ALGEBRAIC COMBINATORICS

Sophia R. Gonet

Iwasawa Theory of Jacobians of Graphs

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ABSTRACT The Jacobian group (also known as the critical group or sandpile group) is an important invariant of a finite, connected graph X; it is a finite abelian group whose cardinality is equal to the number of spanning trees of X (Kirchhoff's Matrix Tree Theorem). A specific type of covering graph, called a *derived graph*, that is constructed from a *voltage graph* with *voltage group* G is the object of interest in this paper.

Towers of derived graphs are studied by using aspects of classical Iwasawa Theory (from number theory). Formulas for the orders of the Sylow p-subgroups of Jacobians in an infinite voltage p-tower, for any prime p, are obtained in terms of classical μ and λ invariants by using the decomposition of a finitely generated module over the Iwasawa Algebra.

1. INTRODUCTION

The Jacobian (or critical group, or sandpile group) is an algebraic invariant of a graph X (in this paper the term graph will mean a simple graph with no loops or multiple edges, unless otherwise explicitly noted) which, for connected X, is a finite abelian group whose size is equal to the number of spanning trees of X (this is well-known as the Matrix Tree Theorem). The study of Jacobians of graphs has a long history, and many applications, as described in [1, 2, 4, 5, 11, 20, 25]. Overall, there are relatively few graphs or families of graphs for which the Jacobian is exactly known: see [3, 6, 8, 12, 13, 19, 22]. In this paper we establish the "asymptotic structure" and orders of the Sylow p-subgroups of the Jacobians of certain covering graphs of a fixed base graph X, namely those that belong to a cyclic voltage p-tower cover of X.

More specifically, we adapt to voltage towers of graphs the classical work of Iwasawa for $\mathbb{Z}_p$-extensions—infinite extensions K_∞ of a number field K with Galois group isomorphic to the additive p-adic integers, $\mathbb{Z}_p$, for some prime p. By using the general theory of $\mathbb{Z}_p[[\Gamma]]$-modules, where $\Gamma = \mathrm{Gal}(K_\infty/K)$, Iwasawa was able to unravel the structure of the inverse limit of the p-Sylow subgroups of the class groups of the finite extension fields in his towers. This enabled him to prove the following theorem, which can be found in [18, 29]: Let K_∞/K be a $\mathbb{Z}_p$-extension. Let p^{e_m} be the exact power of p dividing the order of the class group of K_m, where K_m is the fixed field of the subgroup Γ^{p^m}; then there exist nonnegative integers λ, μ and an integer ν such that $e_m = \mu p^m + \lambda m + \nu$ for all $m \geqslant m_0$ for some $m_0 \geqslant 0$.

The Main Theorem of this paper is the analog in the graph theory setting:

THEOREM 1.1. *Let*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$$

be a cyclic voltage p-tower (see Definition 3.2*), where all X_m are connected. Let $\mathcal{J}_p(X_m)$ be the Sylow p-subgroup of the Jacobian of X_m. Then there are nonnegative integers μ and λ and an integer ν such that*

$$|\mathcal{J}_p(X_m)| = p^{e_m} \qquad where \qquad e_m = \mu p^m + \lambda m + \nu$$

for all $m \geqslant m_0$ for some $m_0 \geqslant 0$.

This theorem gives not only "asymptotic" order formulas for the p-Jacobians of the covering graphs X_m, but also their "asymptotic" invariant factor decompositions, which, in particular give conditions under which the p-ranks grow without bound (which is also analogous to the classical number theoretic results of Iwasawa).

The ideas in [17, 23, 28] inspired the research that culminates in Section 3. However, the work is independent, contemporaneous, and by quite different methods.

The terminology and theory of Jacobians, voltage graphs and their derived covering graphs — including extending these results to infinite towers—is first summarized in Section 2. In Section 3, the theory of finitely generated modules over the Iwasawa algebra, $\mathbb{Z}_p[[\Gamma]]$ is summarized. Consequences of the latter results, that form the essential underpinning of the Main Theorem, are also established. The Main Theorem is then proved using group-theoretic methods, that, in hindsight, illustrate how the decomposition theorem for Iwasawa modules plays the analogous role to the Smith Normal Form decomposition that describes ordinary Jacobians. The p-rank result mentioned above appears as a corollary to the Main Theorem.

This paper comprises the last part of the author's dissertation [16], which contains significantly more details, examples, and an array of additional theoretical and computational material on voltage graphs and their associated derived graphs. We refer to it at points where its material expands on or expedites the development of this paper.

Added after refereeing: Just as this paper was submitted, Daniel Vallières and Kevin McGown circulated a manuscript [24] giving the generalization of Theorem 1.1 to multigraphs. Their work—which is completely independent—uses "analytic" methods (L-series etc.), and so provides a valuable complementary perspective on our result. It seems that the methods herein should also generalize to multigraphs, mutatis mutandis, since the main part of the proof, Section 3, essentially only involves cokernels of Laplacians, and these are well defined for multigraphs.

2. PRELIMINARIES

In Section 2.1, we define the Picard and Jacobian groups. Then in Section 2.2 we define the Laplacian and reduced Laplacian. In Section 2.3 we describe a specific type of covering graph, called a *derived graph*, that arises from what is called a *voltage graph*—where elements from a group (which may be finite or infinite) are assigned to the edges of a fixed base graph X. In Section 2.4, we give the definition of an *intermediate covering graph*. We then state the important result: given a voltage graph with derived graph Y such that Y is connected, Y/X is a normal (i.e. Galois) extension, and conversely, if Y/X is a normal extension with Galois group G, then there exists a voltage assignment such that (X, G, α) is a voltage graph with derived graph Y.

2.1. THE DIVISOR, PICARD, AND JACOBIAN GROUPS. For more details on this section, refer to [10].

DEFINITION 2.1. *A divisor on a graph X (possibly infinite) is an element of the free abelian group on the vertices $V = V(X)$:*

$$\mathrm{Div}(X) = \left\{ \sum_{v \in V(X)} a_v v \mid a_v \in \mathbb{Z} \right\}$$

where each $\sum_{v \in V(X)} a_v v$ is a formal linear combination of the vertices of X with integer coefficients, where only finitely many a_v are nonzero (in the case when X is an infinite graph). The degree *of a divisor is*

$$\deg \left(\sum_{v \in V(X)} a_v v \right) = \sum_{v \in V(X)} a_v.$$

When $V(X) = \{v_1, \ldots, v_n\}$, we may write the elements of $\mathrm{Div}(X)$ as $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$, where each $a_i \in \mathbb{Z}$, and its degree is $a_1 + a_2 + \cdots + a_n$. The *degree map* $\deg : \mathrm{Div}(X) \to \mathbb{Z}$, is a surjective group homomorphism with kernel equal to the subgroup of $\mathrm{Div}(X)$ of *divisors of degree 0*, denoted as $\mathrm{Div}^0(X)$:

$$\mathrm{Div}^0(X) = \{D \in \mathrm{Div}(X) \mid \deg D = 0\}.$$

Next let X be a graph with vertices $\{v_1, \ldots, v_n\}$. For each fixed v_i define the *principal divisor, p_i, based at v_i* by

$$p_i = \deg(v_i) v_i - \sum_{j=1}^{n} \delta_{i,j} v_j$$

where $\delta_{i,j} = 1$ if v_j is adjacent but not equal to v_i and 0 otherwise (and here $\deg(v_i)$ is the valence of vertex v_i in X). Define *principal divisors* to be elements of the $\mathbb{Z}$-submodule of $\mathrm{Div}(X)$ spanned by the principal divisors based at the vertices:

$$\mathrm{Pr}(X) = \mathrm{Span}_{\mathbb{Z}}\{p_i \mid 1 \leqslant i \leqslant n\}.$$

Evidently $\mathrm{Pr}(X)$ is a submodule of $\mathrm{Div}^0(X)$. From this we get the following groups.

DEFINITION 2.2. *The* Picard group *of X is the quotient group*

$$\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{Pr}(X),$$

and the Jacobian group *of X is the subgroup of* $\mathrm{Pic}(X)$

$$\mathcal{J}(X) = \mathrm{Div}^0(X)/\mathrm{Pr}(X).$$

THEOREM 2.3. *If X is connected, then $\mathcal{J}(X)$ is a finite abelian group.*

2.2. THE LAPLACIAN AND REDUCED LAPLACIAN.

DEFINITION 2.4. *Let X be a graph with vertices $\{v_1, \ldots, v_n\}$. The graph* Laplacian $L = L_X$ *is the $n \times n$ matrix given by*

$$L_{i,j} = \begin{cases} \deg(v_i) & \textit{if } i = j \\ -1 & \textit{if } v_i \textit{ is adjacent to } v_j \\ 0 & \textit{if } i \neq j \textit{ and } v_i \textit{ is not adjacent to } v_j. \end{cases}$$

The Laplacian is also the matrix representation of the following group homomorphism $\mathcal{L}$ defined as follows.

$$\mathcal{L} : \mathrm{Div}(X) \to \mathrm{Div}(X) \qquad \text{where} \qquad \mathcal{L}(v_i) = p_i.$$

When extended by $\mathbb{Z}$-linearity to all of $\mathrm{Div}(X)$, this is a $\mathbb{Z}$-linear homomorphism from $\mathrm{Div}(X)$ to itself, whose image is $\mathrm{Pr}(X)$, the group of principal divisors. From this, we get the following important fact:

$$\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{im}(\mathcal{L}) = \mathrm{coker}(\mathcal{L}).$$

A *reduced Laplacian* $\tilde{L}$ is the $(n-1) \times (n-1)$ integer matrix obtained by removing the row and column corresponding to any vertex v from the Laplacian matrix L. So the Jacobian group can be computed as the cokernel of the reduced Laplacian matrix

$$\mathcal{J}(X) \cong \mathbb{Z}^{n-1}/\mathrm{im}(\widetilde{L}) = \mathrm{coker}(\widetilde{L}),$$

where $\mathbb{Z}^{n-1}$ denotes the free $\mathbb{Z}$-module on the set $V(X) - \{v\}$ of rank $n-1$.

2.3. Voltage Graphs. We first give the definition of a general covering graph.

DEFINITION 2.5. *An undirected graph Y is a* covering *of an undirected graph X if, after arbitrarily directing the edges of X, there is an assignment of directions to the edges of Y and an onto graph homomorphism $\pi : Y \to X$ sending neighborhoods of Y one-to-one onto neighborhoods of X which preserve directions. We call such π a* covering map.

DEFINITION 2.6. *A d-sheeted* covering *means every fiber contains exactly d elements, i.e.*

$$|\pi^{-1}(x)| = d \ \forall x \in V(X).$$

DEFINITION 2.7. *Let X be a graph whose edges have been oriented, and let G be a group (finite or infinite). For a fixed orientation of the edges of X, let $E(X)^+$ denote the set of forward-directed edges of X; and let $E(X)^-$ denote the same edges but each with the reverse orientation (so each undirected edge of X becomes two edges in the disjoint union of $E(X)^+$ and $E(X)^-$). An (ordinary)* voltage assignment *is a map*

$$\alpha : E(X)^+ \cup E(X)^- \to G$$

such that if $e_{i,j} \in E(X)^+$ and $\alpha(e_{i,j}) = \alpha_{i,j} \in G$, then $e_{j,i} \in E(X)^-$ and $\alpha(e_{j,i}) = \alpha_{i,j}^{-1}$ (the inverse group element), where $e_{i,j}$ denotes the directed edge from v_i to v_j.

The triple (X, G, α) is called an (ordinary) voltage graph. *The values of α are called the* voltages *and G is called the* voltage group.

Note that a voltage assignment α is uniquely determined by its values on $E(X)^+$, so we will henceforth only specify α on the forward-directed edges of X.

The vertices of X are labeled as $v_1, \ldots, v_n$. This imposes a natural lexicographic orientation on X, namely whenever there is an edge between v_i and v_j, orient the edge $v_i \to v_j$ if $i < j$ (called the *standard orientation*). Note that results on derived graphs do not depend on the choice of orientation by [16], so without further mention, we adopt the standard orientation.

Any such voltage assignment can be codified by its $n \times n$ *voltage adjacency matrix*

$$A_\alpha = \big(A_{i,j} \big)$$

with entries $A_{i,j} \in \mathbb{Z}[G]$ such that $A_{i,j} = 0$ if $i = j$ or there is no edge between v_i and v_j, and $A_{i,j} = \alpha_{i,j}$ otherwise. (Note that the voltage adjacency matrix is also defined in [7, Definition 2.16].)

The purpose of assigning voltages to the graph X, called *the base graph*, is to obtain an object called *the derived graph*, called Y here. To get the vertices of Y, make $d = |G|$ copies of each vertex $x \in V(X)$ labeling them as $x_{\tau_0}, x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_{d-1}}$ where $G = \{\tau_0, \tau_1, \tau_2, \ldots, \tau_{d-1}\}$ has order d (and the same formal construction works even if $|G|$ is uncountable). So there are $|G| \cdot |V(X)|$ vertices in Y. Now create the edges of Y by the following rule: whenever there is an edge from v_i to v_j in the base

graph X with assigned voltage $\alpha_{i,j}$, create edges that go from $v_{i,g}$ to $v_{j,g\alpha_{i,j}}$ in Y, for every $g \in G$, where $g\alpha_{i,j}$ is the group-product of these two group elements in G. If $|G| = d$, then $\pi : Y \to X$ is a d-sheeted covering map (where again, d may be any infinite cardinal too). Note that the degree (valence) of each vertex v_{τ} of Y is the same as the degree of $v = \pi(v_{\tau})$ in X. Also, since our base graph X has no loops (i.e. $i \neq j$ here), no two vertices in the same fiber of π are adjacent in Y.

Many examples as well as computational ways of constructing the ordinary adjacency matrix of Y from the voltage adjacency matrix of X by tensoring with matrices for the regular representation of G appear in [16].

2.4. Galois Theory of Covering Graphs and Voltage Graphs. Refer to [27] for Galois theory of (finite) Galois covers. For proof of the theorems presented below, see [16].

Definition 2.8. *Suppose Y is a covering of X with projection map π. A graph $\widetilde{X}$ is an* intermediate covering *to Y/X if $Y/\widetilde{X}$ is a covering, $\widetilde{X}/X$ is a covering and the projection maps $\pi_1 : \widetilde{X} \to X$ and $\pi_2 : Y \to \widetilde{X}$ have the property that $\pi = \pi_1 \circ \pi_2$. If Y/X is a d-sheeted covering with projection map $\pi : Y \to X$, then it is* normal *or* Galois *if there are exactly d graph automorphisms $\sigma : Y \to Y$ such that $\pi \circ \sigma = \pi$. The* Galois group *is $G = \mathrm{Gal}(Y/X) = \{\sigma : Y \to Y \mid \pi \circ \sigma = \pi\}$.*

Theorem 2.9. *Suppose Y/X is a normal covering with Galois group G and $\widetilde{X}$ an intermediate covering corresponding to the subgroup H of G. Then $\widetilde{X}$ itself is a normal covering of X if and only if H is a normal subgroup of G, in which case $\mathrm{Gal}(\widetilde{X}/X) \cong G/H$.*

Now we put this in terms of voltage graphs.

Theorem 2.10. *Let (X, G, α) be a voltage graph with Y the derived graph. If Y is connected, then Y/X is a normal cover with $\mathrm{Gal}(Y/X) \cong G$. Conversely, given a normal (Galois) cover Y/X, with $G = \mathrm{Gal}(Y/X)$, then Y/X is a voltage cover with the voltage group equal to $\mathrm{Gal}(Y/X)$.*

For any Galois cover $\pi : Y \to X$ of a connected base graph X, the graph Y is necessarily connected except in the case where $G = \mathrm{Gal}(Y/X)$ is the cyclic group of order 2 and Y is two disjoint isomorphic copies of X interchanged by G.

Theorem 2.11. *Let (X, G, α) be a voltage graph with derived graph Y such that Y is connected. If $\widetilde{X}$ is an intermediate cover of Y/X corresponding to the normal subgroup H of G, then $\widetilde{X}/X$ is a voltage graph, whose voltage adjacency matrix is the voltage adjacency matrix of Y/X, but with nonzero entries reduced modulo H (thus has entries in $\mathbb{Z}[G/H]$).*

3. Towers of Voltage Graphs and Iwasawa Theory

We begin Section 3.1 by defining a *cyclic p-tower* of graphs. We then extend this definition to a cyclic *voltage p-tower* of graphs by using Theorems 2.10 and 2.11 from Section 2.4. From this, we get a "universal cover" of the tower by an infinite derived graph that we call X_{p^∞}; it is the derived graph obtained from the voltage graph $(X, \mathbb{Z}_p, \alpha)$, where the voltage group is the additive p-adic integers and the voltage assignment α is determined by the cyclic voltage p-tower. We call X_{p^∞} the *completion of the tower*. In Section 3.2, we present important definitions and results pertaining to Λ-modules. Then in Subsection 3.2.1, we specify the Λ-modules be finitely generated. In Subsection 3.2.2 we construct a finitely generated torsion Λ-module, which we call Pic_Λ. Finally in Section 3.3 we prove the main theorem (Theorem 1.1) of this paper.

3.1. p-Tower Covering Graphs. We begin by defining a cyclic p-tower of graphs. We assume p is a fixed prime.

Definition 3.1. *A cyclic p-tower of graphs above a base graph X is a sequence of covering graphs*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$$

such that for $m \geqslant 0$, the cover X_m/X is normal with $\mathrm{Gal}(X_m/X) \cong \mathbb{Z}/p^m\mathbb{Z}$.

Note that for $m \geqslant 0$, this implies that the cover X_{m+1}/X_m is normal with $\mathrm{Gal}(X_{m+1}/X_m) \cong \mathbb{Z}/p\mathbb{Z}$ by the Fundamental Theorem of Galois Theory, along with the Third Isomorphism Theorem [14]. Now we specialize Definition 3.1 to voltage graphs.

Definition 3.2. *A cyclic voltage p-tower of graphs above a base graph X is a sequence of derived graphs*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$$

such that for $m \geqslant 0$, X_m/X is a derived graph with $\mathrm{Gal}(X_m/X) \cong \mathbb{Z}/p^m\mathbb{Z}$.

Theorems 2.10 and 2.11 in Section 2.4 extend to towers, so we may choose notation that describes the vertices, edges, voltage assignments and Galois actions on the graphs X_m in compatible ways that are determined by the covering maps. In short, each X_m is a derived cover of X defined by a voltage adjacency matrix of fixed degree n whose i,j entries are $\alpha_{i,j}^m$ in the group ring $\mathbb{Z}[\mathbb{Z}/p^m\mathbb{Z}]$; and for each fixed i,j the sequence of such entries has a limit as $m \to \infty$ in the p-adic integral group ring. The $n \times n$ matrix whose entries are these limits forms a voltage adjacency matrix for the "completion graph" that we now describe. This construction is straightforward, and the precise details are given explicitly in [16, Section 5.1]. This leads to a "universal cover" of the tower, by an infinite derived graph that we call X_{p^∞}.

Definition 3.3. *Given a cyclic voltage p-tower as in Definition 3.2, with each X_m the derived graph for the voltage assignment $\alpha_m : E(X)^+ \to \mathbb{Z}/p^m\mathbb{Z}$, let X_{p^∞} be the derived graph obtained from the voltage graph $(X, \mathbb{Z}_p, \alpha)$, where the voltage group is the (additive) p-adic integers and voltage assignment α is determined by the tower. We call X_{p^∞} the* completion of the tower.

Theorem 3.4. *Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$ be a cyclic voltage p-tower, let X_{p^∞} be the completion of the tower, and for each $m \geqslant 0$ let $\overline{X}_m$ be the associated intermediate graphs. Then for all $m \geqslant 0$ there are graph isomorphisms $\overline{X}_m \to X_m$, depicted as the horizontal maps in Figure 1, such that all the maps in that figure commute, and commute with the action of $\mathbb{Z}_p$ as automorphisms of each graph.*
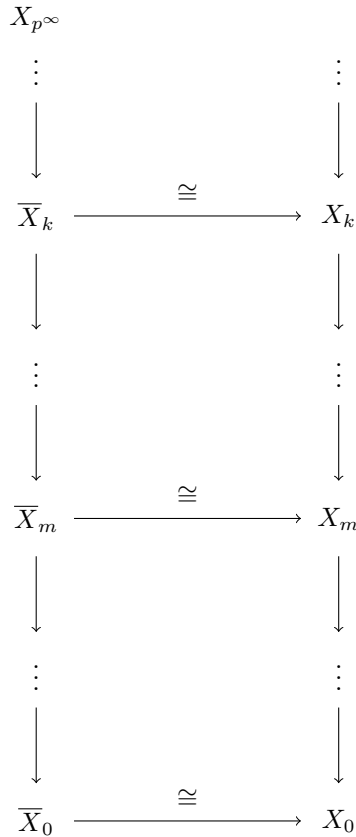
$$X_{p^\infty}$$

FIGURE 1. Cyclic voltage p-tower with completion X_{p^∞}

We will henceforth write the voltage group $\mathbb{Z}_p$ as the *multiplicative* profinite group Γ, where, as a profinite group, it is cyclic: it is the closure of an infinite (multiplicative) cyclic group $\langle \gamma \rangle$ under the p-adic metric topology, for some γ.

3.2. IWASAWA MODULES. Fundamental to Iwasawa's development of p-class groups in $\mathbb{Z}_p$-towers of number fields was his study of certain finitely generated modules over the $\mathbb{Z}_p$-algebra $\Lambda = \mathbb{Z}_p[[\Gamma]]$. Here Λ is the compactification of $\mathbb{Z}_p[\Gamma]$, under the profinite topology defined by the open subgroups Γ^{p^m}. In this section we list and establish some properties of Λ, as well as finitely generated modules over it, that are the underpinnings of our main theorem. The following two useful theorems about Λ can be found in [29].

THEOREM 3.5. *For the indeterminate T, $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ with the isomorphism being induced by $\gamma \mapsto T + 1$.*

THEOREM 3.6. *$\Lambda = \mathbb{Z}_p[[\Gamma]]$ is a Noetherian local ring.*

DEFINITION 3.7. *Two Λ-modules M and M' are said to be* pseudo-isomorphic*, written*

$$M \sim M'$$

if there is a homomorphism $M \to M'$ with finite kernel and co-kernel.

DEFINITION 3.8. *A nonconstant polynomial $P(T) \in \Lambda$*

$$P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$$

is called distinguished *if $p \mid a_i$ for all $0 \leqslant i \leqslant n-1$.*

The following proposition can be obtained immediately from [29, Proposition 7.2 and Lemma 7.5].

PROPOSITION 3.9. *Let $F(T)$ be a distinguished polynomial in $\mathbb{Z}_p[T]$. Then*

$$\mathbb{Z}_p[T]/F(T)\mathbb{Z}_p[T] \cong \mathbb{Z}_p[[T]]/F(T)\mathbb{Z}_p[[T]],$$

where the isomorphism is as $\mathbb{Z}_p[T]$-modules. The isomorphism is the natural one, namely for $r \in \mathbb{Z}_p[T]$, the coset $r + F(T)\mathbb{Z}_p[T]$ maps to $r + F(T)\mathbb{Z}_p[[T]]$.

Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and fix a topological generator γ for Γ; by Theorem 3.5 the map $\gamma \mapsto T + 1$ extends to an isomorphism from Λ to $\mathbb{Z}_p[[T]]$. For $m \geqslant 0$ let $\omega_m = \gamma^{p^m} - 1$.

LEMMA 3.10. *For $m \geqslant 0$ ω_m maps to a distinguished polynomial in $\mathbb{Z}_p[T]$.*

Proof. By definition, ω_m maps to $(T+1)^{p^m} - 1$. Thus

$$(T+1)^{p^m} - 1 \equiv (T^{p^m} + 1^{p^m}) - 1 \equiv T^{p^m} \pmod{p\mathbb{Z}_p[T]},$$

which establishes claim. $\qquad\square$

Let $R = \mathbb{Z}_p[\Gamma]$. Fix $m \geqslant 0$.

DEFINITION 3.11. *Let D be any Λ-module and let B be any subset of D. For every $m \geqslant 0$, define*

$$\Omega_m^D(B) = B \cap \omega_m D.$$

Define $R_m = R/\Omega_m^\Lambda(R) = R/R \cap \omega_m \Lambda$.

In the special case when B is an R-submodule of D (where D is considered as an R-module), we have that $\Omega_m^D(B)$ is an R-submodule of B containing $\omega_m B$.

The sets $\Omega_m^D(B)$ define relatively open subsets of B in the "ω-adic topology" on D. They obey the appropriate transitive property: If B and C are subset of D with $C \subseteq B$, then

$$\Omega_m^D(B) \cap C = \Omega_m^D(C).$$

It is not true in general that $\omega_m B \cap C = \omega_m C$ however.

PROPOSITION 3.12. *Let D be a Λ-module, let A be any Λ-submodule of D and let B be any R-submodule of A, where A is considered as an R-module. Then the map*

$$\phi : B/\Omega_m^D(B) \longrightarrow A/\Omega_m^D(A) \qquad by \qquad \phi(x + \Omega_m^D(B)) = x + \Omega_m^D(A)$$

is a well-defined and injective R-module homomorphism. If B contains a set of Λ-module generators for A, then ϕ is an isomorphism and $A = B + \Omega_m^D(A)$; and if additionally $\omega_m D \subseteq A$ then $A = B + \omega_m D$.

Proof. We first simplify notation by denoting $\Omega_m^D(C)$ by just $\Omega_m(C)$ for every subset C of D throughout the proof. The map $B \to A/\Omega_m(A)$ by $x \mapsto x + \Omega_m(A)$ is a well-defined R-module homomorphisms, and since $\Omega_m(B) \subseteq \Omega_m(A)$, its kernel clearly contains $\Omega_m(B)$. This map therefore factors through $B/\Omega_m(B)$, giving the homomorphism ϕ. We also have

$$\begin{aligned}
\ker\phi &= \{x + \Omega_m(B) \mid x \in B \text{ and } x + \Omega_m(A) = 0 + \Omega_m(A)\} \\
&= \{x + \Omega_m(B) \mid x \in B \text{ and } x \in \Omega_m(A)\} \\
&= (B \cap \Omega_m(A))/\Omega_m(B) \\
&= (B \cap (A \cap \omega_m D))/\Omega_m(B) \\
&= (B \cap \omega_m D)/\Omega_m(B) = \Omega_m(B)/\Omega_m(B) = 1,
\end{aligned}$$

so ϕ is injective. It remains to show if B contains a set of Λ-module generators for A, then ϕ is surjective. Assuming this hypothesis, every $y \in A$ can be written as

$$y = \alpha_1 b_1 + \cdots + \alpha_n b_n, \qquad \text{for some } \alpha_1, \ldots, \alpha_n \in \Lambda \text{ and } b_1, \ldots, b_n \in B.$$

By Proposition 3.9 and Lemma 3.10, for each α_i there is some $r_i \in \mathbb{Z}_p[\gamma] \subseteq R$ such that $\alpha_i - r_i \in \omega_m \Lambda$. Let $y' = r_1 b_1 + \cdots + r_n b_n \in B$. By construction,

$$y - y' = (\alpha_1 - r_1) b_1 + \cdots + (\alpha_n - r_n) b_n \in A \cap \omega_m D = \Omega_m(A).$$

Thus $\phi(y' + \Omega_m(B)) = y' + \Omega_m(A) = y + \Omega_m(A)$, and so ϕ is surjective, hence an isomorphism. Also, surjectivity of ϕ implies that $A = B + \Omega_m(A)$. If $\omega_m D \subseteq A$, then $\Omega_m(A) = \omega_m D$, so the last assertion holds too. $\qquad \square$

COROLLARY 3.13. *For $R = \mathbb{Z}_p[\Gamma]$, we have that ϕ induces an R-module isomorphism*

$$R_m \cong \mathbb{Z}_p[\Gamma_m].$$

Proof. As in Definition 3.11, we have $R_m = R/\Omega_m(R) = R/(R \cap \omega_m \Lambda)$. Now $\mathbb{Z}_p[T]$ corresponds to $\mathbb{Z}_p[\gamma]$ in the isomorphism between $\mathbb{Z}_p[[T]]$ and Λ, where γ is a fixed topological generator for Γ. So we have

$$
\begin{aligned}
R_m &= R/(R \cap \omega_m \Lambda) && \text{(by definition)} \\
&\cong \Lambda/\omega_m \Lambda && \text{(by Proposition 3.12)} \\
&\cong \mathbb{Z}_p[\gamma]/(\omega_m) && \text{(by Proposition 3.9)} \\
&\cong \mathbb{Z}_p[\gamma]/(\gamma^{p^m} - 1) \\
&\cong \mathbb{Z}_p[\Gamma_m],
\end{aligned}
$$

where the last isomorphism follows since $\mathbb{Z}_p[\gamma]/(\gamma^{p^m} - 1)$ is isomorphic to the group ring of the cyclic group $\mathbb{Z}/p^m\mathbb{Z} \cong \Gamma_m$. Hence $R_m \cong \mathbb{Z}_p[\Gamma_m]$. $\qquad \square$

We now record an elementary lemma, which will be used in proving Claim 6(3) in Section 3.3.

LEMMA 3.14. *If $A \cong \mathbb{Z}_p$ as a $\mathbb{Z}_p$-module and B is a $\mathbb{Z}_p$-submodule of A, then either $B = 0$ or A/B is finite.*

Proof. By hypothesis A is isomorphic to the ring $\mathbb{Z}_p$ considered as a module over itself, so its submodules are ideals. If $B \neq 0$, then $B = p^k A$, for some $k \geqslant 0$, and so $A/B \cong \mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}/p^k\mathbb{Z}$, which is finite. $\qquad \square$

In the next subsection, we now specify our Λ-modules to be finitely generated. Theorem 3.22 will be used in Section 3.3 to ultimately prove Theorem 1.1.

3.2.1. *Finitely Generated Λ-Modules.* The following can be found as [29, Theorem 13.12].

THEOREM 3.15 (Structure Theorem for Iwasawa modules). *For any finitely generated Λ-module M, we get the following pseudo-isomorphism:*

$$M \sim \Lambda^r \oplus \left(\oplus_{i=1}^s \Lambda/(p^{k_i}) \right) \oplus \left(\oplus_{j=1}^t \Lambda/(g_j(T)^{m_j}) \right)$$

where $r = rank(M)$, s, t, k_i and $m_j \in \mathbb{Z}$ and $g_i \in \mathbb{Z}_p[T]$ is monic, distinguished and irreducible. This decomposition is uniquely determined by M. If M is a torsion module, then $r = 0$.

The growth formula for the orders of the finite Jacobians in the conclusion of the main result, Theorem 1.1, ultimately comes from the orders of certain finite quotients of the cyclic factors in the Iwasawa Structure Theorem decomposition of a finitely generated, torsion Iwasawa module that we shall construct shortly. The general structure of finite quotients of cyclic Λ-modules is described in [29, Section 13.3].

DEFINITION 3.16. *As in the notation of Theorem* 3.15, *we define the* Iwasawa invariants *of M by*

$$\mu = \sum_{i=1}^{s} k_i \qquad and \qquad \lambda = \sum_j m_j \deg g_j.$$

DEFINITION 3.17. *Let M be any finitely generated torsion Λ-module with p^{k_i} and $g_j^{m_j}$, as in Theorem* 3.15. *The* characteristic polynomial *of M, denoted by* $\operatorname{Char}(M)$, *is the product:*

$$\operatorname{Char}(M) = p^{k_1 + \cdots + k_s} g_1^{m_1} \cdots g_t^{m_t},$$

where $\operatorname{Char}(M) = 1$ *if M is finite.*

We record some basic facts about finitely generated torsion Λ-modules. These may be found in Bence Forrás Master's Thesis [15, Section 1.1]. Part (3) may also be found in [9].

PROPOSITION 3.18. *Let P be any finitely generated torsion Λ-module.*
 (1) *The relation "pseudo-isomorphism" is an equivalence relation on any set of finitely generated torsion Λ-modules.*
 (2) *For any Λ-module M, the characteristic polynomial is an invariant of the pseudo-isomorphism equivalence class of M.*
 (3) *If M is a submodule of P, then* $\operatorname{Char}(P) = \operatorname{Char}(M)\operatorname{Char}(P/M)$. *In particular,* $\operatorname{Char}(M) \big| \operatorname{Char}(P)$.

The following theorem can be found in Romyar Sharifi's online notes [26, Theorem 2.4.7]. In it, $\mathcal{O}$ is a valuation ring of a p-adic field. We simplify his statement by taking $\mathcal{O} = \mathbb{Z}_p$. This can also be obtained from [29, Proposition 13.19 and Lemma 13.21].

THEOREM 3.19. *Let M be a finitely generated, torsion, Λ-module, and let $n_0 \geqslant 0$ be such that $\operatorname{Char}(M)$ and $\omega_{m,n_0} = \omega_m/\omega_{n_0}$ are relatively prime for all $m \geqslant n_0$. Set $\lambda(M) = \lambda$ and $\mu(M) = \mu$. Then there exists an integer ν such that*

$$|M/\omega_{m,n_0}M| = q^{e_m} \qquad where \qquad e_m = \mu p^m + \lambda m + \nu$$

for all sufficiently large $m \geqslant 0$.

This theorem is used to prove Theorem 3.22. First we present two lemmas.

LEMMA 3.20. *Let U be any Unique Factorization Domain and let $d \in U$ with $d \neq 0$. Suppose $\{a_m\}_{m=0}^{\infty}$ is any sequence of nonzero elements of U with $a_m \big| a_{m+1}$ for all $m \geqslant 0$. Then there exists some $n_0 \geqslant 0$ such that*

$$gcd(a_{n_0}, d) = gcd(a_m, d) \qquad for\ all\ m \geqslant n_0, \quad and$$
$$gcd(a_m/a_{n_0}, d) = 1 \qquad for\ all\ m \geqslant n_0.$$

Proof. This is an easy exercise. The key point is that d has only finitely many divisors, so the chain of $gcd(a_m, d)$ must stabilize after finitely many steps. □

LEMMA 3.21. *Let P be a finitely generated Λ-module. Let $P_m = \omega_m P$, for all $m \geqslant 0$. Assume there is a Λ-submodule N of P such that $P_m \subseteq N$ and $|N/P_m| < \infty$, for all $m \geqslant 0$. Assume also that $P/N \cong \mathbb{Z}_p$. Then P is a torsion Λ-module.*

Proof. By hypothesis P/N is a projective (free) $\mathbb{Z}_p$-module, so as $\mathbb{Z}_p$-modules we have

$$(1) \qquad P/\omega_m P \cong (P/N) \times (N/\omega_m P) \cong \mathbb{Z}_p \times finite \qquad for\ all\ m \geqslant 0.$$

If P is not a torsion Λ-module, then in Theorem 3.15 we have $r \geqslant 1$, so P has a Λ-submodule K such that P/K is pseudo isomorphic to Λ (where K is the kernel of

the map $P \sim \Lambda^r \oplus (\Lambda\text{-torsion}) \to \Lambda$). Let overbars denote passage to P/K. Then (as in Claim 1 in Section 3.3 below)

$$\overline{P}/\omega_m \overline{P} \cong \overline{P/\omega_m P} \sim \Lambda/\omega_m \Lambda \cong \mathbb{Z}_p[\Gamma_m].$$

However $\mathbb{Z}_p[\Gamma_m]$ is a free $\mathbb{Z}_p$-module of rank p^m, and so for any $m \geqslant 1$ it cannot be pseudo-isomorphic to a homomorphic image of $P/\omega_m P$ by (1) and the characterization of finitely generated modules over the PID $\mathbb{Z}_p$, a contradiction. $\qquad\square$

THEOREM 3.22. *Let P be a finitely generated Λ-module. Let $P_m = \omega_m P$, for all $m \geqslant 0$. Assume there is a Λ-submodule N of P such that $P_m \subseteq N$ and $|N/P_m| < \infty$, for all $m \geqslant 0$. Assume also that $P/N \cong \mathbb{Z}_p$. Then there are nonnegative integers μ and λ and an integer ν such that*

$$|N/P_m| = p^{e_m} \qquad where \quad e_m = \mu p^m + \lambda m + \nu,$$

for all $m \geqslant m_0$, for some constant $m_0 \geqslant 0$.

Proof. By the preceding lemma, P is a torsion Λ-module. Let $d = \text{Char}(P)$ and apply Lemma 3.20 in $U = \Lambda$ to $a_m = \omega_m$, for all $m \geqslant 0$. Let n_0 be as provided by the conclusion of that lemma. For any $m \geqslant n_0$, define $\omega_{m,n_0} = \omega_m/\omega_{n_0} \in \Lambda$. Let $M = P_{n_0}$.

Note that for all $m \geqslant n_0$ we have

$$\omega_{m,n_0} M = (\omega_m/\omega_{n_0})(\omega_{n_0} P) = \omega_m P = P_m.$$

By hypotheses then, for all $m \geqslant n_0$,

$$|M/\omega_{m,n_0} M| = |P_{n_0}/P_m|$$
$$= \frac{|N/P_m|}{|N/P_{n_0}|} \leqslant |N/P_m| < \infty.$$

By Lemma 3.20 we have that $\omega_{m,n_0} = \omega_m/\omega_{n_0}$ is relatively prime to $\text{Char}(P) = d$, for all $m \geqslant n_0$. By Proposition 3.18(3) we have that ω_{m,n_0} is relatively prime to $\text{Char}(M)$ as well.

We now have the hypotheses of Theorem 3.19 above. This theorem proves that there are μ, λ, and some ν' such that

$$|M/\omega_{m,n_0} M| = p^{e'_m} \qquad where \quad e'_m = \mu p^m + \lambda m + \nu',$$

for all m greater than or equal to some fixed $m_0 \geqslant n_0$.

Now, as noted above, $\omega_{m,n_0} M = P_m$, and so for all $m \geqslant m_0$, by Lagrange we have

$$|N/P_m| = |N/P_{n_0}| \cdot |P_{n_0}/P_m|$$
$$= |N/M| \cdot |M/\omega_{m,n_0} M|$$
$$= p^k \cdot p^{e'_m} \qquad where \quad p^k = |N/M| \quad and \quad e'_m = \mu p^m + \lambda m + \nu'.$$

Finally, let $\nu = k + \nu'$ to obtain the conclusion to the theorem. $\qquad\square$

The goal of the next subsection is to construct a finitely generated torsion Λ-module, $P = \text{Pic}_\Lambda$. We will then apply Theorem 3.22 to Pic_Λ in Section 3.3.

3.2.2. *Constructing a Finitely Generated Λ-Module.* Let $R = \mathbb{Z}_p[\Gamma]$ be the usual group ring of Γ with coefficients from $\mathbb{Z}_p$. For the given voltage p-tower let X_{p^∞} be its completion, so by Theorem 3.4 we may henceforth identify the intermediate graphs of X_{p^∞}/X with corresponding graphs in the tower.

Fix the following subset of X_{p^∞}:

$$\mathcal{B} = \{v_{i,0} \mid 1 \leqslant i \leqslant n\},$$

where 0 is the additive identity of $\mathbb{Z}_p$, so these vertices are taken from the "zeroth sheet". We fix the identification of X and X_0 with $\mathcal{B}$ by v_i is identified with $v_{i,0}$.

We first take the free $\mathbb{Z}$-module on basis $\mathcal{B}$, $\mathrm{Div}_{\mathbb{Z}}(X)$, and extend scalars (see [14, Section 10.4, Corollary 18]) to the free $\mathbb{Z}_p$-module with the same basis, now viewed over $\mathbb{Z}_p$. Denote this module by $\mathrm{Div}_{\mathbb{Z}_p}(X_0)$. We can do likewise for each of the graphs X_m and for X_{p^∞} too. We obtain the free $\mathbb{Z}_p$-modules of divisors

$$\mathrm{Div}_{\mathbb{Z}_p}(X_m) = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Div}_{\mathbb{Z}}(X_m), \quad m \geqslant 0$$
$$\mathrm{Div}_{\mathbb{Z}_p}(X_{p^\infty}) = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Div}_{\mathbb{Z}}(X_{p^\infty}).$$

Now for every $m \geqslant 0$, each $\mathrm{Div}_{\mathbb{Z}}(X_m)$ is a free $\mathbb{Z}[\Gamma_m]$-module on the set $\mathcal{B}$ too, once we consider the group indices for vertices in X_m to be p-adic indices reduced to $\mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}/p^m\mathbb{Z}$; and so $\mathrm{Div}_{\mathbb{Z}_p}(X_m)$ is a free module of rank n over $\mathbb{Z}_p[\Gamma_m]$. We may do likewise for X_{p^∞} to obtain that $\mathrm{Div}_{\mathbb{Z}_p}(X_{p^\infty})$ is a free R-module, also of rank n (on basis $\mathcal{B}$). In order to emphasize the free, rank n nature of these respective modules, we adopt the following notation:

$$\mathrm{Div}_{R_m} = \mathrm{Div}_{\mathbb{Z}_p}(X_m) \qquad \text{and} \qquad \mathrm{Div}_R = \mathrm{Div}_{\mathbb{Z}_p}(X_{p^\infty}).$$

Since $\mathrm{Div}_{R_m} = \mathrm{Div}_{\mathbb{Z}_p}(X_m)$ is a free $\mathbb{Z}_p$-module on the basis of vertices of X_m, $\{v_{i,g} \mid 1 \leqslant i \leqslant n, \ g \in \Gamma_m\}$, we may define the usual degree zero divisors with respect to this $\mathbb{Z}_p$-basis, and denote this by

$$\mathrm{Div}_{\mathbb{Z}_p}^0(X_m) = \left\{ \sum_{i,g} a_{i,g} v_{i,g} \mid a_{i,g} \in \mathbb{Z}_p \text{ and } \sum_{i,g} a_{i,g} = 0 \right\}$$

where these sums are for $1 \leqslant i \leqslant n$ and $g \in \Gamma_m$.

Next we extend scalars from R to Λ. Since Div_R is a free R-module of rank n, its extension is a free Λ-module of rank n, denoted by

$$\mathrm{Div}_\Lambda = \Lambda \otimes_R \mathrm{Div}_R.$$

Since R is a subring of Λ we may simply view the elements of Div_Λ as Λ-linear combinations of $\mathcal{B}$ and Div_R as the subset of these consisting of R-linear combinations of $\mathcal{B}$.

Next we define the Laplacian endomorphism:

$$\mathcal{L}_{p^\infty} : \mathrm{Div}_R \longrightarrow \mathrm{Div}_R \qquad \text{by} \qquad \mathcal{L}_{p^\infty}(v_{i,0}) = p_{i,0} \quad 1 \leqslant i \leqslant n,$$

where $p_{i,0}$, the principal divisor "based at $v_{i,0}$" is, by definition,

$$p_{i,0} = n_i v_{i,0} - \sum_{\substack{j=1 \\ v_i \sim v_j}}^{n} v_{j,0+\alpha_{i,j}},$$

where n_i is the degree of v_i in X. This is extended by R-linearity to all of Div_R. Because Γ acts transitively on vertices in each fiber of X_{p^∞}/X, as usual we have that the image of $\mathcal{L}_{p^\infty}$ is the $\mathbb{Z}_p$-span of the set of *all* principal divisors. We encapsulate this by the following notation (definition):

$$\mathrm{Pr}_R = \mathcal{L}_{p^\infty}(\mathrm{Div}_R).$$

By taking the "same map", but defined on the basis $\mathcal{B}$ of the free Λ-module Div_Λ we denote this by

$$\widehat{\mathcal{L}}_{p^\infty} : \mathrm{Div}_\Lambda \longrightarrow \mathrm{Div}_\Lambda \qquad \text{by} \qquad \widehat{\mathcal{L}}_{p^\infty}(v_{i,0}) = p_{i,0} \quad 1 \leqslant i \leqslant n,$$

extended now by Λ-linearity. (Formally, $\widehat{\mathcal{L}}_{p^\infty} = 1 \otimes \mathcal{L}_{p^\infty}$.) Now we just define

$$\mathrm{Pr}_\Lambda = \widehat{\mathcal{L}}_{p^\infty}(\mathrm{Div}_\Lambda).$$

Likewise, because Γ_m acts transitively on the vertices of X_m, using the same $\mathcal{L}_{p^\infty}$, but instead reading the vertices $v_{i,0}$ as lying in Div_{R_m} (i.e. with the vertex indices reduced to $\mathbb{Z}_p/p^m\mathbb{Z}_p$), and extended by R_m-linearity—call this map $\mathcal{L}_m$—defines the usual Laplacian endomorphism of Div_{R_m}. Its image is the R_m-module of principal divisors of Div_{R_m}, denoted as

$$\mathrm{Pr}_{R_m} = \mathcal{L}_m(\mathrm{Div}_{R_m}).$$

We now define the appropriate Picard groups as follows:

$$
\begin{aligned}
\mathrm{Pic}_{R_m} &= \mathrm{Div}_{R_m}/\mathrm{Pr}_{R_m} && \text{(an } R_m\text{-module)} \\
\mathrm{Pic}_R &= \mathrm{Div}_R/\mathrm{Pr}_R && \text{(an } R\text{-module)} \\
\mathrm{Pic}_\Lambda &= \mathrm{Div}_\Lambda/\mathrm{Pr}_\Lambda && \text{(a } \Lambda\text{-module).}
\end{aligned}
$$

So these modules are *cokernels* of the respective module endomorphisms.

Next, we identify the Λ-submodule that plays the role of "degree zero divisors" in the proof of Theorem 1.1.

DEFINITION 3.23. *Let*

$$
\begin{aligned}
S_1 &= \{v_{i,0} - v_{j,0} \mid 1 \leqslant j < i \leqslant n\} && \text{and} \\
S_2 &= \{p_{i,0} \mid 1 \leqslant i \leqslant n\}
\end{aligned}
$$

Let M_Λ be the Λ-submodule of Div_Λ generated by S_1, S_2 and $(\gamma - 1)\mathrm{Div}_\Lambda$, and let $M_R = \mathrm{Div}_R \cap M_\Lambda$ and $N_\Lambda = M_\Lambda/\mathrm{Pr}_\Lambda$.

It turns out that M_Λ is actually generated by just S_1 and $(\gamma - 1)\mathrm{Div}_\Lambda$ (see Claim 5(1) in the next subsection).

Since Div_Λ is a finitely generated Λ-module and Λ is Noetherian, all of its submodules are finitely generated, and so it follows that the quotient modules $\mathrm{Div}_\Lambda/\mathrm{Pr}_\Lambda = \mathrm{Pic}_\Lambda$ and $M_\Lambda/\mathrm{Pr}_\Lambda = N_\Lambda$ are also finitely generated as Λ-modules.

THEOREM 3.24. *Let Div_Λ, Pr_Λ, Pic_Λ and N_Λ be as above. Then Pic_Λ is a finitely generated module over the Iwasawa Algebra $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and therefore so is its submodule N_Λ.*

3.3. THE MAIN THEOREM. We now go on to prove Theorem 1.1.
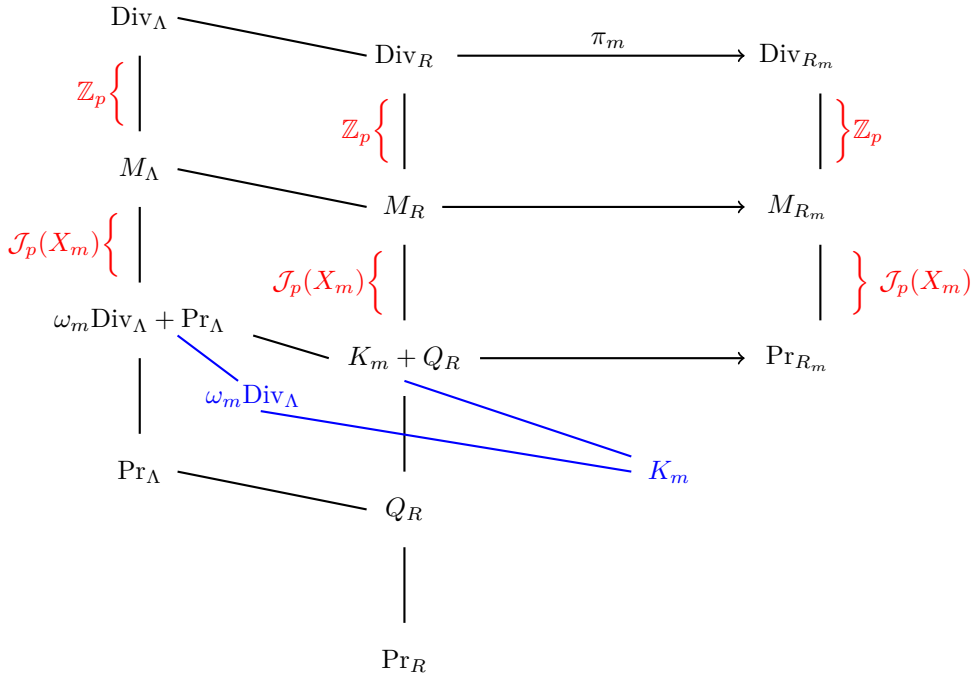
PROPOSITION 3.25. *The following diagram holds.*

FIGURE 2. Lattice and map diagram showing that $M_\Lambda/(\omega_m\mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda) \cong \mathcal{J}_p(X_m)$,

Proposition 3.25 is proved by combining the following six claims 1-6 concerning the columns of Figure 2.

First consider the reduction map

$$\pi_m : \mathrm{Div}_R \to \mathrm{Div}_{R_m} \quad \text{by} \quad v_{i,g} \mapsto v_{i,\overline{g}}$$

where $g \in \Gamma$ and $\overline{g} \in \Gamma_m$ is the reduction of g to $\Gamma/\Gamma^{p^m} \cong \mathbb{Z}_p/p^m\mathbb{Z}_p$, (and recall $\mathrm{Div}_{R_m} = \mathrm{Div}_{\mathbb{Z}_p}(X_m)$). Here we are really defining π_m on the free R-basis vectors on the zeroth sheet, and then extending by R-linearity to all of Div_R. It is helpful to keep in mind that for all $m \geqslant 0$, by the above map and by the previous subsection we have

$$\mathrm{Div}_R \text{ is an } R\text{-submodule of } \mathrm{Div}_\Lambda, \text{ and}$$
$$\mathrm{Div}_{R_m} \text{ is an } R\text{-quotient module of } \mathrm{Div}_R.$$

Now let $D = \mathrm{Div}_\Lambda$ as in Proposition 3.12, but we simplify notation by writing $\Omega_m(\mathrm{Div}_R)$ to denote $\Omega_m^D(\mathrm{Div}_R)$.

Claim 1

The kernel of π_m is $\Omega_m(\mathrm{Div}_R) = \mathrm{Div}_R \cap \omega_m\mathrm{Div}_\Lambda$, where $\omega_m = \gamma^{p^m} - 1$.

Proof. By Proposition 3.12 and Corollary 3.13, we get the following isomorphisms, where the composition of these isomorphisms is the induced map on $\mathrm{Div}_R \bmod \ker \pi_m$:

$$
\begin{aligned}
\mathrm{Div}_R/\Omega_m(\mathrm{Div}_R) &\cong \mathrm{Div}_\Lambda/\omega_m\mathrm{Div}_\Lambda \\
&\cong (\Lambda \oplus \Lambda \oplus \cdots \oplus \Lambda)/(\omega_m(\Lambda \oplus \Lambda \oplus \cdots \oplus \Lambda)) \\
&\cong (\Lambda/(\omega_m)) \oplus \cdots \oplus (\Lambda/(\omega_m)) \\
&\cong R_m \oplus \cdots \oplus R_m \\
&\cong \mathbb{Z}_p[\Gamma_m] \oplus \cdots \oplus \mathbb{Z}_p[\Gamma_m] \\
&\cong \mathrm{Div}_{R_m}
\end{aligned}
$$

the free $\mathbb{Z}_p[\Gamma_m]$-module of rank n. Thus, the kernel of π_m is $\Omega_m(\mathrm{Div}_R)$. $\qquad\square$

Now let

$$K_m = \ker\pi_m \qquad \text{and} \qquad Q_R = \mathrm{Pr}_\Lambda \cap \mathrm{Div}_R.$$

<u>Claim 2:</u>
Columns 1 and 2 have the following intersections:

    (1) $\mathrm{Div}_R \subseteq \mathrm{Div}_\Lambda$,
    (2) $\mathrm{Div}_R \cap M_\Lambda = M_R$,
    (3) $\mathrm{Pr}_\Lambda \cap \mathrm{Div}_R = Q_R$,
    (4) $(\omega_m\mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda) \cap \mathrm{Div}_R = K_m + Q_R = K_m + \mathrm{Pr}_R$.

Proof. (1) holds by Subsection 3.2.2, while (2) and (3) are by definition of M_R and Q_R, respectively. By Proposition 3.12 applied with $D = \mathrm{Div}_\Lambda$, $A = \mathrm{Pr}_\Lambda + \omega_m\mathrm{Div}_\Lambda$ and $B = \mathrm{Pr}_R$, since Pr_R and Pr_Λ are both generated (as R- and Λ-modules, respectively) by the same generators, they both have the same image in $\mathrm{Div}_\Lambda/\omega_m\mathrm{Div}_\Lambda$ as in Claim 1. So, by the last sentence of Proposition 3.12,

$$(2) \qquad\qquad \mathrm{Pr}_R + \omega_m\mathrm{Div}_\Lambda = \mathrm{Pr}_\Lambda + \omega_m\mathrm{Div}_\Lambda.$$

Then since

$$\mathrm{Pr}_R \subseteq Q_R \subseteq \mathrm{Pr}_\Lambda,$$

by Equation (2), we get

$$(3) \qquad\qquad \mathrm{Pr}_R + \omega_m\mathrm{Div}_\Lambda = Q_R + \omega_m\mathrm{Div}_\Lambda = \mathrm{Pr}_\Lambda + \omega_m\mathrm{Div}_\Lambda.$$

Now because Pr_R and Q_R are contained in Div_R, intersecting the subgroups in Equation (3) with Div_R gives

$$
\begin{aligned}
(\mathrm{Pr}_R + \omega_m\mathrm{Div}_\Lambda) \cap \mathrm{Div}_R &= \mathrm{Pr}_R + (\omega_m\mathrm{Div}_\Lambda \cap \mathrm{Div}_R) = \mathrm{Pr}_R + K_m \\
&= Q_R + (\omega_m\mathrm{Div}_\Lambda \cap \mathrm{Div}_R) = Q_R + K_m \\
&= (\mathrm{Pr}_\Lambda + \omega_m\mathrm{Div}_\Lambda) \cap \mathrm{Div}_R,
\end{aligned}
$$

which gives (4). $\qquad\square$

<u>Claim 3:</u>
Columns 1 and 2 have the following containments:

    (1) $\mathrm{Pr}_\Lambda \subseteq \omega_m\mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda \subseteq M_\Lambda \subseteq \mathrm{Div}_\Lambda$,
    (2) $\mathrm{Pr}_R \subseteq Q_R \subseteq K_m + Q_R \subseteq M_R \subseteq \mathrm{Div}_R$.

Proof. (1) is clear. From Claim 1, we have that $\ker\pi_m = K_m = \mathrm{Div}_R \cap \omega_m\mathrm{Div}_\Lambda$. Then since $\omega_m = (\gamma - 1)(1 + \gamma + \cdots + \gamma^{p^m-1})$, we have

$$\omega_m\mathrm{Div}_\Lambda = (\gamma - 1)(1 + \gamma + \cdots + \gamma^{p^m-1})\mathrm{Div}_\Lambda \subseteq (\gamma - 1)\mathrm{Div}_\Lambda \subseteq M_\Lambda.$$

Thus we have $\ker\pi_m \subseteq \mathrm{Div}_R \cap M_\Lambda = M_R$. So all containments in (2) are clear. $\qquad\square$

Claim 4:

Columns 1 and 2 have the following joins:

(1) $M_\Lambda + \mathrm{Div}_R = \mathrm{Div}_\Lambda$,

(2) $(\omega_m \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda) + M_R = M_\Lambda$.

Proof. Apply Proposition 3.12 to $D = A = \mathrm{Div}_\Lambda$ and $B = \mathrm{Div}_R$. Its final assertion gives that $\omega_m \mathrm{Div}_\Lambda + \mathrm{Div}_R = \mathrm{Div}_\Lambda$. Then (1) is immediate because $\omega_m \mathrm{Div}_\Lambda \subseteq M_\Lambda$ by definition. Finally, since $\omega_m \mathrm{Div}_\Lambda + \mathrm{Div}_R = \mathrm{Div}_\Lambda$ and by the latter observation, we have

$$M_\Lambda = M_\Lambda \cap (\omega_m \mathrm{Div}_\Lambda + \mathrm{Div}_R)$$
$$= \omega_m \mathrm{Div}_\Lambda + (M_\Lambda \cap \mathrm{Div}_R)$$
$$= \omega_m \mathrm{Div}_\Lambda + M_R$$

as needed for (2). $\square$

Claim 5:

As R-modules we have the following:

(1) $\mathrm{Div}_\Lambda / M_\Lambda \cong \mathbb{Z}_p$ and M_Λ is the Λ-submodule of Div_Λ generated by

$$S_1 \cup (\gamma - 1)\mathrm{Div}_\Lambda,$$

where S_1 is as in Definition 3.23,

(2) $\mathrm{Div}_R / M_R \cong \mathbb{Z}_p$.

Proof. We first prove (1). We may obtain $\mathrm{Div}_\Lambda / M_\Lambda$ as follows: first factor Div_Λ by $(\gamma - 1)\mathrm{Div}_\Lambda$. By Corollary 3.13 applied with $m = 0$, and as in the proof of Claim 1, we have that

$$\mathrm{Div}_\Lambda / (\gamma - 1)\mathrm{Div}_\Lambda \cong (\Lambda / \omega_0 \Lambda) \oplus \cdots \oplus (\Lambda / \omega_0 \Lambda) \cong \underbrace{\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{n \text{ of these}},$$

where the divisors $v_{1,0}, \ldots, v_{n,0}$ map to a basis of this free $\mathbb{Z}_p$-module of rank n. Now factor out the submodule generated by the images of all $v_{i,0} - v_{j,0} \ \forall i, j$ from the quotient $\mathrm{Div}_\Lambda / (\gamma - 1)\mathrm{Div}_\Lambda$. By doing this, we are simply identifying all the basis vectors with each other, leaving the rank-1 $\mathbb{Z}_p$-module quotient. The latter process is the same as modding $\mathrm{Div}_{\mathbb{Z}_p}(X_0)$ by the degree zero divisors in $\mathrm{Div}_{\mathbb{Z}_p}(X_0)$. So this tells us that $p_{i,0}$ for $1 \leqslant i \leqslant n$ must already be contained in the Λ-submodule generated by just $S_1 \cup (\gamma - 1)\mathrm{Div}_\Lambda$. This proves (1).

To prove (2): First note that by definition $M_R = M_\Lambda \cap \mathrm{Div}_R$. Then by Claim 4(1) we have that

$$\mathrm{Div}_\Lambda / M_\Lambda = (\mathrm{Div}_R + M_\Lambda) / M_\Lambda.$$

Then by the Diamond Isomorphism Theorem (which says that $(\mathrm{Div}_R + M_\Lambda)/M_\Lambda \cong \mathrm{Div}_R / M_R$) and the previous part we see that

$$\mathrm{Div}_R / M_R \cong \mathbb{Z}_p.$$

as desired. $\square$

By the proof of Claim 3(2), it follows that the kernel of the map π_m restricted to M_R (which we simply denote by π_m too) is also equal to $K_m = \mathrm{Div}_R \cap \omega_m \mathrm{Div}_\Lambda$. From the definition in Section 3.2.2 we introduce the new notation:

$$M_{R_m} = \mathrm{Div}^0_{\mathbb{Z}_p}(X_m).$$

Claim 6:

For Columns 2 and 3 the following holds:

(1) $\mathrm{Pr}_{R_m} \subseteq M_{R_m} \subseteq \mathrm{Div}_{R_m}$

(2) $\pi_m : \mathrm{Div}_R \to \mathrm{Div}_{R_m}$ is well-defined and surjective,

(3) $\pi_m(M_R) = M_{R_m}$ and $M_R = \pi_m^{-1}(M_{R_m})$,

(4) $\pi_m(K_m + Q_R) = \mathrm{Pr}_{R_m}$ and $\pi_m^{-1}(\mathrm{Pr}_{R_m}) = K_m + Q_R$.

Proof. Items (1) and (2) are clear by their respective definitions. It is clear that

$$S_1 \cup (\gamma - 1)\mathrm{Div}_R \subseteq \mathrm{Div}_R \cap M_\Lambda = M_R.$$

Next we show that the image of $S_1 \cup (\gamma - 1)\mathrm{Div}_R$ under π_m generates $\mathrm{Div}^0_{\mathbb{Z}_p}(X_m)$ as a $\mathbb{Z}_p$-module. The $\mathbb{Z}_p$-module $\mathrm{Div}^0_{\mathbb{Z}_p}(X_m)$ is generated as a $\mathbb{Z}_p$-module by differences of vertices, $v_{i,r} - v_{j,s}$, where $1 \leqslant i, j \leqslant n$ and $r, s \in \mathbb{Z}/p^m\mathbb{Z}$. Such differences can be written as

$$v_{i,r} - v_{j,s} = (v_{i,r} - v_{i,0}) + (v_{i,0} - v_{j,0}) + (v_{j,0} - v_{j,s}),$$

where the middle term on the right is in S_1. For each i (and j) we may express the first (and third, resp.) differences on the right as telescoping sums of divisors of the form $v_{i,t+\tau} - v_{i,t}$ for all t, where $\tau = \pi_m(\gamma)$ is any additive generator for $\mathbb{Z}/p^m\mathbb{Z}$. The claim then follows since all of the latter differences are in the image of $(\gamma - 1)\mathrm{Div}_R$ under π_m. This argument shows that

(4) $$M_{R_m} \subseteq \pi_m(M_R).$$

To show the reverse containment, let $D = \pi_m^{-1}(M_{R_m})$, (the complete preimage). By basic properties of homomorphisms (part of the Lattice Isomorphism Theorem) and since $\ker \pi_m \subseteq M_R$ we have:

$$\pi_m^{-1}(\pi_m(M_R)) = M_R + \ker \pi_m = M_R.$$

By applying π^{-1} to Equation (4) we get $D \subseteq M_R$. By the Lattice Isomorphism Theorem we have that π_m induces an isomorphism

$$\mathrm{Div}_R/D \cong \mathrm{Div}_{R_m}/\pi_m(D) = \mathrm{Div}_{R_m}/M_{R_m} = \mathrm{Div}_{\mathbb{Z}_p}(X_m)/\mathrm{Div}^0_{\mathbb{Z}_p}(X_m) \cong \mathbb{Z}_p,$$

where the latter follows from $\deg : \mathrm{Div}_{\mathbb{Z}_p}(X_m) \to \mathbb{Z}_p$.

Since $D \subseteq M_R$ we get that Div_R/M_R is a quotient $\mathbb{Z}_p$-module of the $\mathbb{Z}_p$-module Div_R/D. By Claim 5(2) we also have $\mathrm{Div}_R/M_R \cong \mathbb{Z}_p$. This is illustrated in Figure 3.
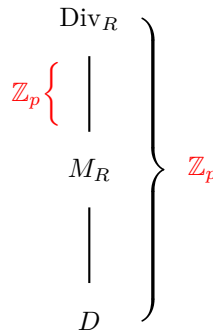

FIGURE 3. $\mathrm{Div}_R/M_R \cong \mathbb{Z}_p$ is a quotient $\mathbb{Z}_p$-module of the $\mathbb{Z}_p$-module Div_R/D

However, the only $\mathbb{Z}_p$-module quotient of $\mathbb{Z}_p$ that is also isomorphic to $\mathbb{Z}_p$ is the quotient by the zero submodule (this follows by Lemma 3.14 in Section 3.2) i.e. we must have $M_R = D$; and so $\pi_m(M_R) = \pi_m(D) = M_{R_m}$. This gives (3).

Now π_m induces the surjective map

$$\overline{\pi}_m : M_R/\mathrm{Pr}_R \to M_{R_m}/\mathrm{Pr}_{R_m}$$

where, by definition, $M_R/\mathrm{Pr}_R = N_R$ and $M_{R_m}/\mathrm{Pr}_{R_m} = \mathcal{J}_p(X_m)$ (the isomorphism $M_{R_m}/\mathrm{Pr}_{R_m} \cong \mathcal{J}_p(X_m)$ is obtained by taking $\otimes \mathbb{Z}_p$ to $\mathrm{Div}^0(X_m)/\mathrm{Pr}(X_m) = \mathcal{J}(X_m)$). This is defined by the following commutative diagram in Figure 4.
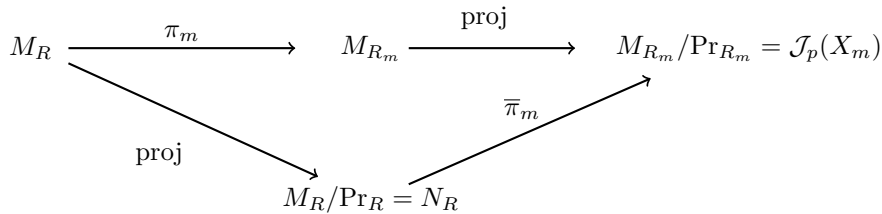

FIGURE 4. The map $\overline{\pi}_m : N_R \to \mathcal{J}_p(X_m)$ commutes with the natural projection map

To prove (4), first invoke Claim 2(4) to obtain that $K_m + Q_R = K_m + \mathrm{Pr}_R$. Since K_m is the kernel of π_m, the subgroups Q_R and Pr_R have the same image under π_m. By definition, Pr_R is generated as an R-module by the principal divisors based at the vertices in the zeroth sheet of X_{p^∞}; and likewise Pr_{R_m} is generated as an R_m-module by the images of these principal divisors in Div_{R_m}. Thus π_m maps Pr_R, hence also $K_m + \mathrm{Pr}_R$, surjectively onto Pr_{R_m}. This gives the first assertion of (4). Furthermore, since $K_m + \mathrm{Pr}_R$ contains the full kernel of π_m and Pr_R maps onto Pr_{R_m}, the Lattice Isomorphism Theorem immediately gives the second assertion of (4). $\qquad\square$

Claims 1–6 prove Proposition 3.25.

Now for each subgroup A of Div_Λ let $\widetilde{A}$ denote the image of A under the natural projection map

$$\sim : \mathrm{Div}_\Lambda \longrightarrow \mathrm{Div}_\Lambda/\mathrm{Pr}_\Lambda$$

(which is both a Λ- and an R-module homomorphism). Since $\sim$ is a Λ-module homomorphism, the image of $\omega_m \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda$ under it is $\omega_m \mathrm{Pic}_\Lambda$. Since Div_R is an R-submodule of Div_Λ, we may apply $\sim$ to it as well, and to its submodules.

By the Diamond Isomorphism Theorem, since we have checked all the appropriate intersections from column 1 to column 2 in Figure 2, this natural projection gives the first two columns in Figure 5 as well as all intersections (depicted, as usual, by horizontal lines) between their subgroups in column 2. To get the third column of Figure 5, factor the third column of Figure 2 by Pr_{R_m}. The horizontal lines—which are homomorphisms—relating column 2 to column 3 in Figure 5 are obtained by taking images of the subgroups in column 2 under $\overline{\pi}_m$.

By the Lattice Isomorphism Theorem, the (already established) quotient groups (in red) are consequently also preserved when passing between columns (thus also transitively from column 1 to column 3). We only need these to be abelian group isomorphisms; but they are, in fact, R- and Λ-module isomorphisms.
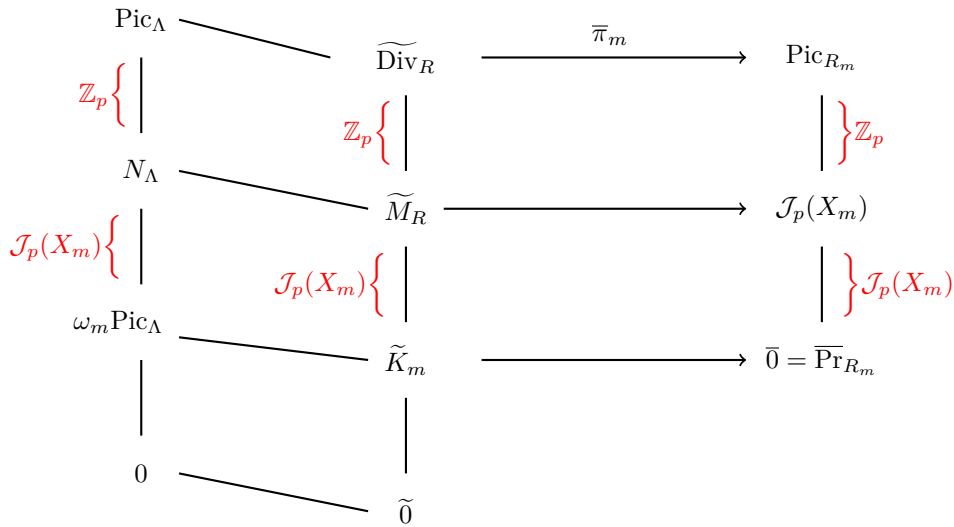
FIGURE 5. The natural projection homomorphism from Div_Λ to Pic_Λ indicated in the first two columns and the passage from π_m to $\overline{\pi}_m$ indicated in the third column

Now we apply Theorem 3.22 with $P = \mathrm{Pic}_\Lambda$ and $N = N_\Lambda = M_\Lambda/\mathrm{Pr}_\Lambda$ used as P and N in its hypotheses. Since X_m is connected and $\omega_m\mathrm{Pic}_\Lambda \subseteq N_\Lambda$ for all m by Figure 5, we have

$$|N_\Lambda/\omega_m\mathrm{Pic}_\Lambda| = |\mathcal{J}_p(X_m)| < \infty.$$

This leads immediately to the conclusion of Theorem 1.1.

An important invariant of any voltage graph with abelian voltage group is *the reduced Stickelberger Element*. With respect to the basis $\mathcal{B}$ of both Div_R (as an R-basis) and Div_Λ (as a Λ-basis) (see Subsection 3.2.2), the two maps, $\mathcal{L}_{p^\infty}$ and $\widehat{\mathcal{L}}_{p^\infty}$, have *the same matrix representation*. Thus we define

$$\Theta_{p^\infty} = \det\mathcal{L}_{p^\infty} = \det\widehat{\mathcal{L}}_{p^\infty},$$

which is an element of $R = \mathbb{Z}_p[\Gamma]$ that plays the role of the *reduced Stickelberger element*.

REMARK 3.26. Θ_{p^∞} annihilates both Pic_R and Pic_Λ— see [16] for discussion and additional uses of the reduced Stickelberger element.

COROLLARY 3.27. *Under the hypothesis and notation of Theorem 1.1, the ranks of $\mathcal{J}_p(X_m)$ are bounded as $m \to \infty$ if and only if p does not divide Θ_{p^∞} in Λ (or in $\mathbb{Z}_p[\Gamma]$).*

Proof. By definition, Pic_Λ is the cokernel of the voltage Laplacian, $\mathcal{L}_{p^\infty} : \mathrm{Div}_\Lambda \to \mathrm{Div}_\Lambda$, where $\Theta_{p^\infty} = \det\mathcal{L}_{p^\infty}$. In the notation of Theorem 3.15, let p^μ be the product of the p^{k_i}. Then the characteristic polynomial, as in Definition 3.17, is equal to

$$(5) \qquad\qquad p^\mu \prod_{j=1}^{t} g_j^{m_j} = \mathrm{Char}(\mathrm{Pic}_\Lambda).$$

Let $M = \omega_{m_0}\mathrm{Pic}_\Lambda$ where $m_0 \geqslant 0$ is fixed. Then since Pic_Λ/M has finite p-rank (fixed, independent of $m \to \infty$), the ranks of Pic_Λ and M differ by a constant, and one is bounded as $m \to \infty$ if and only if the other is bounded.

We now compare μ invariants for Pic_Λ and M, as follows. By Proposition 3.18(3), we have

$$(6) \qquad \mathrm{Char}(M) = \frac{\mathrm{Char}(\mathrm{Pic}_\Lambda)}{\mathrm{Char}(\mathrm{Pic}_\Lambda/M)}.$$

The Λ-module Pic_Λ/M is a quotient of the module $\mathrm{Div}_\Lambda/(\omega_{m_0}\mathrm{Div}_\Lambda)$; and as in Claim 1,

$$\mathrm{Div}_\Lambda/(\omega_{m_0}\mathrm{Div}_\Lambda) \cong \underbrace{(\Lambda/\omega_{m_0}\Lambda) \oplus \cdots \oplus (\Lambda/\omega_{m_0}\Lambda)}_{n \text{ of these}}.$$

But by Lemma 3.10 we know ω_{m_0} maps to a distinguished polynomial in $\mathbb{Z}_p[[T]] \cong \Lambda$, so $(\Lambda/(\omega_{m_0}))^n$ is already in Iwasawa decomposition form, and it clearly has characteristic polynomial $\omega_{m_0}^n$ (again, under the identification $\gamma \mapsto T + 1$). One more usage of Proposition 3.18(3) gives that

$$\mathrm{Char}(\mathrm{Pic}_\Lambda/M) \,\big|\, \omega_{m_0}^n,$$

so $\mathrm{Char}(\mathrm{Pic}_\Lambda/M)$ is relatively prime to p (since the distinguished polynomial ω_{m_0} is). By Equation (6), this shows

$$p \,\big|\, \mathrm{Char}(\mathrm{Pic}_\Lambda) \quad \Longleftrightarrow \quad p \,\big|\, \mathrm{Char}(M).$$

If $\mu(\mathrm{Pic}_\Lambda) = 0$, then $\mathrm{Char}(\mathrm{Pic}_\Lambda) = \Theta_{p^\infty}$ by [21, Proposition 10.23]. In this case, p does not divide Θ_{p^∞} by definition of $\mathrm{Char}(\mathrm{Pic}_\Lambda)$. By [29, Lemma 13.20], we have that the ranks of the finite Λ-module quotients of a finitely generated torsion Λ-module stay bounded if and only if the μ invariant of the Iwasawa decomposition is zero. So if the ranks of $\mathcal{J}_p(X_m)$ stay bounded as $m \to \infty$, then p does not divide Θ_{p^∞}.

Conversely, we show that if the ranks of $\mathcal{J}_p(X_m)$ don't stay bounded as $m \to \infty$, then p does divide Θ_{p^∞} in Λ. So if the rank of $\mathcal{J}_p(X_m) \to \infty$ as $m \to \infty$, then the μ-invariant of the submodule M, and hence also of Pic_Λ, must be nonzero. i.e. the Iwasawa decomposition of Pic_Λ must have at least one factor of the form $\Lambda/(p^a)$, for some $a \geqslant 1$. This forces p to divide Θ_{p^∞} as follows. By [16, Corollary 9] or [29, p. 297], Θ_{p^∞} annihilates Pic_Λ. It follows from the definition of pseudo-isomorphism that there is some submodule of fixed finite index in the Iwasawa Decomposition factor $\Lambda/(p^a)$ that is also annihilated by Θ_{p^∞}; and hence Θ_{p^∞} annihilates a submodule of fixed finite index in every quotient of $\Lambda/(p^a)$. But the latter module has quotient modules, $\Lambda/(p^a, T^k)$, of order p^{ak}, for every $k \geqslant 1$, and none of these possess a nonzero submodule annihilated by an element of Λ that is prime to p. Thus p must divide Θ_{p^∞}. $\qquad\square$

EXAMPLE 3.28. Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$ be a cyclic voltage p-tower of derived graphs over a base graph X whose vertices are $v_1, \ldots, v_n$, with v_1 adjacent to v_2. Assume that the voltage adjacency matrix for X_m/X (with the standard orientation on X) has a generator for the cyclic group of order p^m in entry $1, 2$; its inverse in entry $2,1$; and has a 1 (the identity of the voltage group) in entry i, j whenever v_i is adjacent to v_j for $\{i, j\} \neq \{1, 2\}$; and has zeros elsewhere (including on the diagonal). Then

$$|\mathcal{J}_p(X_m)| = p^{e_m}, \quad \text{where} \quad e_m = \mu p^m + \lambda m + \nu \qquad \text{for all } m \geqslant 0,$$

where p^μ is the largest power of p dividing the reduced Stickelberger element (p^μ can be shown to be independent of m), and $\lambda = 1$. In particular, when X is the complete graph on n vertices, p^μ is the largest power of p dividing $(n - 2)n^{n-3}$, for $n \geqslant 3$.

Starting just with a base graph X as above, we can choose generators for voltage groups $\mathbb{Z}/p^m\mathbb{Z}$ so that the sequence of $1, 2$ entries in such voltage adjacency matrices converges in $\mathbb{Z}_p$. This constructs derived graphs X_m/X that form a cyclic voltage

p-tower (and if X_1 is connected, then all X_m are too) — see [16] for further details on these "single voltage" derived covers.

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Sophia R. Gonet, Rutgers University, Mathematics Department, 110 Frelinghuysen Rd., Piscataway, NJ 08854
E-mail : `sophia.gonet@rutgers.edu`