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Christian Steinert

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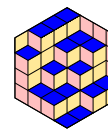
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A diagrammatic approach to string polytopes

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ABSTRACT We prove that for every complex classical group G the string polytope associated to a special reduced decomposition and any dominant integral weight λ will be a lattice polytope if and only if the highest weight representation of the Lie algebra of G with highest weight λ integrates to a representation of G itself. This affirms an earlier conjecture and shows that every partial flag variety of a complex classical group admits a flat projective degeneration to a Gorenstein Fano toric variety.

1. INTRODUCTION

Rational convex polytopes play a crucial role in representation theory for different reasons. First and foremost, there are many polytopes whose lattice points parametrize bases of highest weight representations. The first such polytope for SL_{n+1} was defined by Gel'fand and Tsetlin in [11]. Berenstein and Zelevinsky defined analogous *Gel'fand–Tsetlin polytopes* for all classical Lie algebras in [4]. It should be noted that the corresponding Sp_{2n} -polytopes have been constructed before by Zhelobenko in [23]. These constructions lead to the definition of *string polytopes* by Littelmann in [15]. In contrast to the previous polytopes, these string polytopes give many different parametrizations of bases depending on the choice of a reduced decomposition of the longest word of the Weyl group. They are the main objects of our studies.

It should be noted that there exists a different *string polytope* by Nakashima and Zelevinsky [17] that will not be discussed in this paper. Other famous polytopes have been defined by Lusztig in [16], by Feigin, Fourier and Littelmann in [8] and [9] as well by Gornitskii in [12] and [13]. The latter four polytopes are based on a conjecture by Vinberg.

Although all of these polytopes are united in the fact that their lattice points give parametrizations of bases of highest weight representations, they exhibit different combinatorial properties. Even the most basic question whether a certain polytope is a lattice polytope, yields different answers for the aforementioned polytopes. In fact, even for string polytopes associated to *nice* decompositions (see [15, Section 4]), this question is highly non-trivial.

It has been conjectured before by Alexeev and Brion that for SL_{n+1} every string polytope will be a lattice polytope independent of the reduced decomposition and the weight (see [1, Conjecture 5.8]). Explicit calculations show that this conjecture holds

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for $n \leq 4$. However, in [22, Example 5.5] we were able to provide counterexamples for SL_6 and SL_7 that can be extended to arbitrary rank.

Even worse, one of the very few integrality results for string polytopes [1, Theorem 4.5] is faulty (see [22, Remark 5.9]). However, based on calculations we were able to state a weaker conjecture (see [22, Conjecture 5.10]).

Before stating this conjecture, let us fix some notation. Let G be a complex classical group with Lie algebra $\text{Lie } G$. We write Λ^+ for the set of dominant integral weights of $\text{Lie } G$. Let ϵ_i denote the projection of a matrix in $\text{Lie } G$ onto its i -th diagonal entry. We enumerate the simple roots of $\text{Lie } G$ as $\alpha_i = \epsilon_i - \epsilon_{i-1}$ for $1 \leq i \leq n-1$ and

$$\alpha_n = \begin{cases} \epsilon_n - \epsilon_{n+1} & \text{if } G = SL_{n+1}, \\ \epsilon_n & \text{if } G = SO_{2n+1}, \\ 2\epsilon_n & \text{if } G = Sp_{2n}, \\ \epsilon_{n-1} + \epsilon_n & \text{if } G = SO_{2n}. \end{cases}$$

For simplicity, we will say that the nice decompositions that have been constructed in [15, Sections 5, 6 and 7] are *standard* (see also Definition 2.10).

CONJECTURE 1.1. *Let G be a complex classical group, let $\lambda \in \Lambda^+$ and let w_0^{std} be the standard reduced decomposition of the longest word of the Weyl group of G as stated in [15]. Then the standard string polytope $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ is a lattice polytope if and only if one of the following conditions holds.*

- (i) $G = SL_{n+1}$,
- (ii) $G = SO_{2n+1}$ and $\langle \lambda, \alpha_n^\vee \rangle \in 2\mathbb{Z}$,
- (iii) $G = Sp_{2n}$ or
- (iv) $G = SO_{2n}$ and $\langle \lambda, \alpha_{n-1}^\vee \rangle + \langle \lambda, \alpha_n^\vee \rangle \in 2\mathbb{Z}$ or $n < 4$.

This paper is dedicated to proving said conjecture. By standard results from representation theory (see for example [19, Chapter 10, Theorem 6.1] and [19, Chapter 11, Theorem 6.6]) the following formulation is equivalent.

THEOREM 1.2. *Let G be a complex classical group and λ a dominant integral weight of its Lie algebra $\text{Lie } G$. Then the standard string polytope $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ (in the sense of [15]) is a lattice polytope if and only if the $(\text{Lie } G)$ -representation on $V(\lambda)$ integrates to a representation of G .*

It should be noted that this result has been proved before in types A_n and C_n since the corresponding string polytopes are unimodularly equivalent to the respective Gel'fand–Tsetlin polytopes. These polytopes can be realized as *marked order polytopes*, defined by Ardila, Bliem and Salazar in [2], which readily yields the claim. Alternatively, a result on marked order polytopes by Fang and Fourier could also be used [7].

However, these results do not prove the orthogonal cases (completely). In type B_n one only gets one implication of Theorem 1.2. In type D_n there exists no affine bijection from the string polytope to the Gel'fand–Tsetlin polytope. There exists only a piecewise affine bijection (see [15, Section 7]) which need not preserve vertices. Additionally the Gel'fand–Tsetlin polytope in type D_n is not realized as a marked order polytope.

Our proof will give a visual tool to classify vertices of marked order polytopes via directed graphs that we will call *identity diagrams*. This classification proves the claim of Theorem 1.2 in types A_n , B_n and C_n . The concrete statement is the following.

THEOREM 1.3. *A point in a marked order polytope is a vertex if and only if each connected component in its identity diagram contains a marked element.*

This result is not new. Rather, it is a graphical version of a result by Pegel [18, Proposition 3.2]. But we will reprove it to give an easier understanding of our work in type D_n .

For type D_n we will construct slight deviations of the usual Gel'fand–Tsetlin polytopes. These *tweaked Gel'fand–Tsetlin polytopes* are slightly more complicated as they are realized in a bigger ambient space, creating artificial redundancies in the coordinates. However, we will be able to show that they are better suited for our purpose than the usual Gel'fand–Tsetlin polytopes.

THEOREM 1.4. *Let $G = SO_{2n}$. For every dominant integral weight of the Lie algebra of G , there exists an affine bijection between $\mathbb{R}^{n(n-1)}$ and an affine subspace of $\mathbb{R}^{n^2-2}$ that sends the string polytope onto the tweaked Gel'fand–Tsetlin polytope.*

Furthermore, a point in the tweaked Gel'fand–Tsetlin polytope will correspond to a lattice point in the string polytope if and only if its coordinates are either completely contained in $\mathbb{Z}$ or completely contained in $\mathbb{Z} + \frac{1}{2}$.

These new polytopes will still not be marked order polytopes. But we will construct an analogue of identity diagrams that fulfill the same purpose. These *tweaked Gel'fand–Tsetlin diagrams* give a visual tool to determine whether a given point in the tweaked Gel'fand–Tsetlin polytope is a vertex, although the criterion is slightly more complicated. Ultimately, this yields to the proof of the claim of Theorem 1.2 for D_n .

Together with our main result in [22], this will imply the following.

THEOREM 1.5. *Let G be a complex classical group and λ a dominant integral weight. The standard string polytope $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ (in the sense of [15]) is a reflexive polytope after translation by a lattice vector if and only if λ is the weight of the anticanonical line bundle over some partial flag variety G/P .*

This is important because Batyrev proved in [3] that reflexive polytopes are in one-to-one correspondence with Gorenstein Fano toric varieties. Additionally, Alexeev and Brion were able to construct toric degenerations of spherical varieties using string polytopes in [1]. So we will reach the following conclusion.

THEOREM 1.6. *Let G be a complex classical group. Then every partial flag variety of G admits a flat projective degeneration to a toric Gorenstein Fano variety.*

This observation could lead to a potential application of Batyrev's mirror symmetry construction in [3] to partial flag varieties.

The structure of this paper is as follows. Before proving Theorem 1.2, we will recall the most important definitions regarding string and Gel'fand–Tsetlin polytopes in Section 2 and marked order polytopes in Section 3. We will then construct our identity diagrams in Section 4 and prove Theorem 1.3, which yields to the proof of Theorem 1.2 for types A_n , B_n and C_n . As another potential application we will describe an algorithm that delivers all vertices of a given string polytope in Section 6. In Sections 7 and 8 we will construct our modified Gel'fand–Tsetlin patterns in type D_n and their corresponding diagrams, which leads to the final proof of Theorem 1.2 in Section 9. We will conclude our paper with proofs of Theorem 1.5 and Theorem 1.6 in Section 10.

2. STRING POLYTOPES AND GEL'FAND–TSETLIN POLYTOPES

Let us start by recalling the definition of so-called (*generalized*) *Gel'fand–Tsetlin patterns* introduced by Berenstein and Zelevinsky in [4]. We will widely stick to the notation in [15] although we will make slight adjustments.

NOTATION 2.1. For two numbers $a, b \in \mathbb{R}$ the inequality $a \geq b$ will be written graphically as

$$\begin{array}{ccc} a & & b \\ & \text{or} & \\ b & & a \end{array}.$$

We will now define Gel'fand–Tsetlin patterns for all classical types.

DEFINITION 2.2 (Gel'fand–Tsetlin patterns in type A_n). *Let $G = SL_{n+1}$ and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$ (as before, ϵ_i denotes the projection of a matrix onto its i -th diagonal entry). A Gel'fand–Tsetlin pattern of type λ is a tuple $(y_{i,j}) \in \mathbb{R}^{\frac{n(n+1)}{2}}$, $1 \leq i \leq j \leq n$, such that the coordinates fulfill the relations in Fig. 1.*

$$\begin{array}{cccccccc} \lambda_1 & & \lambda_2 & & \dots & & \dots & & \dots & & \lambda_n & & 0 \\ & y_{1,1} & & y_{1,2} & & \dots & & \dots & & y_{1,n-1} & & y_{1,n} \\ & & y_{2,2} & & y_{2,3} & & \dots & & y_{2,n-1} & & y_{2,n} \\ & & & \ddots & & \ddots & & \ddots & & \ddots & & \\ & & & & y_{n-2,n-2} & & y_{n-2,n-1} & & y_{n-2,n} & & \\ & & & & & y_{n-1,n-1} & & y_{n-1,n} & & \\ & & & & & & y_{n,n} & & \end{array}$$

FIGURE 1. Inequalities of Gel'fand–Tsetlin patterns in type A_n .

REMARK 2.3. Notice that in Littelmann's definition of type A_n Gel'fand–Tsetlin patterns, the top row would be included in the tuple $(y_{i,j})$ as the initial row (i.e. $y_{0,0} = \lambda_1, \dots, y_{0,n-1} = \lambda_n, y_{0,n} = 0$). However, for fixed λ this does only change the embedding of the pattern and not the pattern itself. So we adapted the definition to embed our patterns in a vector space whose dimension equals the number of positive roots of the algebraic group G . Additionally, these entries have a different character (we will call these entries a *marking* later on), so we would like to treat them separately.

DEFINITION 2.4 (Gel'fand–Tsetlin patterns in types B_n and C_n). *Let $G = SO_{2n+1}$ or Sp_{2n} and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. A Gel'fand–Tsetlin pattern of type λ is a pair $(\mathbf{y}, \mathbf{z})$ of tuples $\mathbf{y} = (y_{i,j}) \in \mathbb{R}^{\frac{n(n-1)}{2}}$, $2 \leq i \leq j \leq n$, and $\mathbf{z} = (z_{i,j}) \in \mathbb{R}^{\frac{n(n+1)}{2}}$, $1 \leq i \leq j \leq n$, such that the coordinates fulfill the relations in Fig. 2.*

REMARK 2.5. Notice that in a Gel'fand–Tsetlin pattern $(\mathbf{y}, \mathbf{z})$ of type B_n or C_n the first row as well as the zeroes in the last column are not actually part of the tuple $(\mathbf{y}, \mathbf{z})$. In Littelmann's definition, the first row would be included as $y_{1,1} = \lambda_1, \dots, y_{1,n} = \lambda_n$. The reasons for our change of definition are the same as in type A_n . Otherwise we want to stick with his notation, which yields to the awkward fact that our tuple $(\mathbf{y})$ starts with the index $i = 2$. However, we will not need these indices explicitly, so this should not become a problem.

DEFINITION 2.6 (Gel'fand–Tsetlin patterns in type D_n). *Let $G = SO_{2n}$ and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. A Gel'fand–Tsetlin pattern of type λ is a pair $(\mathbf{y}, \mathbf{z})$ of tuples $\mathbf{y} = (y_{i,j}) \in \mathbb{R}^{\frac{n(n-1)}{2}}$, $2 \leq i \leq j \leq n$, and $\mathbf{z} = (z_{i,j}) \in \mathbb{R}^{\frac{n(n-1)}{2}}$, $1 \leq i \leq j \leq n-1$, such*

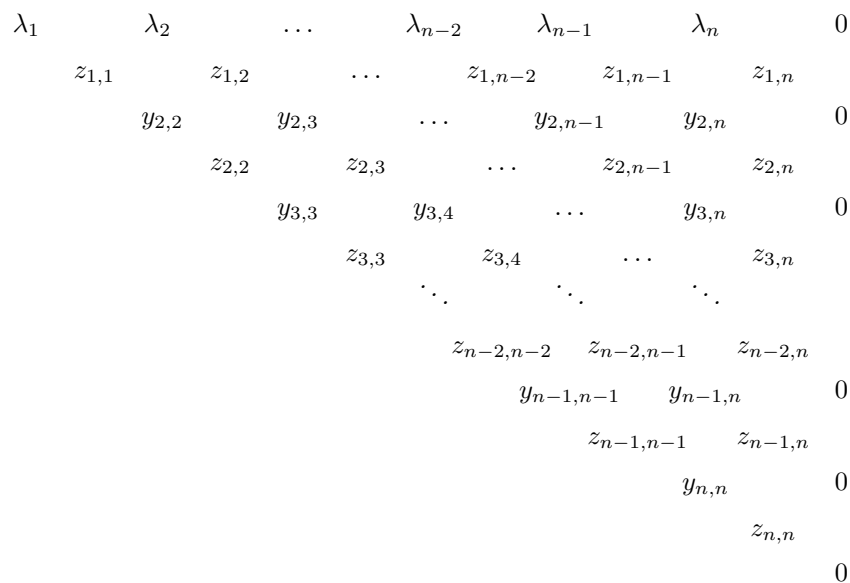


FIGURE 2. Inequalities of Gel'fand–Tsetlin patterns in types B_n and C_n .

that the coordinates fulfill the relations in Fig. 3 and

$$\begin{aligned} z_{1,n-1} &\leq \lambda_n + y_{2,n} + \min\{\lambda_{n-1}, y_{2,n-1}\}, \\ z_{i,n-1} &\leq y_{i,n} + y_{i+1,n} + \min\{y_{i,n-1}, y_{i+1,n-1}\} \text{ for all } 2 \leq i \leq n-2, \\ z_{n-1,n-1} &\leq y_{n-1,n} + y_{n,n} + y_{n-1,n-1}. \end{aligned}$$

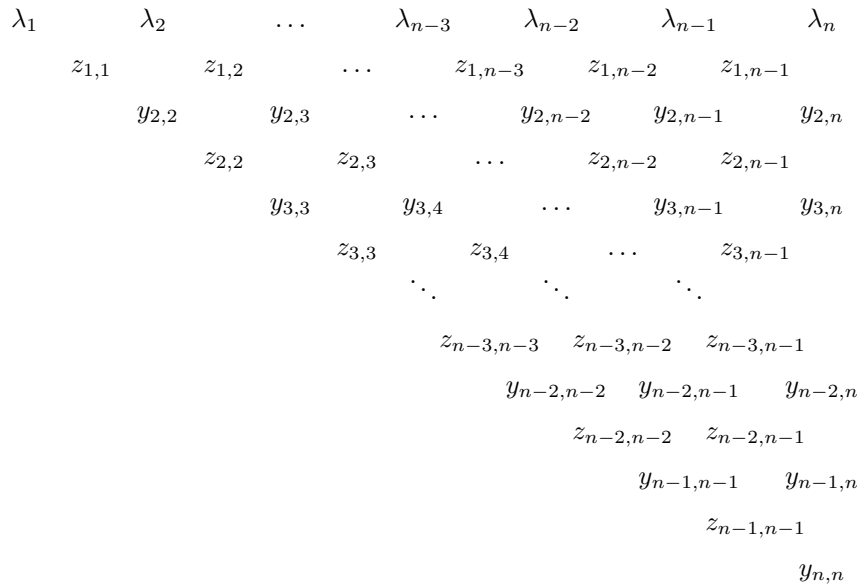


FIGURE 3. Inequalities of Gel'fand–Tsetlin patterns in type D_n .

REMARK 2.7. As before we deviate from Littelmann’s notation by not including the initial row containing the λ_j in our notion of a Gel’fand–Tsetlin pattern. For unified notation of the additional inequalities, we will sometimes write $y_{1,j}$ for λ_j .

One can see quite easily that the set of all Gel’fand–Tsetlin patterns for a given weight is a polytope.

DEFINITION 2.8. Let G be a complex classical group of type X_n and let $\lambda \in \Lambda^+$. Let N denote the number of positive roots of G . The set of all possible Gel’fand–Tsetlin patterns of type λ is called the Gel’fand–Tsetlin polytope $GT_{X_n}(\lambda) \subseteq \mathbb{R}^N$ of X_n and λ . We may omit the subscript X_n if this is clear from the context.

Sometimes we might be interested to use the original definitions instead. So we introduce the following notation.

NOTATION 2.9. An extended Gel’fand–Tsetlin pattern $\hat{x}$ is a Gel’fand–Tsetlin pattern x together with the λ_i in its top row.

We will now recall the connection between Gel’fand–Tsetlin polytopes and standard string polytopes. These are the string polytopes associated to the following reduced decompositions, which are called *nice decompositions* in [15]. Notice that we use Bourbaki enumeration of the simple roots – in contrast to [15].

DEFINITION 2.10.

$$\begin{aligned} \underline{w}_{SL_{n+1}}^{\text{std}} &= s_n(s_{n-1}s_n) \cdots (s_1 \cdots s_n), \\ \underline{w}_{SO_{2n+1}}^{\text{std}} &= \underline{w}_{Sp_{2n}}^{\text{std}} = s_n(s_{n-1}s_n s_{n-1}) \cdots (s_1 \cdots s_n \cdots s_1) \end{aligned}$$

or

$$\underline{w}_{SO_{2n}}^{\text{std}} = s_{n-1}s_n(s_{n-2}s_{n-1}s_n s_{n-2}) \cdots (s_1 \cdots s_{n-2}s_{n-1}s_n s_{n-2} \cdots s_1)$$

respectively. We usually write $\underline{w}_0^{\text{std}}$ instead of $\underline{w}_{0G}^{\text{std}}$ if the group G is fixed.

We will use the following non-standard terminology.

DEFINITION 2.11. Let G be a complex classical group. A Gel’fand–Tsetlin pattern x is called standard if one of the following conditions holds.

- (i) $G = SL_{n+1}$ and all coordinates of $\hat{x}$ are integral.
- (ii) $G = SO_{2n+1}$, the coordinates $z_{1,n}, \dots, z_{n,n}$ are in $\frac{1}{2}\mathbb{Z}$ and the other coordinates of $\hat{x}$ are either all integral or all are in $\frac{1}{2} + \mathbb{Z}$.
- (iii) $G = Sp_{2n}$ and all coordinates of $\hat{x}$ are integral.
- (iv) $G = SO_{2n}$ and the coordinates of $\hat{x}$ are either all integral or all are in $\frac{1}{2} + \mathbb{Z}$.

The following is a combination of [15, Corollary 5 and Corollary 7].

THEOREM 2.12 (Littelmann). Let G be a complex classical group of type $X_n \neq D_n$ and let N denote the number of positive roots. For each dominant integral weight λ there exists an affine bijection $\phi_\lambda: \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $\phi_\lambda(\underline{Q}_{\underline{w}_0^{\text{std}}}(\lambda)) = GT_{X_n}(\lambda)$.

Furthermore, ϕ_λ induces a bijection between the lattice points in $\underline{Q}_{\underline{w}_0^{\text{std}}}(\lambda)$ and the standard X_n -Gel’fand–Tsetlin patterns of type λ .

REMARK 2.13. Interestingly, in type A_n Cho, Kim, Lee and Park gave a combinatorial classification of all reduced decompositions whose string polytope is unimodularly equivalent to the Gel’fand–Tsetlin polytope in [6].

In type D_n the situation is more delicate as can be seen in [15, Corollary 9]. In this case the map ϕ_λ will only be piecewise affine.

THEOREM 2.14 (Littelmann). *Let $G = SO_{2n}$ and let N denote the number of positive roots. For each dominant integral weight λ there exists a piecewise affine bijection $\phi_\lambda: \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $\phi_\lambda(\mathcal{Q}_{w_0^{\text{std}}}(\lambda)) = GT_{D_n}(\lambda)$.*

Furthermore, ϕ_λ induces a bijection between the lattice points in $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ and the standard D_n -Gel'fand–Tsetlin patterns of type λ .

The non-affineness will lead to difficulties in proving Theorem 1.2, which we will overcome by defining new Gel'fand–Tsetlin polytopes for type D_n in Section 8.

3. MARKED ORDER POLYTOPES

We will now study generalized versions of Stanley's order polytopes. The definition of the order polytope associated to a poset is due to Stanley [20]. A *marking* on the poset leads to a generalization by Ardila, Bliem and Salazar in [2]. These polytopes have been studied by Pegel [18], Fang and Fourier [7] and others.

DEFINITION 3.1. *Let $(P, \leq)$ be a finite poset, i.e. P is a finite set with a partial order $\leq$ on P .*

- (i) *The Hasse diagram of P is a directed graph whose set of nodes is P and there is an arrow $p \rightarrow q$ whenever $p < q$ and there exists no r with $p < r < q$.*
- (ii) *A marking on P is a pair (A, λ) where A is a subset of P containing all minimal and maximal elements of P and $\lambda = (\lambda_a)_{a \in A} \in \mathbb{R}^A$ is a real vector such that $\lambda_a \leq \lambda_b$ whenever $a \leq b$. The triplet (P, A, λ) is called a marked poset. We will call the elements of A marked elements.*
- (iii) *Let (A, λ) be a marking on P . The marked order polytope $\mathcal{O}_{P,A}(\lambda)$ associated to (P, A, λ) is defined as*

$$\mathcal{O}_{P,A}(\lambda) := \left\{ x \in \mathbb{R}^{P \setminus A} \left| \begin{array}{l} x_p \leq x_q \text{ for all } p \leq q, \\ \lambda_a \leq x_p \text{ for all } a \leq p, \\ x_p \leq \lambda_b \text{ for all } p \leq b \end{array} \right. \right\}.$$

REMARK 3.2. The Gel'fand–Tsetlin polytopes of types A_n , B_n and C_n are marked order polytopes where the marking is given by the dominant integral weight and some zeroes. The Gel'fand–Tsetlin polytopes of type D_n however are not marked order polytopes because of the additional four-term inequalities in their definition.

The following theorem is due to Pegel, expanding results by Ardila, Bliem and Salazar [2, Lemma 3.5] as well as Fang and Fourier [7, Corollary 2.2]. It can be found in [18, Proposition 3.2].

THEOREM 3.3 (Pegel). *Let (P, A, λ) be any marked poset. The coordinates of every vertex of the marked order polytope $\mathcal{O}_{P,A}(\lambda)$ must lie in the set $\{\lambda_a \mid a \in A\}$. Especially, $\mathcal{O}_{P,A}(\lambda)$ is a lattice polytope if λ is integral.*

This implies the following.

COROLLARY 3.4. *Let G be a complex classical group of type $X_n \neq D_n$ and $\lambda \in \Lambda^+$. Then the Gel'fand–Tsetlin polytope $GT_{X_n}(\lambda)$ is a lattice polytope if $X_n = A_n$ or C_n or if $X_n = B_n$ and $\langle \lambda, \alpha_n^\vee \rangle \in 2\mathbb{Z}$.*

Proof. Set $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$. By Theorem 3.3 it is clear that $GT_{X_n}(\lambda)$ will be a lattice polytope if all coordinates of the marking vector are integral. The set of non-zero coordinates is a subset of $\{\lambda_1, \dots, \lambda_n\}$, so we need to know when the coefficients of a dominant integral weight written in the ϵ_i are integral. By [19, Chapter 10, Section 5.1] this is always the case in types A_n and C_n . In type B_n however, we could get half-integral coefficients. To be more precise, each λ_i can be written as the sum of some

integers plus $\frac{\langle \lambda, \alpha_n^\vee \rangle}{2}$. So we see that the λ_i are integers if and only if $\langle \lambda, \alpha_n^\vee \rangle$ is an even integer, which concludes the proof. $\square$

This corollary would allow us to prove one implication of the claims of Conjecture 1.1 and hence Theorem 1.2 for types A_n , B_n and C_n via Littelmann's affine bijection from Theorem 2.12. The reason is that the inverse of Littelmann's map sends vertices to vertices and lattice points in the Gel'fand–Tsetlin polytope (notice that these are only a proper subset of the standard Gel'fand–Tsetlin patterns in type B_n) to lattice points of the standard string polytope.

Sadly, we cannot prove the *only-if*-part in type B_n directly and we simply cannot use these methods in the case D_n . Firstly, the Gel'fand–Tsetlin polytope in type D_n is not a marked order polytope. Secondly, since Littelmann's bijection of Theorem 2.14 is only piecewise affine, it need not send vertices to vertices. Some vertices could be sent to non-vertices and vice-versa.

From examples I reached the following conjecture which would at least solve this problem. But I could not find a proof.

CONJECTURE 3.5. *Let ϕ_λ be the map from Theorem 2.14, i.e. the piecewise affine bijection with $\phi_\lambda(\mathcal{Q}_{w_0^{\text{std}}}(\lambda)) = GT_{D_n}(\lambda)$. Then ϕ_λ induces a bijection between $\text{vert } \mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ and $\text{vert } GT_{D_n}(\lambda)$.*

So we will develop a tweaked version of Gel'fand–Tsetlin patterns in type D_n that can be studied more easily. Additionally we will introduce a new method to classify vertices of these tweaked Gel'fand–Tsetlin polytopes via diagrammatic combinatorics in Section 8.

We will also apply these methods to the other classical types, thereby reproving Corollary 3.4 and additionally the missing second implication of Conjecture 1.1 in type B_n .

4. IDENTITY DIAGRAMS

We will state our definitions for arbitrary marked posets. The reductions to the Gel'fand–Tsetlin cases A_n , B_n and C_n are obvious.

DEFINITION 4.1. *Let (P, A, λ) be a marked poset and let $x \in \mathcal{O}_{P,A}(\lambda)$. The identity diagram $\mathcal{D}_{P,A}^\lambda(x)$ associated to (P, A, λ, x) is a graph that contains all nodes and arrows of the Hasse diagram of P . Additionally, we draw an arrow $q \rightarrow p$ between two nodes p and q whenever there exists an arrow $p \rightarrow q$ in the Hasse diagram of P and $x_p = x_q$ (if $p, q \notin A$) or $\lambda_p = x_q$ (if $p \in A$) or $x_p = \lambda_q$ (if $q \in A$).*

Whenever we draw these identity diagrams, for simplicity we will represent double arrows $p \rightleftarrows q$ by straight lines and omit single arrows. From this practice we get the following non-standard terminology.

DEFINITION 4.2. *Let (P, A, λ) be a marked poset, let $x \in \mathcal{O}_{P,A}(\lambda)$ and let $\mathcal{D}_{P,A}^\lambda(x)$ be the associated identity diagram. A subset $\mathcal{C}$ of nodes is called connected if it is connected via double arrows, i.e. for any two nodes p and q in $\mathcal{C}$ there exists a (possibly empty) sequence $p_1, \dots, p_t \in \mathcal{C}$ such that*

$$p \rightleftarrows p_1 \rightleftarrows \dots \rightleftarrows p_t \rightleftarrows q.$$

The maximal (with respect to inclusion) connected subsets are called connected components.

Additionally, when drawing identity diagrams for Gel'fand–Tsetlin patterns we will represent the nodes corresponding to marked elements as follows. The zeros in the rightmost column will be drawn as small circles, while the nodes corresponding to

the λ_i in the first row will be drawn as small crosses. This change is made for easier readability as the following example shows.

EXAMPLE 4.3. Let $G = Sp_6$ and $\lambda = 2\epsilon_1 + 2\epsilon_2 + \epsilon_3$. The pattern

$$\begin{array}{ccccccc} 2 & 2 & 1 & 0 & & & \\ & 2 & \frac{3}{2} & 1 & 0 & & \\ & & 2 & 1 & 0 & & \\ & & & \frac{3}{2} & 1 & 0 & \\ & & & & 1 & 0 & \\ & & & & & 0 & \\ & & & & & & 0 \end{array}$$

admits the identity diagram and its visually more appealing drawing depicted in Fig. 4.

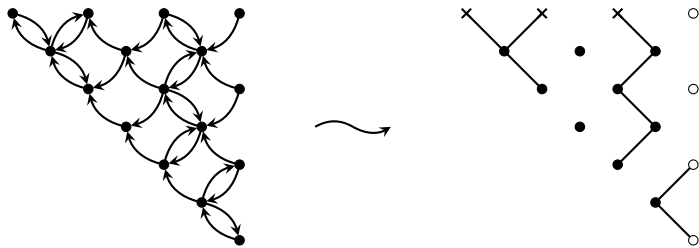


FIGURE 4. Identity diagram and visually more appealing drawing of the Gel'fand–Tsetlin pattern in Example 4.3.

These diagrams give us an easy way to *draw* vertices of marked order polytopes as the following result shows.

THEOREM (Theorem 1.3). *Let (P, A, λ) be a marked poset. A point $x \in \mathcal{O}_{P,A}(\lambda)$ is a vertex of the marked order polytope if and only if every connected component of the associated diagram $\mathcal{D}_{P,A}^\lambda(x)$ contains a marked element, i.e. an element of A .*

Although this theorem sounds rather technical, it is actually quite practical, as the following consequence shows.

COROLLARY 4.4. *A point in a Gel'fand–Tsetlin polytope of type A_n , B_n or C_n is a vertex, if each entry of its Gel'fand–Tsetlin pattern is equal to its upper left or upper right neighbor.*

To prove this result we use the following standard trick.

LEMMA 4.5. *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a convex polytope. Then $x \in \mathcal{P}$ is a vertex of $\mathcal{P}$ if and only if there does not exist a vector $v \in \mathbb{R}^d$, $v \neq 0$, such that $x + v \in \mathcal{P}$ and $x - v \in \mathcal{P}$.*

We will now prove our theorem on the vertices of marked order polytopes.

Proof of Theorem 1.3. Let x be a point in the marked order polytope $\mathcal{O}_{P,A}(\lambda)$. Suppose there exists a vector $v \in \mathbb{R}^{P \setminus A}$ such that $x + v$ and $x - v$ both lie in the marked order polytope. Let $p, q \in P \setminus A$, $p \leq q$, be two nodes of the identity diagram $\mathcal{D}_{P,A}^\lambda(x)$ such that $p \rightleftharpoons q$. Hence we know that $x_p = x_q$.

Since $x + v$ and $x - v$ lie in the marked order polytope, we must have

$$x_p + v_p \leq x_q + v_q \quad \text{and} \quad x_p - v_p \leq x_q - v_q.$$

This implies that $v_p = v_q$. Continuing this argument yields $v_p = v_q$ for any two nodes p and q of the identity diagram lying in the same connected component.

Now let $\mathcal{C}$ be a connected component of $\mathcal{D}_{P,A}^\lambda(x)$. If $\mathcal{C}$ contains an element $a \in A$, we know that $x_p = \lambda_a$ for all $p \in \mathcal{C}$. Suppose that $\mathcal{C}$ is not completely contained in A . Then there exists a pair $p \rightleftharpoons a$ with $p \in \mathcal{C} \setminus A$ and $a \in A$. Without loss of generality let us assume that $p \leq a$. Then we know that

$$x_p + v_p \leq \lambda_a \quad \text{and} \quad x_p - v_p \leq \lambda_a,$$

which implies that $v_p = 0$.

In conclusion we see that $v_p = 0$ for all $p \in P \setminus A$ such that the connected component of p contains an element of A . By Lemma 4.5 this implies that x is a vertex whenever each connected component of $\mathcal{D}_{P,A}^\lambda(x)$ contains a marked element.

For the other implication let us assume there exists a connected component $\mathcal{C}$ of $\mathcal{D}_{P,A}^\lambda(x)$ that does not contain any marked element. By definition of identity diagrams this means that $x_p < x_q$ for any $p \in \mathcal{C}$ and $q \in P \setminus (\mathcal{C} \cup A)$ such that $p \leq q$. Additionally $x_p < \lambda_a$ for any $p \in \mathcal{C}$ and $a \in A$ such that $p \leq a$. Analogous statements hold if $p \geq q$ or $p \geq a$. Since the poset is finite, we can find $\epsilon > 0$ such that

$$\begin{aligned} x_p \pm \epsilon &< x_q \text{ for all } p \in \mathcal{C} \text{ and } q \in P \setminus (\mathcal{C} \cup A) \text{ such that } p \leq q, \\ x_p \pm \epsilon &> x_q \text{ for all } p \in \mathcal{C} \text{ and } q \in P \setminus (\mathcal{C} \cup A) \text{ such that } p \geq q, \\ x_p \pm \epsilon &< \lambda_a \text{ for all } p \in \mathcal{C} \text{ and } a \in A \text{ such that } p \leq a, \\ x_p \pm \epsilon &> \lambda_a \text{ for all } p \in \mathcal{C} \text{ and } a \in A \text{ such that } p \geq a. \end{aligned}$$

Consider the vector $v \in \mathbb{R}^{P \setminus A}$ defined by

$$v_p := \begin{cases} \epsilon & \text{if } p \in \mathcal{C} \\ 0 & \text{else.} \end{cases}$$

Then it is clear that $x + v$ and $x - v$ both lie in $\mathcal{O}_{P,A}(\lambda)$. By Lemma 4.5 this implies that x is not a vertex of $\mathcal{O}_{P,A}(\lambda)$. $\square$

Hence, we recover a proof of Theorem 3.3.

Proof of Theorem 3.3. Let $x \in \mathbb{R}^{P \setminus A}$ be a vertex of $\mathcal{O}_{P,A}(\lambda)$. From Theorem 1.3 we know that every connected component $\mathcal{C}$ of the identity diagram $\mathcal{D}_{P,A}^\lambda(x)$ of x contains an element $a \in A$. By definition this means that $x_p = \lambda_a$ for all $p \in \mathcal{C}$, hence every coordinate of x must be equal to one of the λ_a . $\square$

5. INTEGRALITY OF STANDARD STRING POLYTOPES IN TYPES A_n , B_n , C_n

At first, we would like to make the connection between Conjecture 1.1 and Theorem 1.2 explicit.

REMARK 5.1. If G is simply connected, i.e. G is of type A_n or C_n , it is known (see for example [19, Chapter 10, Theorem 6.1]) that the finite-dimensional, irreducible representations of the Lie algebra of G are in one-to-one correspondence with the finite-dimensional irreducible representations of G . So there are no further restrictions on λ in these types.

If G however is not simply connected, i.e. $G = SO_n$, it is known that not every finite-dimensional irreducible representation of $\mathfrak{so}_n$ integrates to a representation of SO_n . Instead, in general it integrates only to a representation of the spin group $Spin_n$ as the universal covering of SO_n . However, in some cases $V(\lambda)$ will still integrate to a representation of SO_n . By [19, Chapter 11, Theorem 6.6] these cases are precisely the ones listed in Conjecture 1.1.

We can now (re-)prove Theorem 1.2 in three types.

Proof of Theorem 1.2 in types A_n , B_n and C_n . Let G be of types A_n , B_n or C_n . By Theorem 2.12 we know that the vertices of the standard string polytopes are in one-to-one correspondence with the vertices of the Gel'fand–Tsetlin polytopes. We have seen that the Gel'fand–Tsetlin polytopes are marked order polytopes. For $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$ their marking is given by a vector whose coordinates are precisely the λ_i and some zeros.

In types A_n and C_n we know that $\lambda_i \in \mathbb{Z}$ for every $1 \leq i \leq n$. By Theorem 3.3 we know that the vertices of the corresponding Gel'fand–Tsetlin polytope have integral coordinates, so via Littelmann's map in Theorem 2.12 they correspond to lattice points in the standard string polytope.

In type B_n the same argument holds if $\langle \lambda, \alpha_n^\vee \rangle \in 2\mathbb{Z}$. However, if $\langle \lambda, \alpha_n^\vee \rangle$ is an odd integer, we know that $\lambda_i \in \frac{1}{2} + \mathbb{Z}$ for all $1 \leq i \leq n$. Now it is enough to notice that the pattern

$$\begin{array}{cccccc} \lambda_1 & & \lambda_2 & & \dots & & \lambda_n & & 0 \\ & \lambda_2 & & \dots & & & \lambda_n & & 0 \\ & & \ddots & & \ddots & & \ddots & & 0 \\ & & & \lambda_n & & 0 & & & \vdots \\ & & & & 0 & & & & \\ & & & & & 0 & & & \\ & & & & & & \ddots & & 0 \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{array}$$

lies in $GT_{B_n}(\lambda)$ for every $\lambda \in \Lambda^+$. Its identity diagram is drawn in Fig. 5.

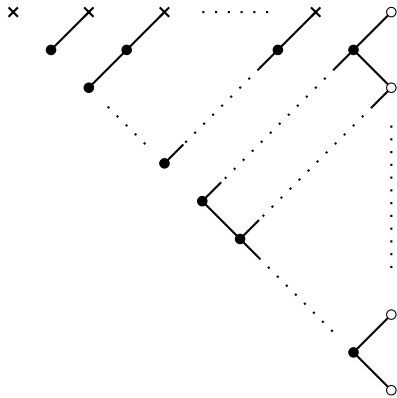


FIGURE 5. Subgraph of the identity diagram of the B_n -Gel'fand–Tsetlin pattern described in the proof of Theorem 1.2 with the usual drawing conventions for readability. If $\lambda_1 > \dots > \lambda_n$, this is precisely the identity diagram. If some of the λ_i coincide, there might be additional edges.

We see that every connected component of the identity diagram contains a node corresponding to a marked element of the poset, hence this pattern must be a vertex of $GT_{B_n}(\lambda)$ by Theorem 1.3.

By construction and since $n > 1$, the coordinate $y_{1,n} = \lambda_n$ of this pattern lies in $\frac{1}{2} + \mathbb{Z}$ while the coordinate $y_{2,n} = 0$ lies in $\mathbb{Z}$. This shows that the pattern is not standard, hence by Theorem 2.12 its preimage under Littelmann's affine bijection is not a lattice point. So we found a non-integral vertex of the standard string polytope $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ in type B_n for every $\lambda \in \Lambda^+$ such that $\langle \lambda, \alpha_n^\vee \rangle$ is odd, which concludes our proof in types A_n , B_n and C_n . $\square$

6. AN ALGORITHMIC APPLICATION

By studying identity diagrams abstractly, we can even give a complete classification of vertices of marked order polytope as the following construction in the Gel'fand–Tsetlin case shows.

CONSTRUCTION 6.1. We want to construct vertices of Gel'fand–Tsetlin polytopes diagrammatically. Fix a dominant integral weight λ and let us consider the poset $(P, A, \tilde{\lambda})$ such that $\mathcal{O}_{P,A}(\tilde{\lambda}) = GT(\lambda)$. Since the marking $\tilde{\lambda}$ is completely determined by the weight λ , we will just write λ for both of them.

An important step in the construction will be the following *completion procedure*. Let $\mathcal{G}$ be a directed graph with node set P that contains every arrow of the Hasse diagram of P . Assume that for every arrow $p \rightarrow q$ in $\mathcal{G}$ we have either an arrow $p \rightarrow q$ or an arrow $q \rightarrow p$ in the Hasse diagram. Additionally assume that $\mathcal{G}$ does not contain double arrows in the same direction. By this we mean that $p \leftrightsquigarrow q$ is allowed but $p \rightrightarrows q$ is forbidden. We say that $\mathcal{G}$ is *complete* if the following two conditions hold.

- (i) Whenever there exists a set of arrows $p \leftrightsquigarrow r \leftrightsquigarrow q$ and $p \rightarrow s \rightarrow q$, there exists a set of arrows $p \leftarrow s \leftarrow q$ as well.
- (ii) Whenever there exists a sequence of arrows $a \rightarrow p_1 \rightarrow \cdots \rightarrow p_t \rightarrow b$ with $a, b \in A$ such that $\lambda_a = \lambda_b$, there exists a reverse sequence of arrows $a \leftarrow p_1 \leftarrow \cdots \leftarrow p_t \leftarrow b$ as well.

Notice that we can complete any graph with the mentioned assumptions by repeatedly adding new arrows, but only those that are strictly necessary, until the graph is complete. Of course, we might have to check every set of arrows repeatedly since we are constantly introducing new arrows in this process. However, since we will never produce a double arrow $p \rightrightarrows q$ and P is finite, this algorithm will eventually stop. Additionally, the completion will be unique, i.e. it does not depend on the order in which we check for and, if necessary, add arrows.

Coming back to our Gel'fand–Tsetlin patterns, we can now describe a process to construct vertices of Gel'fand–Tsetlin polytopes.

We start with the Hasse diagram of the corresponding poset. First of all, the graph might not be complete. So whenever there exists an arrow $a \rightarrow p \rightarrow b$ for some $a, b \in A$ and $p \in P$ we must check whether $\lambda_a = \lambda_b$. Whenever this is the case, we must add the two arrows $b \rightarrow p \rightarrow a$ to the Hasse diagram. The resulting graph might not be complete after this initial step, so finish the completion procedure.

Since we want to create a vertex, we must add more arrows. We can freely introduce new arrows $p \rightarrow q$ whenever there exists an arrow $q \rightarrow p$ and there does not already exist an arrow $p \rightarrow q$. However, we must not add an arrow between two nodes p and q if p is connected (via a possibly empty sequence of double arrows) to an element $a \in A$ and q is connected (via a possibly empty sequence of double arrows) to an element $b \in A$ such that $\lambda_a \neq \lambda_b$. After adding an arrow, we must always complete the graph.

We must keep adding new arrows until we can no longer legally add new arrows. At that point, every vertex will be connected (via a sequence of double arrows) to at least one marked element, i.e. every connected component of the resulting diagram will contain at least one marked element.

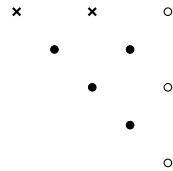
It is clear that we will always reach this stage. But of course the resulting graph is not unique. By adding different arrows, we will in general terminate in a different graph.

By construction, every terminal graph in our algorithm will be the identity graph of a Gel'fand–Tsetlin pattern. Every coordinate x_p of the pattern is given by $x_p = \lambda_a$, where a is a marked element in the connected component of p .

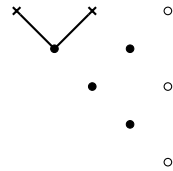
Because of Theorem 1.3 this pattern must be a vertex of the Gel'fand–Tsetlin polytope $GT(\lambda)$. Additionally, we are able to reach every vertex of $GT(\lambda)$ by this procedure (although admittedly it might take some time).

Let us apply this procedure in an example.

EXAMPLE 6.2. Let $G = SO_5$ and consider the weight $\lambda = \omega_2 = \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2$. We will use our usual convention to not draw the arrows of the Hasse diagram but remember their existence by careful positioning of the nodes. Then the Hasse diagram is drawn in the following way.



As an initial step we must search for arrows $a \rightarrow p \rightarrow b$ with $a, b \in A$ such that the marking of a and b coincides. Since $\lambda_1 = \lambda_2 = \frac{1}{2}$ we have one such path in the upper left corner. Hence we must add arrows in the opposite direction. With our usual convention to draw $\rightleftharpoons$ as straight lines we get the following diagram.



Now we can add new arrows as opposites of already existing arrows. As an example, let us add an arrow from the middle vertex of the top row to its right bottom neighbor. Below is the resulting diagram and its completion.

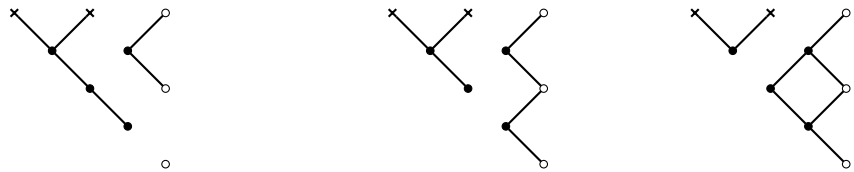


For our next arrow we have three possible choices (all towards the bottom). Two possibilities give the same diagram after completion. The other possibility gives a different diagram. The two distinct complete diagrams are shown below.



Both complete diagrams terminate the procedure. They correspond to the vertices $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ of $GT_{B_2}(\omega_2)$.

However, there are other possibilities by choosing a different arrow in the first addition. They lead to the following three complete diagrams.



Those are the identity diagrams of the vertices $(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, 0, 0)$ respectively. So we have found a visual way to calculate the 5 vertices of $GT_{B_2}(\omega_2)$.

7. TWEAKED GEL'FAND-TSETLIN PATTERNS

Let us now consider the even orthogonal case. Before stating our new construction, let us first underline where the problems arise when trying to copy the previous proof of Theorem 1.2.

In slight deviation of Littelmann's notation in [15], we will enumerate the simple roots of SO_{2n} as

$$\alpha_1 = \epsilon_1 - \epsilon_2, \dots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = \epsilon_{n-1} + \epsilon_n.$$

Additionally we will consider the reduced decomposition

$$w_0^{\text{std}} = (s_{n-1}s_n)(s_{n-2}s_{n-1}s_ns_{n-2}) \cdots (s_1s_2 \cdots s_{n-2}s_{n-1}s_ns_{n-2} \cdots s_2s_1)$$

of the longest word of the Weyl group as the standard one. Notice that this does not completely correspond to Littelmann's standard decomposition since we swap the positions of the (commuting) reflections corresponding to $\epsilon_{n-1} - \epsilon_n$ and $\epsilon_{n-1} + \epsilon_n$. However, since the two reflections commute, the string polytopes will be the same after permutation of some coordinates. So we will sloppily say that these two polytopes are *the same*.

We know that the polytope $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ will be a subset of $\mathbb{R}^{n(n-1)}$. We will denote the coordinates of a vector $a \in \mathbb{R}^{n(n-1)}$ as

$$a = (a_{n-1,n-1}, a_{n-1,n}, a_{n-2,n-2}, a_{n-2,n-1}, a_{n-2,n}, a_{n-2,n+1}, \dots, \dots, a_{1,1}, a_{1,2}, \dots, a_{1,n-2}, a_{1,n-1}, a_{1,n}, a_{1,n+1}, \dots, a_{1,2n-2}).$$

It is understood that $a_{i,j} = 0$ if any of the indices is outside of its allowed range. For every tuple $(a_{i,j})$ with $1 \leq i \leq n-1$ and $i \leq j \leq 2n-1-i$ we will use the notation $\overline{a_{i,j}} := a_{i,2n-1-j}$.

We think of these coordinates as entries of the following triangle.

$$\begin{array}{cccccccccc} a_{1,1} & a_{1,2} & \dots & a_{1,n-2} & a_{1,n-1} & \overline{a_{1,n-1}} & \overline{a_{1,n-2}} & \dots & \overline{a_{1,2}} & \overline{a_{1,1}} \\ & a_{2,2} & \dots & a_{2,n-2} & a_{2,n-1} & \overline{a_{2,n-1}} & \overline{a_{2,n-2}} & \dots & \overline{a_{2,2}} & \\ & & \ddots & & & & & & & \ddots \\ & & & a_{n-2,n-2} & a_{n-2,n-1} & \overline{a_{n-2,n-1}} & \overline{a_{n-2,n-2}} & & & \\ & & & & a_{n-1,n-1} & \overline{a_{n-1,n-1}} & & & & \end{array}$$

Notice that our $(n-1)$ -st column is Littelmann's n -th column and our n -th column is Littelmann's $(n-1)$ -st column. The reason for this change is that the j -th column corresponds to the reflection s_j for $j \leq n-2$. So it is more intuitive if the $(n-1)$ -st

column corresponds to the simple reflection s_{n-1} , and not s_n . However, these changes are rather cosmetic.

The following is a combination of [15, Theorem 7.1 and Corollary 8].

THEOREM 7.1 (Littelmann). *Let $G = SO_{2n}$ and $\lambda = \sum_{i=1}^n \lambda_i \omega_i \in \Lambda^+$. A tuple $(a_{i,j}) \in \mathbb{R}^{n(n-1)}$ is an element of $\mathcal{Q}_{w_0}^{\text{std}}(\lambda)$ if and only if the following two sets of conditions hold.*

$$a_{i,i} \geq a_{i,i+1} \geq \cdots \geq a_{i,n-2} \geq \left\{ \frac{a_{i,n-1}}{a_{i,n-1}} \right\} \geq \overline{a_{i,n-2}} \geq \cdots \geq \overline{a_{i,i+1}} \geq \overline{a_{i,i}} \geq 0,$$

$$a_{n-1,n-1} \geq 0, \quad \overline{a_{n-1,n-1}} \geq 0,$$

for every $1 \leq i \leq n-2$ and

$$a_{i,j} \leq \lambda_j + a_{i,j+1} + \overline{a_{i,j+1}} - 2\overline{a_{i,j}} + \overline{a_{i,j-1}}$$

$$+ \sum_{k=1}^{i-1} (a_{k,j-1} - 2a_{k,j} + a_{k,j+1} + \overline{a_{k,j+1}} - 2\overline{a_{k,j}} + \overline{a_{k,j-1}}),$$

$$\overline{a_{i,j}} \leq \lambda_j + \overline{a_{i,j-1}} + \sum_{k=1}^{i-1} (a_{k,j-1} - 2a_{k,j} + a_{k,j+1} + \overline{a_{k,j+1}} - 2\overline{a_{k,j}} + \overline{a_{k,j-1}}),$$

$$a_{i,n-1} \leq \lambda_{n-1} + \overline{a_{i,n-2}} + \sum_{k=1}^{i-1} (a_{k,n-2} - 2a_{k,n-1} + \overline{a_{k,n-2}}),$$

$$\overline{a_{i,n-1}} \leq \lambda_n + \overline{a_{i,n-2}} + \sum_{k=1}^{i-1} (a_{k,n-2} - 2\overline{a_{k,n-1}} + \overline{a_{k,n-2}})$$

for every $1 \leq i \leq n-1$ and $i \leq j \leq n-2$.

Now we can describe an adapted version of Littelmann's piecewise affine map directly. Notice that we have to make slight adjustments because of our change of reduced decomposition. Let $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$. Notice the base change from ω_i to ϵ_i in contrast to Theorem 7.1. Fix a point $(a_{i,j}) \in \mathcal{Q}_{w_0}^{\text{std}}(\lambda)$. This point is sent via the piecewise affine bijection ϕ_λ to a Gelfand–Tsetlin pattern $x = (\mathbf{y}, \mathbf{z})$ in $\mathbb{R}^{\frac{n(n-1)}{2}} \times \mathbb{R}^{\frac{n(n-1)}{2}}$. By our convention the row index of $\mathbf{y} = (y_{i,j})$ starts with $i = 2$. For easier notation we will set $y_{1,j} := \lambda_j$. Again, in our terminology this row is not actually part of the pattern x . The other rows can be computed reciprocally as

$$y_{i,j} = y_{i-1,j} + a_{i-1,j-1} - a_{i-1,j} - \overline{a_{i-1,j}} + \overline{a_{i-1,j-1}}$$

and

$$y_{i,n} = y_{i-1,n} + a_{i-1,n-1} - \overline{a_{i-1,n-1}}$$

for every $2 \leq i \leq n$ and $i \leq j \leq n-1$. For the z -coordinates we have the formulae

$$z_{i,j} = y_{i,j} + \overline{a_{i,j-1}} - \overline{a_{i,j}},$$

$$z_{i,n-1} = y_{i,n} + \min\{a_{i,n-2} - \overline{a_{i,n-1}}, a_{i,n-1} - \overline{a_{i,n-2}}\}$$

and

$$z_{n-1,n-1} = y_{i,n} + a_{n-1,n-1}$$

for every $1 \leq i \leq n-2$ and $i \leq j \leq n-2$.

So we see that the non-affine part appears in the coordinates $z_{i,n-1}$. Since ϕ_λ is a bijection we are not losing any information when applying ϕ_λ but the minimum function makes it appear that way.

Our goal now is to embed the Gel'fand–Tsetlin pattern in a subspace of a larger vector space to keep track of both values $a_{i,n-2} - \overline{a_{i,n-1}}$ and $a_{i,n-1} - \overline{a_{i,n-2}}$. For that purpose we will introduce new coordinates $z_{i,n-1}^\uparrow$ and $z_{i,n-1}^\downarrow$ to replace the *bad* coordinate $z_{i,n-1}$.

For easier presentation we will use the following notation.

NOTATION 7.2. Let a, b, c, d, e and f be some real numbers. We will write

$$\begin{array}{ccc} a & & b \\ & c & \\ & d & \\ e & & f \end{array}$$

if the numbers fulfill the conditions

$$\begin{aligned} a \geq c \geq \begin{Bmatrix} b \\ f \end{Bmatrix} \quad \text{and} \quad c \leq a + b + f, \\ e \geq d \geq \begin{Bmatrix} b \\ f \end{Bmatrix} \quad \text{and} \quad d \leq e + b + f. \end{aligned}$$

Notice and beware of the asymmetry in this notation! We do not for example require $a \geq d$!

We can now define a modified version of D_n -Gel'fand–Tsetlin patterns.

DEFINITION 7.3 (Tweaked Gel'fand–Tsetlin patterns in type D_n). *Let $G = SO_{2n}$ and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. A tweaked Gel'fand–Tsetlin pattern of type λ is a pair $(\mathbf{y}, \mathbf{z})$ of tuples $\mathbf{y} = (y_{i,j}) \in \mathbb{R}^{\frac{n(n-1)}{2}}$, $2 \leq i \leq j \leq n$, and*

$$\mathbf{z} = (z_{1,1}, \dots, z_{1,n-2}, z_{1,n-1}^\uparrow, z_{1,n-1}^\downarrow, \dots, z_{n-2,n-2}, z_{n-2,n-1}^\uparrow, z_{n-2,n-1}^\downarrow, z_{n-1,n-1})$$

such that

$$(1) \quad y_{i,n-1} - y_{i+1,n-1} = z_{i,n-1}^\uparrow - z_{i,n-1}^\downarrow \quad \text{for all } 1 \leq i \leq n-2$$

and the coordinates fulfill the relations in Fig. 6.

To simplify notation we will sometimes write $z_{n-1,n-1}^\uparrow$ for $z_{n-1,n-1}$.

As in the usual definition, these patterns will define a polytope.

DEFINITION 7.4. *Let $G = SO_{2n}$ and $\lambda \in \Lambda^+$. The tweaked Gel'fand–Tsetlin polytope $\widetilde{GT}(\lambda)$ is defined as the set of all tweaked Gel'fand–Tsetlin patterns of type λ .*

The relation between usual Gel'fand–Tsetlin patterns and tweaked Gel'fand–Tsetlin patterns is given by the following observation.

Let $\mathcal{V}_\lambda$ denote the linear subspace of $\mathbb{R}^{n(n-1)+(n-2)} = \mathbb{R}^{n^2-2}$ defined by the relations in Eq. (1).

THEOREM 7.5. *For every $\lambda \in \Lambda^+$ there exists a bijection $\psi_\lambda: \mathcal{V}_\lambda \rightarrow \mathbb{R}^{n(n-1)}$ given by $(z_{i,n-1}^\uparrow, z_{i,n-1}^\downarrow) \mapsto \min\{z_{i,n-1}^\uparrow, z_{i,n-1}^\downarrow\}$ for all $1 \leq i \leq n-2$ and identity on the other coordinates. Its inverse is given by*

$$\begin{aligned} z_{i,n-1}^\uparrow &:= z_{i,n-1} + y_{i,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\} \\ z_{i,n-1}^\downarrow &:= z_{i,n-1} + y_{i+1,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\} \end{aligned}$$

for every $1 \leq i \leq n-2$ and identity on the other coordinates. Furthermore, ψ_λ induces a bijection between the tweaked Gel'fand–Tsetlin patterns of type λ and the usual Gel'fand–Tsetlin patterns of type λ .

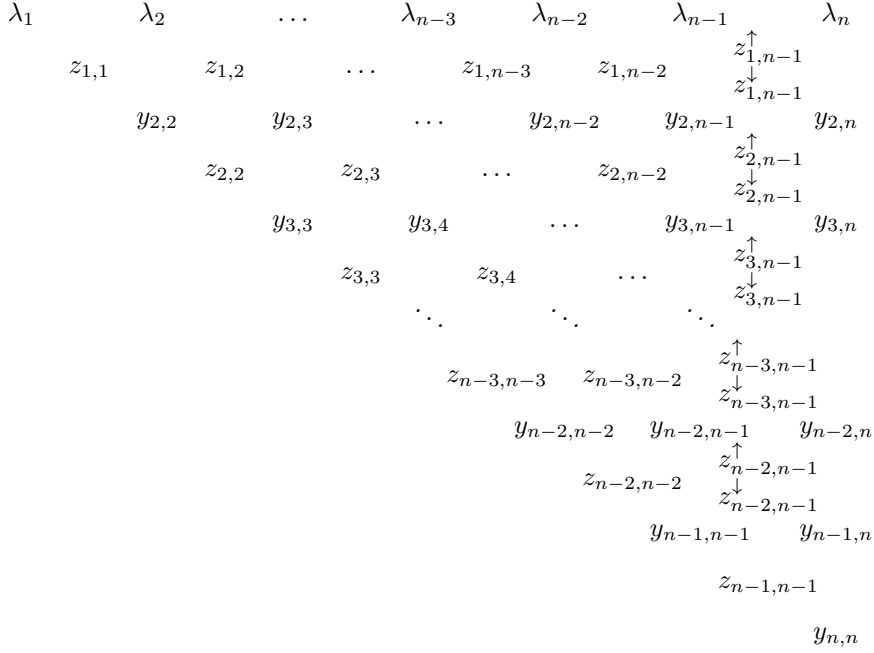


FIGURE 6. Inequalities of tweaked Gel'fand–Tsetlin patterns.

The proof is done by explicit calculation. It can be found in [21, Theorem 6.7.5]. We can now state an analogue of Theorem 2.12 for D_n .

THEOREM (Theorem 1.4). *Let $G = SO_{2n}$ and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. The map $\tilde{\phi}_\lambda := \psi_\lambda^{-1} \circ \phi_\lambda: \mathbb{R}^{n(n-1)} \rightarrow \mathcal{V}_\lambda$ is an affine bijection and $\tilde{\phi}_\lambda(\mathcal{Q}_{w_0^{\text{std}}}(\lambda)) = \widetilde{GT}(\lambda)$. Furthermore, $a \in \mathbb{R}^{n(n-1)}$ is a lattice point if and only if the coordinates of $\tilde{\phi}_\lambda(a)$, including the first row $y_{1,j} = \lambda_j$, are either all integral or all are in $\frac{1}{2} + \mathbb{Z}$.*

Proof. Since ϕ_λ and ψ_λ^{-1} are piecewise affine bijections, the same holds true for $\tilde{\phi}_\lambda$. Since $\phi_\lambda(\mathcal{Q}_{w_0^{\text{std}}}) = GT_{D_n}(\lambda)$ by Theorem 2.14 and $\psi_\lambda(\widetilde{GT}(\lambda)) = GT_{D_n}(\lambda)$ by Theorem 7.5, we have $\tilde{\phi}_\lambda(\mathcal{Q}_{w_0^{\text{std}}}(\lambda)) = \widetilde{GT}(\lambda)$. The claim on lattice points is clear from Theorem 2.14 and the definition of ψ_λ .

It remains to show that $\tilde{\phi}_\lambda$ and $\tilde{\phi}_\lambda^{-1}$ are in fact affine. We only have to check the coordinates $z_{i,n-1}^\uparrow$ and $z_{i,n-1}^\downarrow$ since the map is affine in all other coordinates. Let $x = (\mathbf{y}, \mathbf{z})$ be the image of $(a_{i,j})$ under $\tilde{\phi}_\lambda$. We calculate

$$\begin{aligned} z_{i,n-1}^\uparrow &= z_{i,n-1} + y_{i,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\} \\ &= z_{i,n-1} - \min\{0, y_{i+1,n-1} - y_{i,n-1}\} \\ &= y_{i,n} + \min\{a_{i,n-2} - \overline{a_{i,n-1}}, a_{i,n-1} - \overline{a_{i,n-2}}\} \\ &\quad - \min\{0, a_{i,n-2} - a_{i,n-1} - \overline{a_{i,n-1}} + \overline{a_{i,n-2}}\} \\ &= y_{i,n} + a_{i,n-1} - \overline{a_{i,n-2}}. \end{aligned}$$

This implies that $z_{i,n-1}^\uparrow$ is actually a linear combination of the coordinates of $(a_{i,j})$ plus λ_n . The same holds true for $z_{i,n-1}^\downarrow$ by analogous computation. Hence the map $\tilde{\phi}_\lambda$ is affine, i.e. the concatenation of a linear map and a translation. Since the inverse

of a linear map is linear and the inverse of a translation is a translation we know that $\tilde{\phi}_\lambda^{-1}$ must be affine too. This concludes the proof. $\square$

EXAMPLE 7.6. Let $G = SO_6$ and let $\lambda = \sum_{i=1}^3 \lambda_i \epsilon_i$. The image of the point $(a, b, c, d, e, f) \in \mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ under the map $\tilde{\phi}_\lambda$ is drawn in Fig. 7.

$$\begin{array}{ccccc}
 \lambda_1 & & \lambda_2 & & \lambda_3 \\
 & \lambda_1 - e + f & & \lambda_3 + d - f & \\
 & & & \lambda_3 + c - e & \\
 & \lambda_2 + c - d - e + f & & \lambda_3 + d - e & \\
 & & \lambda_3 + a + d - e & & \\
 & & & \lambda_3 + a - b + d - e &
 \end{array}$$

FIGURE 7. The point $\tilde{\phi}_\lambda(a, b, c, d, e, f)$ from Example 7.6.

8. TWEAKED GEL'FAND-TSETLIN DIAGRAMS

We will now define an analogue of identity diagrams of elements of marked order polytopes for tweaked Gel'fand-Tsetlin patterns. For that purpose we want to define a poset that describes as much of the tweaked Gel'fand-Tsetlin polytope as possible.

CONSTRUCTION 8.1 (Tweaked Gel'fand-Tsetlin poset). Let $G = SO_{2n}$. Let $\mathcal{GT}_n$ be the set of symbols

- (i) $\xi_{i,j}$ for $1 \leq i \leq j \leq n$,
- (ii) $\zeta_{i,j}$ for $1 \leq i \leq j \leq n-2$,
- (iii) $\zeta_{i,n-1}^\uparrow$ and $\zeta_{i,n-1}^\downarrow$ for $1 \leq i \leq n-2$, and
- (iv) $\zeta_{n-1,n-1}$.

For easier notation we sometimes write $\zeta_{n-1,n-1}^\uparrow$ for $\zeta_{n-1,n-1}$. The ordering is given precisely by the inequalities in the definition of tweaked Gel'fand-Tsetlin patterns that do not require addition or subtraction by substituting ξ for y and ζ for z .

Notice that this describes almost all relations defining tweaked Gel'fand-Tsetlin patterns. The ones missing are the four-term relations of type

$$\begin{aligned}
 z_{i,n-1}^\uparrow - z_{i,n-1}^\downarrow &= y_{i,n-1} - y_{i+1,n-1}, \\
 z_{i,n-1}^\uparrow &\leq y_{i,n-1} + y_{i,n} + y_{i+1,n} \text{ and} \\
 z_{i,n-1}^\downarrow &\leq y_{i+1,n-1} + y_{i,n} + y_{i+1,n}.
 \end{aligned}$$

We will return to them later.

EXAMPLE 8.2. The Hasse diagram of the tweaked Gel'fand-Tsetlin poset $\mathcal{GT}_4$ is depicted in Fig. 8.

We would like to define an analogue of the marked order polytope. For this we will use the following non-standard definition.

DEFINITION 8.3. A pseudo-marking on a poset P is a pair (A, λ) where A is a subset of P and $\lambda = (\lambda_a)_{a \in A} \in \mathbb{R}^A$ is a real vector such that $\lambda_a \leq \lambda_b$ whenever $a \leq b$. The triplet (P, A, λ) is called a pseudo-marked poset. We will call the elements of A marked elements.

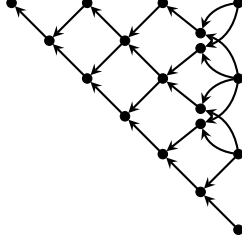


FIGURE 8. Hasse diagram of the tweaked Gel'fand–Tsetlin poset $\mathcal{GT}_4$.

Notice that in contrast to Definition 3.1 we do not require that all minimal and all maximal elements of the poset are marked. As a consequence, the following definition might not give a polytope.

DEFINITION 8.4. Let (P, A, λ) be a pseudo-marked poset. The marked order polyhedron $\mathcal{O}_{P,A}(\lambda)$ associated to (P, A, λ) is defined as

$$\mathcal{O}_{P,A}(\lambda) := \left\{ x \in \mathbb{R}^{P \setminus A} \mid \begin{array}{l} x_p \leq x_q \text{ for all } p \leq q, \\ \lambda_a \leq x_p \text{ for all } a \leq p, \\ x_p \leq \lambda_b \text{ for all } p \leq b \end{array} \right\}.$$

The following observation is clear by construction.

PROPOSITION 8.5. Let $G = SO_{2n}$. Set $A := \{\xi_{1,1}, \dots, \xi_{1,n}\} \subseteq \mathcal{GT}_n$. Let $\lambda \in \mathbb{R}^A$ and denote by $\tilde{\lambda} := \sum_{i=1}^n \lambda_{\xi_{1,i}} \epsilon_i$ the associated weight (not necessarily dominant nor integral). Then

$$\begin{aligned} \widetilde{GT}(\tilde{\lambda}) = \mathcal{O}_{\mathcal{GT}_n, A}(\lambda) \cap \mathcal{V}_{\tilde{\lambda}} \cap \left\{ \begin{array}{l} z_{i,n-1}^{\uparrow} \leq y_{i,n-1} + y_{i,n} + y_{i+1,n} \\ z_{i,n-1}^{\downarrow} \leq y_{i+1,n-1} + y_{i,n} + y_{i+1,n} \end{array} \mid 1 \leq i \leq n-2 \right\} \\ \cap \{z_{n-1,n-1} \leq y_{n-1,n-1} + y_{n-1,n} + y_{n,n}\}. \end{aligned}$$

Notice that we were a bit sloppy with our notation here since $\widetilde{GT}(\tilde{\lambda})$, $\mathcal{O}_{\mathcal{GT}_n, A}(\lambda)$ and $\mathcal{V}_{\tilde{\lambda}}$ are defined as subsets of $\mathbb{R}^{\mathcal{GT}_n \setminus A}$. So we want to understand the fourth polyhedron as a subset of $\mathbb{R}^{\mathcal{GT}_n \setminus A}$ too. Additionally, from now on we simplify notation by using λ for the weight and for the marking simultaneously.

REMARK 8.6. Because of the defining relations of $\mathcal{V}_{\lambda}$ it is clear that the two inequalities $z_{i,n-1}^{\uparrow} \leq y_{i,n-1} + y_{i,n} + y_{i+1,n}$ and $z_{i,n-1}^{\downarrow} \leq y_{i+1,n-1} + y_{i,n} + y_{i+1,n}$ are in fact equivalent. So when checking whether a given point in $\mathcal{V}_{\lambda}$ is a tweaked Gel'fand–Tsetlin pattern, it is sufficient to just verify one of those inequalities for every i . Furthermore, if one of these inequalities happens to be an equality, the other one will be too.

We will now define an analogue of identity diagrams for these special posets.

CONSTRUCTION 8.7. Let (A, λ) be the pseudo-marking on the tweaked Gel'fand–Tsetlin poset $\mathcal{GT}_n$ from Proposition 8.5 and let $x \in \widetilde{GT}(\lambda)$. Let $\mathcal{H}_n$ denote the Hasse diagram of $\mathcal{GT}_n$. The *tweaked Gel'fand–Tsetlin pre-diagram* $\text{pre } \mathcal{D}_{\mathcal{D}_n}^{\lambda}(x)$ associated to x is the colored directed graph whose nodes are labeled by the elements of the tweaked Gel'fand–Tsetlin poset and whose arrows are given by the following construction.

- (i) Add a *black* arrow $p \rightarrow q$ if there exists an arrow $p \rightarrow q$ in $\mathcal{H}_n$ between the corresponding nodes.
- (ii) Add a *black* arrow $p \rightarrow q$ if there exists an opposite arrow $q \rightarrow p$ in $\mathcal{H}_n$ between the corresponding nodes and $x_p = x_q$.

(iii) For every $1 \leq i \leq n-1$ add six (only three if $i = n-1$) *red* arrows

$$\begin{array}{ccccc} \xi_{i,n-1} & \dashrightarrow & \zeta_{i,n-1}^\uparrow & \dashrightarrow & \xi_{i,n} \\ & & \searrow & \nearrow & \\ \xi_{i+1,n-1} & \dashrightarrow & \zeta_{i,n-1}^\downarrow & \dashrightarrow & \xi_{i+1,n} \end{array}$$

if $z_{i,n-1}^\uparrow = y_{i,n-1} + y_{i,n} + y_{i+1,n}$ (or equivalently $z_{i,n-1}^\downarrow = y_{i+1,n-1} + y_{i,n} + y_{i+1,n}$).

For printing reasons we will always draw *red* arrows as black but dashed arrows. This diagram can be further simplified for our purposes. The reason is the following observation.

REMARK 8.8. Let x be a tweaked Gel'fand–Tsetlin pattern. Let $1 \leq i \leq n-1$ be an index such that $z_{i,n-1}^\uparrow = y_{i,n-1} + y_{i,n} + y_{i+1,n}$. Then the following implications hold.

- (i) $y_{i,n-1} = y_{i,n} = y_{i+1,n} \Rightarrow y_{i,n-1} = y_{i,n} = y_{i+1,n} = z_{i,n-1}^\uparrow = 0$.
(This is due to the fact that $y_{i,n-1} \geq z_{i,n-1}^\uparrow \geq y_{i,n}$; hence $y_{i,n-1} = y_{i,n}$ already implies $z_{i,n-1}^\uparrow = y_{i,n-1} = y_{i,n} = y_{i+1,n}$.)
- (ii) $z_{i,n-1}^\uparrow = y_{i,n-1} \Rightarrow y_{i,n} = -y_{i+1,n}$.
- (iii) $z_{i,n-1}^\uparrow = y_{i,n} \Rightarrow y_{i,n-1} = -y_{i+1,n}$.
- (iv) $z_{i,n-1}^\uparrow = y_{i+1,n} \Rightarrow y_{i,n-1} = -y_{i,n}$.

The analogue statements hold true for $z_{i,n-1}^\downarrow$ and $y_{i+1,n-1}$.

We will use these observations to adapt our pre-diagrams. The goal is to indicate whether two entries must be additive inverses of each other because of an equation of the form $z_{i,n-1}^\uparrow = y_{i,n-1} + y_{i,n} + y_{i+1,n}$ or $z_{i,n-1}^\downarrow = y_{i+1,n-1} + y_{i,n} + y_{i+1,n}$.

CONSTRUCTION 8.9. Let x be a tweaked Gel'fand–Tsetlin pattern and let $\text{pre } \mathcal{D}_{\mathbb{D}_n}^\lambda(x)$ be its tweaked Gel'fand–Tsetlin pre-diagram. We will replace *red* arrows as follows.

For all triplets of *red* arrows

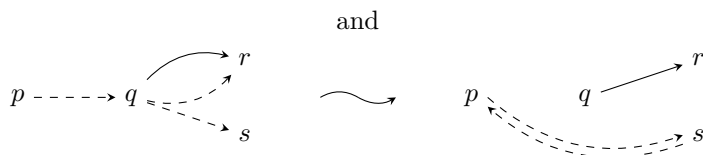
$$p \dashrightarrow q \begin{array}{l} \nearrow r \\ \searrow s \end{array}$$

do the following replacements. (We do not specify whether $r = \xi_{i,n}$ and $s = \xi_{i+1,n}$ or $r = \xi_{i+1,n}$ and $s = \xi_{i,n}$. Both possibilities are allowed!)

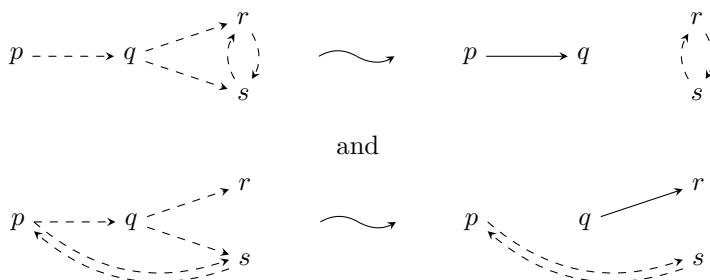
If there exist *black* arrows $p \rightarrow q$, $q \rightarrow r$ and $q \rightarrow s$, delete all three *red* arrows but color the four nodes differently. By default we will call every node *black*. But in this case we chose to color these nodes *white* instead. We will draw them as empty circles, while our normal *black* nodes are filled circles.

For all remaining triplets of *red* arrows do the following two replacements if possible. If both are applicable to a certain triplet, do only one (it does not matter which one, though the first one is usually preferred due to readability).

$$\begin{array}{ccc} \begin{array}{ccc} p & \xrightarrow{\quad} & q \\ & \searrow & \nearrow \\ & & r \\ & & s \end{array} & \rightsquigarrow & \begin{array}{ccc} p & \longrightarrow & q \end{array} \end{array}$$



Our replacement procedure could have produced *red* triangles. We will replace those as follows.

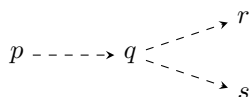


Finally, for every $1 \leq i \leq n-1$ we will add a pair of *black* arrows $\xi_{i,n-1} \rightleftharpoons \xi_{i+1,n-1}$ if one of the following two conditions hold.

- (i) There exist two *black* arrows $\zeta_{i,n-1}^\uparrow \rightarrow \xi_{i,n}$ and $\zeta_{i,n-1}^\downarrow \rightarrow \xi_{i,n}$.
- (ii) There exist two *black* arrows $\zeta_{i,n-1}^\downarrow \rightarrow \xi_{i+1,n}$ and $\zeta_{i,n-1}^\uparrow \rightarrow \xi_{i+1,n}$.

The resulting graph is called the *tweaked Gel'fand–Tsetlin diagram* $\mathcal{D}_{\mathbb{D}_n}^\lambda(x)$ of x .

REMARK 8.10. It is clear by construction that $x_p = x_q$ whenever there exists a pair of *black* arrows $p \rightarrow q$ and $q \rightarrow p$. Analogously, $x_p = -x_q$ whenever there exists a pair of *red* arrows $p \xrightarrow{\text{red}} q$ and $q \xrightarrow{\text{red}} p$. Additionally for every subgraph



we have $x_q = x_p + x_r + x_s$. So our construction really visualizes which of the defining inequalities of $GT(\lambda)$ are actually equalities for the pattern x .

Furthermore, $x_p = 0$ for every *white* node p . The reason is the following. Every white node is part of a quadruplet $\{p, q, r, s\}$ such that $x_p = x_q = x_r = x_s$ and $x_q = x_p + x_r + x_s$ (after proper renaming). But this implies that $x_q = 3x_p$ and hence $0 = x_q = x_p = x_r = x_s$.

For readability of our drawings, sometimes we do not draw single arrows (we will remember their existence from the positions of the nodes), replace *black* double arrows $p \rightarrow q$ and $q \rightarrow p$ by a straight line and represent *red* double arrows $p \xrightarrow{\text{red}} q$ and $q \xrightarrow{\text{red}} p$ by a double straight line. We also usually omit *red* double arrows between white nodes, since they do not contain any new information. Single *red* arrows will be drawn as a dashed line.

DEFINITION 8.11. Let x be a tweaked Gel'fand–Tsetlin pattern and $\mathcal{D}_{\mathbb{D}_n}^\lambda(x)$ its tweaked Gel'fand–Tsetlin diagram. A subset $\mathcal{C}$ of nodes of this diagram is called *connected* if it is connected via double-black and double-red arrows, i.e. for any two nodes p and q in $\mathcal{C}$ there exists a sequence $p_1, \dots, p_t \in \mathcal{C}$ such that $p = p_1$, $q = p_t$ and for every i there either exist two *black* arrows $p_i \rightarrow p_{i+1}$ and $p_{i+1} \rightarrow p_i$ or there exist two *red* arrows $p_i \xrightarrow{\text{red}} p_{i+1}$ and $p_{i+1} \xrightarrow{\text{red}} p_i$. The sequence $(p_1, \dots, p_t)$ is called a *connecting*

sequence *between* p and q . The maximal (with respect to inclusion) connected subsets are called connected components.

One might be inclined to think that a tweaked Gel'fand–Tsetlin pattern is a vertex of the tweaked Gel'fand–Tsetlin polytope if and only if each of the connected components in its tweaked Gel'fand–Tsetlin diagram contains a marked element. However, this is not true in contrast to types A_n , B_n and C_n . There are more possibilities as the following examples show.

EXAMPLE 8.12. Let $G = SO_6$ and $\lambda = \rho = 2\epsilon_1 + \epsilon_2 \in \Lambda^+$. Then the pattern in Fig. 9 is a vertex of $\widetilde{GT}(\lambda)$. However, its tweaked Gel'fand–Tsetlin diagram contains the isolated node $\zeta_{1,2}^\downarrow$.

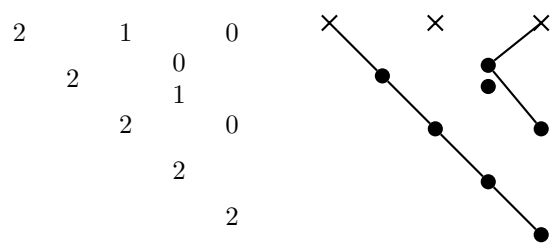


FIGURE 9. Vertex of the tweaked Gel'fand–Tsetlin polytope $\widetilde{GT}(2\epsilon_1 + \epsilon_2)$ and its tweaked Gel'fand–Tsetlin diagram in type D_3 .

EXAMPLE 8.13. Let $G = SO_8$ and $\lambda = \omega_1 + \omega_2 + \omega_3 + 3\omega_4 = 4\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 - \epsilon_4$. Then the pattern in Fig. 10 is a vertex of $\widetilde{GT}(\lambda)$. We see two things happening in this example. First of all, we see a triangular pattern of *white* nodes. Secondly we see two nodes ($\zeta_{1,3}^\uparrow$ and $\zeta_{1,3}^\downarrow$) that are not connected to any other node, although they are connected via single *red* arrows. But these single arrows do not count as connections in our sense.

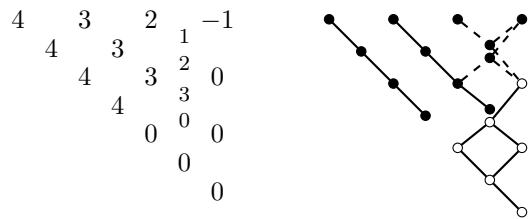
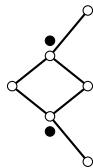


FIGURE 10. Vertex of the tweaked Gel'fand–Tsetlin polytope $\widetilde{GT}(4\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 - \epsilon_4)$ and its tweaked Gel'fand–Tsetlin diagram in type D_4 .

We will systematize these deviations from the A_n , B_n and C_n cases as follows.

DEFINITION 8.14. Let $x \in \widetilde{GT}(\lambda)$ be a tweaked Gel'fand–Tsetlin pattern and let $\mathcal{D}_{D_n}^\lambda(x)$ be its tweaked Gel'fand–Tsetlin diagram.

- (i) An anomaly is a triangle of white nodes of the following form.



- (ii) A single impurity is a node $\zeta_{i,n-1}^\uparrow$ or $\zeta_{i,n-1}^\downarrow$ with $i < n-1$ that is not connected to any other node but the other node $\zeta_{i,n-1}^\downarrow$ (resp. $\zeta_{i,n-1}^\uparrow$) is connected to another node.
- (iii) A double impurity is a pair of nodes $(\zeta_{i,n-1}^\uparrow, \zeta_{i,n-1}^\downarrow)$ with $i < n-1$ such that both nodes are not connected to any other node but they are incident to red arrows.
- (iv) A triviality is a pair of nodes $(\zeta_{i,n-1}^\uparrow, \zeta_{i,n-1}^\downarrow)$ with $i < n-1$ such that both nodes are not connected to any other node and they are not incident to red arrows.

Notice that we do not allow single impurities to happen at $\zeta_{n-1,n-1}$.

We can now state the complete classification of vertices of tweaked Gel'fand–Tsetlin polytopes, as proved in [21, Theorem 6.8.16].

THEOREM 8.15. *A tweaked Gel'fand–Tsetlin pattern is a vertex of the tweaked Gel'fand–Tsetlin polytope if and only if each of the connected components of its tweaked Gel'fand–Tsetlin diagram contains a marked element, contains an anomaly, is a single impurity or is part of a double impurity.*

We will not state a proof of this result as it is quite lengthy and technical. The idea is to use Lemma 4.5 and explicitly show that for each $x \in \widetilde{GT}(\lambda)$ there exists a vector $v \neq 0$ such that $x \pm v \in \widetilde{GT}(\lambda)$ if and only if $\mathcal{D}_{\mathbb{D}_n}^\lambda(x)$ violates the conditions. The construction of v is very similar to the marked order case, one just has to introduce signs for red arrows and carefully treat the double nodes. We invite the reader to verify the explicit calculations in [21, Chapter 6, Section 8].

Let us start our return towards string polytopes. For that purpose we introduce the following notation.

NOTATION 8.16. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ let $\Gamma(\lambda) \subseteq \mathbb{R}$ denote the free abelian group generated by the coefficients $\lambda_1, \dots, \lambda_n$, i.e.

$$\Gamma(\lambda) := \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_n.$$

This will allow us to state the following crucial result.

THEOREM 8.17. *Let $\lambda \in \mathbb{R}^n$ and let $x \in \widetilde{GT}(\sum_{i=1}^n \lambda_i \epsilon_i)$ be a vertex of the tweaked Gel'fand–Tsetlin polytope. Then every coordinate of x lies in $\Gamma(\lambda)$.*

Proof. Let $p \in \mathcal{GT}_n$. We want to calculate x_p via the tweaked Gel'fand–Tsetlin diagram $\mathcal{D}_{\mathbb{D}_n}^\lambda(x)$. By Theorem 8.15 we know that p is connected to a marked element, is connected to an anomaly, is a single impurity or is part of a double impurity.

By construction of $\mathcal{D}_{\mathbb{D}_n}^\lambda(x)$ we know that $x_p = \pm\lambda_j$ if p is connected to $\xi_{1,j}$. If p is connected to an anomaly, we know that $x_p = 0$. So in both cases x_p lies in $\{\pm\lambda_1, \dots, \pm\lambda_n\} \cup \{0\} \subseteq \Gamma(\lambda)$.

We will only prove the impurity case for $p = \zeta_{i,n-1}^\uparrow$ because the proof for $\zeta_{i,n-1}^\downarrow$ is completely analogous. If $p = \zeta_{i,n-1}^\uparrow$ is incident to red arrows, we know that $x_p = z_{i,n-1}^\uparrow$ can be calculated as $z_{i,n-1}^\uparrow = y_{i,n-1} + y_{i,n} + y_{i+1,n}$. Since the latter three coordinates

lie in $\Gamma(\lambda)$, the same holds true for $z_{i,n-1}^\uparrow = x_p$. If $p = \zeta_{i,n-1}^\uparrow$ is not incident to *red* arrows, we know that $\zeta_{i,n-1}^\downarrow$ cannot be an impurity too. So the value $x_p = z_{i,n-1}^\uparrow$ is uniquely determined by the equation $z_{i,n-1}^\uparrow = z_{i,n-1}^\downarrow + y_{i,n-1} - y_{i+1,n-1}$. Since the latter three coordinates lie in $\Gamma(\lambda)$, the same holds true for $z_{i,n-1}^\uparrow = x_p$. This concludes our proof. $\square$

During this proof we have actually shown the following special case.

COROLLARY 8.18. *Let $\lambda \in (\frac{1}{2} + \mathbb{Z})^n$ and let $x \in \widetilde{GT}(\sum_{i=1}^n \lambda_i \epsilon_i)$ be a vertex of the tweaked Gel'fand–Tsetlin polytope. Then*

- (i) $x_p = 0$ if p is connected to an anomaly,
- (ii) $x_p \in \frac{1}{2}\mathbb{Z}$ if p is a single impurity or part of a double impurity, and
- (iii) $x_p \in \frac{1}{2} + \mathbb{Z}$ for any other p .

9. INTEGRALITY OF STANDARD STRING POLYTOPES IN TYPE D_n

We can now prove the following useful translation.

COROLLARY 9.1. *Let $G = SO_{2n}$ and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. Let x be a vertex of the standard string polytope $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$. Then x is a lattice point if and only if*

- (i) $\lambda_i \in \mathbb{Z}$ for all i or
- (ii) $\lambda_i \in \frac{1}{2} + \mathbb{Z}$ for all i and $\mathcal{D}_{D_n}^\lambda(\tilde{\phi}_\lambda(x))$ does not contain any anomaly.

Proof. By Theorem 1.4 we know that x is a vertex of $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ if and only if $\tilde{\phi}_\lambda(x)$ is a vertex of $\widetilde{GT}(\lambda)$. Additionally, x will be a lattice point if and only if the coordinates of $\tilde{\phi}_\lambda(x)$ are either all integers or all in $\frac{1}{2} + \mathbb{Z}$. The claim now follows directly from Theorem 8.17 if $\lambda_i \in \mathbb{Z}$ for all i .

Let us consider the other case $\lambda_i \in \frac{1}{2} + \mathbb{Z}$ for all i . Notice that most of the coordinates will be in $\frac{1}{2} + \mathbb{Z}$ by Corollary 8.18. If $\mathcal{D}_{D_n}^\lambda(x)$ contains an anomaly, there will be coordinates equal to zero. Hence x cannot be a lattice point in this case. If there does not exist an anomaly in $\mathcal{D}_{D_n}^\lambda(x)$, the only nodes whose corresponding coordinate could potentially be not in $\frac{1}{2} + \mathbb{Z}$ would be single or double impurities. But the proof of Theorem 8.17 shows that these coordinates can be calculated as a sum of three coordinates in $\frac{1}{2} + \mathbb{Z}$. Hence they must be in $\frac{1}{2} + \mathbb{Z}$ as well. $\square$

With this result, we can tackle the last remaining case of Theorem 1.2.

Proof of Theorem 1.2 in type D_n . Let $G = SO_{2n}$ and fix a dominant integral weight $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. If $\langle \lambda, \alpha_{n-1}^\vee \rangle + \langle \lambda, \alpha_n^\vee \rangle$ is an even integer, we know that $\lambda_i \in \mathbb{Z}$ for all $1 \leq i \leq n$. So $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ will be a lattice polytope by Corollary 9.1.

If $\langle \lambda, \alpha_{n-1}^\vee \rangle + \langle \lambda, \alpha_n^\vee \rangle$ is an odd integer, we know that $\lambda_i \in \frac{1}{2} + \mathbb{Z}$ for all $1 \leq i \leq n$. Since Theorem 1.4 yields a bijection between the vertices of $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ and the vertices of $\widetilde{GT}(\lambda)$, it is necessary and sufficient to find a vertex x of $\widetilde{GT}(\lambda)$ that contains an anomaly in its tweaked Gel'fand–Tsetlin diagram $\mathcal{D}_{D_n}^\lambda(x)$. Then Corollary 9.1 implies that the vertex $\tilde{\phi}_\lambda^{-1}(x)$ of $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ will not be a lattice point. Notice that for $n < 3$ any tweaked Gel'fand–Tsetlin diagram cannot contain an anomaly. For $n = 3$ an anomaly can only occur if $\lambda_3 = 0$, which implies that $\langle \lambda, \alpha_2^\vee \rangle + \langle \lambda, \alpha_3^\vee \rangle$ is even. So for $n < 4$, all standard string polytopes must be lattice polytopes.

For $n \geq 4$ consider the pattern in Fig. 11. We can verify quite easily that this pattern is indeed a tweaked Gel'fand–Tsetlin pattern for λ . The only nontrivial (in)equality to verify is $\lambda_{n-1} + \lambda_n \geq 0$. But this is true for any dominant integral weight in type D_n . The tweaked Gel'fand–Tsetlin diagram of this pattern is drawn

in Fig. 12. We see that every connected component contains a marked element or an anomaly. Hence this pattern is indeed a vertex of the tweaked Gel'fand–Tsetlin polytope containing an anomaly in its tweaked Gel'fand–Tsetlin diagram. This concludes the proof. $\square$

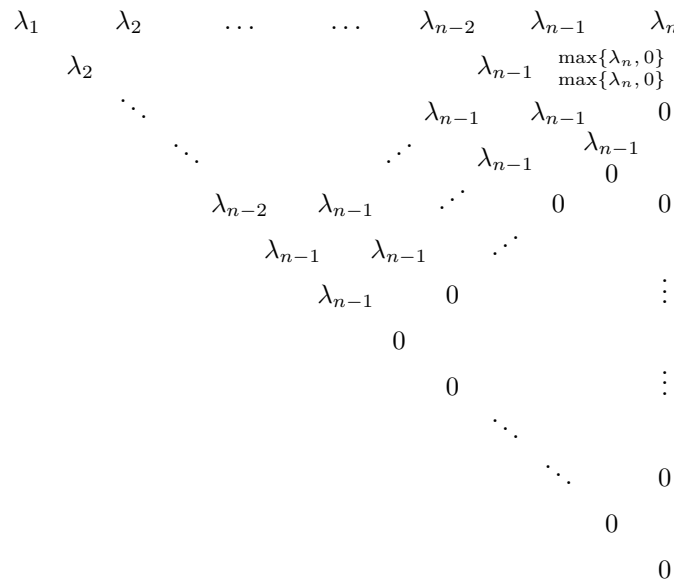


FIGURE 11. A special tweaked Gel'fand–Tsetlin pattern in type D_n for an arbitrary weight λ .

10. GORENSTEIN FANO TORIC DEGENERATIONS

We will conclude our paper with some remarks and consequences.

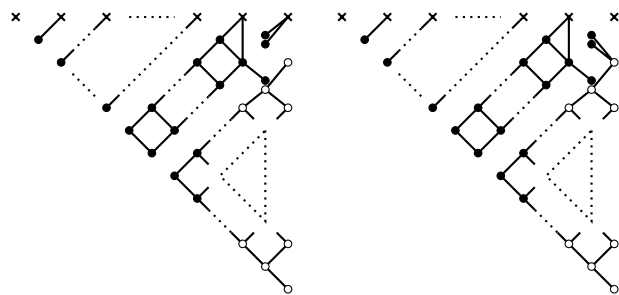


FIGURE 12. Tweaked Gel'fand–Tsetlin diagrams of the tweaked Gel'fand–Tsetlin pattern from Fig. 11. Left hand side for $\lambda_n > 0$, right hand side for $\lambda_n < 0$. We decided to only draw the case where $\lambda_1 > \lambda_2 > \dots > \lambda_{n-1} > \lambda_n \neq 0$ since any equality between the coefficients of λ would only result in more pairs of *black* arrows connecting formerly disjoint connected components, thus not changing any of our arguments.

REMARK 10.1. It should be noted that morally it is absolutely not clear why the string polytope should know *anything* about the representations of the underlying algebraic group. Firstly, its definition and many of its explicit descriptions are done purely from the perspective of the Lie algebra (see [15] and [5]). Secondly, we have already seen in [22, Example 5.5] that Theorem 1.2 does not hold for arbitrary reduced decompositions. So the connection between the standard reduced decomposition and representations of the algebraic group remains mysterious.

We would now like to impose some geometric meaning to our result. Batyrev proved in [3, Proposition 2.2.23] that reflexive polytopes are in one-to-one correspondence to Gorenstein Fano toric varieties. So we want to understand whether a given string polytope is not only integral but reflexive. In [22] we proved the following fact.

THEOREM 10.2. *If the valuation semigroup $\Gamma(\lambda)$ associated to a partial flag variety G/P (G being a semisimple algebraic group and $P \subseteq G$ a parabolic) via the P -regular dominant integral weight λ and full-rank valuation v is finitely generated and saturated, the following properties of the Newton–Okounkov body $\Delta(\lambda)$ are equivalent.*

- (i) $\mathcal{L}_\lambda$ is the anticanonical line bundle over G/P .
- (ii) $\Delta(\lambda)$ contains exactly one lattice point p_λ in its interior.

Furthermore, in this case the polar dual ⁽¹⁾ of the translated Newton–Okounkov body $\Delta(\lambda) - p_\lambda$ is a lattice polytope.

Using this observation we can give a very precise criterion on the reflexivity of string polytopes.

THEOREM (Theorem 1.5). *Let G be a complex classical group and λ a dominant integral weight. The standard string polytope $\mathcal{Q}_{w_0^{\text{std}}}(\lambda)$ (in the sense of [15]) is a reflexive polytope after translation by a lattice vector if and only if λ is the weight of the anticanonical line bundle over some partial flag variety G/P .*

Proof. Let us denote the weight of the anticanonical line bundle over G/P by $\lambda_{G/P}$.

Notice that this anticanonical bundle can be realized as the highest wedge power of the cotangent bundle over G/P , i.e.

$$\mathcal{L}_{\lambda_{G/P}} = \bigwedge^{\dim G/P} (\mathfrak{g}/\mathfrak{p})^*.$$

From this it is clear that $V(\lambda_{G/P})^* \simeq H^0(G/P, \mathcal{L}_{\lambda_{G/P}})$ carries the structure of a G -representation. So by Theorem 1.2 we know that $\mathcal{Q}_{w_0^{\text{std}}}(\lambda_{G/P})$ must be a lattice polytope.

The claim now follows directly from Theorem 10.2 since Kaveh showed in [14, Theorem 1] that every string polytope can be realized as a Newton–Okounkov body for a valuation with sufficient properties. $\square$

These results yield the following geometric interpretation.

THEOREM (Theorem 1.6). *Let G be a complex classical group. Then every partial flag variety admits a flat projective degeneration to a toric Gorenstein Fano variety.*

Proof. By a result of Alexeev and Brion (see [1, Theorem 3.2]) we know that each flag variety G/P admits a flat projective degeneration to the toric variety associated to the string polytope $\mathcal{Q}_{w_0}(\lambda)$ for any reduced decomposition w_0 and any P -regular dominant integral weight λ . Choosing w_0 as the standard reduced decomposition and choosing λ as the weight of the anticanonical line bundle over G/P , we see that the

⁽¹⁾We always refer to the polar dual defined as $S^* := \{y \in \mathbb{R}^N \mid \langle x, y \rangle \leq 1 \text{ for all } x \in S\}$ for an arbitrary set $S \subseteq \mathbb{R}^N$. If S is a polytope with the origin in its interior, then S^* is a polytope.

resulting string polytope will be reflexive by Theorem 1.5. Hence the limit toric variety will be Gorenstein Fano by Batyrev’s result [3, Proposition 2.2.23]. $\square$

We will finally remark that using our diagrammatic approach, it is possible to find the unique interior lattice point from Theorem 10.2. Basically, we just have to find the Gel’fand–Tsetlin pattern with the least amount of *black* double arrows possible in its identity diagram. We will show this idea in the following examples.

EXAMPLE 10.3. Let us first consider the case $G = SL_3$ and the full flag variety G/B . Then the anticanonical weight is given by $\lambda_{G/B} = 2\rho$, where $\rho = \omega_1 + \omega_2$. From Theorem 1.5 we know that the standard string polytope $\mathcal{Q}_{w_0^{\text{std}}}(2\rho)$ can be translated to a reflexive polytope. Hence it contains a unique lattice point in its interior. We can find this lattice point by finding the unique Gel’fand–Tsetlin pattern of type 2ρ such that every entry is neither equal to its left upper neighbor nor its right upper neighbor. This pattern is

$$\begin{array}{ccccc} & & 4 & & 2 & & 0 \\ & & & 3 & & 1 & \\ & & & & 2 & & \end{array}$$

and its preimage under Littelmann’s bijection $\phi_{2\rho}$ is the point $(1, 2, 1)$.

EXAMPLE 10.4. Let $G = SO_5$ and let $P(\alpha_1) \subseteq SO_5$ be the parabolic subgroup corresponding to the first simple root, i.e. $P(\alpha_1) = B\widetilde{s_{\alpha_1}}B$, where $\widetilde{s_{\alpha_1}} \in G$ is a representative of the Weyl group element s_{α_1} . For the partial flag variety $G/P(\alpha_1)$, the anticanonical weight is given by $\lambda_{G/P(\alpha_1)} = 4\omega_2 = 2\epsilon_1 + 2\epsilon_2$. Since the standard string polytope $\mathcal{Q}_{w_0^{\text{std}}}(4\omega_2)$ is not full-dimensional, we will only find points in its relative interior. We can also see this diagrammatically. Every identity diagram in this setting will have three connected nodes in the upper left corner since $2 = \lambda_1 \geq z_{1,1} \geq \lambda_2 = 2$. But if we re-embed the string polytope such that it is indeed full-dimensional, it will contain a unique interior lattice point. This point is associated to the pattern

$$\begin{array}{ccccc} & & 2 & & 2 & & 0 \\ & & & 2 & & \frac{1}{2} & \\ & & & & 1 & & 0 \\ & & & & & \frac{1}{2} & \\ & & & & & & 0 \end{array}$$

whose preimage under Littelmann’s bijection $\phi_{4\omega_2}$ is the point $(1, 2, 3, 0)$.

REMARK 10.5. For the full flag variety, let p_{X_n} denote the unique interior lattice point of the anticanonical standard string polytope $\mathcal{Q}_{w_0^{\text{std}}}(\lambda_{G/B})$. Fujita and Higashitani have calculated p_{A_n} and p_{D_n} , as well as p_{E_6} , p_{E_7} and p_{E_8} in [10, Example 4.6]. This list can be completed as follows.

$$\begin{aligned} p_{A_n} &= (1, 2, 1, 3, 2, 1, \dots, n, n-1, \dots, 2, 1) \\ p_{B_n} &= (1, 2, 3, 1, 4, 3, 5, 2, 1, \dots, 2n-2, 2n-3, \dots, n, 2n-1, n-1, \dots, 2, 1) \\ p_{C_n} &= (1, 3, 2, 1, 5, 4, 3, 2, 1, \dots, 2n-1, 2n-2, \dots, n+1, n, n-1, \dots, 2, 1) \\ p_{D_n} &= (1, 1, 3, 2, 2, 1, 5, 4, 3, 3, 2, 1, \dots, \\ &\quad 2n-3, 2n-4, \dots, n, n-1, n-1, n-2, \dots, 2, 1) \\ p_{E_6} &= (p_{D_5}, 11, 10, 9, 8, 8, 7, 7, 6, 6, 5, 4, 5, 4, 3, 2, 1) \\ p_{E_7} &= (p_{E_6}, 17, 16, 15, 14, 13, 13, 12, 12, 11, 11, \\ &\quad 10, 9, 10, 9, 8, 7, 6, 9, 8, 7, 6, 5, 5, 4, 3, 2, 1) \\ p_{E_8} &= (p_{E_7}, 28, 27, 26, 25, 24, 23, 23, 22, 22, 21, 21, 20, 19, 20, 19, \end{aligned}$$

$$\begin{aligned}
&18, 17, 16, 19, 18, 17, 16, 15, 15, 14, 13, 12, 11, 29, 18, 17, 16, \\
&15, 14, 14, 13, 13, 12, 12, 11, 10, 11, 10, 9, 8, 7, 6, 6, 5, 4, 3, 2, 1), \\
p_{F_4} &= (1, 2, 3, 1, 4, 3, 5, 2, 1, 10, 9, 8, 7, 7, 6, 5, 11, 6, 5, 4, 4, 3, 2, 1) \\
p_{G_2} &= (1, 2, 5, 3, 4, 1) \quad \text{and} \quad p'_{G_2} := (1, 4, 3, 5, 2, 1).
\end{aligned}$$

Here p_{G_2} corresponds to $w_0 = s_1 s_2 s_1 s_2 s_1 s_2$ (α_1 being the short root), whereas p'_{G_2} corresponds to the other reduced decomposition.

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CHRISTIAN STEINERT, RWTH Aachen University, Chair for Algebra and Representation Theory,
Pontdriesch 10–16, 52062 Aachen, Germany
E-mail : `steinert@art.rwth-aachen.de`