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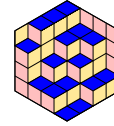
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Intersection density of transitive groups of certain degrees

Ademir Hujdurović, Klavdija Kutnar, Dragan Marušič
& Štefko Miklavič

ABSTRACT Two elements g and h of a permutation group G acting on a set V are said to be *intersecting* if $v^g = v^h$ for some $v \in V$. More generally, a subset \mathcal{F} of G is an *intersecting set* if every pair of elements of \mathcal{F} is intersecting. The *intersection density* $\rho(G)$ of a transitive permutation group G is the maximum value of the quotient $|\mathcal{F}|/|G_v|$ where \mathcal{F} runs over all intersecting sets in G and G_v is a stabilizer of $v \in V$. In this paper the intersection density of transitive groups of degree twice a prime is determined, and proved to be either 1 or 2. In addition, it is proved that the intersection density of transitive groups of prime power degree is 1.

1. INTRODUCTORY REMARKS

For a finite set V let $\text{Sym}(V)$ and $\text{Alt}(V)$ denote the corresponding symmetric group and alternating group on V . (Of course, if $|V| = n$ the standard notations S_n , A_n apply.) Let $G \leq \text{Sym}(V)$ be a permutation group acting on a set V . Two elements $g, h \in G$ are said to be *intersecting* if $v^g = v^h$ for some $v \in V$. Furthermore, a subset \mathcal{F} of G is an *intersecting set* if every pair of elements of \mathcal{F} is intersecting. The *intersection density* $\rho(\mathcal{F})$ of the intersecting set \mathcal{F} is defined to be the quotient

$$\rho(\mathcal{F}) = \frac{|\mathcal{F}|}{\max_{v \in V} |G_v|},$$

and the *intersection density* $\rho(G)$ of a group G , first defined by Li, Song and Pantangi in [8], is the maximum value of $\rho(\mathcal{F})$ where \mathcal{F} runs over all intersecting sets in G , that is,

$$\rho(G) = \max\{\rho(\mathcal{F}) : \mathcal{F} \subseteq G, \mathcal{F} \text{ is intersecting}\} = \frac{\max\{|\mathcal{F}| : \mathcal{F} \subseteq G \text{ is intersecting}\}}{\max_{v \in V} |G_v|}.$$

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KEYWORDS. intersection density, derangement, derangement graph, transitive permutation group.

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Observe that, since G_v is an intersecting set in G , we have $\rho(G) \geq 1$. Observe also that for a transitive group G acting on a set V we have $\rho(G) = 1$ if and only if the maximum cardinality of the intersecting set is $|G|/|V|$, in which case we say that G has the *Erdős-Ko-Rado property* or *EKR-property* in short. Moreover, G has the *strict-EKR-property* if the canonical intersecting sets are the only maximum intersecting sets of G , where a *canonical* intersecting set is an intersecting set of the form gG_v , $v \in V$ and $g \in G$.

Following [12] we define \mathcal{I}_n to be the set of all intersection densities of transitive permutation groups of degree n , that is,

$$\mathcal{I}_n = \{\rho(G) \mid G \text{ transitive of degree } n\},$$

and we let $I(n)$ to be the maximum value in \mathcal{I}_n .

Motivation for this paper comes from [12, Conjectures 6.(3) and 6.(4)] and [13, Question 7.1].

CONJECTURE 1.1. [12, Conjecture 6.6(3)] *If n is a prime power, then $I(n) = 1$.*

CONJECTURE 1.2. [12, Conjecture 6.6(4)] *If $n = 2p$ where p is an odd prime, then $I(n) = 2$.*

Conjecture 1.2 is settled in [13], where an additional problem regarding the possible values of intersection densities in \mathcal{I}_{2p} was posed.

QUESTION 1.3. [13, Question 6.1] *Does there exist an odd prime p and a transitive group G of degree $2p$ such that $\rho(G)$ is not an integer?*

In this paper we settle Conjecture 1.1 and give a negative answer to Question 1.3 by obtaining a complete classification of intersection densities of transitive groups of degree twice a prime. We would like to remark that Conjecture 1.1 has been proved in [8], however the proof presented in this paper is a different one. The main tool we developed in order to prove Conjecture 1.1 is Lemma 3.1, for which we believe that it can be quite useful for other applications as well.

THEOREM 1.4. *For a transitive permutation group G of prime power degree the intersection density $\rho(G)$ is equal to 1.*

THEOREM 1.5. *Let G be a transitive permutation group of degree $2p$, where p is a prime. Then the intersection density $\rho(G)$ is either 1 or 2. More precisely, $\rho(G) = 2$ if and only if either*

- (i) $G \cong K \rtimes H$ acting on a set $V = \{x_i : i \in \mathbb{Z}_p\} \cup \{y_i : i \in \mathbb{Z}_p\}$ where $K \leq E \cap \text{Alt}(V)$, $E \cong \mathbb{Z}_2^p$ is an elementary abelian 2-group generated by the involutions $\epsilon_i = (x_i y_i)$, $i \in \mathbb{Z}_p$, and $H = \langle (x_0 x_1 \dots x_{p-1})(y_0 y_1 \dots y_{p-1}) \rangle \cong \mathbb{Z}_p$, or
- (ii) $G \cong A_5$ acting on a 10-element set of pairs of $\{1, 2, 3, 4, 5\}$.

2. PRELIMINARIES

2.1. (IM)PRIMITIVITY OF TRANSITIVE PERMUTATION GROUPS. Let G be a transitive permutation group G acting on a set V . A partition \mathcal{B} of V is called G -invariant if the elements of G permute the parts, the so called *blocks* of \mathcal{B} , setwise. If the trivial partitions $\{V\}$ and $\{\{v\} : v \in V\}$ are the only G -invariant partitions of V , then G is *primitive*, and is *imprimitive* otherwise. In the latter case the corresponding G -invariant partition will be referred to as the *complete imprimitivity block system* of G . We say that G is *doubly transitive* if given any two ordered pairs (u, v) and (u', v') of elements $u, v, u', v' \in V$, such that $u \neq v$ and $u' \neq v'$, there exists an element $g \in G$ such that $g(u, v) = (u', v')$. Note that a doubly transitive group is primitive. A primitive group which is not doubly transitive is called *simply primitive*.

The following result about normalizers of Sylow p -subgroups in doubly transitive groups of prime degree will be needed in the proof of Theorem 1.5.

LEMMA 2.1. *Let G be a doubly transitive group of prime degree p . Then a Sylow p -subgroup P of G is strictly contained in its normalizer $N_G(P)$.*

Proof. Let G be a doubly transitive group of prime degree p acting on a set V . Consider the action of G on the set \mathcal{P} of all Sylow p -subgroups of G by conjugation. By Sylow theorems this action is transitive with $N = N_G(P)$ as the corresponding stabilizer of P . If $N = P$ then the intersection of any two stabilizers of this action is trivial, and so G acts on \mathcal{P} as a Frobenius group. It follows that G contains a regular normal subgroup T of order $|\mathcal{P}| \equiv 1 \pmod{p}$. Now consider the action of T on the set V . Since T is a normal subgroup of a transitive group G of prime degree it follows that T is transitive on V , a contradiction since $|T|$ is not divisible by p . \square

2.2. DERANGEMENT GRAPHS. The intersection density of a permutation group can be studied via *derangements*, that is, fixed-point-free elements of G . Let \mathcal{D} be the set of all derangements of a permutation group G . Then following [12] we define the *derangement graph* of G to be the graph $\Gamma_G = \text{Cay}(G, \mathcal{D})$ with vertex set G and edge set consisting of all pairs $(g, h) \in G \times G$ such that $gh^{-1} \in \mathcal{D}$. Therefore Γ_G is the Cayley graph of G with connection set \mathcal{D} , which is a loop-less simple graph since \mathcal{D} does not contain the identity element of G and \mathcal{D} is inverse-closed. In the terminology of the derangement graph an intersecting set of G is an independent set or a coclique of Γ_G . Since, by a classical theorem of Jordan [7, Théorème I], a transitive permutation group G on a finite set V of cardinality at least 2 contains derangements, we have $\rho(G) < |V|$. (Note also, that by a theorem of Fein, Kantor and Schacher [1, Theorem 1], every transitive permutation group contains a derangement of prime power order.)

The following classical upper bound on the size of the largest coclique in vertex-transitive graphs turns out to be very useful when considering the intersection densities of permutation groups. Namely, the derangement graph Γ_G of a permutation group G is always vertex-transitive.

PROPOSITION 2.2. [2] *Let Γ be a vertex-transitive graph. Then the largest coclique in Γ is of size $\alpha(\Gamma)$ bounded by*

$$\alpha(\Gamma) \leq \frac{|V(\Gamma)|}{\omega(\Gamma)},$$

where $\omega(\Gamma)$ is the size of a maximum clique in Γ .

2.3. INTERSECTION DENSITY OF TRANSITIVE GROUPS.

PROPOSITION 2.3. *Let G be a transitive permutation group and \mathcal{F} an intersecting set of G . Then there exists an intersecting set \mathcal{F}' such that $|\mathcal{F}| = |\mathcal{F}'|$ and $1 \in \mathcal{F}'$.*

Proof. Take an element $f \in \mathcal{F}$ and let $\mathcal{F}' = f^{-1}\mathcal{F}$. Then $1 \in \mathcal{F}'$ and since for $g_1, g_2 \in \mathcal{F}$ the element $f^{-1}g_1(f^{-1}g_2)^{-1} = f^{-1}g_1g_2^{-1}f$ is not a derangement (as it is a conjugate of a non-derangement) we can conclude that \mathcal{F}' is an intersecting set of G . \square

The following observation regarding intersection density of doubly transitive permutation groups was made in [12].

PROPOSITION 2.4. [12, Lemma 2.1(3)] *If G is a doubly transitive permutation group then $\rho(G) = 1$.*

The following result proved in [12] shows that it suffices to consider minimal transitive subgroups when searching for the maximum value of \mathcal{I}_n .

PROPOSITION 2.5. [12, Lemma 6.5] *If $H \leq G$ are transitive permutation groups then $\rho(G) \leq \rho(H)$.*

PROPOSITION 2.6. *Let G be a transitive permutation group acting on a set V and containing a semiregular subgroup H with k orbits on V . Then $\rho(G) \leq k$. In particular, if H is regular then $\rho(G) = 1$.*

Proof. Since H is semiregular it follows that for any two different elements $g, h \in H$ the element $gh^{-1} \in H$ is a derangement. This implies that $H \subseteq V(\Gamma_G)$ induces a clique in Γ_G of size $|H|$. Consequently, Proposition 2.2 implies that

$$\alpha(\Gamma_G) \leq \frac{|V(\Gamma_G)|}{|H|} = \frac{|G|}{|H|}, \quad \text{and so} \quad \rho(G) = \frac{\alpha(\Gamma_G)}{|G_v|} \leq \frac{|G|}{|H||G_v|} = \frac{|V|}{|H|} = k.$$

If H is regular (that is, if $k = 1$) then the above inequality gives $\rho(G) \leq 1$. But as observed in the introduction the intersection density is at least 1 for any permutation group, and so we conclude that in this case $\rho(G) = 1$. \square

By the above proposition every transitive permutation group admitting a regular subgroup has the EKR-property. Trivial examples of permutation groups with the strict-EKR-property are regular permutation groups. Observe that a transitive permutation group G admitting a regular subgroup of index 2 also has the strict-EKR-property. Namely, if \mathcal{F} is a maximum intersecting set of G containing 1 and $f \in \mathcal{F} \setminus \{1\}$, then f fixes a point v , and therefore $\{1, f\} = G_v$ (since stabilizers have order 2). This shows that every generalized dihedral group has the strict-EKR-property. The same idea cannot be generalized to cases where G has a regular subgroup of index greater than 2. For example, consider $G = A_4$ acting on $\{1, 2, 3, 4\}$. Then G has the EKR-property, as it admits a regular subgroup $\{id, (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ of index 3. However, G does not have the strict-EKR-property since $\{id, (132), (142)\}$ is a maximum non-canonical intersecting set.

In the example below we show that the action of S_4 on 2-element subsets of $\{1, 2, 3, 4\}$ has the EKR-property but not the strict-EKR-property, while the action of A_4 on the same set does not have the EKR-property.

EXAMPLE 2.7. Let $G = S_4$ acting on the set of all 2-element subsets of $\{1, 2, 3, 4\}$. Observe that $B = \{\{1, 2\}, \{3, 4\}\}$ is a block of size 2 for G that induces a complete imprimitivity block system \mathcal{B} with 3 blocks of size 2. The kernel of the action of G on \mathcal{B} is

$$K = \{id, (12)(34), (13)(24), (14)(23)\}.$$

Observe that $\{id, (1234), (132), (142), (1243)\}$ is a clique of size 5 in the derangement graph Γ_G . It follows that $\alpha(\Gamma_G) \leq |V(\Gamma_G)|/\omega(\Gamma_G) \leq 24/5$, and since $\alpha(\Gamma_G)$ is an integer, we have $\alpha(\Gamma_G) \leq 4 = |G_v|$. This shows that G has the EKR-property, that is, $\rho(G) = 1$. Observe that K is an intersecting set of size 4 which is not canonical, and so G does not have the strict-EKR-property.

Also, since $K \leq A_4$ it follows that the action of $H = A_4$ on the set of all 2-element subsets of $\{1, 2, 3, 4\}$ has an intersecting set of size 4, and so H does not have the EKR-property. In fact $\rho(H) = 2$.

3. TRANSITIVE GROUPS OF PRIME POWER DEGREE p^k

The next lemma about intersection densities of transitive permutation groups admitting imprimitivity block systems arising from semiregular subgroups will be used in the proofs of the main results of this paper.

LEMMA 3.1. *Let G be a transitive permutation group admitting a semiregular subgroup H whose orbits form a G -invariant partition \mathcal{B} , and let \overline{G} be the permutation group induced by the action of G on \mathcal{B} . Then $\rho(G) \leq \rho(\overline{G})$.*

Proof. Let G be a transitive permutation group acting on a set V . Let $K = \text{Ker}(G \rightarrow \overline{G})$ be the kernel of the action of G on \mathcal{B} , and let \mathcal{F} be an intersecting set of G . We claim that

$$(1) \quad |\mathcal{F} \cap gH| \leq 1 \text{ for every } g \in G.$$

Let $x, y \in \mathcal{F} \cap gH$. Then $x = gh_1$ and $y = gh_2$ for some $h_1, h_2 \in H$, and $xy^{-1} = g(h_1h_2^{-1})g^{-1}$. Since $x, y \in \mathcal{F}$, it follows that xy^{-1} fixes a point. On the other hand, xy^{-1} is a conjugate of an element $h_1h_2^{-1} \in H$, and thus since H is semiregular, it follows that $h_1 = h_2$, implying that $x = y$, proving (1).

For each $g \in G$ its induced action on \mathcal{B} is denoted by \overline{g} . We now show that $\overline{\mathcal{F}} = \{\overline{f} \mid f \in \mathcal{F}\}$ is an intersecting set of \overline{G} . Let $f, g \in \mathcal{F}$. Then fg^{-1} fixes a point v , and hence $\overline{fg^{-1}} = \overline{f}\overline{g}^{-1}$ fixes the block of \mathcal{B} that contains v , and so $\overline{\mathcal{F}}$ is indeed an intersecting set of \overline{G} .

Let $\overline{f} \in \overline{\mathcal{F}}$ and let $[f] = \{g \in \mathcal{F} \mid \overline{g} = \overline{f}\}$ be the set of all those elements in \mathcal{F} whose image under the homomorphism $G \rightarrow \overline{G}$ is equal to \overline{f} . Of course, $[f] \subseteq fK$. Writing fK as a union of $|K:H|$ cosets of H , and using (1), it follows that $[f]$ contains at most one element from each of the cosets of H , that is, $|[f]| \leq |K|/|H|$. Since

$$\mathcal{F} = \bigcup_{\overline{f} \in \overline{\mathcal{F}}} [f] \text{ it follows that } |\mathcal{F}| \leq \frac{|K||\overline{\mathcal{F}}|}{|H|}.$$

Now $\overline{\mathcal{F}}$ being an intersecting set of \overline{G} , implies that $|\overline{\mathcal{F}}| \leq \rho(\overline{G}) \cdot |\overline{G}_B|$ for $B \in \mathcal{B}$. Since \overline{G} is a transitive permutation group of degree $\frac{|V|}{|H|}$ we have that $|\overline{G}_B| = \frac{|\overline{G}||H|}{|V|}$, and so

$$|\mathcal{F}| \leq \frac{|K|}{|H|} \cdot |\overline{\mathcal{F}}| \leq \frac{|K|}{|H|} \cdot \rho(\overline{G}) \cdot |\overline{G}_B| = \frac{|K|}{|H|} \cdot \rho(\overline{G}) \cdot \frac{|\overline{G}||H|}{|V|} = \rho(\overline{G}) \cdot \frac{|K||\overline{G}|}{|V|} = \rho(\overline{G}) \cdot |G_v|.$$

Hence $|\mathcal{F}|/|G_v| \leq \rho(\overline{G})$, and since \mathcal{F} is an arbitrary intersecting set of G , it follows that $\rho(G) \leq \rho(\overline{G})$. \square

Proof of Theorem 1.4. Let G be a transitive permutation group of degree p^k , where p is a prime and $k \geq 1$, acting on a set V . Let P be a Sylow p -subgroup of G of order $|P| = p^m$. Then, by [16, Theorem 3.4], P is transitive on V . In view of Proposition 2.5 we only need to show that $\rho(P) = 1$.

The proof will be by induction on $|P| = p^m$. If $m = 1$, it follows that P is regular, hence $\rho(P) = 1$ by Proposition 2.6. Suppose that $m > 1$, and that the intersection density of every transitive p -group of order less than p^m is equal to 1. By a well-known result on p -groups, the center $Z = Z(P)$ of P is non-trivial. Observe that, since G acts faithfully on V , the group Z is semiregular on V . Moreover, Z is a normal subgroup of P , hence the orbits of Z form a P -invariant partition. Let Q be the permutation group induced by the action of P on the orbits of Z . Then Q is a transitive p -group of order less than $|P|$, hence by the induction hypothesis $\rho(Q) = 1$. Applying Lemma 3.1 it follows that $\rho(P) \leq \rho(Q) = 1$, hence $\rho(P) = 1$. \square

4. TRANSITIVE GROUPS OF DEGREE $2p$

The intersection density of transitive permutation groups of degree $2p$, p a prime, has first been addressed in [12], with the partial answer that this density is at most 2 given in [13] (see Proposition 4.1). Its proof relies on the fact that a transitive permutation group of degree $2p$, p prime, is either doubly transitive, in which case Proposition 2.4

implies that its intersection density equals 1, or it contains a derangement of order p , in which case the corresponding derangement graph contains a clique of size p , and so Proposition 2.2 applies to get that its intersection density is at most 2.

PROPOSITION 4.1. [13, Theorem 1.10] Let G be transitive permutation group of degree $2p$, p a prime, then $\rho(G) \leq 2$.

Transitive permutation groups of degree $2p$, p a prime, have received a considerable attention over the last decades, mostly within the context of vertex-transitive graphs (see [3, 4, 5, 6, 10, 14, 15]). Such a group is doubly transitive, simply primitive or it admits a complete imprimitivity block system consisting of blocks of size 2 or p . By Proposition 2.4 the intersection density of doubly transitive permutation groups is equal to 1. By the classification of finite simple groups (CFSG) the only simply primitive groups of degree twice a prime are the groups A_5 and S_5 acting on the set of pairs of a 5-element set, see [9]. (It would be of interest to produce a CFSG-free proof of this fact.)

Whereas simply primitive groups and groups admitting a complete imprimitivity block system consisting of blocks of size p are dealt with directly in the proof of Theorem 1.5, some preliminary observations are needed for groups admitting a complete imprimitivity block system with blocks of size 2. So let G be a transitive permutation group acting on the set $V = \{x_0, \dots, x_{p-1}, y_0, \dots, y_{p-1}\}$ with complete imprimitivity block system \mathcal{B} with blocks $\{x_i, y_i\}$ of size 2. We denote by $\overline{G} = G/\mathcal{B}$ the permutation group induced by the action of G on \mathcal{B} , that is, for each $g \in G$ its induced action on \mathcal{B} is denoted by \overline{g} . In this induced action the block $\{x_i, y_i\}$ is identified with i for each $i \in \mathbb{Z}_p$.

LEMMA 4.2. Let p be a prime and G be a transitive permutation group of degree $2p$ acting on a set V and having a complete imprimitivity block system \mathcal{B} with blocks of size 2 such that the kernel $K = \text{Ker}(G \rightarrow \overline{G}) \neq 1$ and the induced action $\overline{G} = G/\mathcal{B}$ is not doubly transitive. Then $\rho(G) = 1$ unless \overline{G} is cyclic and $K \leq \text{Alt}(V)$, in which case $\rho(G) = 2$.

Proof. Let $K = \text{Ker}(G \rightarrow \overline{G})$. Suppose first that K contains an odd permutation k . Let $c \in G$ be such that c acts on \mathcal{B} as a p -cycle. Then c is of order p or $2p$. If c is of order $2p$, then $\langle c \rangle$ is a regular subgroup of G , and it follows that $\rho(G) = 1$ by Proposition 2.6. If c is of order p , then it is easy to see that ck is of order $2p$ since k swaps elements inside a block of \mathcal{B} an odd number of times. Hence $\langle ck \rangle$ is a regular subgroup of G , and again $\rho(G) = 1$ by Proposition 2.6. We may therefore assume that $K \leq \text{Alt}(V)$.

Suppose first that \overline{G} is cyclic. Then $|G| = |K||\overline{G}| = p|K|$. Since $K \leq \text{Alt}(V)$, K contains no derangement, and so K is an intersecting set. Since $|K| = 2|G_v|$ it follows that $\rho(G) \geq 2$, and so, by Proposition 4.1, $\rho(G) = 2$.

Suppose now that \overline{G} is not cyclic. Since \overline{G} is not doubly transitive group of degree p , it follows that $\overline{G} = \langle a \rangle \rtimes \langle b \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_d$ for some divisor $d \geq 2$ of $p - 1$, where we assume that b fixes 0 and has all other cycles of length d in its cycle decomposition. Let \mathcal{F} be an intersecting set of G . By Proposition 2.3 we may assume that $1 \in \mathcal{F}$, and so no element of \mathcal{F} is a derangement. In particular, every element of \mathcal{F} must fix at least one block in \mathcal{B} , and so we can express \mathcal{F} as a union of disjoint sets

$$\mathcal{F} = (\mathcal{F} \cap K) \cup \mathcal{F}_0 \cup \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{p-1},$$

where $\mathcal{F}_i = \{f \in \mathcal{F} \mid \text{fix}(\overline{f}) = \{i\}\}$. Namely, for $k \in K$ we have $\text{fix}(\overline{k}) = \mathbb{Z}_p$, whereas a non-identity element of $\overline{G} \cong \mathbb{Z}_p \rtimes \mathbb{Z}_d$ can have at most one fixed point. Consequently an element of G can belong to at most one of the sets \mathcal{F}_i .

Suppose that $\mathcal{F}_i \neq \emptyset$, for some $i \in \mathbb{Z}_p$. Since no element of \mathcal{F} is a derangement it follows that for each $f \in \mathcal{F}_i$ we have $\text{fix}(f) = \{x_i, y_i\}$. Let $\sigma \in G$ be such that $\bar{\sigma} = b$. In particular, $\sigma \notin K$, and $\bar{\sigma}$ fixes only the block $\{x_0, y_0\}$ and is of order d . Consequently, either $\text{fix}(\sigma) = \emptyset$ or $\text{fix}(\sigma) = \{x_0, y_0\}$. We may assume that $\text{fix}(\sigma) = \{x_0, y_0\}$ for if $\text{fix}(\sigma) = \emptyset$ we can multiply σ with an element of $k \in K$ interchanging x_0 and y_0 (such an element exists since $K \neq 1$), and so $\overline{\sigma k} = b$ and $\text{fix}(\sigma k) = \{x_0, y_0\}$. Choose $\pi \in G$ in such a way that $\bar{\pi} = a$, and let $f_i = \pi^i \sigma \pi^{-i}$. Then $\text{fix}(f_i) = \{x_i, y_i\}$ and $\text{fix}(\overline{f_i}) = \{i\}$. Let $K_i = \{k \in K \mid x_i, y_i \in \text{fix}(k)\}$.

CLAIM 4.3. $\mathcal{F}_i \subseteq f_i K_i \cup f_i^2 K_i \cup \dots \cup f_i^{d-1} K_i$, for every $i \in \mathbb{Z}_p$.

Let $f \in \mathcal{F}_i$ be arbitrary. Then $\text{fix}(\overline{f}) = \{i\}$, which together with the fact that $\overline{G} \cong \mathbb{Z}_p \rtimes \mathbb{Z}_d$ implies that $\overline{f} \in \langle \overline{f_i} \rangle$. Therefore $\overline{f} = \overline{f_i}^t$ for some $t \in \mathbb{Z}_d$, and so $\overline{f_i^{-t} f} = 1$ in \overline{G} . Hence $f_i^{-t} f \in K$, and so $f \in f_i^t K$. Since f is not contained in K we therefore have $f = f_i^t k$ for $t \neq 0$ and $k \in K$. Observe that f as an element of \mathcal{F}_i must have a fixed point, and so $\text{fix}(f) = \{x_i, y_i\}$. Consequently, since $\text{fix}(f_i) = \{x_i, y_i\}$ it follows that k must also fix x_i and y_i . This implies that $f = f_i^t k \in f_i^t K_i$, completing the proof of Claim 4.3.

CLAIM 4.4. If $\mathcal{F} \cap f_i^t K_i \neq \emptyset$ (with $i \in \mathbb{Z}_p$ and $t \in \mathbb{Z}_d \setminus \{0\}$), then $\mathcal{F} \cap f_j^t K_j = \emptyset$ for $j \in \mathbb{Z}_p \setminus \{i\}$.

Let $f = f_i^t k_i \in \mathcal{F} \cap f_i^t K_i$ for some $k_i \in K_i$, and let $g = f_j^t k_j \in \mathcal{F} \cap f_j^t K_j$ for some $j \in \mathbb{Z}_p$ and $k_j \in K_j$. Since \mathcal{F} is an intersecting set, it follows that fg^{-1} has a fixed point. Since K is normal in G we have that $fg^{-1} = f_i^t k_i k_j^{-1} f_j^{-t} = f_i^t f_j^{-t} k$ for some $k \in K$. Consequently, $\overline{fg^{-1}} = \overline{f_i^t f_j^{-t}}$. Observe that $\overline{f_i^t f_j^{-t}} = (\pi^i \sigma^t \pi^{-i})(\pi^j \sigma^{-t} \pi^{-j}) = (\pi^i \sigma^t \pi^{-i}) \pi^{j-i} (\pi^i \sigma^{-t} \pi^{-i}) \pi^{i-j}$, which implies that $\overline{f_i^t f_j^{-t}}$ belongs to the commutator subgroup $[\overline{G}, \overline{G}]$ of \overline{G} . But $[\overline{G}, \overline{G}] = \langle \overline{\pi} \rangle = \langle a \rangle$. By assumption fg^{-1} is not a derangement, and so it follows that $\overline{fg^{-1}} = 1$, that is, $\overline{f} = \overline{g}$. Recall that $\text{fix}(\overline{f}) = \text{fix}(\overline{f_i^t}) = \{i\}$ and $\text{fix}(\overline{g}) = \text{fix}(\overline{f_j^t}) = \{j\}$. It follows that $i = j$, completing the proof of Claim 4.4.

CLAIM 4.5. If exactly m of the sets \mathcal{F}_i are non-empty, then $|\mathcal{F} \cap K| \leq \frac{|K|}{2^m}$.

Let $W = \{i \in \mathbb{Z}_p \mid \mathcal{F}_i \neq \emptyset\}$ and let $|W| = m$. Let $f \in \mathcal{F} \cap K$ be arbitrary and take $g \in \mathcal{F}_i$, $i \in W$. Since g is not a derangement we have that $\text{fix}(g) = \{x_i, y_i\}$. But by assumption gf^{-1} must fix a point, and so $\text{fix}(gf^{-1}) = \{x_i, y_i\}$. Consequently, f must also fix x_i and y_i . This shows that $\mathcal{F} \cap K \subseteq K_i$ for every $i \in W$. It follows that $\mathcal{F} \cap K \subseteq \bigcap_{i \in W} K_i = K_{(W)}$, where $K_{(W)} = \{k \in K \mid k(x_i) = x_i \text{ for every } i \in W\}$. It is easy to see that $|K_{(W)}| = \frac{|K|}{2^m}$, and so $|\mathcal{F} \cap K| \leq \frac{|K|}{2^m}$, proving Claim 4.5.

In the rest of the proof we distinguish two cases. If $\mathcal{F}_i = \emptyset$ for every $i \in \mathbb{Z}_p$ then $\mathcal{F} \subseteq K$, and so $|\mathcal{F}| \leq |K| \leq \frac{|K| \cdot d}{2} = |G_v|$, where the second inequality holds since $d \geq 2$. We conclude that $\rho(G) = 1$.

Suppose now that $\mathcal{F}_i \neq \emptyset$ for some $i \in \mathbb{Z}_p$. Recall that $\mathcal{F} = (\mathcal{F} \cap K) \cup \mathcal{F}_0 \cup \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{p-1}$. By Claim 4.3 it follows that

$$\begin{aligned} \mathcal{F} \subseteq & (\mathcal{F} \cap K) \cup (\mathcal{F} \cap (f_0 K_0 \cup f_1 K_1 \cup \dots \cup f_{p-1} K_{p-1})) \\ & \cup (\mathcal{F} \cap (f_0^2 K_0 \cup f_1^2 K_1 \cup \dots \cup f_{p-1}^2 K_{p-1})) \\ & \dots \\ & \cup (\mathcal{F} \cap (f_0^{d-1} K_0 \cup f_1^{d-1} K_1 \cup \dots \cup f_{p-1}^{d-1} K_{p-1})). \end{aligned}$$

Claim 4.4 implies that $|\mathcal{F} \cap (f_0^t K_0 \cup f_1^t K_1 \cup \dots \cup f_{p-1}^t K_{p-1})| \leq |f_i^t K_i| = |K_i| = |K|/2$. Therefore $|\mathcal{F}| \leq |\mathcal{F} \cap K| + (|K|/2)(d-1)$. Since at least one of the sets \mathcal{F}_i is non-empty,

Claim 4.5 implies that $|\mathcal{F} \cap K| \leq \frac{|K|}{2}$. Consequently, $|\mathcal{F}| \leq \frac{|K|}{2} \cdot d = |G_v|$, and so $\rho(G) = 1$. Completing the proof of Lemma 4.2. \square

COROLLARY 4.6. *Let G be a transitive group of degree $2p$ acting on a set V that admits a complete imprimitivity block system \mathcal{B} with blocks of size 2 such that the kernel $K = \text{Ker}(G \rightarrow \overline{G}) \neq 1$ and $\overline{G} = G/\mathcal{B} \cong \mathbb{Z}_p \rtimes \mathbb{Z}_d$ is not doubly transitive. If $K \leq \text{Alt}(V)$ then G has EKR-property if and only if $d > 1$ and G has the strict-EKR-property if and only if $d > 2$.*

Proof. The claim regarding EKR-property follows directly from Lemma 4.2, since G has EKR-property if and only if $\rho(G) = 1$. If $d = 2$ then $\mathcal{F} = K$ is a maximum intersecting set which is not canonical, implying that G does not have the strict-EKR-property.

Suppose that $d > 2$. Then $|K| < |G_v|$, hence a maximum intersecting set cannot be contained in K . From the proof of Lemma 4.2, it follows that the size of an intersecting set \mathcal{F} is at most $\frac{|K|}{2^m} + \frac{|K|}{2}(d-1)$, where m is the number of sets \mathcal{F}_i (defined in the proof of Lemma 4.2) that are non-empty. Observe that $\frac{|K|}{2^m} + \frac{|K|}{2}(d-1) = \frac{d|K|}{2} = |G_v|$ if and only if $m = 1$. It follows that a maximum intersecting set \mathcal{F} equals $K_i \cup f_i K_i \cup \dots \cup f_i^{d-1} K_i = G_{x_i}$, implying that G has the strict-EKR-property. \square

REMARK 4.7. Note that Example 2.7 is the special case of the situation described in Corollary 4.6 with $p = 3, d = 2, |K| = 4$ for S_4 and $p = 3, d = 1, |K| = 4$ for A_4 .

The following result, which can be extracted from [10, Theorem 6.2] and [11, Lemma 3.4], will be needed in the proof of Theorem 1.5.

PROPOSITION 4.8 ([11, Lemma 3.4]). *Let G be a transitive permutation group of degree $2p$, p a prime, admitting a complete imprimitivity block system \mathcal{B} with blocks of size 2. Then either G also admits blocks of size p , or for any pair $B, B' \in \mathcal{B}$ there exists $g \in K = \text{Ker}(G \rightarrow \overline{G})$ fixing B pointwise and B' setwise but not pointwise.*

We are now ready to prove the main result of this paper.

Proof of Theorem 1.5. Let p be an odd prime, V a set of cardinality $2p$ and G a transitive permutation group acting on V . Since every transitive permutation group of degree mp , where $m \leq p$, contains an (m, p) -semiregular element (see, for example, [10, Theorem 3.6]) it follows that G contains a $(2, p)$ -semiregular automorphism π . Let $P = \langle \pi \rangle$ and let O and O' be the two orbits of P . Applying Proposition 2.6 for the semiregular subgroup P we conclude that $\rho(G) \leq 2$ (see also Proposition 4.1).

Suppose first that G is primitive. Then by CFSG either G is doubly transitive or $p = 5$ and G is isomorphic to A_5 or S_5 acting on a 10-element set of pairs of $\{1, 2, 3, 4, 5\}$. In the first case $\rho(G) = 1$ by Proposition 2.4. As for the second case it was calculated in [13] that $\rho(G) = 2$ if $G = A_5$ and $\rho(G) = 1$ if $G = S_5$. In fact it can be seen that for $G = A_5$ every subgroup $A_4 \leq A_5$ gives rise to an intersecting set of cardinality 12, forcing $\rho(G) = 2$. On the other hand, if $G = S_5$ then in the associated derangement graph a clique of size 10 is obtained from the union of a Sylow 5-subgroup and the coset of this subgroup containing an element of order 4 normalizing this subgroup (see also the more general argument in the next paragraph).

Suppose now that G is imprimitive with \mathcal{B} as the corresponding complete imprimitivity block system. Clearly, \mathcal{B} either consists of two blocks of size p or p blocks of size 2. In the first case $\mathcal{B} = \{O, O'\}$, and Lemma 3.1 implies that $\rho(G) \leq \rho(\overline{G}) = 1$ where $\overline{G} \cong S_2$ is the induced action of G on \mathcal{B} .

We may therefore assume that \mathcal{B} consists of blocks of size 2 and that, furthermore, G admits no blocks of size p . Then, by Proposition 4.8, the kernel $K = \text{Ker}(G \rightarrow \overline{G})$ is

non-trivial. If \overline{G} is not doubly transitive, the result follows by Lemma 4.2. Namely, in this case the condition that \overline{G} is cyclic and that $K \leq \text{Alt}(V)$ is equivalent to part (i) of Theorem 1.5. If \overline{G} is doubly transitive then, by Lemma 2.1, a Sylow p -subgroup P of \overline{G} is strictly contained in $N = N_{\overline{G}}(P)$. Consequently, the preimage H of N under the homomorphism $G \rightarrow \overline{G}$ is a transitive permutation group of degree $2p$ satisfying the assumptions of Lemma 4.2. Since $\overline{H} = N$ is not cyclic it follows that $\rho(H) = 1$. Now Proposition 2.5 implies that $\rho(G) = 1$, too, completing the proof of Theorem 1.5. \square

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