1)))
```

5.2. Understanding the Petrie numbers. Combining Corollary 2.9 with Theorem 2.13 yields an explicit expression of all coefficients in the expansion of a Petrie symmetric function $G(k,m)$ in the Schur basis. It would stand to reason if the identity in Theorem 4.2 **(b)** (whose left hand side is $G(n,2n-1)$) could be obtained from this expression. Surprisingly, we have been unable to do so, which suggests that the description of $\operatorname{pet}_k(\lambda,\varnothing)$ in Theorem 2.13 might not be optimal.

As to $\operatorname{pet}_k(\lambda,\mu)$, we do not have an explicit description at all, unless we count the recursive one that can be extracted from the proof in [13].

5.3. MNable symmetric functions. Combining Theorem 2.15 with Proposition 2.7, we conclude that for any $k>0$ and $m\in\mathbb{N}$, the symmetric function $G(k,m)\in\Lambda$ has the following property: For any $\mu\in\operatorname{Par}$, its product $G(k,m)\cdot s_\mu$ with $s_\mu$ can be written in the form $\sum\limits_{\lambda\in\operatorname{Par}} u_\lambda s_\lambda$ with $u_\lambda\in\{-1,0,1\}$ for all $\lambda\in\operatorname{Par}$. It has this property in common with the symmetric functions $h_m$ and $e_m$ (according to the Pieri rules) and $p_m$ (according to the Murnaghan–Nakayama rule) as well as several others. The study of symmetric functions having this property – which we call *MNable symmetric functions* (in honor of Murnaghan and Nakayama) – has been initiated in [14, §8], but there is much to be done.

5.4. A conjecture of Per Alexandersson. In February 2020, Per Alexandersson suggested the following conjecture:

Conjecture 5.1. *Let $k$ be a positive integer, and $m\in\mathbb{N}$. Then, $G(k,m)\cdot p_2\in\Lambda$ can be written in the form $\sum\limits_{\lambda\in\operatorname{Par}} u_\lambda s_\lambda$ with $u_\lambda\in\{-1,0,1\}$ for all $\lambda\in\operatorname{Par}$.*

For example,

$$G(3,4)\cdot p_2 = s_{(1,1,1,1,1,1)} + s_{(2,2,2)} - s_{(3,1,1,1)} - s_{(3,3)} + s_{(4,2)}.$$

Conjecture 5.1 has been verified for all $k$ and $m$ satisfying $k+m\leqslant 30$.

Note that Conjecture 5.1 becomes false if $p_2$ is replaced by $p_3$. For example,

$$G(3,4)\cdot p_3 = -s_{(1,1,1,1,1,1,1)} + s_{(2,2,1,1,1)} - 2s_{(2,2,2,1)} + s_{(3,2,1,1)} - s_{(4,1,1,1)} - s_{(4,3)} + s_{(5,2)}.$$

5.5. A conjecture of François Bergeron. An even more mysterious conjecture was suggested by François Bergeron in April 2020:

Conjecture 5.2. *Let $k$ and $n$ be positive integers, and $m \in \mathbb{N}$. Let $\nabla$ be the* nabla operator *as defined (e.g. ) in* [5, §3.2.1]. *Then, there exists a sign $\sigma_{n,k,m} \in \{1, -1\}$ such that $\sigma_{n,k,m} \nabla^n (G(k,m))$ is an $\mathbb{N}[q,t]$-linear combination of Schur functions.*

Using SageMath, this conjecture has been checked for $n = 1$ and all $k, m \in \{0, 1, \dots, 9\}$; the signs $\sigma_{1,k,m}$ are given by the following table:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 2 | + | + | + | + | + | + | + | + | + |
| 3 | + | − | − | − | + | + | + | − | − |
| 4 | + | − | + | + | + | − | + | + | + |
| 5 | + | − | + | − | − | − | + | − | + |
| 6 | + | − | + | − | + | + | + | − | + |
| 7 | + | − | + | − | + | − | − | − | + |
| 8 | + | − | + | − | + | − | + | + | + |
| 9 | + | − | + | − | + | − | + | − | − |

(where the entry in the row indexed $k$ and the column indexed $m$ is the sign $\sigma_{1,k,m}$, represented by a "+" sign if it is 1 and by a "−" sign if it is $-1$). I am not aware of a pattern in these signs, apart from the fact that $\sigma_{1,2,m} = 1$ for all $m \in \mathbb{N}$ (a consequence of Haiman's famous interpretation of $\nabla(e_m)$ as a character), and that $\sigma_{1,k,m}$ appears to be $(-1)^{m-1}$ whenever $1 \leqslant m < k$ (which would follow from the conjecture that $(-1)^{m-1} \nabla(h_m)$ is an $\mathbb{N}[q,t]$-linear combination of Schur functions for any $m \geqslant 1$).

5.6. "Petriefication" of Schur functions. Theorem 2.26 shows the existence of a Hopf algebra homomorphism $V_k : \Lambda \to \Lambda$ that sends the complete homogeneous symmetric functions $h_1, h_2, h_3, \dots$ to the Petrie symmetric functions $G(k,1), G(k,2), G(k,3), \dots$. It thus is natural to consider the images of all Schur functions $s_\lambda$ under this homomorphism $V_k$. Experiments with small $\lambda$'s may suggest that these images $V_k(s_\lambda)$ all can be written in the form $\sum_{\lambda \in \mathrm{Par}} u_\lambda s_\lambda$ with $u_\lambda \in \{-1, 0, 1\}$. But this is not generally the case; counterexamples include $V_3\left(s_{(4,4,4)}\right)$, $V_4\left(s_{(4,4)}\right)$ and $V_4\left(s_{(5,1,1,1,1)}\right)$. (Of course, it is true when $\lambda$ is a single row, because of $V_k\left(s_{(m)}\right) = V_k(h_m) = G(k,m)$; and it is also true when $\lambda$ is a single column, because the Hopf algebra homomorphism $V_k$ commutes with the antipode $S$ that sends $h_m \mapsto (-1)^m e_m$ and $s_\lambda \mapsto (-1)^{|\lambda|} s_{\lambda^t}$.)

Note that these images $V_k(s_\lambda)$ are precisely the *modular Schur functions* $s'_\lambda$ studied in [27].

5.7. Postnikov's generalization. At the MIT Algebraic Combinatorics preseminar roundtable (2020), Alexander Postnikov has suggested a generalization of the Petrie symmetric functions that preserves some of their more elementary properties. In this subsection, we shall survey this generalization.

Proofs will be sketched (at best); the reader can find the details in the detailed version [16].

Convention 4. *We fix a formal power series $F \in \mathbf{k}[[t]]$ whose constant term is* 1. *(We will keep this $F$ fixed throughout the present subsection.)*

The notations in the following definition will also be used throughout this subsection:

Definition 5.3.

**(a)** *Let $f_0, f_1, f_2, \ldots$ be the coefficients of the formal power series $F$, so that $F = \sum_{n \in \mathbb{N}} f_n t^n$. Thus, $f_0 = 1$ (by Convention 4).*

**(b)** *We set $f_i = 0$ for each negative integer $i$.*

**(c)** *For any weak composition $\alpha$, we define an element $f_\alpha \in \mathbf{k}$ by*

$$f_\alpha = f_{\alpha_1} f_{\alpha_2} f_{\alpha_3} \cdots .$$

*(Here, the infinite product $f_{\alpha_1} f_{\alpha_2} f_{\alpha_3} \cdots$ is well-defined, since every sufficiently high positive integer $i$ satisfies $\alpha_i = 0$ and thus $f_{\alpha_i} = f_0 = 1$.)*

**(d)** *We define the power series*

$$G_F = \sum_{\alpha \in \mathrm{WC}} f_\alpha \mathbf{x}^\alpha. \tag{72}$$

*This is a formal power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.*

**(e)** *For any $m \in \mathbb{N}$, we define the power series*

$$G_{F,m} = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m}} f_\alpha \mathbf{x}^\alpha. \tag{73}$$

*This is a formal power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.*

Example 5.4. Let us see how these power series $G_F$ and $G_{F,m}$ look for specific values of $F$.

**(a)** Let $F = \dfrac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$. Then, $f_i = 1$ for each $i \in \mathbb{N}$. Hence, $f_\alpha = 1$ for any weak composition $\alpha$. Thus,

$$G_F = \sum_{\alpha \in \mathrm{WC}} \underbrace{f_\alpha}_{=1} \mathbf{x}^\alpha = \sum_{\alpha \in \mathrm{WC}} \mathbf{x}^\alpha$$

and

$$G_{F,m} = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m}} \underbrace{f_\alpha}_{=1} \mathbf{x}^\alpha = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m}} \mathbf{x}^\alpha = h_m \qquad \text{for each } m \in \mathbb{N}.$$

**(b)** Now, let $F = 1$. Then, $f_i = [i = 0]$ for each $i \in \mathbb{N}$ (where we are again using Convention 1). Hence, $f_\alpha = [\alpha = \varnothing]$ for any weak composition $\alpha$. Thus, it is easy to see that $G_F = 1$ and $G_{F,m} = [m = 0]$ for each $m \in \mathbb{N}$.

**(c)** Now, fix a positive integer $k$, and set $F = 1 + t + t^2 + \cdots + t^{k-1}$. Then, $f_i = [i < k]$ for each $i \in \mathbb{N}$. Hence, $f_\alpha = \prod_{i \geqslant 1} [\alpha_i < k] = [\alpha_i < k \text{ for all } i]$ for any weak composition $\alpha$. Thus,

$$\begin{aligned} G_F &= \sum_{\alpha \in \mathrm{WC}} \underbrace{f_\alpha}_{=[\alpha_i < k \text{ for all } i]} \mathbf{x}^\alpha = \sum_{\alpha \in \mathrm{WC}} [\alpha_i < k \text{ for all } i]\, \mathbf{x}^\alpha \\ &= \sum_{\substack{\alpha \in \mathrm{WC};\\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha = G(k). \end{aligned}$$

Likewise, we can see that $G_{F,m} = G(k, m)$ for each $m \in \mathbb{N}$. This shows that the $G_F$ and the $G_{F,m}$ are generalizations of the Petrie symmetric series $G(k)$ and the Petrie symmetric functions $G(k, m)$, respectively.

The next proposition generalizes parts **(a)**–**(c)** of Proposition 2.3:

Proposition 5.5.

**(a)** *The formal power series $G_{F,m}$ is the $m$-th degree homogeneous component of $G_F$ for each $m \in \mathbb{N}$.*

**(b)** *We have*

$$G_F = \sum_{\alpha \in \mathrm{WC}} f_\alpha \mathbf{x}^\alpha = \sum_{\lambda \in \mathrm{Par}} f_\lambda m_\lambda = \prod_{i=1}^{\infty} F(x_i).$$

**(c)** *We have*

$$G_{F,m} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m}} f_\alpha \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = m}} f_\lambda m_\lambda \in \Lambda$$

*for each $m \in \mathbb{N}$.*

**(d)** *The formal power series $G_F$ is symmetric.*

**(e)** *We have $G_{F,0} = 1$.*

*Proof of Proposition 5.5.* Parts **(a)**–**(c)** of Proposition 5.5 generalize the corresponding parts of Proposition 2.3, and are proved more or less analogously. (The only novelty is the use of a fact that says that $f_\alpha = f_\lambda$ whenever a weak composition $\alpha$ is obtained by permuting the entries of a partition $\lambda$. Of course, this fact follows from the definitions of $f_\alpha$ and $f_\lambda$.)

Part **(d)** of Proposition 5.5 is clear from part **(b)**. Part **(e)** follows from $f_\varnothing = 1$. □

Next, let us generalize Definition 2.4:

Definition 5.6. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathrm{Par}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \mathrm{Par}$. Then, the $F$-*Petrie number *$\mathrm{pet}_F(\lambda, \mu)$ of $\lambda$ and $\mu$ is the element of $\mathbf{k}$ defined by*

$$\mathrm{pet}_F(\lambda, \mu) = \det\left(\left(f_{\lambda_i - \mu_j - i + j}\right)_{1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \ell}\right). \tag{74}$$

*Note that this integer does not depend on the choice of $\ell$ (in the sense that it does not change if we enlarge $\ell$ by adding trailing zeroes to the representations of $\lambda$ and $\mu$); this follows from Lemma 5.8 below.*

Example 5.7. For $\ell = 3$, the equality (74) rewrites as

$$\mathrm{pet}_F(\lambda, \mu) = \det\begin{pmatrix} f_{\lambda_1 - \mu_1} & f_{\lambda_1 - \mu_2 + 1} & f_{\lambda_1 - \mu_3 + 2} \\ f_{\lambda_2 - \mu_1 - 1} & f_{\lambda_2 - \mu_2} & f_{\lambda_2 - \mu_3 + 1} \\ f_{\lambda_3 - \mu_1 - 2} & f_{\lambda_3 - \mu_2 - 1} & f_{\lambda_3 - \mu_3} \end{pmatrix}.$$

We can now state the generalization of Lemma 2.6 that is needed to justify Definition 5.6:

Lemma 5.8. *Let $\lambda \in \mathrm{Par}$ and $\mu \in \mathrm{Par}$. Let $\ell \in \mathbb{N}$ be such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$. Then, the determinant $\det\left(\left(f_{\lambda_i - \mu_j - i + j}\right)_{1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \ell}\right)$ does not depend on the choice of $\ell$.*

The slickest way to prove Lemma 5.8 is using a $\mathbf{k}$-algebra homomorphism $\alpha_F : \Lambda \to \mathbf{k}$ that generalizes the $\alpha_k$ from Definition 3.5. Let us introduce this $\alpha_F$:

Definition 5.9. *The $h$-universal property of $\Lambda$ (see Subsection 3.4) shows that there is a unique $\mathbf{k}$-algebra homomorphism $\alpha_F : \Lambda \to \mathbf{k}$ that sends $h_i$ to $f_i$ for all positive integers $i$. Consider this $\alpha_F$.*

For future use, we state some elementary properties of $\alpha_F$.

Lemma 5.10.

**(a)** *We have*

$$\alpha_F(h_i) = f_i \qquad \text{for all } i \in \mathbb{N}.$$

**(b)** *We have*

$$\alpha_F(h_i) = f_i \qquad \textit{for all } i \in \mathbb{Z}.$$

**(c)** *Let $\lambda$ be a partition. Define $h_\lambda$ as in Definition* 3.1*. Then,*

$$\alpha_F(h_\lambda) = f_\lambda.$$

*Proof of Lemma 5.10.* Analogous to the proof of Lemma 3.6. □

*Proof of Lemma 5.8.* Adapt the proof of Lemma 2.6, using $\alpha_F$ instead of $\alpha_k$. □

We now come to more substantive properties of $G_F$ and $G_{F,m}$.

The following theorem generalizes Theorem 2.8:

Theorem 5.11. *We have*

$$G_F = \sum_{\lambda \in \mathrm{Par}} \mathrm{pet}_F(\lambda, \varnothing)\, s_\lambda.$$

*(Recall that $\varnothing$ denotes the empty partition $() = (0,0,0,\ldots)$.)*

The following corollary (which already appeared in [25, Exercise 7.91 (d)]) generalizes Corollary 2.9:

Corollary 5.12. *Let $m \in \mathbb{N}$. Then,*

$$G_{F,m} = \sum_{\lambda \in \mathrm{Par}_m} \mathrm{pet}_F(\lambda, \varnothing)\, s_\lambda.$$

The following theorem generalizes Theorem 2.15:

Theorem 5.13. *Let $\mu \in \mathrm{Par}$. Then,*

$$G_F \cdot s_\mu = \sum_{\lambda \in \mathrm{Par}} \mathrm{pet}_F(\lambda, \mu)\, s_\lambda.$$

The following corollary generalizes Corollary 2.16:

Corollary 5.14. *Let $m \in \mathbb{N}$. Let $\mu \in \mathrm{Par}$. Then,*

$$G_{F,m} \cdot s_\mu = \sum_{\lambda \in \mathrm{Par}_{m+|\mu|}} \mathrm{pet}_F(\lambda, \mu)\, s_\lambda.$$

*Proofs of Theorem 5.13, Corollary 5.14, Theorem 5.11 and Corollary 5.12.* These proofs are analogous to the proofs of Theorem 2.15, Corollary 2.16, Theorem 2.8 and Corollary 2.9, respectively (but using $\alpha_F$ instead of $\alpha_k$). □

Proposition 5.5 **(c)** shows that $G_{F,m} \in \Lambda$ for each $m \in \mathbb{N}$. Hence, we can apply the comultiplication $\Delta$ of the Hopf algebra $\Lambda$ to $G_{F,m}$. The next theorem (which generalizes Theorem 2.17) gives a simple expression for the result of this:

Theorem 5.15. *Let $m \in \mathbb{N}$. Then,*

$$\Delta(G_{F,m}) = \sum_{i=0}^{m} G_{F,i} \otimes G_{F,m-i}.$$

*Proof of Theorem 5.15.* Proposition 5.5 **(b)** tells us that $G_F = \prod_{i=1}^{\infty} F(x_i)$. However, Proposition 5.5 **(a)** yields $G_F = \sum_{k \in \mathbb{N}} G_{F,k} = \sum_{k \in \mathbb{N}} G_{F,k}(\mathbf{x})$. Comparing these two equalities, we find

$$\sum_{k \in \mathbb{N}} G_{F,k}(\mathbf{x}) = \prod_{i=1}^{\infty} F(x_i). \tag{75}$$

Substituting $y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ in this equality, we obtain

$$\sum_{k\in\mathbb{N}} G_{F,k}(\mathbf{y}) = \prod_{i=1}^{\infty} F(y_i). \tag{76}$$

Substituting the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ on both sides of the equality $G_F = \prod_{i=1}^{\infty} F(x_i)$, we obtain

$$G_F(\mathbf{x}, \mathbf{y}) = \left(\prod_{i=1}^{\infty} F(x_i)\right)\left(\prod_{i=1}^{\infty} F(y_i)\right) = \left(\sum_{k\in\mathbb{N}} G_{F,k}(\mathbf{x})\right)\left(\sum_{k\in\mathbb{N}} G_{F,k}(\mathbf{y})\right)$$

(by (75) and (76)). Comparing the $m$-th degree homogeneous components of both sides of this equality, we find

$$G_{F,m}(\mathbf{x}, \mathbf{y}) = \sum_{i\in\{0,1,\ldots,m\}} G_{F,i}(\mathbf{x})\, G_{F,m-i}(\mathbf{y})$$

(since Proposition 5.5 **(a)** shows that the $m$-th degree homogeneous component of $G_F(\mathbf{x}, \mathbf{y})$ is $G_{F,m}(\mathbf{x}, \mathbf{y})$, whereas the homogeneity of the $G_{F,k}$'s shows that the $m$-th degree homogeneous component of $\left(\sum_{k\in\mathbb{N}} G_{F,k}(\mathbf{x})\right)\left(\sum_{k\in\mathbb{N}} G_{F,k}(\mathbf{y})\right)$ is $\sum_{i\in\{0,1,\ldots,m\}} G_{F,i}(\mathbf{x})\, G_{F,m-i}(\mathbf{y})$). Hence, (10) holds for $f = G_{F,m}$, $I = \{0, 1, \ldots, m\}$, $(f_{1,i})_{i\in I} = (G_{F,i})_{i\in\{0,1,\ldots,m\}}$ and $(f_{2,i})_{i\in I} = (G_{F,m-i})_{i\in\{0,1,\ldots,m\}}$. Therefore, (9) (applied to these $f$, $I$, $(f_{1,i})_{i\in I}$ and $(f_{2,i})_{i\in I}$) yields

$$\Delta(G_{F,m}) = \sum_{i\in\{0,1,\ldots,m\}} G_{F,i} \otimes G_{F,m-i} = \sum_{i=0}^{m} G_{F,i} \otimes G_{F,m-i}.$$

This proves Theorem 5.15. $\square$

The next few results we will state rely on the following definition:

Definition 5.16. *Let $F'$ be the derivative of the formal power series $F \in \mathbf{k}[[t]]$. Let us write the formal power series $\frac{F'}{F} \in \mathbf{k}[[t]]$ (which is well-defined, since $F$ has constant term 1) in the form $\frac{F'}{F} = \sum_{n\in\mathbb{N}} \gamma_n t^n$ for some $\gamma_0, \gamma_1, \gamma_2, \ldots \in \mathbf{k}$.*

Example 5.17. Let us see how $F'$ and $\gamma_n$ look for specific values of $F$.

**(a)** Let $F = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$. Then, $F' = \frac{1}{(1-t)^2}$, so that

$$\frac{F'}{F} = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n\in\mathbb{N}} t^n.$$

Therefore, $\gamma_n = 1$ for each $n \in \mathbb{N}$.

**(b)** Now, let $F = 1$. Then, $F' = 0$, so that $\frac{F'}{F} = 0 = \sum_{n\in\mathbb{N}} 0t^n$. Therefore, $\gamma_n = 0$ for each $n \in \mathbb{N}$.

**(c)** Now, fix a positive integer $k$, and set $F = 1 + t + t^2 + \cdots + t^{k-1}$. Then, $F = \frac{1-t^k}{1-t}$, and thus a simple calculation using the quotient rule shows that

$F' = \dfrac{1+(k-1)\,t^k - kt^{k-1}}{(1-t)^2}$. Hence,

$$\frac{F'}{F} = \frac{1+(k-1)\,t^k - kt^{k-1}}{(1-t)\,(1-t^k)} = \underbrace{\frac{1}{1-t}}_{=\sum_{n\in\mathbb{N}} t^n} - kt^{k-1} \cdot \underbrace{\frac{1}{1-t^k}}_{=\sum_{n\in\mathbb{N}} (t^k)^n}$$

$$= \sum_{n\in\mathbb{N}} t^n - \underbrace{kt^{k-1}\cdot \sum_{n\in\mathbb{N}} \left(t^k\right)^n}_{= \sum_{\substack{n\in\mathbb{N};\\ k\mid n+1}} kt^n} = \sum_{n\in\mathbb{N}} t^n - \sum_{\substack{n\in\mathbb{N};\\ k\mid n+1}} kt^n$$

$$= \sum_{n\in\mathbb{N}} \left(1 - [k|n+1]\,k\right) t^n.$$

Therefore, $\gamma_n = 1 - [k|n+1]\,k$ for each $n \in \mathbb{N}$.

The next proposition is easily seen to generalize Proposition 2.22:

Proposition 5.18. *Let $m$ be a positive integer. Then, $\langle p_m, G_{F,m}\rangle = \gamma_{m-1}$.*

The proof of this proposition relies on the following property of the $\mathbf{k}$-algebra homomorphism $\alpha_F : \Lambda \to \mathbf{k}$ from Definition 5.9:

Lemma 5.19. *We have $\alpha_F(p_m) = \gamma_{m-1}$ for each positive integer $m$.*

*Proof of Lemma 5.19.* Consider the ring $\Lambda[[t]]$ of formal power series in one indeterminate $t$ over $\Lambda$. Consider also the analogous ring $\mathbf{k}[[t]]$ over $\mathbf{k}$.

The map $\alpha_F : \Lambda \to \mathbf{k}$ is a $\mathbf{k}$-algebra homomorphism, and therefore induces a $\mathbf{k}[[t]]$-algebra homomorphism

$$\alpha_F[[t]] : \Lambda[[t]] \to \mathbf{k}[[t]]$$

that sends each formal power series $\sum_{n\geqslant 0} a_n t^n \in \Lambda[[t]]$ (with $a_n \in \Lambda$) to $\sum_{n\geqslant 0} \alpha_F(a_n)\, t^n$. Consider this $\alpha_F[[t]]$.

Define the formal power series

$$H(t) = \prod_{i=1}^{\infty} (1 - x_i t)^{-1} \in (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]. \tag{77}$$

Then, from [17, (2.4.1)], we know that

$$H(t) = \sum_{n\geqslant 0} \underbrace{h_n(\mathbf{x})}_{=h_n} t^n = \sum_{n\geqslant 0} h_n t^n \in \Lambda[[t]].$$

It is now easy to see that

$$(\alpha_F[[t]])(H(t)) = F. \tag{78}$$

(Indeed, this follows by straightforward computations using the definition of $\alpha_F[[t]]$ from $H(t) = \sum_{n\geqslant 0} h_n t^n$ and from Lemma 5.10 **(a)**.)

Also, it is easy to see that the map $\alpha_F[[t]]$ respects derivatives: i.e. any power series $u \in \Lambda[[t]]$ satisfies $(\alpha_F[[t]])(u') = ((\alpha_F[[t]])(u))'$. Applying this to $u = H(t)$, we obtain

$$(\alpha_F[[t]])(H'(t)) = \Bigl(\underbrace{(\alpha_F[[t]])(H(t))}_{=F}\Bigr)' = F'. \tag{79}$$

From [17, Exercise 2.5.21], we know that

$$\sum_{m\geqslant 0} p_{m+1}t^m = \frac{H'(t)}{H(t)}. \tag{80}$$

Applying the map $\alpha_F[[t]]$ to both sides of this equality, we find

$$\begin{aligned}(\alpha_F[[t]])\left(\sum_{m\geqslant 0} p_{m+1}t^m\right) &= (\alpha_F[[t]])\left(\frac{H'(t)}{H(t)}\right) = \frac{(\alpha_F[[t]])(H'(t))}{(\alpha_F[[t]])(H(t))} \\ &\qquad\text{(since } \alpha_F[[t]] \text{ is a } \mathbf{k}\text{-algebra homomorphism)} \\ &= \frac{F'}{F} \qquad \text{(by (78) and (79))} \\ &= \sum_{n\in\mathbb{N}} \gamma_n t^n.\end{aligned}$$

Comparing this with

$$(\alpha_F[[t]])\left(\sum_{m\geqslant 0} p_{m+1}t^m\right) = \sum_{m\geqslant 0} \alpha_F(p_{m+1})\,t^m \qquad \text{(by the definition of } \alpha_F[[t]]),$$

we obtain

$$\sum_{m\geqslant 0} \alpha_F(p_{m+1})\,t^m = \sum_{n\in\mathbb{N}} \gamma_n t^n.$$

Comparing $t^n$-coefficients on both sides of this equality, we find

$$\alpha_F(p_{n+1}) = \gamma_n \qquad \text{for each } n\in\mathbb{N}.$$

In other words, $\alpha_F(p_m) = \gamma_{m-1}$ for each positive integer $m$. This proves Lemma 5.19. ☐

*Proof of Proposition 5.18.* Proposition 5.5 **(c)** yields $G_{F,m} = \sum_{\lambda\in\mathrm{Par};\ |\lambda|=m} f_\lambda m_\lambda$. Hence,

$$\langle p_m, G_{F,m}\rangle = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=m}} f_\lambda \langle p_m, m_\lambda\rangle \tag{81}$$

(since the Hall inner product is $\mathbf{k}$-bilinear).

Now, recall that the bases $(m_\lambda)_{\lambda\in\mathrm{Par}}$ and $(h_\lambda)_{\lambda\in\mathrm{Par}}$ of $\Lambda$ are dual to each other with respect to the Hall inner product $\langle\cdot,\cdot\rangle$. Hence, every $a\in\Lambda$ satisfies

$$a = \sum_{\lambda\in\mathrm{Par}} \langle m_\lambda, a\rangle h_\lambda$$

(by a general property of dual bases with respect to symmetric bilinear forms). Applying this to $a = p_m$, we obtain

$$\begin{aligned}p_m &= \sum_{\lambda\in\mathrm{Par}} \langle m_\lambda, p_m\rangle h_\lambda = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=m}} \underbrace{\langle m_\lambda, p_m\rangle}_{\substack{=\langle p_m, m_\lambda\rangle \\ \text{(since the Hall inner} \\ \text{product is symmetric)}}} h_\lambda + \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|\neq m}} \underbrace{\langle m_\lambda, p_m\rangle}_{\substack{=0 \\ \text{(by (2), since } m_\lambda \text{ and } p_m \\ \text{are homogeneous} \\ \text{of degrees } |\lambda| \text{ and } m \\ \text{(and since } |\lambda|\neq m))}} h_\lambda \\ &= \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=m}} \langle p_m, m_\lambda\rangle h_\lambda.\end{aligned}$$

Applying the map $\alpha_F$ to both sides of this equality (and recalling that this map is $\mathbf{k}$-linear), we find

$$\alpha_F\left(p_m\right)=\sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=m}}\left\langle p_m,m_\lambda\right\rangle \underbrace{\alpha_F\left(h_\lambda\right)}_{\substack{=f_\lambda\\ \text{(by Lemma 5.10 }\mathbf{(c)})}}=\sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=m}}f_\lambda\left\langle p_m,m_\lambda\right\rangle .$$

Comparing this with (81), we obtain

$$\left\langle p_m,G_{F,m}\right\rangle=\alpha_F\left(p_m\right)=\gamma_{m-1}\qquad\text{(by Lemma 5.19)}.$$

This proves Proposition 5.18. $\square$

We can now generalize Theorem 2.20:

Theorem 5.20. *Assume that all the elements $\gamma_0,\gamma_1,\gamma_2,\ldots$ are invertible in $\mathbf{k}$. Then, the family $\left(G_{F,m}\right)_{m\geqslant 1}=\left(G_{F,1},G_{F,2},G_{F,3},\ldots\right)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra $\Lambda$. (In other words, the canonical $\mathbf{k}$-algebra homomorphism*

$$\begin{aligned}\mathbf{k}\left[u_1,u_2,u_3,\ldots\right]&\to\Lambda,\\ u_m&\mapsto G_{F,m}\end{aligned}$$

*is an isomorphism.)*

*Proof of Theorem 5.20.* Analogous to the proof of Theorem 2.20, but using Proposition 5.18 (and Proposition 5.5) instead of Proposition 2.22 (and Proposition 2.3). $\square$

Remark 5.21. It is not hard to verify that the converse of Theorem 5.20 also holds: If the family $\left(G_{F,m}\right)_{m\geqslant 1}=\left(G_{F,1},G_{F,2},G_{F,3},\ldots\right)$ generates the $\mathbf{k}$-algebra $\Lambda$, then all the elements $\gamma_0,\gamma_1,\gamma_2,\ldots$ are invertible in $\mathbf{k}$. We omit the proof of this.

The next theorem generalizes parts of Theorem 2.26 (specifically, it generalizes the properties of the map $V_k$ stated in Theorem 2.26, even though it defines this map differently):[31]

Theorem 5.22. *The h-universal property of $\Lambda$ shows that there is a unique $\mathbf{k}$-algebra homomorphism $V_F:\Lambda\to\Lambda$ that sends $h_i$ to $G_{F,i}$ for all positive integers $i$ (since $G_{F,i}\in\Lambda$ for each positive integer $i$). Consider this $V_F$.*

- **(a)** *This map $V_F$ is a $\mathbf{k}$-Hopf algebra homomorphism.*
- **(b)** *We have $V_F\left(h_m\right)=G_{F,m}$ for each $m\in\mathbb{N}$.*
- **(c)** *We have $V_F\left(p_n\right)=\gamma_{n-1}p_n$ for each positive integer $n$. (See Definition 5.16 for the meaning of $\gamma_{n-1}$.)*

*Proof of Theorem 5.22.* **(b)** When $m$ is positive, this follows from the very definition of $V_F$. It remains to prove this for $m=0$. However, this boils down to showing that $V_F\left(1\right)=1$, which is clear (since $V_F$ is a $\mathbf{k}$-algebra homomorphism).

**(a)** Let $\Delta$ and $\varepsilon$ be the comultiplication and the counit of the Hopf algebra $\Lambda$. Both $\Delta$ and $\varepsilon$ are $\mathbf{k}$-algebra homomorphisms. It suffices to show that $\Delta\circ V_F=\left(V_F\otimes V_F\right)\circ\Delta$ and $\varepsilon\circ V_F=\varepsilon$. We shall show that $\Delta\circ V_F=\left(V_F\otimes V_F\right)\circ\Delta$ only; the proof of $\varepsilon\circ V_F=\varepsilon$ is similar but much simpler (since $\varepsilon$ sends any homogeneous symmetric function of positive degree to 0).

Recall that the family $\left(h_n\right)_{n\geqslant 1}$ generates $\Lambda$ as a $\mathbf{k}$-algebra. Thus, in order to prove that $\Delta\circ V_F=\left(V_F\otimes V_F\right)\circ\Delta$, it suffices to prove the equality $\left(\Delta\circ V_F\right)\left(h_n\right)=$

[31] We recall the “h-universal property of $\Lambda$”, which we stated in Subsection 3.4.

$((V_F \otimes V_F) \circ \Delta)(h_n)$ for each $n \geqslant 1$ (since both $\Delta \circ V_F$ and $(V_F \otimes V_F) \circ \Delta$ are $\mathbf{k}$-algebra homomorphisms). In view of Theorem 5.22 **(b)**, this equality rewrites as $\Delta(G_{F,n}) = \sum_{i=0}^{n} G_{F,i} \otimes G_{F,n-i}$. But this follows directly from Theorem 5.15.

**(c)** This is best proved using the notion of a logarithmic derivative. Let us first define it in full generality, without any assumptions on $\mathbf{k}$.

If $R$ is a commutative ring, and if $F \in R[[t]]$ is any formal power series whose constant term is 1 (or, more generally, any formal power series that has a multiplicative inverse), then the *logarithmic derivative* of $F$ is defined to be the formal power series $\frac{F'}{F} \in R[[t]]$ (this is well-defined, since $F$ is invertible). This logarithmic derivative is denoted by $\operatorname{lder} F$.

The following properties of logarithmic derivatives are easy to prove[32]:

(1) Let $R$ be a commutative ring. Let $u, v \in R[[t]]$ be two formal power series whose constant terms are 1. Then, $\operatorname{lder}(uv) = \operatorname{lder} u + \operatorname{lder} v$.

(*Proof:* Just recall the definition of logarithmic derivatives and the Leibniz law $(uv)' = u'v + uv'$.)

(2) Let $R$ be a commutative topological ring. Let $(u_n)_{n\in\mathbb{N}} = (u_0, u_1, u_2, \ldots) \in R[[t]]^{\mathbb{N}}$ be a sequence of formal power series whose constant terms are 1. Let $u \in R[[t]]$ be a formal power series whose constant term is 1. Assume that $\lim_{n\to\infty} u_n = u$ (with respect to the standard topology on $R[[t]]$ induced by the topology on $R$). Then, $\lim_{n\to\infty} (\operatorname{lder} u_n) = \operatorname{lder} u$ (with respect to the same topology on $R[[t]]$).

(*Proof:* Let $R[[t]]_1$ be the set of power series in $R[[t]]$ whose constant term is 1. Argue that $\lim_{n\to\infty} (u_n') = u'$ first; then argue that the map

$$\begin{aligned} R[[t]] \times R[[t]]_1 &\to R[[t]], \\ (v, w) &\mapsto \frac{v}{w} \end{aligned}$$

is continuous.)

(3) Let $R$ be a commutative ring. Let $u_1, u_2, \ldots, u_n \in R[[t]]$ be finitely many formal power series whose constant terms are 1. Then,

$$\operatorname{lder}\left(\prod_{i=1}^{n} u_i\right) = \sum_{i=1}^{n} \operatorname{lder} u_i.$$

(*Proof:* Induction on $n$, using Property 1 in the induction step.)

(4) Let $R$ be a commutative topological ring. Let $u_1, u_2, u_3, \ldots \in R[[t]]$ be infinitely many formal power series whose constant terms are 1. Assume that the infinite product $\prod_{i=1}^{\infty} u_i$ converges (with respect to the standard topology on $R[[t]]$ induced by the topology on $R$). Then, the infinite sum $\sum_{i=1}^{\infty} \operatorname{lder} u_i$ converges as well, and we have

$$\operatorname{lder}\left(\prod_{i=1}^{\infty} u_i\right) = \sum_{i=1}^{\infty} \operatorname{lder} u_i.$$

[32] If $R$ is a commutative $\mathbb{Q}$-algebra, then the logarithmic derivative $\operatorname{lder} F$ of a power series $F \in R[[t]]$ equals the derivative of $\log F$. This trivializes many of the properties stated below; but this shortcut is not available when $R$ is merely an arbitrary commutative ring.

(*Proof:* This is the "$n \to \infty$" limit of Property 3. Use Property 2 to pass to this limit.)

(5) Let $R$ be a commutative ring. Let $u \in R[[t]]$ be a formal power series whose constant term is 1. Let $\lambda \in R$. Then,

$$\operatorname{lder}(u(\lambda t)) = \lambda \cdot (\operatorname{lder} u)(\lambda t).$$

(*Proof:* This follows from the equality $(u(\lambda t))' = \lambda \cdot u'(\lambda t)$, which is an easy consequence of the chain rule but also easy to check directly.)

(6) Let $R$ and $S$ be two commutative $\mathbf{k}$-algebras. Let $\alpha : R \to S$ be a $\mathbf{k}$-algebra homomorphism. As we know, $\alpha$ induces a $\mathbf{k}[[t]]$-algebra homomorphism

$$\alpha[[t]] : R[[t]] \to S[[t]]$$

that sends each power series $\sum_{n \geqslant 0} a_n t^n \in R[[t]]$ (with $a_n \in R$) to $\sum_{n \geqslant 0} \alpha(a_n) t^n \in S[[t]]$.

Let $u \in R[[t]]$ be a formal power series whose constant term is 1. Then, the constant term of the power series $(\alpha[[t]])(u)$ is 1, and we have

$$\operatorname{lder}((\alpha[[t]])(u)) = (\alpha[[t]])(\operatorname{lder} u).$$

(*Proof:* This is essentially saying that the logarithmic derivative is functorial with respect to the base ring. The proof is straightforward.)

Now, let us resume proving Theorem 5.22 **(c)**:

Consider the ring $(\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]$ of formal power series in one indeterminate $t$ over $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This ring is a topological ring, where the topology is the standard one induced by the standard topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (not the discrete topology!). This topological ring $(\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]$ is, of course, isomorphic to $\mathbf{k}[[x_1, x_2, x_3, \ldots, t]]$. The ring $\Lambda[[t]]$ is a subring of $(\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]$.

Now, for each $m \in \mathbb{N}$, we know that $G_{F,m}$ is homogeneous of degree $m$ (by Proposition 5.5 **(a)**), and therefore satisfies

$$G_{F,m}(tx_1, tx_2, tx_3, \ldots) = t^m \cdot G_{F,m} \tag{82}$$

(since any formal power series $u \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ that is homogeneous of degree $m$ satisfies $u(tx_1, tx_2, tx_3, \ldots) = t^m \cdot u$).

On the other hand, from (75), we obtain

$$\prod_{i=1}^{\infty} F(x_i) = \sum_{k \in \mathbb{N}} G_{F,k}(\mathbf{x}) = \sum_{m \in \mathbb{N}} G_{F,m}(\mathbf{x}).$$

Substituting $tx_1, tx_2, tx_3, \ldots$ for $x_1, x_2, x_3, \ldots$ on both sides of this equality, we obtain

$$\prod_{i=1}^{\infty} F(tx_i) = \sum_{m \in \mathbb{N}} \underbrace{G_{F,m}(tx_1, tx_2, tx_3, \ldots)}_{\substack{= t^m \cdot G_{F,m} \\ \text{(by (82))}}} = \sum_{m \in \mathbb{N}} t^m \cdot G_{F,m}. \tag{83}$$

The map $V_F : \Lambda \to \Lambda$ is a $\mathbf{k}$-algebra homomorphism. Hence, it induces a $\mathbf{k}[[t]]$-algebra homomorphism

$$V_F[[t]] : \Lambda[[t]] \to \Lambda[[t]]$$

that sends each formal power series $\sum_{n \geqslant 0} a_n t^n \in \Lambda[[t]]$ (with $a_n \in \Lambda$) to $\sum_{n \geqslant 0} V_F(a_n) t^n$. Consider this $V_F[[t]]$.

Define the formal power series $H(t)$ as in (77). Then, from [17, (2.4.1)], we know that

$$H(t) = \sum_{n\geqslant 0} \underbrace{h_n(\mathbf{x})}_{=h_n} t^n = \sum_{n\geqslant 0} h_n t^n \in \Lambda[[t]].$$

Moreover, $H(t) = \sum_{n\geqslant 0} h_n t^n$ shows that the constant term of $H(t)$ is $h_0 = 1$. Thus, $\operatorname{lder}(H(t))$ is well-defined.

Applying the map $V_F[[t]]$ to both sides of the equality $H(t) = \sum_{n\geqslant 0} h_n t^n$, we obtain

$$\begin{aligned}(V_F[[t]])(H(t)) &= (V_F[[t]])\left(\sum_{n\geqslant 0} h_n t^n\right) = \underbrace{\sum_{n\geqslant 0}}_{=\sum_{n\in\mathbb{N}}} \underbrace{V_F(h_n)}_{\substack{=G_{F,n} \\ \text{(by Theorem 5.22 (b))}}} t^n \\ &\qquad \text{(by the definition of } V_F[[t]]) \\ &= \sum_{n\in\mathbb{N}} G_{F,n} t^n = \sum_{n\in\mathbb{N}} t^n \cdot G_{F,n} = \sum_{m\in\mathbb{N}} t^m \cdot G_{F,m}.\end{aligned}$$

Comparing this with (83), we find

$$(V_F[[t]])(H(t)) = \prod_{i=1}^{\infty} F(tx_i) = \prod_{i=1}^{\infty} F(x_i t). \tag{84}$$

Now, the definition of $\operatorname{lder}(H(t))$ yields

$$\begin{aligned}\operatorname{lder}(H(t)) = \frac{H'(t)}{H(t)} &= \sum_{m\geqslant 0} p_{m+1} t^m \qquad \text{(by (80))} \\ &= \sum_{n\geqslant 0} p_{n+1} t^n.\end{aligned}$$

Applying the map $V_F[[t]]$ to both sides of this equality, we find

$$\begin{aligned}(V_F[[t]])(\operatorname{lder}(H(t))) &= (V_F[[t]])\left(\sum_{n\geqslant 0} p_{n+1} t^n\right) = \sum_{n\geqslant 0} V_F(p_{n+1}) t^n \\ &\qquad \text{(by the definition of } V_F[[t]]) \\ &= \sum_{n\in\mathbb{N}} V_F(p_{n+1}) t^n. \qquad (85)\end{aligned}$$

Now is the time to use our above-listed properties of logarithmic derivatives. Recall that the constant term of $H(t)$ is 1. Hence, Property 6 of logarithmic derivatives shows that the constant term of the power series $(V_F[[t]])(H(t))$ is 1, and that we have

$$\operatorname{lder}((V_F[[t]])(H(t))) = (V_F[[t]])(\operatorname{lder}(H(t))). \tag{86}$$

Now, (85) yields

$$\begin{aligned}\sum_{n\in\mathbb{N}} V_F(p_{n+1}) t^n &= (V_F[[t]])(\operatorname{lder}(H(t))) \\ &= \operatorname{lder}((V_F[[t]])(H(t))) \qquad \text{(by (86))} \\ &= \operatorname{lder}\left(\prod_{i=1}^{\infty} F(x_i t)\right) \qquad \text{(by (84))}. \qquad (87)\end{aligned}$$

Now, the infinite product $\prod_{i=1}^{\infty} F(x_i t)$ converges (as we know from (84)). Hence, Property 4 of logarithmic derivatives yields that the infinite sum $\sum_{i=1}^{\infty} \operatorname{lder}(F(x_i t))$ converges as well, and that we have

$$\operatorname{lder}\left(\prod_{i=1}^{\infty} F(x_i t)\right) = \sum_{i=1}^{\infty} \operatorname{lder}(F(x_i t)) . \tag{88}$$

The definition of $\operatorname{lder} F$ yields

$$\operatorname{lder} F = \frac{F'}{F} = \sum_{n\in\mathbb{N}} \gamma_n t^n .$$

Hence, for each $i \in \{1, 2, 3, \ldots\}$, we have

$$(\operatorname{lder} F)(x_i t) = \sum_{n\in\mathbb{N}} \gamma_n \underbrace{(x_i t)^n}_{=x_i^n t^n} = \sum_{n\in\mathbb{N}} \gamma_n x_i^n t^n . \tag{89}$$

Now, (87) becomes

$$\begin{aligned}
\sum_{n\in\mathbb{N}} V_F(p_{n+1}) t^n &= \operatorname{lder}\left(\prod_{i=1}^{\infty} F(x_i t)\right) = \sum_{i=1}^{\infty} \underbrace{\operatorname{lder}(F(x_i t))}_{\substack{=x_i\cdot(\operatorname{lder} F)(x_i t) \\ \text{(by Property 5 of} \\ \text{logarithmic derivatives)}}} \qquad \text{(by (88))} \\
&= \sum_{i=1}^{\infty} x_i \cdot \underbrace{(\operatorname{lder} F)(x_i t)}_{\substack{=\sum_{n\in\mathbb{N}} \gamma_n x_i^n t^n \\ \text{(by (89))}}} = \sum_{i=1}^{\infty} x_i \cdot \sum_{n\in\mathbb{N}} \gamma_n x_i^n t^n = \sum_{i=1}^{\infty} \sum_{n\in\mathbb{N}} \gamma_n x_i^{n+1} t^n \\
&= \sum_{n\in\mathbb{N}} \gamma_n \underbrace{\left(\sum_{i=1}^{\infty} x_i^{n+1}\right)}_{=p_{n+1}} t^n = \sum_{n\in\mathbb{N}} \gamma_n p_{n+1} t^n .
\end{aligned}$$

Comparing coefficients before $t^n$ in this equality, we conclude that

$$V_F(p_{n+1}) = \gamma_n p_{n+1} \qquad \text{for each } n \in \mathbb{N}.$$

In other words, $V_F(p_n) = \gamma_{n-1} p_n$ for each positive integer $n$. This proves Theorem 5.22 **(c)**. ☐

Our next (and last) few results are not generalizations of any properties of Petrie functions. To state them, we take a somewhat more high-level point of view. We forget that we fixed the power series $F$. Instead, for **every** power series $F \in \mathbf{k}[[t]]$ whose constant term is 1, we define a power series $G_F$ according to Definition 5.3 **(d)**. Moreover, for **every** power series $F \in \mathbf{k}[[t]]$ whose constant term is 1, and for **every** $m \in \mathbb{N}$, we define a power series $G_{F,m}$ according to Definition 5.3 **(e)**. We then have the following:

Proposition 5.23. *Let $A$ and $B$ be two power series in $\mathbf{k}[[t]]$ whose constant terms are 1. Then:*

- **(a)** *We have $G_{AB} = G_A G_B$.*
- **(b)** *Let $n \in \mathbb{N}$. We have $G_{AB,n} = \sum_{i=0}^{n} G_{A,i} G_{B,n-i}$.*

*Proof of Proposition 5.23.* The power series $AB$ has constant term 1 (since $A$ and $B$ have constant term 1). Thus, $G_{AB}$ is well-defined, as is $G_{AB,n}$ for each $n \in \mathbb{N}$.

**(a)** Proposition 5.5 **(b)** yields that $G_F = \prod_{i=1}^{\infty} F(x_i)$ for any power series $F \in \mathbf{k}[[t]]$ whose constant term is 1. Applying this to $F = A$ and to $F = B$ and to $F = AB$ yields $G_A = \prod_{i=1}^{\infty} A(x_i)$ and $G_B = \prod_{i=1}^{\infty} B(x_i)$ and

$$G_{AB} = \prod_{i=1}^{\infty} \underbrace{(AB)(x_i)}_{=A(x_i)B(x_i)} = \prod_{i=1}^{\infty} (A(x_i) B(x_i)) = \underbrace{\left(\prod_{i=1}^{\infty} A(x_i)\right)}_{=G_A} \underbrace{\left(\prod_{i=1}^{\infty} B(x_i)\right)}_{=G_B} = G_A G_B.$$

This proves Proposition 5.23 **(a)**.

**(b)** Proposition 5.23 **(a)** yields $G_{AB} = G_A G_B$. Thus, the $n$-th degree homogeneous components of $G_{AB}$ and of $G_A G_B$ are equal. But this means precisely that $G_{AB,n} = \sum_{i=0}^{n} G_{A,i} G_{B,n-i}$ (by Proposition 5.5 **(a)**). This proves Proposition 5.23 **(b)**. □

Finally, we can express the image of the symmetric function $G_{F,n}$ under the antipode of $\Lambda$ (a result suggested by Sasha Postnikov):

Theorem 5.24. *Let $S$ be the antipode of the Hopf algebra $\Lambda$. Let $F \in \mathbf{k}[[t]]$ be a formal power series whose constant term is 1. Then, for each $n \in \mathbb{N}$, we have*

$$S(G_{F,n}) = G_{F^{-1},n}. \tag{90}$$

*Proof of Theorem 5.24.* Let $\Delta$ and $\varepsilon$ be the comultiplication and the counit of the Hopf algebra $\Lambda$. Let $\eta : \mathbf{k} \to \Lambda$ be the map that sends each $u \in \mathbf{k}$ to $u \cdot 1_\Lambda \in \Lambda$. It is easy to see that each positive integer $n$ satisfies

$$\varepsilon(G_{F,n}) = 0. \tag{91}$$

(Indeed, $\varepsilon$ sends each homogeneous symmetric function of positive degree to 0; but $G_{F,n}$ is a homogeneous symmetric function of degree $n$.) Also, Proposition 5.5 **(e)** yields $G_{F,0} = 1$ and thus $\varepsilon(G_{F,0}) = 1$.

We shall use the convolution $\star$ introduced in Definition 2.25. The antipode $S$ of $\Lambda$ is the $\star$-inverse of the map $\mathrm{id}_\Lambda : \Lambda \to \Lambda$ (by the definition of the antipode of a Hopf algebra). In other words,

$$S \star \mathrm{id}_\Lambda = \mathrm{id}_\Lambda \star S = \eta \circ \varepsilon$$

(since $\eta \circ \varepsilon : \Lambda \to \Lambda$ is the neutral element with respect to $\star$). We also have $S(1) = 1$ (by one of the fundamental properties of the antipode of a Hopf algebra).

Now, for each $n \in \mathbb{N}$, we have

$$\Delta(G_{F,n}) = \sum_{i=0}^{n} G_{F,i} \otimes G_{F,n-i}$$

(by Theorem 5.15) and therefore

$$(S \star \mathrm{id}_\Lambda)(G_{F,n}) = \sum_{i=0}^{n} S(G_{F,i}) \cdot G_{F,n-i} \qquad \text{(by the definition of convolution)},$$

so that

$$\begin{aligned} \sum_{i=0}^{n} S(G_{F,i}) \cdot G_{F,n-i} &= \underbrace{(S \star \mathrm{id}_\Lambda)}_{=\eta\circ\varepsilon}(G_{F,n}) = (\eta \circ \varepsilon)(G_{F,n}) \\ &= [n = 0] \end{aligned} \tag{92}$$

(the last equality sign here follows easily from (91) and from $\varepsilon(G_{F,0}) = 1$).

On the other hand, the constant term of the power series $F^{-1}$ is 1 (since the constant term of $F$ is 1). Hence, $G_{F^{-1},n}$ is well-defined for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we have

$$G_{F^{-1}F,n} = \sum_{i=0}^{n} G_{F^{-1},i} G_{F,n-i}$$

(by Proposition 5.23 **(b)**, applied to $A = F^{-1}$ and $B = F$) and thus

$$\sum_{i=0}^{n} G_{F^{-1},i} G_{F,n-i} = G_{F^{-1}F,n} = G_{1,n} = [n = 0] \tag{93}$$

(the last equality sign here has been shown in Example 5.4 **(b)**).

Recall that $G_{F,0} = 1$. Hence, the equalities (93) (for all $n \in \mathbb{N}$) can be recursively solved for $G_{F^{-1},0}, G_{F^{-1},1}, G_{F^{-1},2}, \ldots$ (starting with $G_{F,0}, G_{F,1}, G_{F,2}, \ldots$); we obtain

$$G_{F^{-1},n} = [n = 0] - \sum_{i=0}^{n-1} G_{F^{-1},i} G_{F,n-i} \qquad \text{for each } n \in \mathbb{N}.$$

The same argument, but using the equalities (92) instead of (93), yields

$$S(G_{F,n}) = [n = 0] - \sum_{i=0}^{n-1} S(G_{F,i}) \cdot G_{F,n-i} \qquad \text{for each } n \in \mathbb{N}.$$

Comparing these two recursive formulas for $G_{F^{-1},n}$ and $S(G_{F,n})$, we see that they are the same. Thus, by strong induction on $n$, we conclude that

$$S(G_{F,n}) = G_{F^{-1},n} \qquad \text{for each } n \in \mathbb{N}.$$

This completes the proof of Theorem 5.24. $\square$

As a consequence of Theorem 5.24, we obtain a formula for the antipode of a Petrie symmetric function:

Corollary 5.25. *Let $k$ be a positive integer such that $k > 1$. A weak composition $\alpha$ will be called* $k$-friendly *if each $i \in \{1, 2, 3, \ldots\}$ satisfies $\alpha_i \equiv 0 \bmod k$ or $\alpha_i \equiv 1 \bmod k$. If $\alpha$ is a weak composition, then $w(\alpha)$ shall denote the number of all $i \in \{1, 2, 3, \ldots\}$ satisfying $\alpha_i \equiv 1 \bmod k$.*

*Let $S$ be the antipode of the Hopf algebra $\Lambda$. Then, for each $n \in \mathbb{N}$, we have*

$$S(G(k,n)) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = n; \\ \alpha \text{ is } k\text{-friendly}}} (-1)^{w(\alpha)} \mathbf{x}^{\alpha} = \sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = n; \\ \lambda \text{ is } k\text{-friendly}}} (-1)^{w(\lambda)} m_{\lambda}.$$

*Proof of Corollary 5.25.* Let $F = 1 + t + t^2 + \cdots + t^{k-1} \in \mathbf{k}[[t]]$. Then, $F$ is a power series whose constant term is 1. Hence, its reciprocal $F^{-1}$ is well-defined and again has constant term 1. Let us denote this reciprocal $F^{-1}$ by $Q$; thus, $Q = F^{-1}$.

Let $q_0, q_1, q_2, \ldots$ be the coefficients of the formal power series $Q$, so that $Q = \sum_{n \in \mathbb{N}} q_n t^n$. Thus, $q_0 = 1$ (since the constant term of $Q$ is 1).

On the other hand,

$$\begin{aligned}
Q = F^{-1} &= \left(\frac{1-t^k}{1-t}\right)^{-1} \qquad \left(\text{since } F = 1+t+t^2+\cdots+t^{k-1} = \frac{1-t^k}{1-t}\right) \\
&= \frac{1-t}{1-t^k} = (1-t)\cdot \underbrace{\left(1-t^k\right)^{-1}}_{=t^0+t^k+t^{2k}+t^{3k}+\cdots} = (1-t)\cdot\left(t^0+t^k+t^{2k}+t^{3k}+\cdots\right) \\
&= t^0 - t^1 + t^k - t^{k+1} + t^{2k} - t^{2k+1} \pm \cdots \\
&= \sum_{n\in\mathbb{N}} \left([n \equiv 0 \bmod k] - [n \equiv 1 \bmod k]\right) t^n.
\end{aligned}$$

Comparing coefficients on both sides of this equality, we find

$$q_n = [n \equiv 0 \bmod k] - [n \equiv 1 \bmod k] \qquad \text{for each } n \in \mathbb{N}. \tag{94}$$

For any weak composition $\alpha$, we define an element $q_\alpha \in \mathbf{k}$ by

$$q_\alpha = q_{\alpha_1} q_{\alpha_2} q_{\alpha_3} \cdots .$$

(This infinite product is well-defined, since every sufficiently high positive integer $i$ satisfies $\alpha_i = 0$ and thus $q_{\alpha_i} = q_0 = 1$.)

It is now easy to see (using (94)) that

$$q_\alpha = [\alpha \text{ is } k\text{-friendly}] \cdot (-1)^{w(\alpha)} \tag{95}$$

for any weak composition $\alpha$.

Now, let $n \in \mathbb{N}$. Recall that our scalars $q_i$ and $q_\alpha$ were defined in the exact same way as the scalars $f_i$ and $f_\alpha$ were defined in Definition 5.3, but using the power series $Q$ instead of $F$. Hence, Proposition 5.5 **(c)** (applied to $Q$, $q_i$, $q_\alpha$ and $n$ instead of $F$, $f_i$, $f_\alpha$ and $m$) yields that

$$G_{Q,n} = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n}} q_\alpha \mathbf{x}^\alpha = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=n}} q_\lambda m_\lambda \in \Lambda.$$

Hence,

$$\begin{aligned}
G_{Q,n} &= \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n}} \underbrace{q_\alpha}_{\substack{=[\alpha \text{ is } k\text{-friendly}]\cdot(-1)^{w(\alpha)}\\ \text{(by (95))}}} \mathbf{x}^\alpha = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n}} [\alpha \text{ is } k\text{-friendly}] \cdot (-1)^{w(\alpha)}\, \mathbf{x}^\alpha \\
&= \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n;\\ \alpha \text{ is } k\text{-friendly}}} (-1)^{w(\alpha)}\, \mathbf{x}^\alpha
\end{aligned}$$

and

$$\begin{aligned}
G_{Q,n} &= \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=n}} \underbrace{q_\lambda}_{\substack{=[\lambda \text{ is } k\text{-friendly}]\cdot(-1)^{w(\lambda)}\\ \text{(by (95),}\\ \text{applied to } \lambda \text{ instead of } \alpha)}} m_\lambda = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=n}} [\lambda \text{ is } k\text{-friendly}] \cdot (-1)^{w(\lambda)}\, m_\lambda \\
&= \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=n;\\ \lambda \text{ is } k\text{-friendly}}} (-1)^{w(\lambda)}\, m_\lambda.
\end{aligned}$$

Combining these two equalities, we find

$$G_{Q,n} = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n;\\ \alpha \text{ is } k\text{-friendly}}} (-1)^{w(\alpha)}\, \mathbf{x}^\alpha = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=n;\\ \lambda \text{ is } k\text{-friendly}}} (-1)^{w(\lambda)}\, m_\lambda. \tag{96}$$

This will immediately yield the claim of Corollary 5.25 if we can show that $S(G(k,n)) = G_{Q,n}$. But this is easy: In Example 5.4 **(c)**, we have seen that $G_{F,m} = G(k,m)$ for each $m \in \mathbb{N}$. Applying this to $m = n$, we obtain $G_{F,n} = G(k,n)$. On the other hand, Theorem 5.24 yields $S(G_{F,n}) = G_{F^{-1},n} = G_{Q,n}$ (since $F^{-1} = Q$). In view of $G_{F,n} = G(k,n)$, this rewrites as $S(G(k,n)) = G_{Q,n}$. This completes our proof of Corollary 5.25. □

One last property shall be noted in passing:

Proposition 5.26. *For any power series $F \in \mathbf{k}[[t]]$ whose constant term is 1, we define a $\mathbf{k}$-algebra homomorphism $V_F : \Lambda \to \Lambda$ as in Theorem 5.22. Then:*

- **(a)** *If $A$ and $B$ are two power series in $\mathbf{k}[[t]]$ whose constant terms are 1, then $V_{AB} = V_A \star V_B$.*
- **(b)** *We have $V_1 = \eta \circ \varepsilon$.*
- **(c)** *For any power series $F \in \mathbf{k}[[t]]$ whose constant term is 1, we have $V_{F^{-1}} = V_F \circ S$, where $S$ is the antipode of $\Lambda$.*

This follows easily from Proposition 5.23; we leave the details to the reader.

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