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Forbidden subgraphs in generating graphs of finite groups

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Abstract Let G be a 2-generated finite group. The generating graph $\Gamma(G)$ is the graph whose vertices are the elements of G and where two vertices g_1 and g_2 are adjacent if $G = \langle g_1, g_2\rangle$. This graph encodes the combinatorial structure of the distribution of generating pairs across G. In this paper we study some graph theoretic properties of $\Gamma(G)$, with particular emphasis on those properties that can be formulated in terms of forbidden induced subgraphs. In particular we investigate when the generating graph $\Gamma(G)$ is a cograph (giving a complete description when G is soluble) and when it is perfect (giving a complete description when G is nilpotent and proving, among other things, that $\Gamma(S_n)$ and $\Gamma(A_n)$ are perfect if and only if $n \leqslant 4$). Finally we prove that for a finite group G, the properties that $\Gamma(G)$ is split, chordal or C_4-free are equivalent.

1. Introduction

If a finite group G can be generated by d elements, then the problem of determining the d-element generating sets for G is non-trivial. The simplest interesting case is when G is 2-generated. One tool developed to study generators of a 2-generated finite group G is the generating graph $\Gamma(G)$ of G. This is the graph which has the elements of G as vertices and an edge between two elements g_1 and g_2 if G is generated by g_1 and g_2. Some authors exclude the identity element in the set of vertices of $\Gamma(G)$; there is no substantial difference if G is is non-cyclic, but we choose to include the identity because we will also consider cyclic groups. Note that the generating graph may be defined for any group G, but it only has edges if G is 2-generated.

Several strong structural results about $\Gamma(G)$ are known in the case where G is simple, and this reflects the rich group theoretic structure of these groups. For example, if G is a non-abelian simple group, then the only isolated vertex of $\Gamma(G)$ is the identity [13] and the graph $\Delta(G)$ obtained by removing the isolated vertex is connected with diameter two [2] and, if $|G|$ is sufficiently large, admits a Hamiltonian cycle [3] (it is conjectured that the condition on $|G|$ can be removed). Moreover, in recent years there has been considerable interest in attempting to classify the groups G for which $\Gamma(G)$ shares the strong properties of the generating graphs of simple groups. Recently, the following remarkable result has been proved in [4]: the identity is the unique isolated vertex of $\Gamma(G)$ if and only if all proper quotients of G are cyclic. An open question is whether the subgraph $\Delta(G)$ of $\Gamma(G)$ induced by the non-isolated vertices is connected, for every finite group G. The answer is positive if G is soluble [7] and in this case the diameter of $\Delta(G)$ is at most three [16]. In [14] it is proved that

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when G is nilpotent, then $\Delta(G)$ is maximally connected, i.e. the connectivity of the graph $\Delta(G)$ equals its minimum degree (recall that the connectivity of a finite graph Γ is the least size of a subset X of the set $V(\Gamma)$ of the vertices such that the induced subgraph on $V(\Gamma) \setminus X$ is disconnected).

The subgraph of a graph Γ induced by a subset X of the vertex set is the graph whose vertices are the elements of X and where the edges are the edges of Γ with both endpoints in X. A number of important classes of graphs can be defined either structurally or in terms of forbidden induced subgraphs, i.e. by specifying a family of graphs that cannot appear as induced subgraphs. The aim of this paper is to investigate some properties of the forbidden subgraphs of the generating graphs of finite groups.

A perfect graph is a graph in which the chromatic number of every induced subgraph equals the order of the largest clique of that subgraph (clique number). A hole in a graph Γ is an induced subgraph of Γ isomorphic to a chordless cycle of length at least 4. An antihole is an induced subgraph Δ of Γ, such that $\overline{\Delta}$ is a hole of the complement graph $\overline{\Gamma}$. A hole (resp. an antihole) is odd or even according to the number of its vertices. The strong perfect graph theorem is a forbidden graph characterization of perfect graphs as being exactly the graphs that have neither odd holes nor odd antiholes. It was conjectured by Claude Berge in 1961. A proof by Maria Chudnovsky, Neil Robertson, Paul Seymour and Robin Thomas was announced in 2002 and published by them in 2006 [6]. Motivated by the strong perfect graph theorem we analyze the existence of m-holes or m-antiholes in the generating graph of a finite group G. The first result that can be proved with this approach is a complete characterization of the 2-generated finite nilpotent groups with a perfect generating graph.

Theorem 1.1. *Let G be a finite 2-generated nilpotent group. Then $\Gamma(G)$ is perfect if and only if the index of the Frattini subgroup is the product of at most four (not necessarily distinct) primes.*

In general the condition on the number of prime divisors of the index of the Frattini subgroup is neither necessary nor sufficient to ensure that the generating graph is perfect, as it follows for example from the study of the generating graph of dihedral groups.

Theorem 1.2. *Let D_n be the dihedral group of order $2n$. Then $\Gamma(D_n)$ is perfect if and only if one of the following occurs:*

(1) *n is even;*
(2) *n is odd and divisible by at most two distinct primes.*

An interesting and surprising consequence of Theorem 1.2 is that if G is a 2-generated finite group and N is a normal subgroup of G, then the fact that $\Gamma(G)$ is perfect does not imply that $\Gamma(G/N)$ is also perfect. For example let $m = p_1 \cdot p_2 \cdot p_3$ be the product of three distinct odd primes and let $G = D_{2m}$ be the dihedral group of order $4m$. By Theorem 1.2, $\Gamma(G)$ is perfect. However, G has a normal subgroup N of order 2 such that $G/N \cong D_m$ and, again by Theorem 1.2, $\Gamma(G/N)$ is not perfect.

We will prove (see Theorem 3.30) that the alternating group A_5 is the smallest 2-generated finite group whose generating graph is not perfect. Moreover:

Theorem 1.3. *$\Gamma(A_n)$ and $\Gamma(S_n)$ are perfect if and only if $n < 5$.*

The behaviour of the generating graph of the alternating groups suggests the following conjecture.

Conjecture 1.4. *If G is a finite non-abelian simple group, then $\Gamma(G)$ is not perfect.*

Indeed, the proof of Theorem 1.3 shows that if $n \geqslant 5$ then $\Gamma(A_n)$ and $\Gamma(S_n)$ contain a 5-hole, so we may also formulate a stronger conjecture.

Conjecture 1.5. *If G is a finite non-abelian simple group, then there exists a subset X of G such that the subgraph of $\Gamma(G)$ induced by X is a 5-hole.*

With the use of GAP [10], we have checked the existence of a 5-hole in $\Gamma(G)$ when G is the Tits group or one of the sporadic simple groups with the exception of the Janko group J_4, the Thompson group, the Lyons group, the Baby Monster group and the Monster group. Moreover Conjecture 1.5 is true when G is a rank one group of Lie type, so we have:

Theorem 1.6. *If G is a simple group of Lie type of rank one, then $\Gamma(G)$ is not perfect.*

A path graph is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n-1$. A path graph with n-vertices is usually denoted by P_n. A graph Γ is called a cograph if Γ has no induced subgraph isomorphic to the four-vertex path P_4. Several alternative characterizations of cographs can be given:

(1) a cograph is a graph all of whose induced subgraphs have the property that any maximal clique intersects any maximal independent set in a single vertex;
(2) a cograph is a graph in which every non-trivial induced subgraph has at least two vertices with the same neighbourhoods;
(3) a cograph is a graph in which every connected induced subgraph has a disconnected complement;
(4) a cograph is a graph all of whose connected induced subgraphs have diameter at most 2.

We will prove that if N is a normal subgroup of a 2-generated finite group G and $\Gamma(G/N)$ contains an induced subgraph isomorphic to P_n, then so does $\Gamma(G)$ (see Lemma 2.2). Thus, in contrast to perfectness, the property that $\Gamma(G)$ is a cograph is inherited by the epimorphic images of G. This is a considerable advantage in the study of groups whose generating graph is a cograph and allows us to obtain some quite general results. For example we can completely characterize the 2-generated finite soluble groups whose generating graph is a cograph.

Theorem 1.7. *Let G be a 2-generated finite soluble group. Then $\Gamma(G)$ is a cograph if and only if one of the following occurs.*

(1) *G is cyclic and $|G|$ is divisible by at most two distinct primes.*
(2) *G is a p-group.*
(3) *$G/\operatorname{Frat}(G) \cong V \rtimes \langle x \rangle$ where x has prime order and V is a faithful irreducible $\langle x \rangle$-module.*

Moreover we will prove the following theorems.

Theorem 1.8. *Let G be a finite group and assume that the identity element is the unique isolated vertex of $\Gamma(G)$. If $\Gamma(G)$ is a cograph, then G is soluble.*

Theorem 1.9. *Let G be a 2-generated finite group. If $\Gamma(G)$ is a cograph and N is a maximal normal subgroup of G, then G/N is abelian.*

Corollary 1.10. *Let G be a non-trivial 2-generated finite group. If G is perfect, then $\Gamma(G)$ is not a cograph.*

The previous result suggests the following stronger conjecture.

Conjecture 1.11. *Let G be a 2-generated finite group. If $\Gamma(G)$ is a cograph, then G is soluble.*

A graph is chordal if it contains no induced cycle of length greater then 3. A graph is called split if its vertex set is the disjoint union of two subsets A and B so that A induces a complete graph and B induces an empty graph. In the final part of the paper, we will prove the following result.

Theorem 1.12. *Let G be a 2-generated finite group. Then the following conditions are equivalent.*

(1) $\Gamma(G)$ *is split.*
(2) $\Gamma(G)$ *is chordal.*
(3) $\Gamma(G)$ *is C_4-free, i.e. no induced subgraph of* $\Gamma(G)$ *is isomorphic to a cyclic graph with four vertices.*
(4) *Either G is a cyclic p-group or $|G| = 2p$ for some prime p.*

2. Cographs

Our first result is that if $\Gamma(G)$ is a cograph, then $\Gamma(G/N)$ is also a cograph, for every normal subgroup N of G. In order to prove a more general statement which implies the previous sentence, we need to recall an auxiliary result, which generalizes an argument due to Gaschütz [11]. Given a subset X of a finite group G, we will denote by $d_X(G)$ the smallest cardinality of a set of elements of G generating G together with the elements of X. In the particular case when $X = \varnothing$, $d_\varnothing(G) = d(G)$ is the smallest cardinality of a generating set of G.

Lemma 2.1. [7, Lemma 6] *Let X be a subset of G and N a normal subgroup of G and suppose that $\langle g_1, \dots, g_r, X, N\rangle = G$. If $r \geqslant d_X(G)$, then we can find $n_1, \dots, n_r \in N$ so that $\langle g_1n_1, \dots, g_rn_r, X\rangle = G$.*

Lemma 2.2. *Let G be a 2-generated finite group and N a normal subgroup of G and let $t \in \mathbb{N}$ with $t \geqslant 2$. If $\Gamma(G/N)$ contains an induced subgraph isomorphic to P_t, then so does $\Gamma(G)$.*

Proof. Assume that $(a_1N, a_2N, \dots, a_tN)$ is a t-vertex path in $\Gamma(G/N)$. By Lemma 2.1 there exist $n_1, n_2 \in N$ such that $\langle a_1n_1, a_2n_2\rangle = G$. In particular $d_{\{a_2n_2\}}(G) \leqslant 1$, so, again by Lemma 2.1, if $t \geqslant 3$ then there exists $n_3 \in N$ such that $\langle a_2n_2, a_3n_3\rangle = G$. By repeating this argument, we can find $n_1, \dots, n_t \in N$ such that $\langle a_in_i, a_{i+1}n_{i+1}\rangle = G$ for $1 \leqslant i \leqslant t-1$. If $(r,s) \neq (i, i+1)$ for some $i \in \{1, \dots, t-1\}$, then $\langle a_r, a_s\rangle N \neq G$, and consequently $\langle a_rn_r, a_sn_s\rangle \neq G$. So $(a_1n_1, \dots, a_tn_t)$ is a t-vertex path in $\Gamma(G)$. □

Proof of Theorem 1.8. This can be proved with the same argument used by Cameron in [5, Theorem 8.8]. Let $\Delta(G)$ be the subgraph of $\Gamma(G)$ obtained by deleting the identity element. By [4, Theorem 1] the graph $\Delta(G)$ is connected. The join graph of G is the graph whose vertices are the non-trivial proper subgroups of G and in which two vertices H and K are adjacent if and only if $H \cap K \neq 1$. By [20] if G is not soluble, then this graph is connected. It can be easily seen that this implies that the complement graph $\overline{\Delta(G)}$ is connected. Since the graph complement of a connected cograph is disconnected, it follows that $\Delta(G)$ (and consequently $\Gamma(G)$) is not a cograph when G is not soluble. □

Proof of Theorem 1.9. Let N be a maximal normal subgroup of G. If G/N is non-abelian, then it is isomorphic to a non-abelian simple group and by [13] the identity element is the unique isolated vertex of $\Gamma(G/N)$. So the conclusion follows immediately by combining Lemma 2.2 and Theorem 1.8. □

Let $\mathrm{Frat}(G)$ be the Frattini subgroup of G.

Lemma 2.3. *Let G be a 2-generated finite nilpotent group. If $\Gamma(G)$ is a cograph, then $|G/\operatorname{Frat}(G)|$ is the product of at most two primes.*

Proof. Assume that $|G/\operatorname{Frat}(G)|$ is divisible by $p_1p_2p_3$, with p_1, p_2, p_3 prime numbers. Since $d(G) \leqslant 2$, we cannot have $p_1 = p_2 = p_3$ and so we may assume $p_3 \notin \{p_1, p_2\}$. Consider $X = \langle x_1\rangle \times \langle x_2\rangle \times \langle x_3\rangle$ with $|x_i| = p_i$ for $1 \leqslant i \leqslant 3$. It can be easily checked that $(x_1, 1, 1), (1, x_2, x_3), (x_1, 1, x_3), (1, x_2, 1)$ is a four-vertex path in $\Gamma(X)$. Since X is an epimorphic image of G, Lemma 2.2 would imply that $\Gamma(G)$ is not a cograph. □

Before we state the following lemma, let us recall some definitions that will be used in the statement. A chief factor X/Y of a finite group H is said to be complemented in H if X/Y has a complement in H/Y. If V is a finite irreducible H-module, then a chief factor X/Y of H is H-isomorphic to V if there exists a group isomorphism $\phi : X/Y \to V$ with the property that $\phi(x^hY) = \phi(xY)^h$, for any $x \in X$ and $h \in H$.

Lemma 2.4. *Let H be a 2-generated finite soluble group and V a finite non-trivial irreducible H-module. Assume that there exist $a, b \in H$ such that*

(1) *$H = \langle a, b\rangle$;*
(2) *$H \neq \langle a\rangle, H \neq \langle b\rangle$;*
(3) *$a \notin C_H(V), b \notin C_H(V)$.*

Consider the semidirect product $G = V \rtimes H$. If no complemented chief factor of H is H-isomorphic to V, then $\Gamma(G)$ contains a subgraph isomorphic to the four-vertex path P_4.

Proof. Let $|V| = p^t$, with p a prime. Define

$$\Omega_a = \{v \in V \mid \langle a, bv\rangle = G\}, \quad \Omega_b = \{v \in V \mid \langle av, b\rangle = G\}.$$

Assume $v \notin \Omega_a$. Then $\langle a, bv\rangle$ is a complement of V in G. The fact that no complemented chief factor of H is H-isomorphic to V ensures that all the complements of V in G form a single conjugacy class (see [12, Satz 3]), so there exists $w \in V$ such that $(a, bv) = (a^w, b^w)$. In particular $w \in C_V(a)$ and $v = [b, w]$. This implies $|V \smallsetminus \Omega_a| \leqslant |[b, C_V(a)]| \leqslant |C_V(a)|$. Since we are assuming $C_V(a) < V$, we deduce

$$|\Omega_a| \geqslant |V| - |C_V(a)| \geqslant p^t - p^{t-1}. \tag{1}$$

For the same reason

$$|\Omega_b| \geqslant |V| - |C_V(b)| \geqslant p^t - p^{t-1}. \tag{2}$$

Let $\Omega = \{(v_1, v_2) \in V^2 \mid \langle av_1, bv_2\rangle = G\}$. The number of pairs (v_1, v_2) in $V^2 \smallsetminus \Omega$ coincides with the number of complements of V in G, so

$$|\Omega| = |V^2| - |V|. \tag{3}$$

If $(v_1, v_2) \in \Omega \cap (\Omega_b \times \Omega_a)$ then (a, bv_2, av_1, b) is a four-vertex path in $\Gamma(G)$. In particular, if $|\Omega_a \times \Omega_b| + |\Omega| > |V|^2$, then $(\Omega_a \times \Omega_b) \cap \Omega \neq \varnothing$, and $\Gamma(G)$ contains P_4. So we may assume

$$|\Omega_a||\Omega_b| \leqslant |V^2| - |\Omega| = |V|. \tag{4}$$

In particular it follows from (1), (2) and (3), that $(p^t - p^{t-1})^2 \leqslant p^t$, i.e.

$$p^t \leqslant \left(\frac{p}{p-1}\right)^2. \tag{5}$$

This implies $p = 2$ and $t = 2$, i.e. $V \cong C_2 \times C_2$. We have two possibilities:

a) $H/C_H(V) \cong \mathrm{GL}(2, 2) \cong S_3$. In this case $G/C_H(V) \cong S_4$. Since

$$((1, 2), (2, 3, 4), (1, 4), (1, 2, 3))$$

is a four-vertex path in S_4, the conclusion follows from Lemma 2.2.

b) $H/C_H(V) \cong C_3$. In this case $C_V(a) = C_V(b) = \{0\}$, but then, by (1) and (2), $|\Omega_a|, |\Omega_b| \geqslant 3$, in contradiction with (4). □

Lemma 2.5. *Let G be a non-nilpotent 2-generated finite soluble group. If $\Gamma(G)$ is a cograph, then $G/\operatorname{Frat}(G) \cong N \rtimes H$, where N is a faithful irreducible H-module and H is cyclic of prime order.*

Proof. Assume that $\Gamma(G)$ is a cograph. Then also $\Gamma(G/\operatorname{Frat}(G))$ is a cograph. Moreover $G/\operatorname{Frat}(G)$ is not nilpotent (otherwise G would be nilpotent) so it is not restrictive to assume $\operatorname{Frat}(G) = 1$. Since G is not nilpotent, there exists a minimal normal subgroup of G, say N, which is not central in G. Set $H = G/C_G(N)$. Then N is a faithful irreducible H-module and the semidirect product $N \rtimes H$ is an epimorphic image of G. By Lemma 2.2, $\Gamma(N \rtimes H)$ is a cograph, so it follows from Lemma 2.4 that H is a cyclic group and consequently $\dim_{\operatorname{End}_H(N)} N = 1$.

Let K be a complement of N in G. Since G is 2-generated and $\dim_{\operatorname{End}_H(N)} N = 1$, it follows from [11, Satz 4] that no complemented chief factor of K is K-isomorphic to N. Since $G/C_G(N)$ is cyclic, there exists $x \in K$ such that $K = \langle x, C_K(N)\rangle$. Moreover, since K is 2-generated, by Lemma 2.1 there exist $c_1, c_2 \in C_K(N)$ such that $\langle xc_1, xc_2\rangle = K$. If K is not cyclic, then the two elements $a = xc_1$ and $b = xc_2$ satisfy the assumptions of Lemma 2.4. But this would imply that $\Gamma(G)$ is not a cograph, a contradiction. With a similar argument we can prove that $K/C_K(N)$ is a p-group. Indeed assume $|K/C_K(N)| = rs$ with $r, s \geqslant 2$ and $(r, s) = 1$. There exist $y_1, y_2 \in K$ such that $\langle y_1, y_2\rangle = K$, $|y_1C_K(N)| = r$ and $|y_2C_K(N)| = s$. We take y_1, y_2 in the role of a, b in Lemma 2.4 and we deduce that $\Gamma(G)$ is not a cograph. So we may assume $K = \langle xy\rangle$ where $|x|$ is a p-power, $y \in C_K(N)$ and $(|y|, p) = 1$. If $y \neq 1$, then, for any $1 \neq n \in N$, (n, xy, ny, x) is a four-vertex path in $\Gamma(G)$. So $y = 1$ and K is a cyclic p-group. In particular $C_K(N) \leqslant \operatorname{Frat}(K)$. However $\operatorname{Frat}(K) \cap C_K(N) \leqslant \operatorname{Frat}(G) = 1$, so we deduce that $C_K(N) = 1$.

We have now proved that $K = \langle x\rangle$ is cyclic of order p^t, for some $t \in \mathbb{N}$ and N is a faithful irreducible K-module. In particular K acts fixed-point-freely on N. Choose $1 \neq n \in N$. Then K and K^n are two maximal subgroups of G with trivial intersection. If $t > 1$, then $(x^p, x^n, x, (x^n)^p)$ is a four-vertex path in $\Gamma(G)$. Since $\Gamma(G)$ is a cograph we conclude $t = 1$. □

Proof of Theorem 1.7. Assume that $\Gamma(G)$ is a cograph. If G is nilpotent then, by Lemma 2.3, $G/\operatorname{Frat}(G)$ is either a p-group or a cyclic group of order p_1p_2, where p_1 and p_2 are two different primes. In the first case G is a p-group, in the second G is a cyclic group and p_1, p_2 are the only prime divisors of $|G|$. If G is not nilpotent, then, by Lemma 2.5, $G/\operatorname{Frat}(G) \cong V \rtimes \langle x\rangle$ where x has prime order and V is a faithful irreducible $\langle x\rangle$-module.

Conversely we have to prove that if G satisfies (1), (2) or (3), then $\Gamma(G)$ is a cograph. If (g_1, g_2, g_3, g_4) is a four-vertex path in $\Gamma(G)$, then either $(g_1 \operatorname{Frat}(G), g_2 \operatorname{Frat}(G), g_3 \operatorname{Frat}(G), g_4 \operatorname{Frat}(G))$ is a four-vertex path in $\Gamma(G/\operatorname{Frat}(G))$ or there exist $1 \leqslant i < j \leqslant 4$ with $g_i \operatorname{Frat}(G) = g_j \operatorname{Frat}(G)$. However if the second possibility occurs, then there exists $k \in \{1, 2, 3, 4\} \smallsetminus \{i, j\}$ such that g_k is adjacent to g_j but not to g_i. This implies $G = \langle g_k, g_j\rangle = \langle g_k, g_j, \operatorname{Frat}(G)\rangle = \langle g_k, g_i, \operatorname{Frat}(G)\rangle < G$, a contradiction. Therefore, to complete the proof of the theorem we may assume $\operatorname{Frat}(G) = 1$.

Assume by contradiction that (g_1, g_2, g_3, g_4) is a four-vertex path in $\Gamma(G)$. There are four possibilities to consider.

a) $G \cong C_p$. There exists $i \in \{1, 2, 3, 4\}$ such that $|g_i| = p$, but then g_i is adjacent to g_j for any $j \neq i$, a contradiction.
b) $G \cong C_p \times C_p$. In this case $|g_1| = |g_2| = |g_3| = |g_4| = p$. Since g_1 and g_3 are not adjacent in $\Gamma(G)$, $\langle g_1\rangle = \langle g_3\rangle$. Moreover, since g_3 and g_4 are adjacent

$\Gamma(G)$, $\langle g_3\rangle \neq \langle g_4\rangle$. But then $\langle g_1\rangle \neq \langle g_4\rangle$ and g_1 and g_4 are adjacent in $\Gamma(G)$, a contradiction.

c) $G \cong C_{p_1} \times C_{p_2}$, with $p_1 \neq p_2$. There is no $i \in \{1,2,3,4\}$ such that $|g_i| = p_1p_2$, since this would imply g_i adjacent to g_j for any $j \neq i$. It is not restrictive to assume $|g_1| = p_1$. This would imply $|g_2| = p_2, |g_3| = p_1, |g_4| = p_2$, and consequently that g_1 and g_4 are adjacent, a contradiction.

d) $G \cong V \rtimes \langle x\rangle$ where x has order p and V is a faithful irreducible $\langle x\rangle$-module. There exists a prime $q \neq p$ such that V is an elementary abelian q-group and every non-trivial element g of G has order p or q. Assume that $|g_1| = p$. Then $\langle g_1\rangle$ is the unique maximal subgroup of G containing g_1. Since g_3 and g_4 are not adjacent to g_1, we must have $g_3, g_4 \in \langle g_1\rangle$, but then g_3, g_4 are not adjacent in $\Gamma(G)$. So $|g_1| = q$. For the same reason $|g_4| = q$ and consequently $|g_2| = |g_3| = p$. But this would imply that g_2 and g_4 are adjacent.

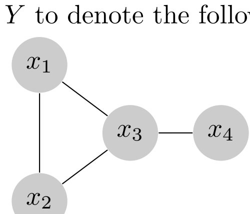
□

3. Perfect graphs

3.1. Preliminary results. The results of this section strongly depend on the strong perfect graph theorem, that has been already mentioned in the introduction and can be stated in the following way [6].

Theorem 3.1. *A graph is perfect if and only if it admits neither odd holes nor antiholes as induced subgraph.*

In the following, we will use Y to denote the following graph:



Recall that the tensor product $\Gamma_1 \wedge \Gamma_2$ of two graphs Γ_1 and Γ_2 is the graph whose vertex set coincides with the cartesian product of the vertex sets of Γ_1 and Γ_2 and where (x_1, y_1) and (x_2, y_2) are adjacent if and only if x_1, x_2 are adjacent in Γ_1 and y_1, y_2 are adjacent in Γ_2. If G_1 and G_2 are 2-generated finite groups, then $\Gamma(G_1 \times G_2)$ is a subgraph of $\Gamma(G_1) \wedge \Gamma(G_2)$, and $\Gamma(G_1 \times G_2) \cong \Gamma(G_1) \wedge \Gamma(G_2)$ if $|G_1|$ and $|G_2|$ are coprime (see [14, Lemma 2.5]).

Theorem 3.2 ([18, Theorem 3.2]). *The tensor product $\Gamma_1 \wedge \Gamma_2$ of two graphs Γ_1 and Γ_2 is perfect if and only if either*

(1) *Γ_1 or Γ_2 is bipartite, or*
(2) *neither Γ_1 nor Γ_2 contain Y or an odd n-hole with $n \geqslant 5$, as an induced subgraph.*

Remark 3.3. Let $\Gamma_1 \cong Y$ be a graph with vertex-set $\{x_1, x_2, x_3, x_4\}$ and $\Gamma_2 \cong K_3$ be a complete graph with vertex-set $\{y_1, y_2, y_3\}$. Then

$$((x_1, y_1), (x_2, y_3), (x_3, y_1), (x_4, y_2), (x_3, y_3))$$

is a 5-hole in the tensor product $\Gamma_1 \wedge \Gamma_2$.

Another remark that will be used in some of the proofs is that a 5-hole in a graph Γ is also a 5-antihole. Indeed if $\{x_1, x_2, x_3, x_4, x_5\}$ is a subset of the vertices of a graph Γ inducing a 5-hole and $(x_1, x_2, x_3, x_4, x_5)$ is a 5-cycle in Γ, then $(x_1, x_3, x_5, x_2, x_4)$ is a 5-cycle in the complement graph.

In this and in the following sections we will use the notations $g_1 \sim g_2$ and $g_1 \nsim g_2$ to denote that g_1 and g_2 are adjacent, or non-adjacent, in $\Gamma(G)$.

Lemma 3.4. *$\Gamma(G)$ is perfect if and only if $\Gamma(G/\operatorname{Frat}(G))$ is perfect.*

Proof. By the strong perfect graph theorem, it suffices to prove that if $m \geqslant 5$, then $\Gamma(G)$ contains an m-hole or an m-antihole if and only if $\Gamma(G/\operatorname{Frat}(G))$ has the same property. Since $\langle g_1 \operatorname{Frat}(G), g_2 \operatorname{Frat}(G)\rangle = G/\operatorname{Frat}(G)$ if and only if $\langle g_1, g_2\rangle = G$, if the subset $\{x_1 \operatorname{Frat}(G), \dots, x_m \operatorname{Frat}(G)\}$ induces an m-hole or an m-antihole in $\Gamma(G/\operatorname{Frat}(G))$, then so does $\{x_1, \dots, x_m\}$ in $\Gamma(G)$. Conversely, assume that $\{x_1, \dots, x_m\}$ induces an m-hole or an m-antihole in $\Gamma(G)$. If $1 \leqslant i < j \leqslant m$, then there exists $k \in \{1, \dots, m\} \smallsetminus \{i, j\}$ such that x_k is adjacent to x_i but not to x_j. In particular $x_i \operatorname{Frat}(G) \neq x_j \operatorname{Frat}(G)$ and $\{x_1 \operatorname{Frat}(G), \dots, x_m \operatorname{Frat}(G)\}$ induces an m-hole or an m-antihole in $\Gamma(G/\operatorname{Frat}(G))$. □

Let $I_n = \{1, \dots, n\}$ and consider the graph Δ_n whose vertices are the subsets of I_n and where J_1 and J_2 are adjacent if and only if $J_1 \cup J_2 = I_n$.

Lemma 3.5. *The graph Δ_n is perfect if and only if $n \leqslant 4$.*

Proof. If $n \geqslant 5$, then $(\{1,2,4,6,\dots,n\}, \{1,3,5,6,\dots,n\}, \{2,4,5,6,\dots,n\}, \{1,3,4,6,\dots,n\}, \{2,3,5,6,\dots,n\})$ is a 5-hole in Δ_n so Δ_n is not perfect. We may assume $n \leqslant 4$. Let $m \geqslant 5$ be an odd integer and assume that X is a subset of the vertex-set of Δ_n inducing an m-hole or an m-antihole. Clearly $I_n \notin X$. As a consequence, $\varnothing \notin X$. Moreover if $\{i\}$ is a singleton, then $I_n \smallsetminus \{i\}$ is the unique proper subset of I_n adjacent to $\{i\}$, so $\{i\} \notin X$. So we have at most $2^n - n - 2$ possible choices for an element of X. This implies that $n = 4$ and X consists of sets of cardinality 2 or 3. Since I_4 contains only four subsets of cardinality 3, it is not restrictive to assume $\{1,2\} \subseteq X$. Note that a subset of cardinality 3 is adjacent to all the other subsets of cardinality 3. So if X induces an m-hole, then X contains at most 2 (adjacent) subsets of cardinality 3. This implies that X contains at least 3 subsets of cardinality 2, inducing a 3-vertex path. But this is impossible since a subset of cardinality 2 is adjacent to only one subset of cardinality 2. If X induces an m-antihole, then it contains at least one subset of cardinality 2, say Y, and this must be adjacent to another $m-3$ elements of X. However there is a unique subset of cardinality 2 and two subsets of cardinality 3 adjacent to Y, hence $m - 3 \leqslant 3$. But this implies $m = 5$ and we may exclude this possibility since a 5-antihole is also a 5-hole, as noted above. □

Lemma 3.6. *Let G be a finite group and let $g \in G$ be an element which is contained in a unique maximal subgroup of G. Then g cannot be the vertex of an m-hole or m-antihole in $\Gamma(G)$ with $m \geqslant 5$.*

Proof. Let $M \leqslant G$ be the unique maximal subgroup containing g and let $m \geqslant 5$.

Let $(g, a_2, \dots, a_m)$ be an m-hole. We have $g \nsim a_3, a_4$, which implies $a_3, a_4 \in M$, so they cannot be adjacent in $\Gamma(G)$, a contradiction.

Let $(g, a_2, \dots, a_m)$ be an m-antihole. We have $g \nsim a_2, a_m$, which implies $a_2, a_m \in M$, so they cannot be adjacent in $\Gamma(G)$, a contradiction. □

Lemma 3.7. *Let $m \geqslant 5$ and suppose $(a_1, \dots, a_m)$ is an m-hole or an m-antihole in $\Gamma(G)$. If $\langle a_i\rangle = \langle a_j\rangle$, then $i = j$.*

Proof. Let $i \neq j$ and $\langle a_i\rangle = \langle a_j\rangle$. We can assume without loss of generality that $i = 1$ and $2 \leqslant j \leqslant \frac{m+1}{2}$. If $(a_1, \dots, a_m)$ is an m-hole, then $a_m \sim a_1$, and this implies $a_m \sim a_j$ and consequently $j = m - 1$. But then $m - 1 \leqslant \frac{m+1}{2}$, hence $m \leqslant 3$, a contradiction. If $(a_1, \dots, a_m)$ is an m-antihole, then $a_m \nsim a_1$, and so $a_m \nsim a_j$ and we argue as before. □

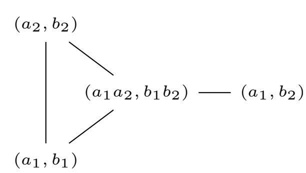
3.2. Nilpotent groups. The aim of this subsection is to prove Theorem 1.1. First we prove the statement in the special case where G is cyclic.

Lemma 3.8. *Let G be a finite cyclic group. Then $\Gamma(G)$ is perfect if and only if $|G|$ is divisible by at most four different primes.*

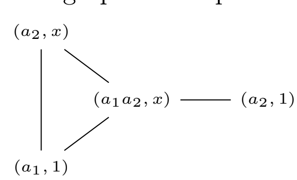
Proof. By Lemma 3.4, we may assume $\mathrm{Frat}(G) = 1$, so $|G| = p_1 \cdots p_t$ where $p_1, \dots, p_t$ are distinct primes. Assume that $(a_1, \dots, a_m)$ is an m-hole or an m-antihole in $\Gamma(G)$. Let $\pi = \{p_1, \dots, p_t\}$ and for any $i \in \{1, \dots, t\}$, let π_i be the set of prime divisors of $|a_i|$. By Lemma 3.7, if $i \neq j$, then $\pi_i \neq \pi_j$, moreover a_i and a_j are adjacent in $\Gamma(G)$ if and only if $\pi_i \cup \pi_j = \pi$. This implies that $\Gamma(G)$ is perfect if and only if Δ_t is perfect, and the conclusion follows from Lemma 3.5. □

The proof of the general case requires some preliminary lemmas and remarks.

Remark 3.9. Let p and q be two different primes. If $P = \langle a_1, a_2 \rangle$ is a finite, 2-generated, non-cyclic p-group and $Q = \langle b_1, b_2 \rangle$ is a finite, 2-generated, non-cyclic q-group, then $\Gamma(P \times Q)$ contains an induced subgraph isomorphic to Y :



Remark 3.10. If $P = \langle a_1, a_2 \rangle$ is a finite, 2-generated, non-cyclic finite p-group and $C = \langle x \rangle$ is a non-trivial finite cyclic group whose order is not divisible by p, then $\Gamma(P \times C)$ contains an induced subgraph isomorphic to Y :



Remark 3.11. If G is a 2-generated finite group of order at least 3, then $\Gamma(G)$ contains an induced subgraph isomorphic to K_3. In particular $\Gamma(G)$ is not a bipartite graph.

Proof. If $G = \langle a, b \rangle$ is not cyclic, then we can take the subgraph of $\Gamma(G)$ induced by $\{a, b, ab\}$. If $G = \langle x \rangle$, we can take the subgroup induced by $\{1, x, x^{-1}\}$. □

Lemma 3.12. *Let G be a 2-generated finite nilpotent group. If $\Gamma(G)$ is perfect, then the order of $G/\mathrm{Frat}(G)$ is the product of at most four (not necessarily distinct) primes.*

Proof. By Lemma 3.4 we may assume $\mathrm{Frat}(G) = 1$. For any prime divisor p of $|G|$, the Sylow p-subgroup of G is either cyclic of order p or elementary abelian of order p^2. If all the Sylow subgroups of G are cyclic, then G is cyclic and the conclusion follows from Lemma 3.8. So we may assume that G contains a non-cyclic Sylow p-subgroup, say P, of order p^2. Let K be a complement of P in G. Assume, by contradiction, that $|K|$ is the product of at least three primes. If K is not cyclic, then $K = Q_1 \times Q_2 \times H$ where Q_1, Q_2 are Sylow subgroups, Q_1 is non-cyclic and $Q_2 \neq 1$. By Remarks 3.10 and 3.11, $\Gamma(P \times Q_2)$ and $\Gamma(Q_1 \times H)$ contain an induced subgraph isomorphic, respectively, to Y and K_3. But then we deduce from Remark 3.3 that $\Gamma(G) \cong \Gamma(P \times Q_2) \wedge \Gamma(Q_1 \times H)$ is not perfect. So we may assume that $K = \langle x \rangle$ and that $|x|$ is divisible by at least three different primes q_1, q_2, q_3. Let Ω be the set of the vertices y of $\Gamma(K)$ with the property that $\langle y \rangle \neq K$ and let Λ be the subgraph of $\Gamma(K)$ induced by Ω. Notice that the subgraph of $\Gamma(G)$ induced by the subset $P \times \Omega$ is isomorphic to $\Gamma(P) \wedge \Lambda$ and that $\{x^{q_1}, x^{q_2}, x^{q_3}, x^{q_1 q_2}\}$ induces a subgraph of Λ isomorphic to Y. But then, again by Remark 3.3, $\Gamma(P) \wedge \Lambda$, and consequently $\Gamma(G)$, contains a 5-hole. □

Lemma 3.13. *Let G be a non-cyclic 2-generated finite p-group. Then $\Gamma(G)$ is perfect and does not contain an induced subgraph isomorphic to Y.*

Proof. We have $G/\operatorname{Frat}(G) \cong C_p \times C_p$. If g is a non-isolated vertex of $\Gamma(G)$, then $|g \operatorname{Frat}(G)| = p$ and $\langle g \rangle$ is the unique maximal subgroup of G containing g. It follows from Lemma 3.6 that $\Gamma(G)$ contains no m-hole or m-antihole with $m \geqslant 5$, so it follows from the strong perfect graph theorem that $\Gamma(G)$ is perfect. Now assume by contradiction that $\{g_1, g_2, g_3, g_4\}$ induces a subgraph of $\Gamma(G)$ isomorphic to Y. We may order these four vertices in such a way that g_1 and g_2 are adjacent, while g_4 is not adjacent to g_1 nor to g_2. The latter condition implies $\langle g_4 \operatorname{Frat}(G) \rangle = \langle g_1 \operatorname{Frat}(G) \rangle = \langle g_2 \operatorname{Frat}(G) \rangle$, in contradiction with $\langle g_1, g_2 \rangle = G$. □

Proof of Theorem 1.1. By Lemma 3.4 we may assume that $\operatorname{Frat}(G) = 1$. By Lemma 3.12, the condition that $|G|$ is the product of at most 4 primes is necessary for $\Gamma(G)$ to be perfect. We have to prove that this condition is also sufficient. By Lemma 3.8, we may assume that G is not cyclic. This means that $G = P \times K$, where $P \cong C_p \times C_p$ for a suitable prime p and K is a nilpotent group whose order is coprime with p and is the product of at most two primes. By Lemma 3.13, we may assume $K \neq 1$.

If K is not cyclic, then $K \cong C_q \times C_q$, for a prime $q \neq p$ and $\Gamma(G) \cong \Gamma(P) \wedge \Gamma(Q)$ is perfect, as a consequence of Theorem 3.2 and Lemma 3.13.

The previous argument does not work if $K = \langle g \rangle$ is cyclic. Indeed, $\Gamma(G)$ and $\Gamma(P) \wedge \Gamma(K)$ are not isomorphic. For example, if $P = \langle a_1, a_2 \rangle$, then (a_1, g) and (a_2, g) are adjacent in $\Gamma(G)$ but not in $\Gamma(P) \wedge \Gamma(K)$. However we can argue in the following way. Assume that $X \subseteq G$ induces an m-hole or an m-antihole, with $m \geqslant 5$. If $K = \langle g \rangle$ and $y \in P$, then either (y, g) is an isolated vertex of $\Gamma(G)$ (when $y = 1$), or $\langle (y, g) \rangle$ is the unique maximal subgroup of G containing (y, g). In both the cases, since the vertices of an m-hole or an m-antihole are not isolated, Lemma 3.6 implies that $(y, g) \notin X$. In particular, this implies that K has composite order (no element of K could belong to X), so it remains to handle the case where K is cyclic of order $r \cdot s$ for distinct primes r and s. In this case, we consider the subgraph Δ of $\Gamma(K)$ induced by the elements of K of prime order. From what we said above, it follows that X induces an m-hole or m-antihole in $\Gamma(G)$ if and only if it induces an m-hole or m-antihole in $\Gamma(P) \wedge \Delta$. This would imply that $\Gamma(P) \wedge \Delta$ is not perfect, and consequently, by Theorem 3.2 and Lemma 3.8, that Δ contains an induced subgraph isomorphic to Y. So assume by contradiction that $\{g_1, g_2, g_3, g_4\}$ induces a subgraph of Δ isomorphic to Y. We may order these four vertices in such a way that g_1 and g_2 are adjacent while g_4 is not adjacent to g_1 nor to g_2. The latter condition implies $\langle g_4 \rangle = \langle g_1 \rangle = \langle g_2 \rangle$, in contradiction with $\langle g_1, g_2 \rangle = K$. □

3.3. The dihedral group.

In this subsection we determine when the dihedral group

$$D_n = \left\langle \rho, \iota \mid \rho^n = \iota^2 = 1, \rho^\iota = \rho^{-1} \right\rangle$$

has a perfect generating graph. We start with a preliminary lemma.

Lemma 3.14. *Let N be a normal subgroup of G such that $G/N \cong C_2 \times C_2$. Then $\Gamma(G)$ has no m-antihole with $m \geqslant 7$.*

Proof. Let $a, b, c \in G$ be such that $G/N \coloneqq \{N, aN, bN, cN\}$. Suppose $(a_1, \dots, a_m)$ is an m-antihole in $\Gamma(G)$. Since a_1 and a_3 are adjacent vertices of $\Gamma(G)$, we may assume without loss of generality that $a_1 N = aN$ and $a_3 N = bN$. Since $a_5, \dots, a_{m-1}$ are adjacent to both a_1 and a_3, it follows that $a_5 N, \dots, a_{m-1} N$ are all equal to cN. In particular, if $m > 7$, then $a_5N = a_7N$ implies $a_5 \nsim a_7$, a contraction. So we may assume $m = 7$. Since $a_4 \sim a_1$, $a_4 \sim a_6$, $a_1N = aN$ and $a_6N = cN$, we must have $a_4N = bN$. Analogously, from $a_2 \sim a_4$ and $a_2 \sim a_6$ it follows that $a_2N = aN$. But now consider a_7N : $a_7N \neq aN$ since $a_7 \sim a_2$, $a_7N \neq bN$ since $a_7 \sim a_3$ and $a_7N \neq cN$ since $a_7 \sim a_5$. This would imply $a_7 \in N$, and consequently that a_7 is an isolated vertex of $\Gamma(G)$, a contradiction. □

Proof of Theorem 1.2. Let $m \geqslant 5$ be odd. We start with two general remarks.

(a) No rotation ρ^i can appear in an m-hole or m-antihole. Indeed if $|\rho^i| < n$, then ρ^i is an isolated vertex of $\Gamma(D_n)$. If $|\rho^i| = n$, then $\langle\rho\rangle$ is the unique maximal subgroup of D_n containing ρ^i and we conclude using Lemma 3.6. So every m-hole or m-antihole in $\Gamma(D_n)$ must be of the form

$$(a_1, \dots, a_m) = (\rho^{x_1}\iota, \dots, \rho^{x_m}\iota)$$

for some $x_i \in \mathbb{Z}$.

(b) $\left\langle \rho^a\iota, \rho^b\iota \right\rangle = D_n$ if and only if $(a - b, n) = 1$.

First we prove that if n is odd, then (2) is a necessary condition for $\Gamma(D_n)$ to be perfect. Suppose n is odd and $n = p^aq^br^ck$ with p, q, r distinct primes and $k \geqslant 1$ coprime to these primes. Consider the elements $\alpha_1, \dots, \alpha_4$, obtained by solving the following systems (note that the existence of solutions is guaranteed by the Chinese Reminder Theorem):

$$\begin{cases} \alpha_1 \equiv 1 & \mod p \\ \alpha_1 \equiv b & \mod q \\ \alpha_1 \equiv -1 & \mod r \\ (\alpha_1 \equiv 1 & \mod k) \end{cases} \qquad \begin{cases} \alpha_2 \equiv -1 & \mod p \\ \alpha_2 \equiv -1-b & \mod q \\ \alpha_2 \equiv c & \mod r \\ (\alpha_2 \equiv 1 & \mod k) \end{cases}$$

$$\begin{cases} \alpha_3 \equiv 1 & \mod p \\ \alpha_3 \equiv 1 & \mod q \\ \alpha_3 \equiv d & \mod r \\ (\alpha_3 \equiv 1 & \mod k) \end{cases} \qquad \begin{cases} \alpha_4 \equiv a & \mod p \\ \alpha_4 \equiv -1 & \mod q \\ \alpha_4 \equiv -c-d & \mod r \\ (\alpha_4 \equiv 1 & \mod k) \end{cases}$$

where the conditions in the round brackets are considered only when $k \neq 1$, and a, b, c, d are such that

$$a \not\equiv 0, -1 \mod p, \quad b \not\equiv 0, -1 \mod q, \quad c, d \not\equiv 0 \mod r, \quad c + d \not\equiv 0 \mod r.$$

It can be easily checked that $(\iota, \rho^{\alpha_1}\iota, \rho^{\alpha_1+\alpha_2}\iota, \rho^{\alpha_1+\alpha_2+\alpha_3}\iota, \rho^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\iota)$ is a 5-hole in $\Gamma(D_n)$.

Now we prove that if (1) and (2) are satisfied, then $\Gamma(D_n)$ is perfect. We distinguish three cases according to n.

First assume n is even. Since D_n has an epimorphic image isomorphic to $C_2 \times C_2$, by Lemma 3.14 the graph $\Gamma(D_n)$ has no m-antihole with $m \geqslant 7$. Suppose that $\Gamma(D_n)$ contains an m-hole $(a_1, \dots, a_m)$, as described in (a). Since $D_n = \langle\rho^{x_i}\iota, \rho^{x_{i+1}}\iota\rangle$ for every i (where $m + 1$ is considered to be 1), by (b) we should have $x_{i+1} - x_i$ odd for every $1 \leqslant i \leqslant m$. Then, consider

$$0 = \sum_{i=1}^{m} x_{i+1} - x_i.$$

The right hand side should be odd, because it is a sum of an odd number of odd terms, contradiction. This shows that $\Gamma(D_n)$ has no m-holes nor m-antiholes for all $m \geqslant 5$ (recall that a 5-hole is also a 5-antihole), so $\Gamma(D_n)$ is perfect.

Next suppose $n = p^a$ is a prime power. Suppose that $\Gamma(D_n)$ contains an m-hole $(a_1, \dots, a_m)$. Since $a_1 \nsim a_3, a_4$, we should have that $x_4 - x_1$ and $x_3 - x_1$ are divisible by p, hence their difference (i.e. $x_4 - x_3$) should be divisible by p and therefore $a_3 \nsim a_4$, a contradiction. Suppose now there is an m-antihole $(a_1, \dots, a_m)$. Since $a_2 \nsim a_1, a_3$, the prime p should divide $x_3 - x_2$ and $x_2 - x_1$ and so p should divide their sum (i.e. $x_3 - x_1$), which means $a_1 \nsim a_3$, a contradiction.

Finally let us assume $n = p^a q^b$, where p, q are distinct primes. Suppose there is an m-hole $(a_1, \dots, a_m)$ in $\Gamma(D_n)$. Since $a_1 \nsim a_3, a_4$, the differences $x_3 - x_1$ and $x_4 - x_1$ are divisible by at least one of p or q. We may assume without loss of generality that $x_3 - x_1$ is divisible by p. Then $x_4 - x_1$ is divisible by q, otherwise $a_3 \nsim a_4$. Similarly $x_4 - x_2$ is divisible by p and $x_m - x_3$ is divisible by q. From the fact that p divides $x_4 - x_2$, arguing as before we deduce that if $5 \leqslant i \leqslant m$, then $x_i - x_2$ is divisible by q when i is odd and by p when i is even. In particular $x_m - x_2$ is divisible by q and since q also divides $x_m - x_3$, we have $a_2 \nsim a_3$, a contradiction. Suppose now there is an m-antihole $(a_1, \dots, a_m)$ in $\Gamma(D_n)$. Since $a_i \nsim a_{i+1}$, the difference $x_{i+1} - x_i$ is divisible by at least one of p or q. We have an odd number of possible i, so there must be a k such that $x_{k+1} - x_k$ and $x_k - x_{k-1}$ are both divisible by the same prime, which means that $x_{k+1} - x_{k-1}$ is also divisible by this prime, hence $a_{k+1} \nsim a_{k-1}$, a contradiction. □

3.4. Groups of order $p^a q^b$ and pqr. We have seen in the previous subsections that if G is a dihedral group or a 2-generated nilpotent group and $|G|$ is divisible by at most three distinct primes, then $\Gamma(G)$ is perfect. However there exist 2-generated finite groups whose generating graph is not perfect, although their order is divisible only by two distinct primes.

Example 3.15. Let $H = C_2^2$ and let h_1, h_2, h_3 be the non-trivial elements of H. Let p be an odd prime number and consider $N = \langle x_1, x_2, x_3 \rangle \cong C_p^3$. We may define an action of H on N by setting

$$\begin{aligned} x_1^{h_1} &= x_1, \quad & x_1^{h_2} &= x_1^{-1}, \quad & x_1^{h_3} &= x_1^{-1}, \\ x_2^{h_1} &= x_2^{-1}, \quad & x_2^{h_2} &= x_2, \quad & x_2^{h_3} &= x_2^{-1}, \\ x_3^{h_1} &= x_3^{-1}, \quad & x_3^{h_2} &= x_3^{-1}, \quad & x_3^{h_3} &= x_3. \end{aligned}$$

Let G be the semidirect product $N \rtimes H$. Then G is 2-generated and it can be easily checked that

$$(x_1 h_1, x_2 x_3 h_2, x_1 x_3 h_3, x_1^2 x_2 h_2, x_2 x_3 h_3)$$

is a 5-hole in $\Gamma(G)$.

Lemma 3.16. *Let G be a 2-generated finite group and let m be an odd integer, with $m \geqslant 5$. Let $X \subseteq G$. If there exist two maximal subgroups M_1 and M_2 of G such that $X \subseteq M_1 \cup M_2$, then X does not induce an m-hole nor an m-antihole.*

Proof. Suppose that $(a_1, \dots, a_m)$ is an m-hole induced by X. We may assume $a_1 \in M_1$. Since $G = \langle a_i, a_{i+1} \rangle$, it follows $a_i \in M_2 \setminus M_1$ if i is even, $a_i \in M_1 \setminus M_2$ if i is odd. In particular, since m is odd, $G = \langle a_1, a_m \rangle \leqslant M_1$, a contradiction. Now suppose that $(a_1, \dots, a_m)$ is an m-antihole induced by X. Again we may assume $a_1 \in M_1$. If $3 \leqslant i \leqslant m - 1$, then $G = \langle a_1, a_i \rangle$ implies $a_i \in M_2$ and therefore $m = 5$, otherwise $G = \langle a_3, a_{m-1} \rangle \leqslant M_2$. We may exclude this possibility since a 5-antihole is also a 5-hole. □

Lemma 3.17. *Suppose that $G = (\langle x \rangle \times \langle y \rangle) \rtimes \langle z \rangle$, with $|x| = p_1$, $|y| = p_2$, $|z| = p_3$, where p_1, p_2, p_3 are primes. If G is 2-generated, then $\Gamma(G)$ is perfect.*

Proof. If G is abelian, then the conclusion follows from Theorem 1.1. So we may assume $x \notin Z(G)$. Let $m \geqslant 5$ be an odd integer and suppose that $X \subseteq G$ induces an m-hole or an m-antihole in $\Gamma(G)$.

First we claim that if $y \in Z(G)$, then $p_2 = p_3$. Indeed assume $y \in Z(G)$ and $p_2 \neq p_3$ and let $g = x^i y^j z^k \in G$. If $|y^j z^k| = p_2 p_3$, then $|g| = p_2 p_3$, so $\langle g \rangle$ is the unique maximal subgroup of G containing g and $g \notin X$ by Lemma 3.6. But then $X \subseteq M_1 \cup M_2$, with $M_1 = \langle x, z \rangle$ and $M_2 = \langle x, y \rangle$, in contradiction with Lemma 3.16.

Our second claim is that $\langle x \rangle$ and $\langle y \rangle$ are not $\langle z \rangle$-isomorphic. This is obvious if $p_1 \neq p_2$, otherwise it is a necessary condition for G being 2-generated.

The two previous claims imply that for every $r, s, u, v \in \mathbb{Z}$, $\langle x^r y^s, x^u y^v z \rangle = G$ if and only if $x^r, y^s \neq 1$. In particular consider $w = x^r y^s \in \langle x, y \rangle$. If either $x^r = 1$ or $y^s = 1$, then w is an isolated vertex in $\Gamma(G)$. Moreover, the fact that $\langle x \rangle$ and $\langle y \rangle$ are not $\langle z \rangle$-isomorphic implies that $\langle x, y \rangle$ is the unique maximal subgroup of G containing w. In any case, w cannot be an element of X.

Let $(a_1, \ldots, a_m)$ be an m-hole or an m-antihole in $\Gamma(G)$ induced by X. By what we have said above, it is not restrictive to assume $a_i = x^{r_i} y^{s_i} z$ with $r_i, s_i \in \mathbb{Z}$ and in particular we may assume that $a_1 = z$. Notice that $\langle x^{r_i} y^{s_i} z, x^{r_j} y^{s_j} z \rangle = G$ if and only if $r_i \not\equiv r_j \mod p_1$ and $s_i \not\equiv s_j \mod p_2$.

If $(a_1, \ldots, a_m)$ is an m-hole, then $a_1 \not\sim a_j$ for any $j \in \{3, \ldots, m-1\}$. This implies that either $a_j \in \langle x, z \rangle$ or $a_j \in \langle y, z \rangle$. On the other hand $a_j \sim a_{j+1}$, so it is not restrictive to assume

$$s_3 \equiv 0 \mod p_2,\ r_4 \equiv 0 \mod p_1, \quad \ldots \quad, s_{m-2} \equiv 0 \mod p_2,\ r_{m-1} \equiv 0 \mod p_1.$$

Notice that $a_1 \sim a_2$ and $a_1 \sim a_m$ implies $r_2, r_m \not\equiv 0 \mod p_1$ and $s_2, s_m \not\equiv 0 \mod p_2$. By Lemma 3.7, $r_3 \not\equiv 0 \mod p_1$ and $s_{m-1} \not\equiv 0 \mod p_2$. Since $a_2 \not\sim a_{m-1}$ and $a_3 \not\sim a_m$, we deduce $s_2 \equiv s_{m-1} \mod p_2$ and $r_3 \equiv r_m \mod p_1$. Since $a_2 \sim a_3$ and $a_{m-1} \sim a_m$, it follows $r_2 \not\equiv r_3 \mod p_1$ and $s_{m-1} \not\equiv s_m \mod p_2$, but then $r_2 \not\equiv r_m \mod p_1$ and $s_2 \not\equiv s_m \mod p_2$. This implies $a_2 \sim a_m$, a contradiction.

Now suppose that $(a_1, \ldots, a_m)$ is an m-antihole. We may assume $m \geqslant 7$ since a 5-antihole is isomorphic to a 5-hole. From the conditions $a_1 \not\sim a_2$ and $a_1 \not\sim a_m$ it follows that it is not restrictive to assume $s_2 = 0$ and $r_m = 0$. Since $a_i \not\sim a_{i+1}$, it follows

$$a_1 = z,\ a_2 = x^{r_2} z,\ a_3 = x^{r_2} y^{s_3} z,\ a_4 = x^{r_4} y^{s_3} z,\ a_5 = x^{r_4} y^{s_5} z,\ a_6 = x^{r_6} y^{s_5} z \ldots$$

In particular

$$a_{m-2} = x^{r_{m-3}} y^{s_{m-2}} z,\ a_{m-1} = x^{r_{m-1}} y^{s_{m-2}} z,\ a_m = y^{s_m} z.$$

From $a_{m-1} \sim a_1$, it follows $r_{m-1} \not\equiv 0 \mod p_1$ and from $a_m \sim a_{m-2}$, it follows $s_m \not\equiv s_{m-2} \mod p_2$. However this implies that $a_{m-1} \sim a_m$, a contradiction. ☐

Proposition 3.18. *If $|G| = pq$ with p, q primes, then $\Gamma(G)$ is perfect.*

Proof. This follows immediately from Lemma 3.6. ☐

Proposition 3.19. *If $|G| = pqr$, where p, q and r are three distinct primes, then G is 2-generated and $\Gamma(G)$ is perfect.*

Proof. By [19, 10.1.10], G is 2-metacyclic. We may assume that G is non-abelian, so $G = \langle x \rangle \rtimes \langle y \rangle$, with $y \neq 1$ and $C_{\langle y \rangle}(x) = 1$. If $|x|$ is the product of two different primes, then the conclusion follows from Lemma 3.17. So we may assume that $|x| = p$. Assume that X induces an m-hole or an m-antihole in $\Gamma(G)$, where $m \geqslant 5$ is an odd integer. By Lemma 3.6, X contains only elements of prime order; moreover an element of order p is adjacent in $\Gamma(G)$ only to elements of order $q \cdot r$, so X can contain only elements of order q or r. On the other hand two elements of the same order q or r

are not adjacent in $\Gamma(G)$, and it is easy to see that this implies that X cannot induce neither an m-hole nor an m-antihole. □

Proposition 3.20. *If $|G| = p^2 q$, where p and q are distinct primes, then $\Gamma(G)$ is perfect.*

Proof. First assume G has a unique Sylow p-subgroup P. If $P \cong C_p \times C_p$, then the conclusion follows from Lemma 3.17. If $P \cong C_{p^2}$, then $\mathrm{Frat}(G)$ has order p, so $\Gamma(G/\mathrm{Frat}(G))$ is perfect by Proposition 3.18 and consequently $\Gamma(G)$ is perfect by Lemma 3.4. So we may assume that the Sylow p-subgroups are not normal, which implies that the Sylow q-subgroup, say Q, is normal. Either $G \cong (C_p \times C_q) \rtimes C_p$ or $G \cong C_q \rtimes C_{p^2}$. In the first case the conclusion follows again from Lemma 3.17. In the second case the only non-trivial elements of G that are contained in at least two different maximal subgroups are those of order p, so, by Lemma 3.6, if $X \subseteq G$ induces an m-hole or an m-antihole, with $m \geqslant 5$ an odd integer, then X contains only elements of order p. However no two elements of order p are adjacent in $\Gamma(G)$, so we reached a contradiction. □

3.5. The symmetric and alternating group. In this subsection we prove Theorem 1.3, determining the values of n for which the generating graphs of the symmetric and alternating groups of degree n are perfect. In the proofs we will need the following elementary lemmas:

Lemma 3.21. *Let $H \leqslant S_n$ be a transitive permutation group. If $\sigma \in H$ is a cycle of length $n-1$, then H is primitive.*

Proof. We may assume without loss of generality that the fixed point of σ is 1. Suppose H is imprimitive. Let B be the imprimitivity block which contains 1, then $B^\sigma = B$ since $1^\sigma = 1$. By the imprimitivity assumption, there exists $1 \neq i \in B$. But then $i, i^\sigma, i^{\sigma^2}, \dots, i^{\sigma^{n-2}}$ are all distinct elements, so $B = \{1, \dots, n\}$, a contradiction. □

Lemma 3.22. *Let $n \geqslant 3$ be an odd natural number and $H \leqslant S_n$ be a transitive permutation group. If $\sigma \in H$ is an $(n-2)$-cycle, then H is primitive.*

Proof. Suppose, without loss of generality, that the fixed points of σ are 1 and 2. Suppose H is imprimitive. As in the proof of the previous lemma, take B to be the block containing 1. Since n is odd, $|B| \geqslant 3$, so there is at least an element i in $B \smallsetminus \{1, 2\}$. Arguing as in the proof of the previous lemma, we obtain that $|B| \geqslant n-1$ and, since $|B|$ divides n, we conclude $B = \{1, \dots, n\}$, a contradiction. □

Theorem 3.23. *$\Gamma(S_n)$ is perfect if and only if $n \leqslant 4$.*

Proof. If $n \in \{2, 3\}$, then $\Gamma(S_n)$ is perfect, indeed $\Gamma(S_2) \cong K_2$ while $S_3 \cong D_3$, in which case we may apply Theorem 1.2.

Assume $n = 4$. Suppose there is an m-hole $(a_1, \dots, a_m)$ in $\Gamma(S_4)$, with $m \geqslant 5$. Two consecutive vertices a_i and a_{i+1} are adjacent and therefore they cannot both belong to A_4. Since m is odd, there must be two consecutive vertices which are in $S_4 \smallsetminus A_4$. Since two elements of order 2 do not generate the group, one of these two vertices should be a 4-cycle. However a 4-cycle is contained in a unique maximal subgroup, so we have a contradiction by Lemma 3.6. Suppose now that there is an m-antihole $(a_1, \dots, a_m)$ in $\Gamma(S_4)$, with $m \geqslant 7$. Since 4-cycles cannot occur in an m-antihole and elements of the Klein subgroup cannot generate with another element, each vertex in the antihole must be a transposition or a 3-cycle. There are at most two 3-cycles among the vertices of the antihole. Indeed if we pick three elements in an m-antihole, at least two of them are adjacent but two 3-cycles do not generate S_4. So, at least $m-2$

of the vertices of the antihole $(a_1, \dots, a_m)$ are transpositions. Since two transpositions do not generate S_4, we have a contradiction. We conclude that $\Gamma(S_4)$ is perfect.

If $n = 5, 6, 7$, then $\Gamma(S_n)$ is not perfect. Indeed it can be easily checked that the following are 5-holes in $\Gamma(S_n)$:

$$\Gamma(S_5): \quad ((1,2,3,4,5),(2,4),(1,2,3,5,4),(2,4,5,3),(1,2,4,5));$$

$$\Gamma(S_6): \quad ((1,3,2,4),(3,4,6,5),(1,2,3,4,5),(1,3,4,6),(2,3,4,5,6));$$

$$\Gamma(S_7): \quad ((1,5,4,7,2,3),(2,6,5,7,3,4),(1,2,3,4,5,7,6),(4,5),(1,2,3,4,5,6,7)).$$

We remain with two cases: $n \geqslant 8$ even and $n \geqslant 9$ odd.

Assume $n \geqslant 8$ even. In this case we claim that

$$\begin{aligned} a_1 &= (1, \dots, n-2) \\ a_2 &= (3, \dots, n) \\ a_3 &= (1, \dots, n-1) \\ a_4 &= (1, 3, 4, n) \\ a_5 &= (2, \dots, n) \end{aligned}$$

is a 5-hole in $\Gamma(S_n)$.

Notice that $\langle a_1, a_3\rangle$, $\langle a_1, a_4\rangle$, $\langle a_2, a_4\rangle$, $\langle a_2, a_5\rangle$ are intransitive subgroups and $\langle a_3, a_5\rangle \leqslant A_n$, so the pairs of corresponding vertices are not joined by an edge. Since a_3 and a_5 are $(n-1)$-cycles, the transitive subgroups $\langle a_2, a_3\rangle$, $\langle a_3, a_4\rangle$, $\langle a_4, a_5\rangle$, $\langle a_5, a_1\rangle$ are also primitive by Lemma 3.21. Let us now prove that the transitive subgroup $\langle a_1, a_2\rangle$ is also primitive. Let B be an imprimitive block which contains 1. Clearly $B^{a_2} = B$. If $B \cap \{3, \dots, n\} \neq \varnothing$, then $\{1, 3, \dots, n\} \subseteq B$, a contradiction. So $B = \{1, 2\}$, but then $B \cap B^{a_1} = \{2\}$, another contradiction and we conclude that $\langle a_1, a_2\rangle$ is primitive. Moreover

$$\begin{aligned} a_1a_2^{-1} &= (1, 2, n, n-1, n-2) && \in \langle a_1, a_2\rangle \\ a_3a_2^{-1} &= (1, 2, n, n-1) && \in \langle a_2, a_3\rangle \\ a_4 &= (1, 3, 4, n) && \in \langle a_3, a_4\rangle \\ a_4 &= (1, 3, 4, n) && \in \langle a_4, a_5\rangle \\ a_1a_5^{-1} &= (1, n, n-1, n-2) && \in \langle a_5, a_1\rangle \end{aligned}$$

But then, by [15, Corollary 1.3], the five subgroups $\langle a_1, a_2\rangle, \langle a_2, a_3\rangle, \langle a_3, a_4\rangle, \langle a_4, a_5\rangle$ and $\langle a_5, a_1\rangle$ contain A_n. Since they also contain elements outside A_n, they must be equal to S_n and so the corresponding pairs of vertices are joined by an edge.

Finally assume $n \geqslant 9$ odd. In this case we claim that

$$\begin{aligned} a_1 &= (1, \dots, n-3) \\ a_2 &= (4, \dots, n) \\ a_3 &= (1, \dots, n-2) \\ a_4 &= (1, 2, 4, 5, n-1, n-2) \\ a_5 &= (3, \dots, n) \end{aligned}$$

is a 5-hole in $\Gamma(S_n)$.

As in the discussion of the previous case, $\langle a_1, a_3\rangle$, $\langle a_1, a_4\rangle$, $\langle a_2, a_4\rangle$, $\langle a_2, a_5\rangle$ are intransitive subgroups and $\langle a_3, a_5\rangle \leqslant A_n$, so the pairs of corresponding vertices are not joined by an edge. Since a_3 and a_5 are $(n-2)$-cycles, the transitive subgroups $\langle a_2, a_3\rangle$, $\langle a_3, a_4\rangle$, $\langle a_4, a_5\rangle$, $\langle a_5, a_1\rangle$ are also primitive by Lemma 3.22. We claim that the transitive subgroup $\langle a_1, a_2\rangle$ is also primitive. Let B be an imprimitivity block which contains 1, so that $B^{a_2} = B$. If $B \cap \{4, \dots, n\} \neq \varnothing$, then $\{1, 4, \dots, n\} \subseteq B$, a contradiction. Since $|B| \geqslant 3$, the only possibility is $B = \{1,2,3\}$, but this leads to a contradiction since $B \cap B^{a_1} = \{2,3\}$. Finally, observe that

$$\begin{aligned} a_1 &= (1,\dots,n-3) && \in \langle a_1,a_2\rangle \\ a_3a_2^{-1} &= (1,2,3,n,n-1,n-2) && \in \langle a_2,a_3\rangle \\ a_4 &= (1,2,4,5,n-1,n-2) && \in \langle a_3,a_4\rangle \\ a_4 &= (1,2,4,5,n-1,n-2) && \in \langle a_4,a_5\rangle \\ a_1 &= (1,\dots,n-3) && \in \langle a_5,a_1\rangle \end{aligned}$$

and, as in the previous case, we deduce from [15, Corollary 1.3] that $\langle a_1,a_2\rangle = \langle a_2,a_3\rangle = \langle a_3,a_4\rangle = \langle a_4,a_5\rangle = \langle a_5,a_1\rangle = S_n$. □

Theorem 3.24. *$\Gamma(A_n)$ is perfect if and only if $n \leqslant 4$.*

Proof. If $n = 3$, then $\Gamma(A_3) \cong K_3$ is perfect.

Assume $n = 4$. Let m be an odd positive integer, with $m \geqslant 5$. In an m-hole or in an m-antihole, at least one vertex should be a 3-cycle, since in a pair of generators one should be outside the Klein subgroup. However a 3-cycle is contained in a unique maximal subgroup, so we conclude using Lemma 3.6 that there is neither an m-hole nor an m-antihole.

For $n \geqslant 5$, it remains to show that $\Gamma(A_n)$ is not perfect.

First assume $n \geqslant 5$ odd. In this case we claim that

$$\begin{aligned} a_1 &= (1,2,3,6\dots,n) \\ a_2 &= (2,4,5,6\dots,n) \\ a_3 &= (1,3,5,6\dots,n) \\ a_4 &= (2,3,4,6\dots,n) \\ a_5 &= (1,4,5,6\dots,n) \end{aligned}$$

is a 5-hole in $\Gamma(A_n)$. The cases $n = 5,7,9$ can be easily checked by hand, so we assume $n \geqslant 11$. Notice that $\langle a_1,a_3\rangle$, $\langle a_1,a_4\rangle$, $\langle a_2,a_4\rangle$, $\langle a_2,a_5\rangle$ and $\langle a_3,a_5\rangle$ are intransitive subgroups, so the pair of corresponding vertices are not joined by an edge. Since $a_1,\dots,a_5$ are $(n-2)$-cycles, the transitive subgroups $\langle a_1,a_2\rangle$, $\langle a_2,a_3\rangle$, $\langle a_3,a_4\rangle$, $\langle a_4,a_5\rangle$ and $\langle a_5,a_1\rangle$ are also primitive from Lemma 3.22. Moreover

$$\begin{aligned} a_1^2a_2^{-2} &= (1,3,5,2,4,n,n-1) && \in \langle a_1,a_2\rangle \\ a_2a_3^{-1} &= (1,n,2,4,3) && \in \langle a_2,a_3\rangle \\ a_3^2a_4^{-2} &= (1,5,4,2,n-1) && \in \langle a_3,a_4\rangle \\ a_4^2a_5^{-2} &= (1,n-1,2,n,3,4,5) && \in \langle a_4,a_5\rangle \\ a_5a_1^{-1} &= (1,4,5,3,2) && \in \langle a_5,a_1\rangle \end{aligned}$$

and we can use [15, Corollary 1.3] to conclude that $\langle a_1,a_2\rangle = \langle a_2,a_3\rangle = \langle a_3,a_4\rangle = \langle a_4,a_5\rangle = \langle a_5,a_1\rangle = A_n$.

Finally, assume $n \geqslant 6$ even. Notice $\Gamma(A_6)$ contains the following 5-hole:

$$((1,2,3,4,5),(1,3)(5,6),(1,2,4,5,6),(1,4,2,3,5),(1,2,6)).$$

We claim that if $n \geqslant 8$, then

$$\begin{aligned} a_1 &= (1,2,3,4,5,9,\dots,n) \\ a_2 &= (1,3,6,7,8,9,\dots,n) \\ a_3 &= (2,7,8,4,5,9,\dots,n) \\ a_4 &= (1,6,3,4,5,9,\dots,n) \\ a_5 &= (1,2,6,7,8,9,\dots,n) \end{aligned}$$

is a 5-hole in $\Gamma(A_n)$. The subgroups $\langle a_1, a_3\rangle$, $\langle a_1, a_4\rangle$, $\langle a_2, a_4\rangle$, $\langle a_2, a_5\rangle$ and $\langle a_3, a_5\rangle$ are intransitive, so the pair of corresponding vertices are not joined by an edge. We prove that the transitive subgroup $\langle a_1, a_2\rangle$ is primitive (a similar argument works for the subgroups $\langle a_2, a_3\rangle$, $\langle a_3, a_4\rangle$, $\langle a_4, a_5\rangle$ and $\langle a_5, a_1\rangle$). Suppose it is imprimitive. Let B be an imprimitive block containing 6, so that $B^{a_1} = B$. We must have $B \subseteq \{6,7,8\}$, otherwise we would have $\{1,2,3,4,5,6,9,\dots,n\} \subseteq B$. Moreover, since $B \cap B^{a_2} = \{7,8\}$, $B \neq \{6,7,8\}$, so either $B = \{6,7\}$ or $B = \{6,8\}$. In the first case the block containing 8 also contains an element different from $6,7,8$ and we get a contradiction as before. A similar argument applies in the second case, working with the block containing 7. Since $a_1,\dots,a_5$ are $(n-3)$-cycles, we can conclude, using [15, Corollary 1.3], that $\langle a_1, a_2\rangle = \langle a_2, a_3\rangle = \langle a_3, a_4\rangle = \langle a_4, a_5\rangle = \langle a_5, a_1\rangle = A_n$. □

3.6. Rank one groups of Lie type. In the previous subsection we have proved that $\Gamma(A_n)$ contains a 5-hole when $n \geqslant 5$ and we conjecture that this could be true for every finite non-abelian simple group. In this subsection we prove that Conjecture 1.5 is true when G is isomorphic to one of the groups

$$\mathrm{PSL}_2(q), \mathrm{PSU}_3(q), {}^2B_2(q), {}^2G_2(q),$$

i.e. when G is a rank one group of Lie type. We need the following elementary observation.

Lemma 3.25. *Let G be a permutation group on the set Ω. Let $\omega_1,\dots,\omega_5 \in \Omega$ such that $\mathrm{Stab}_G(\omega_i) < G$ for $i = 1,\dots,5$. Let*

$$\begin{aligned} a &\in \mathrm{Stab}_G(\omega_1) \cap \mathrm{Stab}_G(\omega_2), \\ b &\in \mathrm{Stab}_G(\omega_3) \cap \mathrm{Stab}_G(\omega_4), \\ c &\in \mathrm{Stab}_G(\omega_5) \cap \mathrm{Stab}_G(\omega_1), \\ d &\in \mathrm{Stab}_G(\omega_2) \cap \mathrm{Stab}_G(\omega_3), \\ e &\in \mathrm{Stab}_G(\omega_4) \cap \mathrm{Stab}_G(\omega_5). \end{aligned}$$

If $\langle a,b\rangle = \langle b,c\rangle = \langle c,d\rangle = \langle d,e\rangle = \langle e,a\rangle = G$, then (a,b,c,d,e) is a 5-hole in $\Gamma(G)$.

Proposition 3.26. *Let $G = \mathrm{PSL}_2(q)$, with $q > 3$. Then $\Gamma(G)$ contains a 5-hole.*

Proof. We may assume $q \notin \{4,5,9\}$, since $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong A_5$ and $\mathrm{PSL}_2(9) \cong A_6$. The group G has a faithful 2-transitive action on the $q+1$ points of the 1-dimensional projective space $\mathrm{PG}(1,q)$ over the field $\mathbb{F}_q$ with q elements. Let A, B, C, D be four distinct points of $\mathrm{PG}(1,q)$. The subgroups $H = \mathrm{Stab}_G(A) \cap \mathrm{Stab}_G(B)$ and $K = \mathrm{Stab}_G(C) \cap \mathrm{Stab}_G(D)$ are cyclic of order $u = (q-1)/(q-1,2)$. Notice that $\langle H, K\rangle$ connot be contained in the stabilizer of an element of $\mathrm{PG}(1,q)$, since the only element of G which fixes three distinct points is the identity. The list of the maximal subgroups of G is well-known (see for example [1, Tables 8.1, 8.2]). In particular if $q \notin \{7,11\}$, then no maximal subgroup of G, except from a point stabilizer, contains two distinct cyclic subgroups of order u. This implies $G = \langle H, K\rangle$ and we can use Lemma 3.25 to conclude.

Finally, if $q=7$, then $G\leqslant S_8$ and the following is a 5-hole in $\Gamma(G)$:

$$((2,3,4)(5,8,7),(1,4,5)(3,7,6),(2,7,8)(3,6,5),(1,2,4)(6,7,8),(1,2,5,7)(3,8,6,4)).$$

Similarly, if $q=11$, then $G\leqslant S_{12}$ and the following is a 5-hole in $\Gamma(G)$:

$$((3,9,5,11,7)(4,10,6,12,8),(1,6,3,4,12)(2,11,9,10,7),(1,3,8,5,4)(6,7,9,12,10),$$
$$(2,12,11,8,3)(4,7,9,10,6),(1,9,6,7,5)(2,4,12,3,10)).\quad\square$$

Proposition 3.27. *Let $G=\mathrm{PSU}_3(q)$, with $q>2$. Then $\Gamma(G)$ contains a 5-hole.*

Proof. Let $d=(q{+}1,3)$. The group G is a 2-transitive group of permutations of the set Ω of the $q^3{+}1$ points of the corresponding polar space. If A_1,A_2 are two distinct points of Ω, then $\mathrm{Stab}_G(A_1)\cap\mathrm{Stab}_G(A_2)$ is a cyclic group of order $(q^2{-}1)/d$. Moreover A_1 and A_2 are the only points fixed by $\mathrm{Stab}_G(A_1)\cap\mathrm{Stab}_G(A_2)$ and $\mathrm{Stab}_G(A_1)\cap\mathrm{Stab}_G(A_2)$ acts on the remaining q^3-1 point with $q\cdot d$ orbits of size $(q^2-1)/d$ and one orbit of size $q-1$ (this information can be deduced for example from the description of the action of G on Ω given in [8, Section 7.7, pages 248-249]).

The statement can be directly proved using GAP if $q\leqslant 5$, by searching elements in the intersections of stabilizers, in such a way to reproduce the situation of Lemma 3.25; so we may assume $q\geqslant 7$.

Let A_1,A_2,A_3,A_4 be four distinct points of Ω and consider $H=\mathrm{Stab}_G(A_1)\cap\mathrm{Stab}_G(A_2)$ and $K=\mathrm{Stab}_G(A_3)\cap\mathrm{Stab}_G(A_4)$. The list of the maximal subgroups of G is well-known (see for example [1, Tables 8.5, 8.6]). In particular, since $q\geqslant 7$, if M is a maximal subgroup of G containing an element of order $(q^2-1)/d$, then either M is a point-stabilizer or $M=X/Y$ with $X\cong\mathrm{GU}_2(q)$ and Y cyclic of order d. In the first case M cannot contain both H and K, since H fixes only A_1 and A_2 and K fixes only A_3 and A_4. In the second case $Z(M)$ is cyclic of order $(q+1)/d$ and fixes precisely $q+1$ elements of Ω and any element of order q^2-1 contained in M acts on the set of these $q+1$ elements with two fixed points and an orbit of cardinality $q-1$. In particular, if we choose A_3,A_4 such that they don't belong to the orbit of size $q-1$ of H, then $G=\langle H,K\rangle$. With this choice of A_3 and A_4, choose A_5 distinct from A_1,A_2,A_3,A_4 and not contained in the orbit of size $q-1$ of $\mathrm{Stab}_G(A_i)\cap\mathrm{Stab}_G(A_j)$, for any $1\leqslant i<j\leqslant 4$. We can use Lemma 3.25 to conclude. $\square$

Proposition 3.28. *Let $q=2^{2n+1}$ with $n\geqslant 1$. If $G={}^2B_2(q)$ is a Suzuki group, then $\Gamma(G)$ contains a 5-hole.*

Proof. The group G has a faithful 2-transitive action on an ovoid Ω in a 4-dimensional symplectic geometry over $\mathbb{F}_q$. Up to conjugacy, the maximal subgroups of G are as follows (for example, see [1, Table 8.16]):

(1) the stabilizer of $\omega\in\Omega$ (the Borel subgroup of order $q^2(q-1)$);
(2) the dihedral group of order $2(q-1)$;
(3) $C_{q+\sqrt{2q}+1}\rtimes C_4$;
(4) $C_{q-\sqrt{2q}+1}\rtimes C_4$;
(5) ${}^2B_2(q_0)$, where $q=q_0^r$, r is prime and $q_0>2$.

If ω_i and ω_j are distinct elements of Ω, then $\mathrm{Stab}_G(\omega_i)\cap\mathrm{Stab}_G(\omega_j)$ is cyclic of order $q-1$. Let x be a generator of this cyclic group. Next choose ω_l and ω_k in Ω such that $\omega_i,\omega_j,\omega_k,\omega_l$ are all distinct, and let y be a generator of $\mathrm{Stab}_G(\omega_k)\cap\mathrm{Stab}_G(\omega_l)$. Since the only element fixing three points is the identity, we have that $\langle x\rangle\neq\langle y\rangle$. Consider the subgroup $H:=\langle x,y\rangle$. If H is a proper subgroup, it is contained in a maximal subgroup. However H cannot be contained in subgroups of type (3), (4) and (5), since they do not contain elements of order $q-1$. Since $\langle x\rangle\neq\langle y\rangle$ we can also rule out the possibility that H is contained in a subgroup of type (2). Finally, if $H\leqslant\mathrm{Stab}_G(\omega)$, for some $\omega \in \Omega$, then either x or y must fix three different points, which is impossible. Therefore $H = G$ and we can use Lemma 3.25 to construct a 5-hole. □

PROPOSITION 3.29. *Let $q = 3^{2n+1}$ with $n \geqslant 1$. If $G \coloneqq {}^2G_2(q)$, then $\Gamma(G)$ contains a 5-hole.*

Proof. The group G has a faithful 2-transitive action on an ovoid Ω in a 7-dimensional orthogonal geometry over $\mathbb{F}_q$. The maximal subgroups of G are as follows, up to conjugacy (see for example [1, Table 8.43]):

(1) the stabilizer of $\omega \in \Omega$ (the Borel subgroup of order $q^3(q-1)$);
(2) the centralizer of an involution, which is isomorphic to $C_2 \times \mathrm{PSL}_2(q)$;
(3) the normalizer of a four-group, which is isomorphic to $(2^2 \times D_{(q+1)/4}) \rtimes 3$;
(4) $C_{q+\sqrt{3q}+1} \rtimes C_6$;
(5) $C_{q-\sqrt{3q}+1} \rtimes C_6$;
(6) ${}^2G_2(q_0)$, where $q = q_0^r$ and r prime.

The intersection of two different point-stabilizers is cyclic with order $q-1$. Moreover any involution t in G fixes precisely $q+1$ points in Ω, and the set of these $q+1$ elements is called the block of t. Any two blocks can intersect in at most 1 point and any two points are pointwise fixed by a unique involution.

Choose $\omega_1, \dots, \omega_5 \in \Omega$ all distinct in the following way: ω_1, ω_2 and ω_3 are chosen randomly and let $\Omega_{i,j}$ be the unique block which contains ω_i and ω_j. Since $|\Omega| = q^3+1$ and a block has cardinality $q+1$, it is possible to choose $\omega_4 \in \Omega \smallsetminus \Omega_{2,3}$ and $\omega_5 \in \Omega \smallsetminus (\Omega_{3,4} \cup \Omega_{1,2})$. Since the block containing two elements is unique, we have that four of these five elements never belong to the same block. Let $\omega_i, \omega_j, \omega_k, \omega_l$ be four of these five elements. Let x be a generator of $\mathrm{Stab}_G(\omega_i) \cap \mathrm{Stab}_G(\omega_j)$ and y a generator of $\mathrm{Stab}_G(\omega_l) \cap \mathrm{Stab}_G(\omega_k)$ and consider $H \coloneqq \langle x, y\rangle$. The subgroup H cannot be contained in maximal subgroups of type (3), (4), (5) and (6), since these maximal subgroups do not contain elements of order $q-1$. There are no elements in G of order $q-1$ which fix three distinct elements on Ω, so H is not contained in maximal subgroups of type (1). Therefore, if $H < G$ is proper, then $H \leqslant C_G(t)$ for a suitable involution t of G. This occurs when t is contained in the intersection of the four stabilizers of $\omega_i, \omega_j, \omega_k, \omega_l$, as can be deduced from [17, Lemma 3.2, 3], but in this case $\omega_i, \omega_j, \omega_k, \omega_l$ belong to the same block, which is incompatible with our choice of $\omega_1, \dots, \omega_5$. So $H = G$ and we may conclude by applying Lemma 3.25. □

3.7. SMALL GROUPS. Here we prove that A_5 is the smallest 2-generated finite group with a non-perfect generating graph.

THEOREM 3.30. *Let G be a 2-generated finite group, with $|G| \leqslant 60$. Then $\Gamma(G)$ is perfect if and only if $G \neq A_5$.*

Proof. By Theorem 3.24, we only have to prove that if $|G| \leqslant 60$ and $G \neq A_5$, then $\Gamma(G)$ is perfect. By Theorem 1.1, $C_{30} \times C_6$ is the smallest 2-generated finite nilpotent group whose generating graph is not perfect. So we may assume that G is not nilpotent, and by the results in subsection 3.4 we may exclude $|G| \in \{pq, pqr, p^2q\}$ with p, q, r different primes. Hence $|G| \in \{24, 36, 40, 48, 54, 56, 60\}$. This requires a case by case analysis.

As an example we consider $G \cong C_6 \times D_5$, which is the case that requires more attention. The other cases can be handled with similar, but in general shorter, arguments.

Let $m \geqslant 5$ be odd. Set $\langle c\rangle = C_6$ and $\langle \rho, \iota\rangle = D_5$, with ρ a rotation of order 5 and ι a reflection. Since $C_2 \times C_2$ is an epimorphic image of G, it follows from Lemma 3.14 that $\Gamma(G)$ does not contain m-antiholes with $m \geqslant 7$, so we only have to check the non-existence of m-holes. To prove this, we need the list of maximal subgroups of G:

- $M_1 = \langle c^2, \iota, \rho \rangle$;
- $M_2 = \langle c^3, \iota, \rho \rangle$;
- $M_3 = \langle c, \rho \rangle$;
- $M_4 = \langle \rho, c\iota \rangle$;
- $M_{5+\alpha} = \langle c, \rho^\alpha \iota \rangle$ with $\alpha \in \{0,1,2,3,4\}$.

Suppose $(a_1, \dots, a_m)$ is an m-hole. Consider the two projections $\pi_1 : G \to \langle c \rangle$, $\pi_2 : G \to \langle \rho, \iota \rangle$. Notice that $\langle \pi_1(a_i) \rangle = \langle c \rangle$ for some $i \in \{1, \dots, m\}$. Indeed, if this fails to hold then $|\pi_1(a_j)| \in \{2,3\}$ for every $1 \leqslant j \leqslant m$, and, since m is odd, there would exist two consecutive vertices a_k and a_{k+1} with $|\pi_1(a_k)| = |\pi_1(a_{k+1})| = t \in \{2,3\}$, and consequently $G = \langle a_k, a_{k+1} \rangle \leqslant \langle c^{6/t} \rangle \times D_5$. So without loss of generality we may assume that $\pi_1(a_1) = c$. Next observe that $\pi_2(a_1) \neq 1$ (otherwise a_1 would be an isolated vertex of $\Gamma(G)$). Moreover $\pi_2(a_1) \notin \langle \rho \rangle$, otherwise M_3 would be the unique maximal subgroup of G containing a_1, which contradicts Lemma 3.6. So we may assume $a_1 = c\iota$. Let $3 \leqslant j \leqslant m-1$. Since $\langle a_1, a_j \rangle \neq G$ and M_4, M_5 are the only maximal subgroups of G containing a_1, it follows that $a_j \in M_4 \cup M_5$. Two consecutive vertices of $(a_1, \dots, a_m)$ generate G, so they cannot belong to the same maximal subgroup. Hence we can label the vertices of the m-hole so that $a_3 \in M_5$ (and consequently $a_4 \in M_4$). So $a_3 = c^\alpha \iota$ with $\alpha \in \{0,1,2,3,4,5\}$. Moreover $\alpha \neq 3$ (otherwise $\langle a_3, a_4 \rangle \leqslant M_4$) and $\alpha \notin \{1,5\}$ (since, by Lemma 3.7, $\langle a_3 \rangle \neq \langle a_1 \rangle$), and so we have $\alpha = \pm 2$. Notice that M_1 and M_5 are the only maximal subgroups of G containing a_3, so $a_m \in M_1 \cup M_5$. On the other hand, from $\langle a_1, a_m \rangle = G$ and $a_1 \in M_5$, it follows that $a_m \notin M_5$ and therefore $a_m \in M_1$. Now let $a_2 = c^x \rho^y \iota^z$, with $0 \leqslant x \leqslant 5$, $0 \leqslant y \leqslant 4$ and $0 \leqslant z \leqslant 1$. We have $x \notin \{0,2,4\}$, otherwise $\langle a_2, a_3 \rangle \leqslant M_1$. If $x \in \{1,5\}$, then $z = 1$ (otherwise a_2 would have order 30 and consequently would be contained in a unique maximal subgroup), but this would imply $\langle a_1, a_2 \rangle \leqslant M_4$, a contradiction. So we must have $x = 3$. If $z = 1$, then again $\langle a_1, a_2 \rangle \leqslant M_4$, a contradiction, so $z = 0$ and therefore $a_2 = c^3 \rho^y$ with $y \neq 0$. In particular M_2 and M_3 are the only maximal subgroups of G containing a_2, and, since a_m and a_2 are not adjacent, $a_m \in M_2 \cup M_3$. We have already proved that $a_m \in M_1$ so $a_m \in (M_1 \cap M_2) \cup (M_1 \cap M_3)$. Since $M_1 \cap M_3 \leqslant M_4$ and $a_1 \in M_4$, if $a_m \in M_1 \cap M_3$, then $G = \langle a_1, a_m \rangle \leqslant M_4$, a contradiction. So $a_m \in M_1 \cap M_2 = \langle \rho, \iota \rangle$, and consequently we may assume $a_{m-1} = c\rho^s \iota^t$. We have $t = 1$, otherwise a_{m-1} is contained in a unique maximal subgroup. Then $\langle a_2, a_{m-1} \rangle = \langle c\rho^s\iota, c^3\rho^y \rangle = \langle c\rho^s\iota, c^3, \rho \rangle = \langle c\iota, c^3, \rho \rangle = \langle c, \iota, \rho \rangle = G$, a contradiction. ☐

4. Other forbidden graphs

The main aim of this final section is to give the proof of Theorem 1.12, stated in the introduction. As a preliminary auxiliary result, we classify the 2-generated finite groups whose generating graphs do not contain the path P_3 of length 3 as an induced subgraph.

Proposition 4.1. *Let G be a non-trivial 2-generated finite group. Then $\Gamma(G)$ does not contain an induced subgraph isomorphic to P_3 if and only if either $G \cong C_2 \times C_2$ or $G \cong C_p$ for some prime p.*

Proof. Suppose that G satisfies the following property:

(*) there exist $a, b \in G$ such that $G = \langle a, b \rangle, G \neq \langle a \rangle, G \neq \langle b \rangle$ and $a \neq a^{-1}$.

Then (a, b, a^{-1}) is a three-vertex path in $\Gamma(G)$.

First assume $G = \langle a, b \rangle$ is not cyclic. If G is not a dihedral group, then $(|a|, |b|) \neq (2,2)$. If $G \cong D_n$ is a dihedral group of order $2n$, then we may choose a, b such that $(|a|, |b|) = (n, 2)$. So if G is not cyclic, then either G satisfies (*) or $G \cong C_2 \times C_2$. In this latter case, assume that (x_1, x_2, x_3) is a three-vertex path in $\Gamma(G)$: then $x_i \neq 1$ for any $i \in \{1,2,3\}$, but then $\{x_1, x_2, x_3\}$ induces a complete graph K_3.

Finally, suppose $G = \langle x \rangle \cong C_n$. If $n = rs$ and $(r, s) = 1$, then $G = \langle x^r, x^s \rangle$ and $(|x^r|, |x^s|) = (s, r) \neq (2, 2)$ so G satisfies (*). If $n = p^t$ with p a prime and $t \geqslant 2$, then $(1, x, x^p)$ is a three-vertex path in $\Gamma(G)$. If $n = p$ is a prime and x_1, x_2, x_3 are three distinct elements of G, then $\{x_1, x_2, x_3\}$ induces a complete graph K_3. ☐

Lemma 4.2. *Let p be a prime and assume that either G is a cyclic p-group or $|G| = 2p$. Suppose that the subgraph of $\Gamma(G)$ induced by four distinct non-isolated vertices contains at least one edge. Then at least one of the four vertices is adjacent to all the others.*

Proof. Assume that $X = \{g_1, g_2, g_3, g_4\}$ induces a non empty-edges subgraph of $\Gamma(G)$. If $G \cong C_{p^n}$ is cyclic of prime power order, then there exists i with $|g_i| = p^n$ (otherwise all the elements of X belong to the unique maximal subgroup of G). But then g_i is adjacent to g_j whenever $j \neq i$.

Now assume $|G| = 2p$ with p a prime. If X contains an element of order $2p$ then this element generates G so it is adjacent to all the others. Moreover we cannot have $|g_j| = p$ for every $j \in \{1, \dots, 4\}$, since all the elements of order p belong to the same maximal subgroup. Thus there exists $g_i \in X$ with $|g_i| = 2$, but again this implies that g_i is adjacent to g_j whenever $j \neq i$. ☐

Proof of Theorem 1.12. Clearly (2) implies (3).

Assume that (3) holds. If there exist $a, b \in G$ so that $G = \langle a, b \rangle$, $\langle a \rangle \neq G$, $\langle b \rangle \neq G$, $|a| \neq 2$, $|b| \neq 2$, then the subgraph of $\Gamma(G)$ induced by $\{a, b, a^{-1}, b^{-1}\}$ is a four-vertex cycle. If G is cyclic of order n, then we can find a, b with these properties except when n is a prime-power or $n = 2p$ with p a prime. So we may assume that G is non-cyclic and $G = \langle a, b \rangle$ with $|a| = 2$. Moreover either b or ab has order 2, otherwise (b, ab) is a generating pair with the previous properties. Hence $G = \langle a, b \rangle = \langle a, ab \rangle$ can be generated by two involutions, so G is isomorphic to a dihedral group D_n of order $2n$ and we may assume $|b| = n$. If n is not a prime and p is a prime divisor of n, then the subgraph of $\Gamma(G)$ induced by $\{a, b, ab^p, b^{-1}\}$ is a four-vertex cycle.

It follows from Lemma 4.2 that (4) implies (2).

It was shown in [9] that a graph is split if and only if it does not have an induced subgraph isomorphic to one of the three forbidden graphs, C_4, C_5 or $2K_2$ (here $2K_2$ denotes the graph with four vertices, two disjoint edges, and no further edges connecting the vertices). In particular (1) implies (3) and we may immediately deduce from Lemma 4.2 that (4) implies (1). ☐

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