ALGEBRAIC COMBINATORICS

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Abstract Howe dualities lead to diagrammatic categories which describe the representations of Lie-type objects as a monoidal category (that is, via generators and relations). Applying this philosophy to the type Q Howe duality of Cheng–Wang and Sergeev, we introduce diagrammatic web supercategories of type Q via generators and relations and show they describe the full subcategory of supermodules for the Lie superalgebra of type Q given by the tensor products of supersymmetric tensor powers of the natural supermodule.

1. Introduction

1.1. Background. The enveloping algebra of a complex Lie algebra is a Hopf algebra and, hence, the category of finite-dimensional $U(\mathfrak{g})$-modules is naturally a monoidal $\mathbb{C}$-linear category. In trying to understand the representation theory of $U(\mathfrak{g})$ it is natural to ask if we can describe this category (or a well-chosen subcategory) as a monoidal category. Even when the category in question is semisimple it is not obvious how to describe such a category by, say, generators and relations or combinatorially. The rank one case follows from work of Rumer, Teller, and Weyl [31]. Kuperberg completed the rank 2 case by giving a diagrammatic presentation for the monoidal subcategory generated by the fundamental representations [24]. For $\mathfrak{g} = \mathfrak{sl}(n)$ a diagrammatic presentation for the monoidal category generated by the fundamental representations was conjectured for $n = 4$ by Kim [23] and for general n by Morrison [25].

In 2014, Cautis–Kamnitzer–Morrison gave a complete combinatorial description of the monoidal category generated by the fundamental representations of $U(\mathfrak{sl}(n))$ [11]; that is, the full subcategory of all modules of the form $\Lambda^{k_1}(V_n)\otimes\cdots\otimes\Lambda^{k_t}(V_n)$, where V_n is the natural module for $\mathfrak{sl}(n)$. Perhaps of greater significance is the method of proof used therein. They show that a diagrammatic description of the category follows from a skew Howe duality between $\mathfrak{gl}(m)$ and $\mathfrak{sl}(n)$. More generally, centralizing actions are now understood to naturally lead to presentations of monoidal categories. This philosophy has since been applied in a number of settings. For example, Tubbenhauer–Vaz–Wedrich give a presentation of the monoidal category of $\mathfrak{gl}(n)$-modules generated by the exterior and symmetric powers of the natural module [37]. See [28, 29, 32] for further examples. In turn, these combinatorial presentations allow for explicit

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calculations, bases, and cellular structures on the representation theory side [19] as well as constructing categorifications and Khovanov–Rozansky homology theories via foams (e.g. [27, 30]) and interesting decategorifications via traces (e.g. [1]).

1.2. Webs of Type Q. The setting of this paper is within $\mathbb{C}$-linear monoidal *supercategories*. A supercategory is a category enriched over the category of superspaces (that is, $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces) that satisfies graded versions of the usual axioms for a $\mathbb{C}$-linear monoidal category. For example, the interchange law now requires a sign according to the parity of the morphisms (see (3) for a diagrammatic version). Unless otherwise specified, all categories will be assumed to be $\mathbb{C}$-linear monoidal supercategories and all functors will be assumed to be $\mathbb{C}$-linear monoidal even superfunctors as defined in [9, Section 2].

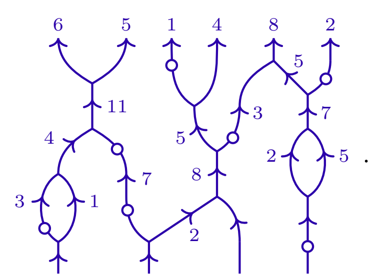
In Definition 4.1 we introduce the supercategory of upward oriented webs of type Q, $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$. The objects are monoidally generated by the set

$$\{\uparrow_k \mid k \in \mathbb{Z}_{\geqslant 0}\}.$$

That is, objects are words in symbols from this set. The generating morphisms are certain diagrams which we call dots, merges, and splits.

The $\mathbb{Z}/2\mathbb{Z}$-grading is given by declaring dots to be of degree $\bar{1}$ and the merges and splits to be of degree $\bar{0}$. As customary for diagrammatic categories, composition is given by vertical concatenation (read bottom to top) and the monoidal (or tensor) product is given by horizontal concatenation (read left to right). A diagram obtained by a finite sequence of these operations is called a *web* and a general morphism is a linear combination of webs. For example, the following web is a morphism from $\uparrow_4\uparrow_9\uparrow_6\uparrow_7$ to $\uparrow_6\uparrow_5\uparrow_1\uparrow_4\uparrow_8\uparrow_2$:

(1)



Morphisms are subject to an explicit list of diagrammatic relations. See Definition 4.1 for details.

In Definition 6.1 we introduce the oriented version, $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. The objects are monoidally generated by the set

$$\{\uparrow_k, \downarrow_k \mid k \in \mathbb{Z}_{\geqslant 0}\}.$$

The generating morphisms of $\mathfrak{q}(n)$-$\mathbf{Web}_{\uparrow\downarrow}$ are the dots, merges, and splits from before along with two new morphisms called cups and caps. Cups and caps are declared to have even parity. Again morphisms are linear combinations of webs and are subject to an explicit list of diagrammatic relations. By definition, $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ is a rigid supercategory with $\downarrow_k$ being left and right dual to $\uparrow_k$.

In Sections 4 and 6 we describe the structure of these categories. Let $\uparrow_1^k$ denote the k-fold tensor product $\uparrow_1\uparrow_1 \cdots \uparrow_1$. An important ingredient in our analysis is a thorough understanding of the endomorphism algebra

$$\mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\uparrow_1^k).$$

In Proposition 5.6 we show it is isomorphic to the Sergeev superalgebra and that it admits a natural analogue of the Jones–Wenzl projector which we call the *clasp* *idempotent* (see Definition 4.8). Using these we introduce braiding isomorphisms for general objects and show these categories are symmetric braided monoidal supercategories.

1.3. The Lie superalgebra of type Q. Let $\mathfrak{q}(n)$ be the Lie superalgebra of $2n\times 2n$ complex matrices of the form

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\},$$

where A and B are $n \times n$ matrices and where the Lie bracket is given by the graded matrix commutator (see Section 3.2 for details).

The representations of $\mathfrak{q}(n)$ do not have a classical analogue. Despite the important early work done by Penkov–Serganova, Brundan, and others to obtain character formulas and other information (see [26, 3] and references therein), the representation theory in type Q remains mostly mysterious. For example, only very recently the structure of category $\mathcal{O}$ for $\mathfrak{q}(n)$ became clear thanks to the work of Chen [13], Cheng–Kwon–Wang [14], and Brundan–Davidson [7, 8].

If V_n denotes the natural $\mathfrak{q}(n)$-supermodule given by column vectors of height $2n$, then we can consider the degree k supersymmetric tensors $S^k(V_n)$. Let $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ and $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$ denote the monoidal supercategory generated by the supersymmetric tensors, and supersymmetric tensors and their duals, respectively. That is, $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ is the full subcategory of all $\mathfrak{q}(n)$-supermodules of the form $S^{k_1}(V_n)\otimes\cdots\otimes S^{k_t}(V_n)$ for $t \geqslant 0$ and various nonnegative integers $k_1,\dots,k_t$. Similarly, $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$ is the full subcategory of all $\mathfrak{q}(n)$-supermodules which are of a similar form but where some symmetric powers are replaced with their duals. The category $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ is semisimple but the category $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$ and the category of all finite-dimensional $\mathfrak{q}(n)$-supermodules are not.

We allow all morphisms, not just grading preserving ones. This causes new phenomena to arise. For example, V_n is a simple $\mathfrak{q}(n)$-supermodule and yet Schur's Lemma in this setting implies $\mathrm{End}_{\mathfrak{q}(n)}(V_n)$ is two-dimensional. Similarly, for all $k \geqslant 1$ there is a degree reversing isomorphism between $S^k(V_n)$ and $\Lambda^k(V_n)$ and a degree preserving isomorphism between $S^k(V_n)^*$ and $S^k(V_n^*)$. Hence, $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ also contains the exterior powers of the natural supermodule and $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$ also contains the symmetric and exterior powers of the dual of the natural supermodule. The main goal of the present work is to show the supercategories $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ and $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ provide combinatorial models for $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ and $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$, respectively, in the spirit of the work discussed in Section 1.1.

1.4. Main Results. Using a Howe duality for $\mathfrak{q}(m)$ and $\mathfrak{q}(n)$ first introduced by Cheng–Wang [15] and Sergeev [35] and the strategy of Cautis–Kamnitzer–Morrison, our first main result provides essentially surjective full functors of monoidal supercategories for all $n \geqslant 1$:

$$\Psi_n^{\uparrow} : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}},$$
$$\Psi_n^{\uparrow\downarrow} : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}.$$

The functor $\Psi_n^{\uparrow\downarrow}$ is also a functor of rigid monoidal supercategories.

The functors $\Psi_n^{\uparrow}$ and $\Psi_n^{\uparrow\downarrow}$ are given explicitly on objects and morphisms. On objects $\Psi_n^{\uparrow}(\uparrow_k) = \Psi_n^{\uparrow\downarrow}(\uparrow_k) = S^k(V_n)$ and $\Psi_n^{\uparrow\downarrow}(\downarrow_k) = S^k(V_n)^*$. For example,

$$\Psi_n^{\uparrow\downarrow}(\uparrow_2\downarrow_3\uparrow_5\uparrow_3) = S^2(V_n) \otimes S^3(V_n)^* \otimes S^5(V_n) \otimes S^3(V_n).$$

The description of $\Psi_n^{\uparrow}$ and $\Psi_n^{\uparrow\downarrow}$ on generating morphisms is equally explicit, see Proposition 5.3 and Theorem 6.8.

We further show these functors fit into a commuting square of monoidal functors:

$$\begin{tikzcd} \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \arrow[r, "\Psi_n^{\uparrow}"] \arrow[d, hook] & \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}} \arrow[d, hook] \\ \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow} \arrow[r, "\Psi_n^{\uparrow\downarrow}"] & \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}, \end{tikzcd}$$

where the vertical functors are fully faithful inclusion functors. It should also be emphasized that the categories $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ and $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ do not depend on n. In this sense $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ and $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ provide universal combinatorial models for $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ and $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$ for all $n \geqslant 1$.

For $n \geqslant 1$ and $k = (n+1)(n+2)/2$, we follow Sergeev and introduce an explicit quasi-idempotent $e_{\lambda(n)} \in \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\uparrow_1^k) \cong \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\uparrow_1^k)$. Define $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow}$ and $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}$ to be, respectively, the monoidal supercategories given by the same generators and relations as $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ and $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ along with the additional relation

$$e_{\lambda(n)} = 0.$$

The second main result of the paper is given in Theorem 7.2. We show the above functors induce equivalences of monoidal supercategories

$$\Psi_n^{\uparrow} : \mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow} \xrightarrow{\cong} \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}},$$
$$\Psi_n^{\uparrow\downarrow} : \mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow} \xrightarrow{\cong} \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}.$$

These functors also fit into a commuting square of monoidal functors:

$$\begin{tikzcd} \mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow} \arrow[r, "\Psi_n^{\uparrow}"] \arrow[d, hook] & \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}} \arrow[d, hook] \\ \mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow} \arrow[r, "\Psi_n^{\uparrow\downarrow}"] & \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}, \end{tikzcd}$$

where the vertical functors are fully faithful inclusion functors.

In short, the supercategories $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow}$ and $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}$ provide complete combinatorial models for their respective categories of $\mathfrak{q}(n)$-supermodules. More generally, these functors induce equivalences between the additive and idempotent completions of $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow}$ and $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}$ and the respective full subcategories of all finite-dimensional $\mathfrak{q}(n)$-supermodules which are isomorphic to a direct summand of a direct sum of tensor products of symmetric powers of the natural supermodule (resp. natural supermodule and its dual). In this way $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow}$ completely describes what can be seen to be the full category of polynomial representations of $\mathfrak{q}(n)$ (a semisimple category) and $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}$ completely describes the full subcategory of $\mathfrak{q}(n)$-supermodules which are obtained from tensor products of polynomial representations and their duals (a non-semisimple category). In either case all such $\mathfrak{q}(n)$-supermodules can, in principle, be studied using the combinatorics of webs.

1.5. Related Work. This paper is part of a larger program to develop diagrammatic categories of type Q. That is, monoidal supercategories which have degree $\bar{1}$ morphisms which square to a non-zero scalar multiple of the identity. In [5] Brundan, Comes, and the second author introduced the oriented Brauer–Clifford category $\mathcal{OBC}$ and its affine analogue $\mathcal{AOBC}$. The category $\mathcal{OBC}$ can be viewed as the special case of oriented webs where all edges have label 1. One corollary of the present work is an analogue for $\mathcal{OBC}$ of this paper's main results. In other work the first author uses the category $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ to give a complete combinatorial description of the full subcategory of permutation supermodules for the Sergeev algebra [3]. In [12] Chang–Wang introduce a quantum Howe duality of type Q. This is done by defining an action of quantized enveloping algebras of type Q on a quantized symmetric algebra which also appeared in [2]. In [4] the authors and Davidson show this Howe duality can be used to obtain quantum analogues of the results presented here. In that context the braidings are no longer be symmetric and we show that one can obtain several interesting knot invariants from the quantum web supercategory of type Q and its relatives. Finally, in a somewhat different direction, Comes and the second author intend to show diagrammatic categories of type Q can be used to categorify the twisted Heisenberg algebra at arbitrary integral level. One can consider the additive and Karoubi envelopes of $\mathcal{OBC}$, $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$, and $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. The Grothendieck ring and the trace decategorification of these monoidal supercategories will define superalgebras. It would be interesting to determine them.

2. Monoidal supercategories

In this section we give a brief introduction to monoidal supercategories following [9, Section 2]. We refer the reader to *op. cit.* for further details.

2.1. Superspaces. Throughout the ground field will be the complex numbers, $\mathbb{C}$. A *superspace* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-vector space where elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are said to have parity $\bar{0}$ or to be *even* (resp. parity $\bar{1}$ or *odd*). Given a homogeneous element $v \in V$ we write $|v| \in \mathbb{Z}/2\mathbb{Z}$ for the parity of the element. Given two superspaces V and W, the set of all linear maps $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is naturally $\mathbb{Z}/2\mathbb{Z}$-graded by declaring that $f : V \to W$ has parity $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ if $f(V_{\varepsilon'}) \subseteq V_{\varepsilon+\varepsilon'}$ for all $\varepsilon' \in \mathbb{Z}/2\mathbb{Z}$. Let $\mathfrak{svec}$ denote the category of all superspaces with $\mathrm{Hom}_{\mathfrak{svec}}(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)$. Note that we allow maps which do not preserve the $\mathbb{Z}/2\mathbb{Z}$-grading. Let $\underline{\mathfrak{svec}}$ denote the underlying purely even category; that is, the subcategory of $\mathfrak{svec}$ consisting of all $\mathbb{C}$-superspaces, but only the grading preserving linear maps.

Given superspaces V and W, the tensor product $V \otimes W$ as vector spaces is also naturally a superspace with $\mathbb{Z}/2\mathbb{Z}$-grading given by declaring $|v \otimes w| = |v| + |w|$ for all homogeneous $v \in V$ and $w \in W$. The tensor product of linear maps between superspaces is defined via $(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$. This gives $\underline{\mathfrak{svec}}$ (but *not* $\mathfrak{svec}$) the structure of a monoidal category with unit object $\mathbb{1} = \mathbb{C}$ (viewed as superspace concentrated in even parity). The graded flip map $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ gives $\underline{\mathfrak{svec}}$ the structure of a symmetric monoidal category. Here and elsewhere we write formulas only for homogeneous elements with the general case given by extending linearly.

2.2. Monoidal supercategories. We generally follow the conventions of [9] and only give a brief summary here. Details can be found therein. By a *supercategory* we mean a category enriched in $\underline{\mathfrak{svec}}$. That is, each morphism space is a superspace, and composition and tensor product of morphisms each induces an even linear map. Similarly, a *superfunctor* $F : \mathcal{A} \to \mathcal{B}$ between supercategories is a functor enriched in $\underline{\mathfrak{svec}}$. That is, the maps $\mathrm{Hom}_{\mathcal{A}}(\mathsf{a}, \mathsf{a}') \to \mathrm{Hom}_{\mathcal{B}}(F(\mathsf{a}), F(\mathsf{a}')), f \mapsto F(f)$ are even linear maps for objects $\mathsf{a}, \mathsf{a}' \in \mathcal{A}$. Given two supercategories $\mathcal{A}$ and $\mathcal{B}$, there is a supercategory $\mathcal{A} \boxtimes \mathcal{B}$ whose objects are pairs (a, b) of objects $\mathsf{a} \in \mathcal{A}$ and $\mathsf{b} \in \mathcal{B}$ and whose morphisms are given by the tensor product of superspaces $\mathrm{Hom}_{\mathcal{A}\boxtimes\mathcal{B}}((\mathsf{a}, \mathsf{b}), (\mathsf{a}', \mathsf{b}')) = \mathrm{Hom}_{\mathcal{A}}(\mathsf{a}, \mathsf{a}') \otimes \mathrm{Hom}_{\mathcal{B}}(\mathsf{b}, \mathsf{b}')$ with composition given by

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|} (f \circ h) \otimes (g \circ k). \tag{2}$$

By a *monoidal supercategory* we mean a supercategory $\mathcal{A}$ equipped with a functor $- \otimes - : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$, a unit object $\mathbb{1}$, and even supernatural isomorphisms $(- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ and $\mathbb{1} \otimes - \xrightarrow{\sim} - \xleftarrow{\sim} - \otimes \mathbb{1}$ called *coherence maps* satisfying certain axioms analogous to the ones for a monoidal category (see [22, Chapter 1] for the coherence axioms in the enriched setting). A monoidal supercategory is called *strict* if its coherence maps are identities. A *monoidal functor* between two monoidal supercategories $\mathcal{A}$ and $\mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ equipped with an even supernatural isomorphism $(F-) \otimes (F-) \xrightarrow{\sim} F(- \otimes -)$ and an even isomorphism $\mathbb{1}_{\mathcal{B}} \xrightarrow{\sim} F\mathbb{1}_{\mathcal{A}}$ satisfying axioms analogous to the ones for a monoidal functor.

A *braided monoidal supercategory* is a monoidal supercategory $\mathcal{A}$ equipped with a $\underline{\mathfrak{svec}}$-enriched version of a braiding. More precisely, let $T : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ denote the functor defined on objects by $(\mathsf{a}, \mathsf{b}) \mapsto \mathsf{b} \otimes \mathsf{a}$ and on morphisms by $f \otimes g \mapsto (-1)^{|f||g|} g \otimes f$. A braiding on $\mathcal{A}$ is a supernatural isomorphism $\gamma : - \otimes - \to T$ satisfying the usual hexagon axioms. A *symmetric monoidal supercategory* is a braided monoidal supercategory $\mathcal{A}$ with $\gamma^{-1}_{\mathsf{a},\mathsf{b}} = \gamma_{\mathsf{b},\mathsf{a}}$ for all objects $\mathsf{a}, \mathsf{b} \in \mathcal{A}$. In this paper all braidings will be even.

Given a monoidal supercategory $\mathcal{A}$ and an object $\mathsf{a} \in \mathcal{A}$, by a *(left) dual* to a we mean an object a^* equipped with *evaluation* and *coevaluation* morphisms $\mathrm{ev}_{\mathsf{a}} : \mathsf{a}^* \otimes \mathsf{a} \to \mathbb{1}$ and $\mathrm{coev}_{\mathsf{a}} : \mathbb{1} \to \mathsf{a} \otimes \mathsf{a}^*$, respectively, in which ev_{a} and $\mathrm{coev}_{\mathsf{a}}$ have the same parity and satisfy the super version of the usual adjunction axioms. In our case the evaluation and coevaluation maps will be even and, hence, satisfy the usual adjunction axioms (see (38) for a diagrammatic example). For example, given a finite-dimensional superspace V with homogeneous basis $\{v_i \mid i \in I\}$, then $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with the evaluation and coevaluation given by $f \otimes v \mapsto f(v)$ and $1 \mapsto \sum_{i \in I} v_i \otimes v_i^*$ respectively, where $v_i^* \in V^*$ is defined by $v_i^*(v_j) = \delta_{i,j}$. Note that the coevaluation map is independent of choice of homogenous basis. A monoidal supercategory in which every object has a (left) dual is called *(left) rigid*.

The following two examples will be relevant for this paper.

Example 2.1. The tensor product and braiding defined in Section 2.1 give $\mathfrak{svec}$ the structure of a symmetric monoidal supercategory with $\mathbb{1} = \mathbb{C}$ (viewed as a superspace concentrated in parity $\bar{0}$). The symmetric braiding $M \otimes N \to N \otimes M$ is given by the graded flip map. The full subcategory of finite-dimensional superspaces is rigid.

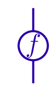
Example 2.2. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a complex Lie superalgebra. That is, a superspace with a Lie bracket which satisfies graded versions of the Lie algebra axioms. See Section 3.2 where the Lie superalgebra $\mathfrak{g} = \mathfrak{q}(n)$ is defined. Let $\mathfrak{g}$-smod denote the category of all $\mathfrak{g}$-supermodules. That is, superspaces $M = M_{\bar{0}} \oplus M_{\bar{1}}$ with an action by $\mathfrak{g}$ which respects the grading in the sense that $\mathfrak{g}_{\varepsilon}.M_{\varepsilon'} \subseteq M_{\varepsilon+\varepsilon'}$. The tensor product $M \otimes N$ has action given by $x.(m \otimes n) = (x.m) \otimes n + (-1)^{|x||m|} m \otimes (x.n)$ for all homogeneous $x \in \mathfrak{g}$, $m \in M$, and $n \in N$ and the graded flip map provides a symmetric braiding. The unit object $\mathbb{1}$ is the ground field $\mathbb{C}$ concentrated in parity $\bar{0}$ and with trivial $\mathfrak{g}$-action. In this way $\mathfrak{g}$-smod is a symmetric monoidal supercategory. The full subcategory of finite-dimensional $\mathfrak{g}$-supermodules is rigid with the action given on M^* by $(x.f)(m) = -(-1)^{|x||f|} f(x.m)$ for homogeneous elements. We will frequently omit the prefix "super" from supermodule in what follows.

When working with monoidal supercategories it will sometimes be convenient to use the following notation. Given objects a and b in a monoidal supercategory, we write $\mathsf{ab} := \mathsf{a} \otimes \mathsf{b}$. We will also write $\mathsf{a}^r := \underbrace{\mathsf{a} \otimes \cdots \otimes \mathsf{a}}_{r \text{ times}}$.

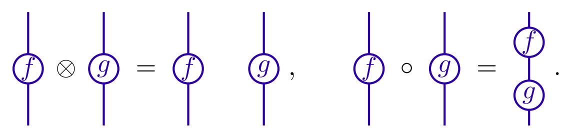
2.3. String calculus.

There is an established string calculus for strict monoidal supercategories which generalizes that of strict monoidal categories (cf. [20]). We briefly describe it here and refer the reader to [9] for details. A morphism $f : \mathsf{a} \to \mathsf{b}$ is drawn as

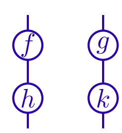
or simply as



when the objects are left implicit. (Recall that the convention used in this paper is to read diagrams from bottom to top.) The products of morphisms $f \otimes g$ and $f \circ g$ are given by horizontal and vertical stacking respectively:

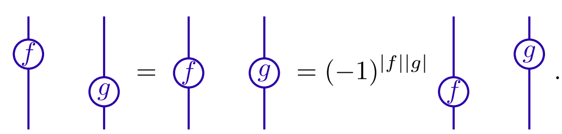


Pictures involving multiple products should be interpreted by *first composing horizontally, then composing vertically.* For example,



should be interpreted as $(f\otimes g)\circ(h\otimes k)$. In general, this is *not* the same as $(f\circ h)\otimes(g\circ k)$ because of the *super interchange law* given in (2). Diagrammatically:

(3)



2.4. Supercategories and idempotent superalgebras. Let $\mathcal{C}$ be a (small) $\mathbb{C}$-linear supercategory and let Λ be a complete irredundant set of objects. To $\mathcal{C}$ we can associate an associative $\mathbb{C}$-superalgebra $A_{\mathcal{C}}$. As a superspace,

$$A_{\mathcal{C}} = \bigoplus_{\lambda,\mu\in\Lambda} \mathrm{Hom}_{\mathcal{C}}(\lambda,\mu),$$

and the multiplication is given by extending composition of morphisms linearly. In particular, the set of identity morphisms, $\{1_\lambda : \lambda \to \lambda \mid \lambda \in \Lambda\}$, is a distinguished set of orthogonal even idempotents which make $A_{\mathcal{C}}$ a locally unital superalgebra. For each even $\mathbb{C}$-linear functor $F : \mathcal{C} \to \mathfrak{svec}$ one can define an $A_{\mathcal{C}}$-supermodule by setting

$$M = \oplus_{\lambda\in\Lambda} M_\lambda,$$

where $M_\lambda := F(\lambda)$. Given $g \in \mathrm{Hom}_{\mathcal{C}}(\lambda,\mu)$ we have the $\mathbb{C}$-linear morphism $F(g) : F(\lambda) \to F(\mu)$. The action of $g \in \mathrm{Hom}_{\mathcal{C}}(\lambda,\mu)$ on M_γ is given by $g.x = F(g)(x)$ if $\gamma = \lambda$ and $g.x = 0$ if $\gamma \neq \lambda$; the action is given in general by extending linearly. This makes M into an $A_{\mathcal{C}}$-supermodule. Moreover, we see that M is locally unital since 1_λ acts on M by projecting onto M_λ for each $\lambda \in \Lambda$.

Conversely, if A is a locally unital superalgebra with a distinguished set of orthogonal even idempotents $\{1_\lambda \mid \lambda \in \Lambda\}$, then one can define a supercategory $\mathcal{C}_A$ as follows. The objects are the elements of the set Λ and $\mathrm{Hom}_{\mathcal{C}_A}(\lambda,\mu) := 1_\mu A 1_\lambda$ for all $\lambda,\mu \in \Lambda$. The composition of morphisms is given by multiplication in A. If M is a locally unital A-supermodule, then there is an even k-linear functor $F_M : \mathcal{C}_A \to \mathfrak{svec}$ given by $F(\lambda) = 1_\lambda M$ and, given $g \in \mathrm{Hom}_{\mathcal{C}_A}(\lambda,\mu) = 1_\mu A 1_\lambda$, $F(g) : 1_\lambda M \to 1_\mu M$ is the linear map given by the action of g.

These mutually inverse constructions are the super analogues of the of the classical (non-super) relation between categories and idempotent algebras. We will freely switch between the categorical and superalgebra points of view when convenient.

2.5. Monoidal supercategories and tensor ideals. The notion of a tensor ideal in a monoidal category also has a natural super analogue. Suppose $\mathcal{C}$ is a monoidal supercategory. A *(two-sided) tensor ideal* $\mathcal{I}$ of $\mathcal{C}$ consists of a subsuperspace $\mathcal{I}(\mathsf{a},\mathsf{b}) \subseteq \mathrm{Hom}_{\mathcal{C}}(\mathsf{a},\mathsf{b})$ for each pair of objects a, b in $\mathcal{C}$, such that for all objects a, b, c, d we have $h \circ g \circ f \in \mathcal{I}(\mathsf{a},\mathsf{d})$ whenever $f \in \mathrm{Hom}_{\mathcal{C}}(\mathsf{a},\mathsf{b})$, $g \in \mathcal{I}(\mathsf{b},\mathsf{c})$, $h \in \mathrm{Hom}_{\mathcal{C}}(\mathsf{c},\mathsf{d})$ and $g \otimes \mathrm{id}_c \in \mathcal{I}(\mathsf{a}\otimes\mathsf{c},\mathsf{b}\otimes\mathsf{c})$ and $\mathrm{id}_c \otimes g \in \mathcal{I}(\mathsf{c}\otimes\mathsf{a},\mathsf{c}\otimes\mathsf{b})$.

The quotient $\mathcal{C}/\mathcal{I}$ of $\mathcal{C}$ by the tensor ideal $\mathcal{I}$ is the supercategory with the same objects as $\mathcal{C}$ and morphisms given by $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(\mathsf{a},\mathsf{b}) = \mathrm{Hom}_{\mathcal{C}}(\mathsf{a},\mathsf{b})/\mathcal{I}(\mathsf{a},\mathsf{b})$. It is straightforward to check that the quotient of a monoidal supercategory by a tensor ideal is again a monoidal supercategory. Moreover, the quotient of a braided (resp. symmetric) monoidal supercategory is again braided (resp. symmetric).

Since the intersection of tensor ideals is again a tensor ideal, there is a unique minimal tensor ideal which contains a given fixed collection of morphisms. We call this the tensor ideal generated by this collection of morphisms. If $F : \mathcal{C} \to \mathcal{D}$ is an even functor of monoidal supercategories, then the kernel, $\mathcal{K}$, of F given by setting $\mathcal{K}(\mathsf{a},\mathsf{b}) = \{f \in \mathrm{Hom}_{\mathcal{C}}(\mathsf{a},\mathsf{b}) \mid F(f) = 0\}$ is a tensor ideal of $\mathcal{C}$ and there is an induced even functor $\tilde{F} : \mathcal{C}/\mathcal{K} \to \mathcal{D}$.

If $\mathcal{C}$ is a supercategory without a monoidal structure, then one can instead consider an ideal $\mathcal{I}$ as above which satisfies only the condition on composition. One can then form the quotient supercategory $\mathcal{C}/\mathcal{I}$ as above and similar non-monoidal statements still hold.

3. Lie superalgebras of type Q

In this section we summarize the results we will need from the structure and representation theory of associative superalgebras and Lie superalgebras. The more diagrammatically inclined reader may choose to skip to Section 4 and return to this section as needed.

3.1. Representations of superalgebras. We give a brief overview of the representation theory of associative unital superalgebras over $\mathbb{C}$. Details can be found in, for example, [16, Section 3.1] or [10, Section 2]. In what follows we say a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ has dimension $m|n$ if $\dim_{\mathbb{C}} V_{\bar{0}} = m$ and $\dim_{\mathbb{C}} V_{\bar{1}} = n$. Let A be a finite-dimensional superalgebra and let M be an A-supermodule. The *parity shift* of M is the A-supermodule ΠM obtained by reversing the grading (i.e. $(\Pi M)_\varepsilon = M_{\varepsilon+\bar{1}}$ for $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$). Because we allow for odd morphisms an A-supermodule is isomorphic to its parity shift and we do not distinguish between the two.

Let A be a finite-dimensional superalgebra and let S be a simple A-supermodule. We call S type M if it remains irreducible as an A-*module* (i.e. if $\mathbb{Z}/2\mathbb{Z}$-gradings are ignored). Otherwise we call S type Q. In the former case $\mathrm{End}_A(S)$ is $1|0$-dimensional and in the latter case $\mathrm{End}_A(S)$ is $1|1$-dimensional and S admits an odd involution. This dichotomy is the analogue of Schur's Lemma in the $\mathbb{Z}/2\mathbb{Z}$-graded setting. Correspondingly, there are two types of simple finite-dimensional associative superalgebras: the superalgebra of linear endomorphisms of an $m|n$-dimensional superspace V, $M(V)$, and the superalgebra of linear endomorphisms of an $n|n$-dimensional superspace V which preserve an odd involution, $Q(V)$. In both cases V is the unique simple supermodule and is of type M and type Q, respectively. More generally, if A is a semisimple finite-dimensional associative unital superalgebra, then

$$A \cong \left(\bigoplus_{V \text{ of type M}} M(V) \right) \bigoplus \left(\bigoplus_{V \text{ of type Q}} Q(V) \right)$$

as superalgebras where the direct sum is over a complete irredundant set of simple supermodules for A.

If A and B are associative superalgebras and S and T are simple A- and B-supermodules, respectively, then $A \otimes B$ is naturally a superalgebra and the outer tensor product $S \boxtimes T$ is naturally a supermodule for $A \otimes B$. If both S and T are of type M, then $S \boxtimes T$ is a simple $A \otimes B$-supermodule of type M. If exactly one of S and T are of type Q, then $S \boxtimes T$ is a simple $A \otimes B$-supermodule of type Q. If both S and T are of type Q, then $S \boxtimes T$ is a direct sum of two isomorphic $A \otimes B$-supermodules of type M. We write $S \circledast T$ for this simple $A \otimes B$-supermodule. We extend notation by declaring $S \circledast T$ to be the simple $A \otimes B$-supermodule $S \boxtimes T$ in the other cases.

3.2. The Lie superalgebra $\mathfrak{q}(n)$. In the next several sections we summarize what we need to know about $\mathfrak{q}(n)$, Sergeev superalgebras, and Sergeev and Howe dualities. The reader may consult [16] for further details. Fix an $n|n$-dimensional superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Fix a homogeneous basis $v_1, \dots, v_n, v_{\bar{1}}, \dots, v_{\bar{n}}$ with $|v_i| = \bar{0}$ and $|v_{\bar{i}}| = \bar{1}$ for $i = 1, \dots, n$. We write $I = I(n|n)$ for the index set $\{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ and $I_0 = I_0(n|n)$ for the set $\{1, \dots, n\}$. There is an involution on I given by $i \mapsto \bar{i}$ and $\bar{i} \mapsto i$ for any $i = 1, \dots, n$. It is convenient to adopt the convention that $\bar{\bar{i}} = i$ for all $i \in I$. Let $c : V \to V$ be the odd linear map given by $c(v_i) = (-1)^{|v_i|}\sqrt{-1}v_{\bar{i}}$ for all $i \in I$, where $\sqrt{-1}$ is a fixed square root of -1.

The vector space of all linear endomorphisms of V, $\mathfrak{gl}(V)$, is naturally $\mathbb{Z}/2\mathbb{Z}$-graded as in Section 2.1. Furthermore, $\mathfrak{gl}(V)$ is a Lie superalgebra under the graded commutator bracket; this, by definition, is given by $[x,y] = xy - (-1)^{|x||y|}yx$ for all homogeneous $x, y \in \mathfrak{gl}(V)$. For $i, j \in I$ we write $e_{i,j} \in \mathfrak{gl}(V)$ for the linear map $e_{i,j}(v_k) = \delta_{j,k}v_i$. These are the matrix units and they form a homogeneous basis for $\mathfrak{gl}(V)$ with $|e_{i,j}| = |v_i| + |v_j|$.

By definition $\mathfrak{q}(V)$ is the Lie subsuperalgebra of $\mathfrak{gl}(V)$ given by

$$\mathfrak{q}(V) = \{x \in \mathfrak{gl}(V) \mid [x,c] = 0\}.$$

Then $\mathfrak{q}(V)$ has a homogenous basis given by $e^{\bar{0}}_{i,j} := e_{i,j} + e_{\bar{i},\bar{j}}$ and $e^{\bar{1}}_{i,j} := e_{\bar{i},j} + e_{i,\bar{j}}$ for $1 \leqslant i, j \leqslant n$. Note that $|e^{\varepsilon}_{i,j}| = \varepsilon$ for all $1 \leqslant i, j \leqslant n$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$.

Remark 3.1. Realized as matrices with respect to our choice of basis for V, $\mathfrak{gl}(V) = \mathfrak{gl}(n|n)$ is all $2n \times 2n$ matrices with entries from $\mathbb{C}$. In this matrix realization $\mathfrak{q}(V) = \mathfrak{q}(n)$ is the subspace

$$\mathfrak{q}(V) = \mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\} \subseteq \mathfrak{gl}(n|n). \tag{4}$$

Then $\mathfrak{q}(n)_{\bar{0}}$ (resp. $\mathfrak{q}(n)_{\bar{1}}$) is the subspace of all such matrices with $B = 0$ (resp. $A = 0$).

Fix the Cartan subalgebra of $\mathfrak{h} \subseteq \mathfrak{q}(n)$ consisting of matrices as in (4) with A and B both diagonal. For $i = 1, \dots, n$, let $\varepsilon_i : \mathfrak{h}_{\bar{0}} \to \mathbb{C}$ be defined by $\varepsilon_i(e^{\bar{0}}_{j,j}) = \delta_{i,j}$. Set $X(T) = X(T_n) = \oplus_{i=1}^{n} \mathbb{Z}\varepsilon_i \subseteq \mathfrak{h}^*_{\bar{0}}$. Fix the Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{q}(n)$ consisting of matrices with A and B both upper triangular. Corresponding to this choice, the set of roots, positive roots, and simple roots are $\{\alpha_{i,j} = \varepsilon_i - \varepsilon_j \mid 1 \leqslant i, j \leqslant n\}$, $\{\alpha_{i,j} = \varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}$, and $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leqslant i \leqslant n-1\}$, respectively.

A $\mathfrak{q}(n)$-supermodule is a superspace $W = W_{\bar{0}} \oplus W_{\bar{1}}$ with a linear action by $\mathfrak{q}(n)$ which respects the $\mathbb{Z}/2\mathbb{Z}$-grading and $[x,y].m = x.(y.m) - (-1)^{|x||y|}y.(x.m)$ for all homogeneous $x, y \in \mathfrak{q}(n)$, $w \in W$. A *weight supermodule* for $\mathfrak{q}(n)$ is a $\mathfrak{q}(n)$-supermodule W for which there is a superspace decomposition $W = \bigoplus_{\lambda \in \mathfrak{h}^*_{\bar{0}}} W_\lambda$ where W_λ is the λ-*weight space*,

$$W_\lambda := \{w \in W \mid h.w = \lambda(h)w \text{ for all } h \in \mathfrak{h}_0\}.$$

We identify $X(T)$ with $\mathbb{Z}^n$ via the map $\sum_i a_i\varepsilon_i \leftrightarrow (a_1, \dots, a_n)$. All supermodules considered in this paper will be weight supermodules with weights lying in $X(T)$.

Given a superspace W and $d \geqslant 0$, let $S^d(W)$ denote the dth supersymmetric power of W and $\Lambda^d(W)$ the dth skew supersymmetric power. If W is a supermodule for a Lie superalgebra, then $S^d(W)$ and $\Lambda^d(W)$ are naturally supermodules for the Lie superalgebra via the coproduct (see Example 2.2). Furthermore, if ΠW denotes the parity shift of W, then $S^d(W) \cong \Lambda^d(\Pi W)$ as supermodules.

Remark 3.2. In this paper we will be interested in the case when $W = V_n$ is the natural module for $\mathfrak{q}(n)$. In this case $V_n \cong \Pi V_n$ and, more generally, $S^d(V_n) \cong S^d(\Pi V_n) \cong \Lambda^d(V_n)$. Consequently, while we only consider the symmetric powers, our results implicitly include the exterior powers as well.

3.3. The Sergeev algebra. For each $k \geqslant 1$ the *Sergeev superalgebra* Ser_k is defined to be the associative unital superalgebra generated by the even elements $s_1, \dots, s_{k-1}$ and odd elements $c_1, \dots, c_k$ subject to the relations:

$$\begin{gathered} c_i^2 = 1, \quad c_ic_j = -c_jc_i, \\ s_i^2 = 1, \quad s_is_j = s_js_i \quad \text{if } i \neq j \pm 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \\ c_is_j = s_jc_i \quad \text{if } i \neq j \pm 1, \quad s_ic_i = c_{i+1}s_i, \quad s_ic_{i+1} = c_is_i \end{gathered} \tag{5}$$

for all admissible i, j. As a superspace $\mathrm{Ser}_k \cong C_k \otimes \mathbb{C}\Sigma_k$, where C_k is the Clifford superalgebra on odd generators $c_1, \dots, c_k$, and $\mathbb{C}\Sigma_k$ is the group algebra of the symmetric group viewed as a superalgebra concentrated in parity $\bar{0}$. Then $C_k \cong C_k \otimes 1$ and $\mathbb{C}\Sigma_k \cong 1 \otimes \mathbb{C}\Sigma_k$ as superalgebras and we have the mixed relation $sc_i = c_{s(i)}s$ for all $s \in \Sigma_k$.

A *strict partition* of k is a non-increasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ such that $\sum_i \lambda_i = k$ and $\lambda_i = \lambda_{i+1}$ implies $\lambda_i = 0$. Let $\mathcal{SP}(k)$ denote the set of all strict partitions of k. For a partition $\lambda \in \mathcal{SP}(k)$, set $|\lambda| = \sum_i \lambda_i$, $\ell(\lambda)$ equal to the number of nonzero parts in λ, and

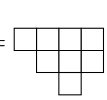
$$\delta(\lambda) = \begin{cases} 0, & \ell(\lambda) \text{ is even;} \\ 1, & \ell(\lambda) \text{ is odd.} \end{cases}$$

It is known (e.g. see [10, Lemma 3.6]) that Ser_k is semisimple and the simple supermodules are labeled by the strict partitions of k. If we write T^λ for the simple supermodule labeled by $\lambda \in \mathcal{SP}(k)$, then T^λ is of type M or Q if $\delta(\lambda) = 0$ or 1, respectively. That is, as discussed in Section 3.1,

$$\mathrm{Ser}_k \simeq \bigoplus_{\substack{\lambda \in \mathcal{SP}(k) \\ \delta(\lambda)=0}} M(T^\lambda) \oplus \bigoplus_{\substack{\lambda \in \mathcal{SP}(k) \\ \delta(\lambda)=1}} Q(T^\lambda) \tag{6}$$

as superalgebras.

We next recall the definition from [34] of certain quasi-idempotents $e_\lambda \in \mathrm{Ser}_k$ parameterized by $\mathcal{SP}(k)$. To every strict partition λ we associate to λ the *shifted frame* $[\lambda]$, the array of squares with λ_i squares in row i for $1 \leqslant i \leqslant l(\lambda)$, such that row i has been shifted to the right $i-1$ units. Here we use the English convention of reading Young diagrams from top to bottom. For example,

$[(4,3,1)] =$  .

For strict partitions λ, μ, we write $\lambda \subseteq \mu$ if $\lambda_i \leqslant \mu_i$ for all i, or, equivalently, if $[\lambda]$ is a shifted subframe of $[\mu]$.

For $1 \leqslant i < j \leqslant k-1$, define the elements $s_{i,j}, \tau_{i,j}, \pi_j \in \mathrm{Ser}_k$ by letting $s_{i,j} \in \Sigma_k$ be the transposition interchanging i and j and setting $\pi_1 = 0$, and

$$\tau_{i,j} = \frac{1}{\sqrt{2}}(c_i - c_j)s_{i,j}, \tag{7}$$

$$\pi_j = \tau_{1,j} + \tau_{2,j} + \cdots + \tau_{j-1,j}. \tag{8}$$

The π_j are the *odd Jucys–Murphy elements* of [34].

For $\lambda \in \mathcal{SP}(k)$, let T_λ be the *canonical filling* of $[\lambda]$ obtained by numbering the squares $1, 2, 3, \dots$ left-to-right in each row, starting from the top and working down; for example,

$$T_{(4,3,1)} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & 7 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}\ .$$

Given T_λ, let $\mathrm{col}(i)$ be the number of the column occupied by i in T_λ. Define $a_\lambda \in \mathrm{Ser}_k$ by

$$a_\lambda = \prod_{i=1}^{k} \left(\frac{\mathrm{col}(i)(\mathrm{col}(i)+1)}{2} - \pi_i^2 \right).$$

For example,

$$a_{(4,3,1)} = 1 \cdot (3 - \pi_2^2)(6 - \pi_3^2)(10 - \pi_4^2)(3 - \pi_5^2)(6 - \pi_6^2)(10 - \pi_7^2)(6 - \pi_8^2).$$

Let $R_\lambda \subseteq \Sigma_k$ denote the row stabilizer of T_λ and set $b_\lambda \in \mathrm{Ser}_k$ to be

$$b_\lambda = \sum_{\sigma \in R_\lambda} \sigma.$$

Finally, define $e_\lambda \in \mathrm{Ser}_k$ to be

$$e_\lambda := a_\lambda b_\lambda. \tag{9}$$

By [34, Corollary 3.3.4], each e_λ is quasi-idempotent. Moreover, $\mathrm{Ser}_k\, e_\lambda$ is a direct summand of the T^λ-isotypic component of the regular supermodule, Ser_k. In particular, e_λ lies in the summand of (6) corresponding to T^λ.

One of Sergeev's quasi-idempotents will play a distinguished role in what follows. For $n \in \mathbb{Z}_{>0}$, define the strict partition $\lambda(n) := (n+1, n, n-1, \dots, 3, 2, 1)$. The shifted frame of this partition will be an inverted staircase. For example,

$$[\lambda(2)] = \begin{array}{|c|c|c|} \hline & & \\ \hline \multicolumn{1}{c|}{} & & \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \\ \cline{3-3} \end{array}\ .$$

For $k = (n+1)(n+2)/2$, there is the corresponding quasi-idempotent

$$e_{\lambda(n)} \in \mathrm{Ser}_k\,. \tag{10}$$

3.4. Sergeev Duality. We have the following *Sergeev duality* established by Sergeev [33] (see also [16, Section 3.4.1]).

Theorem 3.3. *Let $V = V_n$ be the natural $\mathfrak{q}(n)$-supermodule and let $V^{\otimes k}$ denote its k-fold tensor product. Then:*

(a) *If $r \neq s$, then* $\mathrm{Hom}_{\mathfrak{q}(n)}\left(V^{\otimes r}, V^{\otimes s}\right) = 0$.

(b) *There is a homomorphism*

$$\psi : \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}(n)}\left(V^{\otimes k}\right)$$

given by:

$$\psi(s_i)(v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k,$$

$$\psi(c_i)(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k) = (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \otimes \cdots \otimes c(v_i) \otimes \cdots \otimes v_k.$$

(c) *The homomorphism ψ is surjective.*

(d) *There is an isomorphism of* $U(\mathfrak{q}(n)) \otimes \mathrm{Ser}_k$*-supermodules:*

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \in \mathcal{SP}(k) \\ \ell(\lambda) \leqslant n}} L_n(\lambda) \circledast T^{\lambda}.$$

Here $L_n(\lambda)$ *is the simple* $\mathfrak{q}(n)$*-supermodule of highest weight* $\lambda \in X(T_n)$ *and* T^{λ} *the simple* Ser_k*-supermodule labeled by* $\lambda \in \mathcal{SP}(k)$*. These simple supermodules are both of type* M *if* $\delta(\lambda) = 0$ *and are both of type* Q *otherwise.*

We have the following description of the kernel of the homomorphism ψ.

Proposition 3.4. *For each* $k, n \geqslant 1$ *the kernel of the homomorphism* $\psi : \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}(n)}(V_n^{\otimes k})$ *is generated as a (two-sided) ideal by*

$$\{e_\lambda \mid \lambda \in \mathcal{SP}(k), \ell(\lambda) > n\}\,. \tag{11}$$

Moreover, ψ *is an isomorphism if and only if* $k < (n+1)(n+2)/2$*.*

Proof. Since Ser_k is a semisimple superalgebra by Section 3.3, we may apply the results of Section 3.1. In light of the decomposition into simple superalgebras given in Section 3.1, a direct summand given there will be in the kernel of ψ if and only if T^λ does not appear as a summand of the Ser_k-supermodule $V_n^{\otimes k}$. By (6) this happens if and only if $\ell(n) > n$. On the other hand, since e_λ lies in the simple summand corresponding to T^λ, it follows that the kernel is generated by $\{e_\lambda \mid \lambda \in \mathcal{SP}(k), \ell(\lambda) > n\}$, as claimed. The last statement follows from Theorem 3.3 and observing there exist strict partitions of length greater than n if and only if k is greater than or equal to the given bound. ☐

3.5. The shifted Littlewood–Richardson rule. The proof of Theorem 7.2 will need the description of the kernel of ψ given in Proposition 3.4 to be extended to a description of the morphisms which vanish under the functors $\Psi_n^{\uparrow}$ and $\Psi_n^{\uparrow\downarrow}$. To do so requires Corollary 3.9, a claim about the appearance of certain composition factors in tensor products of $\mathfrak{q}(m)$-modules. This follows from the Littlewood–Richardson rule formulated by Stembridge for shifted strict tableaux, as we now explain.

Let $\mathbb{A}$ denote the ordered alphabet $\mathbb{A} = \{1' < 1 < 2' < 2 < \cdots\}$. We say the letters $1', 2', 3', \ldots$ are *marked*, and use the notation $\underline{a}$ to denote the unmarked version of any $a \in \mathbb{A}$.

Definition 3.5. *Given a strict partition* λ*, a* shifted tableau of shape λ *is a filling of the boxes of the shifted frame* $[\lambda]$ *with elements of* $\mathbb{A}$ *in such a way that*

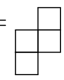
- *the entries in each row are nondecreasing,*
- *the entries in each column are nondecreasing,*
- *each row has at most one* a' *for* $a = 1, 2, 3, \ldots,$ *and*
- *each column has at most one* a *for* $a = 1, 2, 3, \ldots.$

An example of a shifted tableau of shape $(4, 3, 1)$ is

$$\begin{array}{|c|c|c|c|}\hline 1 & 2' & 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2' & 4' & 4 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{5'} \\ \cline{3-3}\end{array}\,.$$

Given a shifted tableau T of shape λ and $i = 1, 2, 3, \ldots$, let ν_i be the number of entries a in T such that $\underline{a} = i$. The *content* of T is then defined to be $\nu = (\nu_1, \nu_2, \ldots)$. When writing contents we often choose to supress the trailing zeros. For example, the content of the shifted tableau above is $(1, 2, 2, 2, 1)$.

If $\lambda \subseteq \mu$ are strict partitions, then the *skew shifted frame* $[\mu/\lambda]$ is the array of boxes obtained by removing $[\lambda]$ from $[\mu]$. For example, if $\mu = (4, 3, 1)$ and $\lambda = (3, 1)$, then we have

$$[\mu/\lambda] = $$  .

A *shifted tableau of shape* μ/λ is a filling of the boxes of $[\mu/\lambda]$ with elements of $\mathbb{A}$ in such a way that the four conditions of Definition 3.5 are satisfied.

The *word* $w = w(T) = w_1 w_2 \cdots$ associated to a (possibly skew) shifted tableau T is the sequence of elements of $\mathbb{A}$ obtained by reading the rows of T from left to right, starting with the bottom row and working up. For example, the word of the shifted tableau above is $5'2'4'412'33$.

Given a word $w = w_1 \cdots w_n$ in the alphabet $\mathbb{A}$, define a series of statistics $m_i(j)$ for $i \in \mathbb{A}$ and $j = 1, \ldots, 2n$ as follows:

- $m_i(j) =$ multiplicity of i among $w_{n-j+1}, \ldots, w_n$, for $0 \leqslant j \leqslant n$, and
- $m_i(n+j) = m_i(n)$ + multiplicity of i' among $w_1, \ldots, w_j$, for $0 < j \leqslant n$.

In particular, $m_i(0)$ will be zero for all $i \in \mathbb{A}$.

Definition 3.6. *We say a word* $w = w_1 \cdots w_n$ *has the* lattice property *if whenever* $m_i(j) = m_{i-1}(j)$ *we have*

(1) $w_{n-j} \neq i, i'$ *if* $0 \leqslant j < n$, *and*
(2) $w_{j-n+1} \neq i-1, i'$ *if* $n \leqslant j < 2n$.

Let $\underline{w} := \underline{w_1} \cdots \underline{w_n}$ denote the unmarked version of w. We are now ready to state the shifted Littlewood–Richardson rule. For a strict partition λ, let P_λ denote the Schur P-function labeled by λ as in [36]. Define $f^{\lambda}_{\mu,\nu}$ by

$$P_\lambda P_\nu = \sum_\mu f^{\mu}_{\lambda,\nu} P_\mu.$$

Stembridge provides the following combinatorial rule for computing these structure constants.

Theorem 3.7. [36, Theorem 8.3] *The coefficient* $f^{\mu}_{\lambda,\nu}$ *is the number of shifted tableaux* T *of shape* μ/λ *and content* ν *such that*

(1) *the word* $w = w(T)$ *satisfies the lattice property, and*
(2) *the leftmost* i *of* $\underline{w}$ *is unmarked in* w *for* $1 \leqslant i \leqslant l(\nu)$.

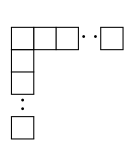
We call a shifted tableau T *satisfying* (1) *and* (2) *a* shifted Littlewood–Richardson tableau.

Recall the strict "staircase" partition $\lambda(n) = (n+1, n, \ldots, 2, 1)$ as defined in Section 3.3. It is easy to verify every $\mu \in \mathcal{SP}$ with $\ell(\mu) > n$ has $\mu \supseteq \lambda(n)$.

Proposition 3.8. *For every strict partition* μ *with* $l(\mu) > n$, *there exists a strict partition* ν *of shape* $\mu/\lambda(n)$ *such that the shifted Littlewood–Richardson coefficient* $f^{\mu}_{\lambda(n),\nu}$ *is nonzero.*

Proof. For every such μ, we construct a shifted Littlewood–Richardson tableaux $T_{\mu,n}$ of shape $\mu/\lambda(n)$ whose content ν is also a strict partition, proving the proposition.

First, we define a *hook* to be a left-justified array of boxes in which only the first row may have more than one box:



Given the shape of $\lambda(n)$, the skew shape $[\mu/\lambda(n)]$ is an ordinary partition and, hence, can be thought of as consisting of a series of hooks wedged inside each other with the corner of each hook lying on the diagonal of $[\mu/\lambda(n)]$. We number the hooks of $[\mu/\lambda(n)]$ from the upper left to the lower right.

We define $T_{\mu,n}$ to be the shifted tableau of shape $[\mu/\lambda(n)]$ whose i-th hook has the form

$$\begin{array}{ccccc} \boxed{i'} & \boxed{i} & \boxed{i} & \cdot\cdot & \boxed{i} \\ \boxed{i'} & & & & \\ \boxed{i'} & & & & \\ \vdots & & & & \\ \boxed{i'} & & & & \\ \boxed{i} & & & & \end{array}.$$

The unmarked i at the very bottom takes priority over all of the i', so that if the i-th hook has only one row then every entry will be i. For example, if $\mu = (8, 5, 4, 2)$ and $n = 2$, then we have

$$T_{\mu,2} = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & \multicolumn{2}{c}{} \\ \cline{1-3} 1' & 2' & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} 1 & 2 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

whose corresponding word is $w(T_{\mu,2}) = 121'2'31'2'21'1111$.

Clearly $T_{\mu,n}$ satisfies the conditions of Definition 3.5 and is a shifted tableau. Since ν_i is just the number of boxes in the i-th hook, the content $\nu = (\nu_1, \nu_2, \dots)$ of $T_{\mu,n}$ is a strict partition. Indeed, at its largest, the $(i+1)$-st hook extends as far to the right and one box farther down than the i-th hook, which is precisely when $\nu_{i+1} = \nu_i - 1$; otherwise $\nu_{i+1} < \nu_i - 1$. And, as was previously observed, property (2) of the shifted Littlewood–Richardson rule is satisfied.

It remains to show that $T_{\mu,n}$ has the lattice property. Let $i \geqslant 2$ and let l denote the length of the word $w = w(T_{\mu,n})$. Suppose $m_i(j) = m_{i-1}(j)$ for $0 \leqslant j < l$. If $m_i(j) = m_{i-1}(j) = 0$, then $w_{l-j} \in \{1, 1', 2, 2', \dots, i-1, (i-1)'\}$ so $w_{l-j} \neq i, i'$. The case of $m_i(j) = m_{i-1}(j) > 0$ occurs precisely if the width of the $(i-1)$-st hook exceeds that of the i-th hook by one box (e.g. the 2nd and 3rd hooks in the example). In that case w_{l-j} lies in the $(i-1)$-st hook, so $w_{l-j} \in \{i-1, (i-1)'\}$ and $w_{l-j} \neq i, i'$. Thus $T_{\mu,n}$ satisfies condition (1) of the lattice property. Since $m_i(l) \leqslant m_{i-1}(l) - 1$, and between every pair of i' in w is at least one $(i-1)'$, we have $m_i(j) \neq m_{i-1}(j)$ for $l \leqslant j \leqslant 2l$. Thus $T_{\mu,n}$ satisfies condition (2) of the lattice property, completing the proof. $\square$

The simple $\mathfrak{q}(m)$-supermodule labeled by the strict partition λ, $L_m(\lambda)$, has character given by a power of two multiple of the Schur P-function $P_\lambda(x_1, \dots, x_m)$ (e.g. see [16, Theorem 3.48]). Furthermore, tensor products of $\mathfrak{q}(m)$-supermodules corresponds to the multiplication of functions. Consequently, $f^{\mu}_{\lambda,\nu} \neq 0$ if and only $L_m(\mu)$ is a direct summand of $L_m(\lambda) \otimes L_m(\nu)$.

Corollary 3.9. *Fix a strict partition μ with $n < \ell(\mu)$ and fix a $m \geqslant \ell(\mu)$. Then there is a strict partition ν such that the simple $\mathfrak{q}(m)$-supermodule $L_m(\mu)$ is a direct summand of the tensor product $L_m(\lambda(n)) \otimes L_\mu(\nu)$ and, hence, is a direct summand of $L_m(\lambda(n)) \otimes V_m^{|\nu|}$.*

Proof. By Proposition 3.8 there exists a strict partition ν with $|\nu| = |\mu| - |\lambda(n)|$, of shape $\mu/\lambda(n)$, and with $f^{\mu}_{\lambda(n),\nu} \neq 0$. This implies $L_m(\mu)$ is a direct summand of $L_m(\lambda(n)) \otimes L_m(\nu)$, as asserted. The final statement follows from the fact that $L_m(\nu)$ is a direct summand of $V_m^{|\nu|}$ by Theorem 3.3. $\square$

3.6. Howe duality of type Q. We now describe a Howe duality of type Q first introduced by Cheng–Wang [15]. Let V_m and V_n denote the natural supermodules of $\mathfrak{q}(m)$ and $\mathfrak{q}(n)$, respectively. Recall, $c : V_m \to V_m$ and $c : V_n \to V_n$ are the odd involutions used to define $\mathfrak{q}(m)$ and $\mathfrak{q}(n)$.

Let $V_m \boxtimes V_n$ denote the $U(\mathfrak{q}(m)) \otimes U(\mathfrak{q}(n))$-supermodule given by the outer tensor product of V_m and V_n. Since both are of type Q, $V_m \boxtimes V_n \cong V_m \circledast V_n \oplus V_m \circledast V_n$ as discussed in Section 3.1. This decomposition can be made overt as follows. There is an even supermodule involution $p := \sqrt{-1}c \boxtimes c : V_m \boxtimes V_n \to V_m \boxtimes V_n$ given on homogeneous pure tensors by $p(v_a \otimes v_b) = (-1)^{|v_a|}\sqrt{-1}c(v_a) \otimes c(v_b)$. Since this map is an involution we may decompose $V_m \boxtimes V_n$ into ± 1-eigenspaces. This decomposition is as $U(\mathfrak{q}(m)) \otimes U(\mathfrak{q}(n))$-supermodules and it provides the decomposition described above. We choose $V_m \circledast V_n$ to be the $+1$ eigenspace of p.

We can describe $V_m \circledast V_n$ completely explicitly as follows. Set

$$\begin{aligned} x_{i,j} &= v_i \otimes v_j + \sqrt{-1}v_{\bar{i}} \otimes v_{\bar{j}}, \\ y_{i,j} &= v_i \otimes v_{\bar{j}} - \sqrt{-1}v_{\bar{i}} \otimes v_j. \end{aligned} \tag{12}$$

Then we have the following:

- The superspace $V_m \circledast V_n$ has a homogeneous basis $\{x_{i,j}, y_{i,j} \mid i \in I_0(m|m), j \in I_0(n|n)\}$ with the parity of $x_{i,j}$ (resp. $y_{i,j}$) equal to $\bar{0}$ (resp. $\bar{1}$);
- As a $U(\mathfrak{q}(m))$-supermodule $V_m \circledast V_n \cong V_m^{\oplus n}$ with the $U(\mathfrak{q}(m))$ acting on $x_{i,j}$ (resp. $y_{i,j}$) as on v_i (resp. $-\sqrt{-1}v_{\bar{i}}$);
- As a $U(\mathfrak{q}(n))$-supermodule $V_m \circledast V_n \cong V_n^{\oplus m}$ with $U(\mathfrak{q}(n))$ acting on $x_{i,j}$ (resp. $y_{i,j}$) as on v_j (resp. $v_{\bar{j}}$).

Note that since $U(\mathfrak{q}(m))$ and $U(\mathfrak{q}(n))$ are Hopf superalgebras, there is a natural Hopf superalgebra structure on $U(\mathfrak{q}(m)) \otimes U(\mathfrak{q}(n))$. In particular, the symmetric superalgebra

$$\mathcal{S} := S\left(V_m \circledast V_n\right) = \bigoplus_{k \geqslant 0} S^k\left(V_m \circledast V_n\right)$$

is a $U(\mathfrak{q}(m)) \otimes U(\mathfrak{q}(n))$-supermodule via the coproduct. As a superalgebra $\mathcal{S}$ is the free supercommutative superalgebra generated by $\{x_{i,j}, y_{i,j} \mid i \in I_0(m|m), j \in I_0(n|n)\}$, where we recall that a superalgebra A is supercommutative if $ab = (-1)^{|a||b|}ba$ for all homogeneous $a, b \in A$.

A direct calculation verifies that $\mathcal{S}$ is a weight supermodule for $\mathfrak{q}(m)$ with weights lying in

$$X(T_m)_{\geqslant 0} := \left\{ \lambda = \sum_{i=1}^{m} \lambda_i \varepsilon_i \in X(T_m) \;\middle|\; \lambda_i \geqslant 0 \text{ for } i = 1, \ldots, m \right\}.$$

Since the actions of $U(\mathfrak{q}(m))$ and $U(\mathfrak{q}(n))$ commute, the decomposition of $\mathcal{S}$ into weight spaces for $U(\mathfrak{q}(m))$ is a decomposition into $\mathfrak{q}(n)$-supermodules. Given $t \geqslant 0$ and a tuple of nonnegative integers, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$, we set the shorthand

$$S^{\lambda} := S^{\lambda_1}(V_n) \otimes \cdots \otimes S^{\lambda_t}(V_n).$$

A straightforward computation using (12) verifies the following.

Lemma 3.10. *Let*

$$\mathcal{S} = \bigoplus_{\lambda \in X(T_m)} \mathcal{S}_{\lambda}$$

be the decomposition into weight spaces with respect to the Cartan subalgebra of $\mathfrak{q}(m)$. *Then* $\mathcal{S}_{\lambda} \neq 0$ *if and only if* $\lambda \in X(T_m)_{\geqslant 0}$. *Furthermore, if* $\lambda = \sum_{i=1}^{m} \lambda_i \varepsilon_i \in X(T_m)_{\geqslant 0}$, *then as a* $\mathfrak{q}(n)$*-supermodule*

$$\mathcal{S}_{\lambda} \cong S^{\lambda} = S^{\lambda_1}(V_n) \otimes \cdots \otimes S^{\lambda_m}(V_n).$$

For the full statement of Howe duality in type Q see [15, Theorem 3.1].

3.7. THE SUPERCATEGORY $\dot{\mathbf{U}}(\mathfrak{q}(m))$. We now introduce a $\mathbb{C}$-linear supercategory $\dot{\mathbf{U}}(\mathfrak{q}(m))$ via generators and relations. The reader may find it helpful to compare with the $\mathbb{C}(q)$-linear category $\dot{\mathbf{U}}_q(\mathfrak{gl}(m))$ given in [11, Section 4.1]. When writing compositions of morphisms we often write them as products (e.g. $fg = f \circ g$). To lighten notation we use the same name for morphisms between different objects and leave the objects implicit when there is no risk of confusion (e.g. when a statement is true for all morphisms for which it makes sense regardless of domain and range). With this convention in mind, if f is a family of morphisms of the same name which can be composed and $k \geqslant 1$, then $f^{(k)}$ denotes the morphism $f^k/k!$. Finally, when a statement requires that we specify the domain and/or range, we do so by pre/post-composing with the relevant identity morphisms. In what follows recall $\{\alpha_i \mid i = 1, \dots, m-1\} \subset X(T_m)$ denotes the set of simple roots for $\mathfrak{q}(m)$ (see Section 3.2).

DEFINITION 3.11. *Let $\dot{\mathbf{U}}(\mathfrak{q}(m))$ be the $\mathbb{C}$-linear supercategory with set of objects $X(T_m)$ and with the morphisms generated by $e_i, e_{\bar{i}} : \lambda \to \lambda + \alpha_i$, $f_i, f_{\bar{i}} : \lambda \to \lambda - \alpha_i$, and $h_{\bar{j}} : \lambda \to \lambda$ for all $\lambda \in X(T_m)$, $i = 1, \dots, m-1$, $j = 1, \dots, m$. For all i, j the parities of e_i and f_i are even and the parities of $e_{\bar{i}}$, $f_{\bar{i}}$, and $h_{\bar{j}}$ are odd. We write $1_\lambda : \lambda \to \lambda$ for the identity morphism.*

The morphisms in $\dot{\mathbf{U}}(\mathfrak{q}(m))$ are subject to the following relations for all objects λ and all admissible i, j:

$$\begin{aligned} h_{\bar{i}}^2 1_\lambda &= \lambda_i 1_\lambda, \\ h_{\bar{i}} h_{\bar{j}} &= -h_{\bar{j}} h_{\bar{i}} \quad \text{if } i \neq j, \end{aligned} \tag{Q1}$$

$$\begin{aligned} h_{\bar{i}} e_j - e_j h_{\bar{i}} &= \delta_{i,j} e_{\bar{j}} - \delta_{i,j+1} e_{\bar{j}}, \\ h_{\bar{i}} f_j - f_j h_{\bar{i}} &= -\delta_{i,j} f_{\bar{j}} + \delta_{i,j+1} f_{\bar{j}}, \\ h_{\bar{i}} e_{\bar{j}} + e_{\bar{j}} h_{\bar{i}} &= (\delta_{i,j} + \delta_{i,j+1}) e_j, \\ h_{\bar{i}} f_{\bar{j}} + f_{\bar{j}} h_{\bar{i}} &= (\delta_{i,j} + \delta_{i,j+1}) f_j, \end{aligned} \tag{Q3}$$

$$\begin{aligned} (e_i f_j - f_j e_i) 1_\lambda &= \delta_{i,j} (\lambda_i - \lambda_{i+1}) 1_\lambda, \\ (e_{\bar{i}} f_{\bar{j}} + f_{\bar{j}} e_{\bar{i}}) 1_\lambda &= \delta_{i,j} (\lambda_i + \lambda_{i+1}) 1_\lambda, \\ e_{\bar{i}} f_j - f_j e_{\bar{i}} &= \delta_{i,j} (h_{\bar{i}} - h_{\overline{i+1}}), \\ e_i f_{\bar{j}} - f_{\bar{j}} e_i &= \delta_{i,j} (h_{\bar{i}} - h_{\overline{i+1}}), \end{aligned} \tag{Q4}$$

$$\begin{aligned} e_i e_{\bar{j}} - e_{\bar{j}} e_i = e_{\bar{i}} e_{\bar{j}} + e_{\bar{j}} e_{\bar{i}} &= 0 \quad \text{if } i \neq j \pm 1, \\ f_i f_{\bar{j}} - f_{\bar{j}} f_i = f_{\bar{i}} f_{\bar{j}} + f_{\bar{j}} f_{\bar{i}} &= 0 \quad \text{if } i \neq j \pm 1, \\ e_i e_j - e_j e_i = f_i f_j - f_j f_i &= 0 \quad \text{if } |i - j| > 1, \end{aligned} \tag{Q5}$$

$$\begin{aligned} e_i e_{i+1} - e_{i+1} e_i &= e_{\bar{i}} e_{\overline{i+1}} + e_{\overline{i+1}} e_{\bar{i}}, \\ e_i e_{\overline{i+1}} - e_{\overline{i+1}} e_i &= e_{\bar{i}} e_{i+1} - e_{i+1} e_{\bar{i}}, \\ f_i f_{i+1} - f_{i+1} f_i &= f_{\bar{i}} f_{\overline{i+1}} + f_{\overline{i+1}} f_{\bar{i}}, \\ f_i f_{\overline{i+1}} - f_{\overline{i+1}} f_i &= f_{\bar{i}} f_{i+1} - f_{i+1} f_{\bar{i}}, \end{aligned} \tag{Q6}$$

$$\begin{aligned} e_i^{(2)} e_j - e_i e_j e_i + e_j e_i^{(2)} &= 0 \quad \text{if } i = j \pm 1, \\ e_{\bar{i}} e_i e_j - e_{\bar{i}} e_j e_i - e_i e_j e_{\bar{i}} + e_j e_i e_{\bar{i}} &= 0 \quad \text{if } i = j \pm 1, \\ f_i^{(2)} f_j - f_i f_j f_i + f_j f_i^{(2)} &= 0 \quad \text{if } i = j \pm 1, \\ f_{\bar{i}} f_i f_j - f_{\bar{i}} f_j f_i - f_i f_j f_{\bar{i}} + f_j f_i f_{\bar{i}} &= 0 \quad \text{if } i = j \pm 1 \end{aligned} \tag{Q7}$$

where

$$e_i^{(k)} := \frac{e_i^k}{k!},\ f_i^{(k)} := \frac{f_i^k}{k!}$$

are the divided powers*.*

Remark 3.12. As discussed in Section 2.4, the supercategory $\dot{\mathbf{U}}(\mathfrak{q}(m))$ defines a locally unital superalgebra

$$\dot{U}(\mathfrak{q}(m)) = \bigoplus_{\lambda,\mu\in X(T_m)} \mathrm{Hom}_{\dot{\mathbf{U}}(\mathfrak{q}(m))}(\lambda,\mu).$$

Using the presentation given in [18, Proposition 2.1] one sees that $\dot{U}(\mathfrak{q}(m))$ is an idempotent version of the enveloping superalgebra $U(\mathfrak{q}(m))$.

Definition 3.13. *Let $\dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0}$ be the quotient of $\dot{\mathbf{U}}(\mathfrak{q}(m))$ given by setting $1_\lambda = 0$ for all $\lambda \notin X(T_m)_{\geqslant 0}$ (i.e. it is $\dot{\mathbf{U}}(\mathfrak{q}(m))/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by $\{1_\lambda \mid \lambda \notin X(T_m)_{\geqslant 0}\}$).*

Applying the discussion in Section 2.4, the corresponding superalgebra $\dot{U}(\mathfrak{q}(m))_{\geqslant 0}$ is the quotient of $\dot{U}(\mathfrak{q}(m))$ by the ideal generated by $\{1_\lambda \mid \lambda \notin X(T_m)_{\geqslant 0}\}$.

Define $\mathfrak{q}(n)$-$\mathbf{Mod}_{\mathcal{S}}$ to be the monoidal supercategory of $\mathfrak{q}(n)$-supermodules monoidally generated by $\{S^p(V_n) \mid p \geqslant 0\}$. That is, it is the full subcategory of $\mathfrak{q}(n)$-supermodules consisting of objects of the form

$$S^{p_1}(V_n) \otimes \cdots \otimes S^{p_t}(V_n),$$

where $t, p_1, \ldots, p_t \in \mathbb{Z}_{\geqslant 0}$. In particular, by convention, $S^0(V_n)$ is the trivial supermodule, $\mathbb{C}$.

Since $\mathcal{S}$ is a weight supermodule for $U(\mathfrak{q}(m))$ with all weights lying in $X(T_m)$, it is a supermodule for the idempotent version of the enveloping superalgebra, $\dot{U}(\mathfrak{q}(m))$. Moreover, since the weight space $\mathcal{S}_\lambda$ equals zero whenever $\lambda \notin X(T_m)_{\geqslant 0}$, this representation factors through and defines a representation of $\dot{U}(\mathfrak{q}(m))_{\geqslant 0}$. In an abuse of notation we write

$$\phi : \dot{U}(\mathfrak{q}(m))_{\geqslant 0} \to \mathrm{End}_{\mathfrak{q}(n)}(\mathcal{S})$$

for this representation. As discussed in Section 3.7, the notion of a supermodule for a locally unital superalgebra with a distinguished set of idempotents is equivalent to having a functor $\dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0} \to \mathfrak{svec}$. The fact that the action of $\dot{U}(\mathfrak{q}(m))_{\geqslant 0}$ on $\mathcal{S}$ commutes with the action of $\mathfrak{q}(n)$ implies that this functor can be viewed as having codomain the category of $\mathfrak{q}(n)$-supermodules. The existence of this functor is summarized in the following result.

Proposition 3.14. *For every $m, n \geqslant 1$ there exists a functor of supercategories*

$$\Phi_{m,n} : \dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}.$$

On objects,

$$\Phi_{m,n}(\lambda) = S_\lambda \cong S^{\lambda_1}(V_n) \otimes \cdots \otimes S^{\lambda_m}(V_n),$$

for all $\lambda = \sum_{i=1}^m \lambda_i \varepsilon_i \in X(T_m)_{\geqslant 0}$. On a morphism $x \in \mathrm{Hom}_{\dot{\mathbf{U}}(\mathfrak{q}(m))}(\lambda,\mu)$,

$$\Phi_{m,n}(x) = \phi(x).$$

Remark 3.15. Let m', m be positive integers with $m' \geqslant m$. Given an element $\lambda = (\lambda_1, \ldots, \lambda_m) \in X(T_m)$ we can view λ as the element $\lambda = (\lambda_1, \ldots, \lambda_m, 0, \ldots, 0) \in X(T_{m'})$ by extending by $m' - m$ zeros. With this identification in mind, there is a functor of supercategories,

$$\Theta_{m,m'} : \dot{\mathbf{U}}(\mathfrak{q}(m)) \to \dot{\mathbf{U}}(\mathfrak{q}(m')),$$

given by sending the objects and generating morphisms of $\dot{\mathbf{U}}(\mathfrak{q}(m))$ to the objects and morphisms of the same name in $\dot{\mathbf{U}}(\mathfrak{q}(m'))$. Moreover this functor defines a functor, which we call by the same name,

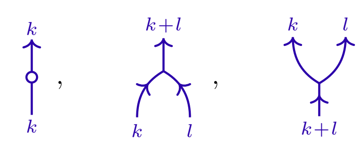
$$\Theta_{m,m'} : \dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0} \to \dot{\mathbf{U}}(\mathfrak{q}(m'))_{\geqslant 0}.$$

For any fixed $n \geqslant 1$ the functors $\Phi_{m,n}$ and $\Phi_{m',n}$ are compatible in the sense that $\Phi_{m',n} \circ \Theta_{m,m'}$ and $\Phi_{m,n}$ are canonically isomorphic.

4. Upward Webs

4.1. Upward Webs of Type Q. In what follows we introduce the diagrammatic supercategory of webs of type Q. Much of our exposition parallels that of [37].

Definition 4.1. *Let* $\mathfrak{q}$-$\mathbf{Web}_\uparrow$ *be the monoidal supercategory generated by the objects* $\{\uparrow_k \mid k \in \mathbb{Z}_{\geqslant 0}\}$, *and with generating morphisms*

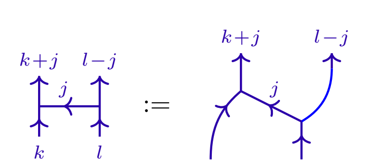


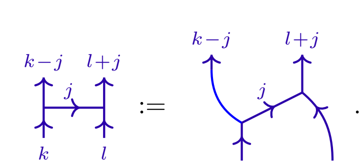
for $k, l \in \mathbb{Z}_{>0}$. *We call these* dots, merges, *and* splits, *respectively. The* $\mathbb{Z}/2\mathbb{Z}$*-grading is given by declaring merges and splits to have parity* $\bar{0}$ *and dots to have parity* $\bar{1}$. *The morphisms in* $\mathfrak{q}$-$\mathbf{Web}_\uparrow$ *are subject to the relations* (13)–(18), (20), *and* (21).

We use the diagrammatic calculus described in Section 2.3 to work with morphisms in $\mathfrak{q}$-$\mathbf{Web}_\uparrow$. In particular, diagrams are read bottom to top. Vertical concatenation is composition and horizontal concatenation is the monoidal product. In this way any finite sequence of these operations applied to merges, splits, and dots yields a diagram which is a morphism in $\mathfrak{q}$-$\mathbf{Web}_\uparrow$. We call such a diagram a *web*. If $\mathsf{a} = (a_1, \dots, a_r)$ is a tuple of nonnegative integers, then we write $\uparrow_\mathsf{a}$ for the object $\uparrow_{a_1}\uparrow_{a_2} \cdots \uparrow_{a_r}$. If $\mathsf{a} = (a_1, \dots, a_r)$ and $\mathsf{b} = (b_1, \dots, b_s)$ are tuples of positive integers, then we say a web is of type $\mathsf{a} \to \mathsf{b}$ if it is a morphism from $\uparrow_\mathsf{a} \to \uparrow_\mathsf{b}$. For example, the web in (1) is of type $(4, 9, 6, 7) \to (6, 5, 1, 4, 8, 2)$.

We follow the convention that an edge labeled by zero is understood to mean the edge is omitted. We declare any web containing an edge labeled by a negative integer to be the zero morphism. When no confusion is possible, we sometimes choose to suppress edge labels.

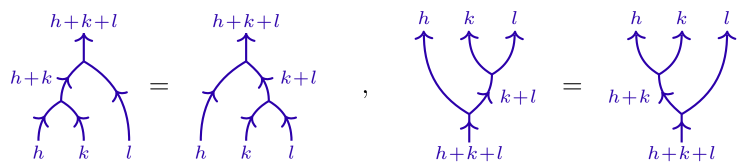
An arbitrary morphism from a to b is a linear combination of webs of type $\mathsf{a} \to \mathsf{b}$. To write the relations for $\mathfrak{q}$-$\mathbf{Web}_\uparrow$ it is convenient to set the following shorthand. A *ladder* is a web which is finite sequence of monoidal products and compositions of identities, dots, and webs of the form

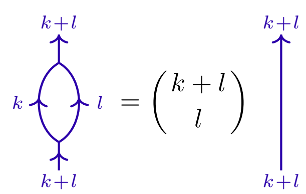


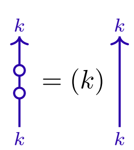


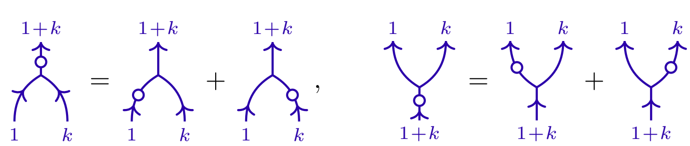
for $k, l \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{\geqslant 0}$. The edge which connects two vertical strands is called a *rung*. Note that by choosing a suitable k, l, and j every merge and split is itself a ladder (for example, by choosing $j = l$ and $k = 0$ in the first ladder).

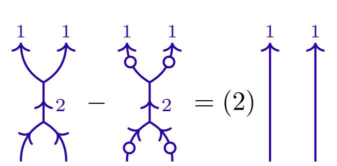
The morphisms in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ are subject to the following relations for all nonnegative integers h, k, l, along with the relations obtained by reflecting the webs in (16) across a vertical axis, and by reversing all rung orientations of the ladders in (20) and (21), where edge labels at the top of the diagram are changed as needed to make a valid web. In what follows, to avoid confusing scalars with edge labels we usually choose to write the scalars in parentheses. Also, as usual, $\binom{k+l}{l} = \frac{(k+l)!}{k!l!}$.

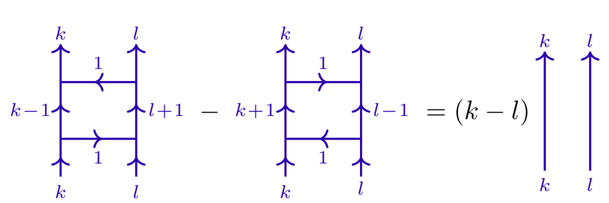
(13) 

(14) 

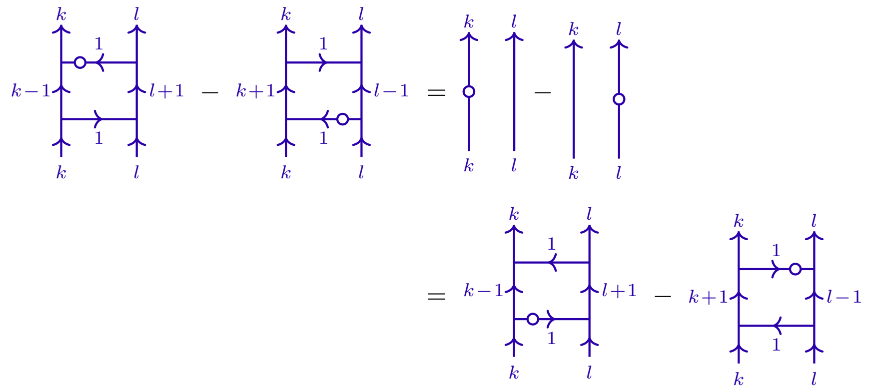
(15) 

(16) 

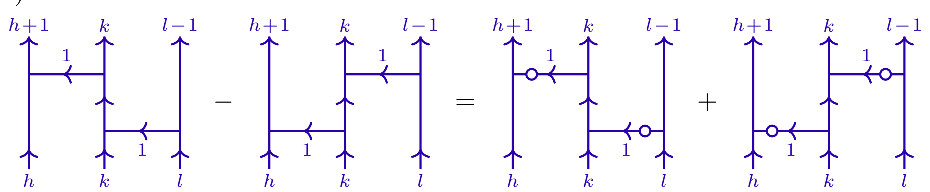
(17) 

(18) 

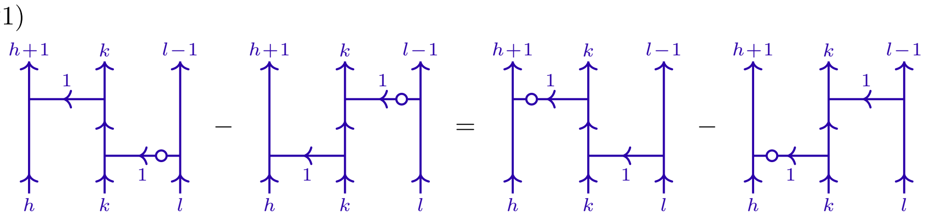
(19)



(20)



(21)



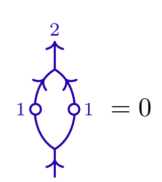
Remark 4.2. While we choose to work over the complex numbers, the definition of $\mathfrak{q}(n)$-$\mathbf{Web}_{\uparrow}$ makes sense over the integers and, more generally, any commutative ring with 1.

4.2. Additional Relations. From the defining relations of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ we deduce the following useful consequences.

Lemma 4.3. *For all $k \geqslant 1$ the following relations holds in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$:*

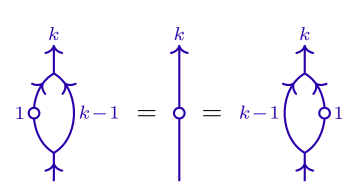
(a)

(22)

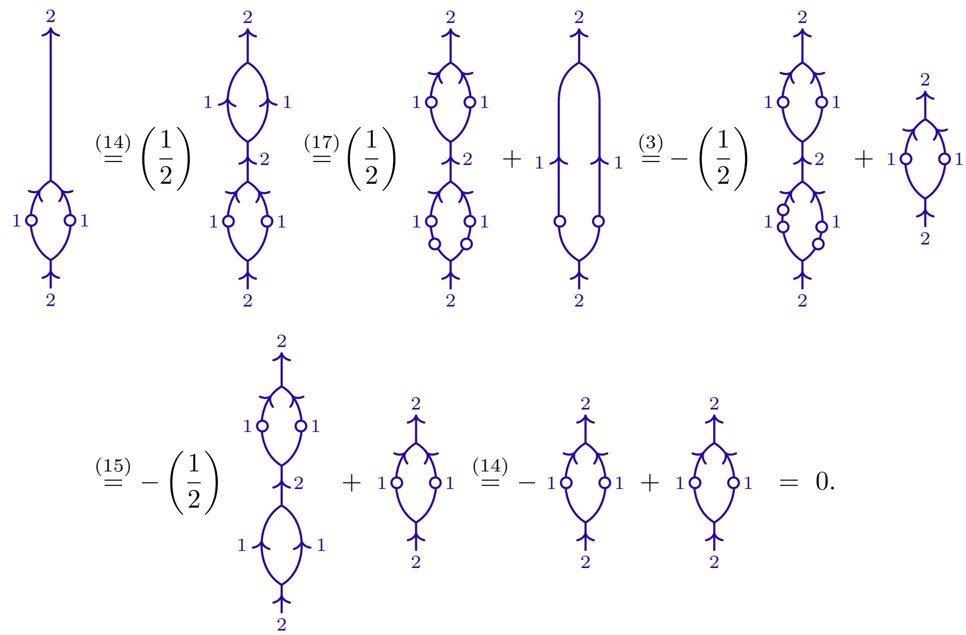


(b)

(23)

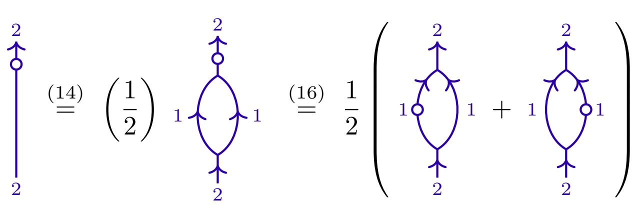


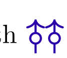
Proof. To prove (22) is a direct calculation similar to the analogous result for type A [37, Lemma 2.9] (keeping in mind the super interchange law) and is as follows:

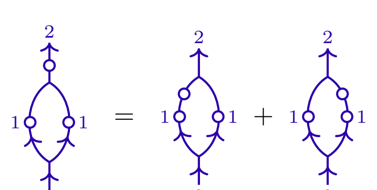


To prove (23), we first prove the case of $k=2$. For this, we start by computing that

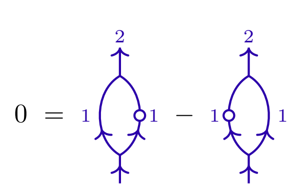
(24)



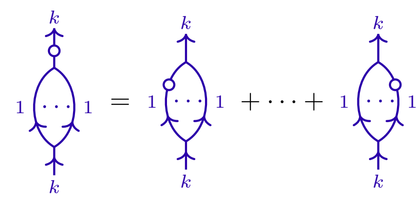
Next, we compose (16) on bottom with  followed by a split to get



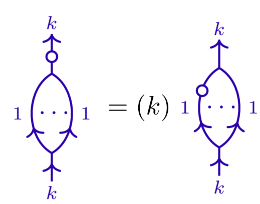
Using (22) on the left, and (2) and (15) on the right, this becomes



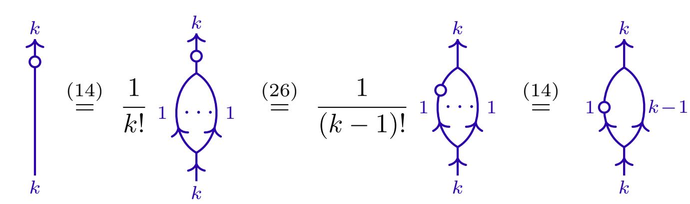
Combining the above with (24) and symmetry, we have (23) in case $k = 2$. For general k, we use (16) repeatedly to get

(25) 

where the small dots indicate k-strands which have been completely “exploded” into k separate 1-strands. By (13) the order this is done does not matter. The sum is over the k different webs with a dot on a unique 1-strand. By (13) and the $k = 2$ case, the summands are pairwise equal and we have, for example,

(26) 

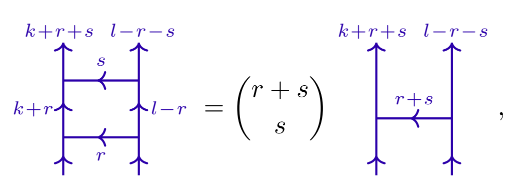
where, on the right, only the leftmost 1-strand has a dot. We finish the proof by computing that



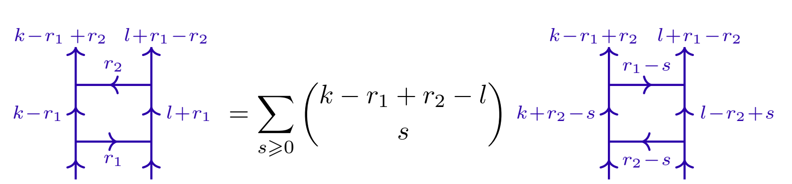
and noting that the other side of (23) follows by symmetry. ☐

Lemma 4.4. *For $h, k, l, r, s \geqslant 0$,*

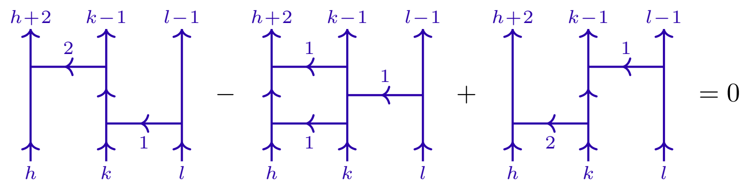
(a)

(27) 

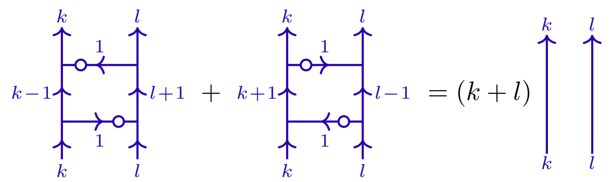
(b)

(28) 

(c)

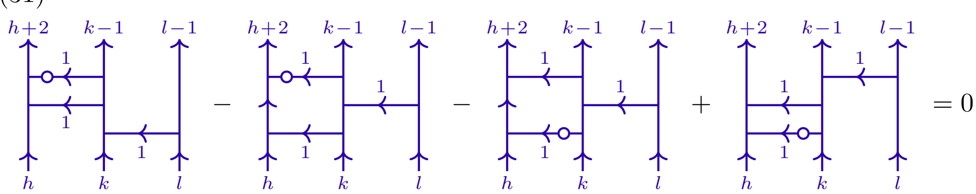
(29) 

(d)

(30) 

(e)

(31)



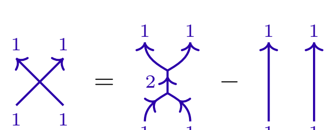
In addition, we have the equations obtained by reversing all rung orientations of the ladders in (27) *(relabeling the tops of diagrams as needed to make a valid web). We also have the equations obtained from* (29) *and* (31) *by reversing all rung orientations, reflecting across the vertical line through the middle arrow, and both reversing and reflecting.*

Proof. Since the relations (27)–(29) do not have dots and since the defining relations of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ without dots are a version of type A web relations, the type A proofs of these statements apply. The reader who would like to verify these relations will find (27) follows from (13) and (14); (28) follows from (30) by an induction argument; and (29) can be proven arguing as in the proof of [37, Lemma 2.10(c)].

The proof of (30) is similar to [37, Lemma 2.10(b)]. It involves first applying (18) on the edges labeled $l+1$ (viewed as a square with left side labeled by 0) and $k+1$ (viewed as square with right side labeled by zero) in the webs on the left. The result is four diagrams, two of which can be reduced using (14) and (15) and two of which can be reduced using (14) and (17). After simplifying one obtains the right hand side of (30).

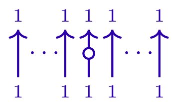
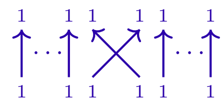
To prove (31) one can use (20) to rewrite the middle two diagrams on the left hand side into a linear combination of six diagrams. Combining (13) and (22) shows the two diagrams which have neighboring rungs with dots are equal to zero. Two of the four remaining diagrams cancel out and the remaining two cancel with the other two diagrams given on the left side of (31). $\square$

4.3. Crossings and the Sergeev Algebra. We introduce the following additional diagram as a shorthand. Define the upward crossing morphism in $\mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\uparrow_1^2)$ by



More generally, we introduce the following notation.

Definition 4.5. *Fix $k \in \mathbb{Z}_{>0}$. For $i = 1, \dots, k$ and $j = 1, \dots, k-1$, define the morphisms $c_i, s_j \in \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow_1^k)$ by*

$c_i =$  , $s_j =$ 

where the dot on c_i is on the ith strand and s_j is the crossing of the jth and $(j+1)$st strands.

Drawing these morphisms as crossings is justified by the following lemma.

Lemma 4.6. *For any $k \geqslant 2$ the following relations hold in $\mathfrak{q}$-$\mathbf{Web}_\uparrow$ for all admissible i, j:*

(a) $s_j^2 = 1$;
(b) $s_i s_j = s_j s_i$ *if* $|i-j| > 1$;
(c) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$;
(d) $s_i c_j = c_j s_i$ *if* $|i-j| > 1$;
(e) $s_i c_i = c_{i+1} s_i$.

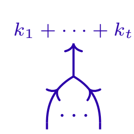
Proof. Statements (b) and (d) follow immediately from the super interchange law, while the rest can be verified by direct calculations involving (14), (18), and (19). ☐

Comparing the previous lemma to the defining relations of the Sergeev algebra given in Section 3.4 we see there is a homomorphism of superalgebras

$$\xi_k : \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow_1^k) \tag{32}$$

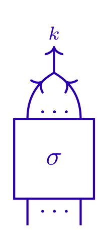
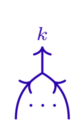
for any $k \geqslant 1$ which sends the generators of Ser_k to the morphisms in $\mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow_1^k)$ of the same name. In Section 5.3 we will see that this map is an isomorphism. Meanwhile, for $x \in \mathrm{Ser}_k$ we abuse notation by writing $x \in \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow_1^k)$ for $\xi_k(x)$. In particular, in what follows we do this for elements of the symmetric group Σ_k via our identification of $\mathbb{C}\Sigma_k$ as a subalgebra of Ser_k in Section 3.3.

In what follows we allow ourselves to draw merges and splits of multiple strands. For example, define

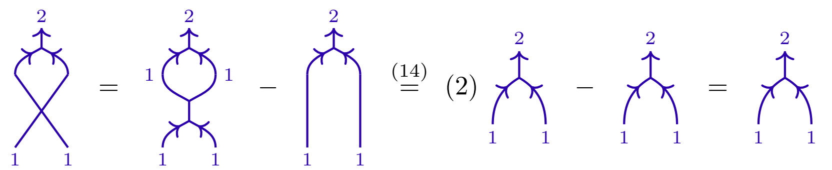


to be the vertical concatenation of $t-1$ merge diagrams. By (13) the resulting morphism is independent of how this is done. We define the split into multiple strands similarly.

Lemma 4.7. *Let $k \in \mathbb{Z}_{>0}$ and let $\sigma \in \Sigma_k$. Then,*

(33)  $=$  .

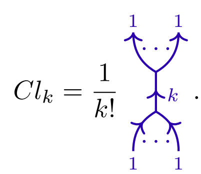
Proof. First, note that

(34) 

The first equality is by definition and the second follows from (14). Since every permutation is a product of simple transpositions, a straightforward calculation using (34) and (13) proves the statement in general. ☐

4.4. The clasp idempotents.

Definition 4.8. *For $k \in \mathbb{Z}_{>1}$, the k-th* clasp $Cl_k \in \mathrm{End}_{\mathbf{q\text{-}Web}_{\uparrow}}(\uparrow_1^k)$ *is given by*

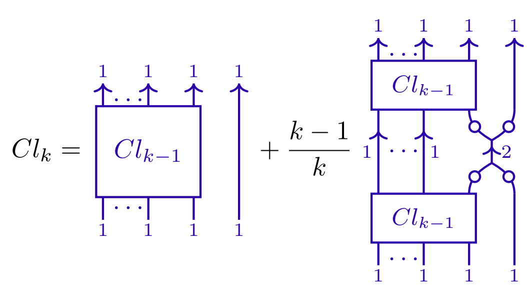


In addition, $Cl_1 \in \mathrm{End}_{\mathbf{q\text{-}Web}_{\uparrow}}(\uparrow_1)$ is understood to be the identity.

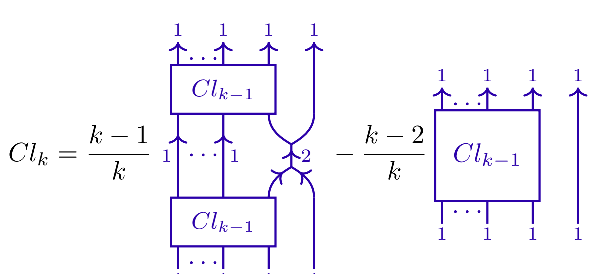
Note that a calculation using (14) shows Cl_k is an idempotent for all $k \in \mathbb{Z}_{\geqslant 1}$.

The following lemma shows clasps admit recursion formulas similar to those of the *Jones–Wenzl projectors* in the Temperley–Lieb algebra (e.g. see [40]).

Lemma 4.9. *For $k \in \mathbb{Z}_{>1}$,*

(a) 

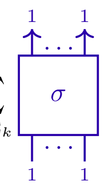
and

(b) 

Proof. The proof of the first recursion identical to [29, Lemma 2.13]. See also [37, Lemma 2.12]. The second recursion follows from the first by applying (17) to the rightmost webs. ☐

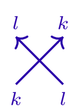
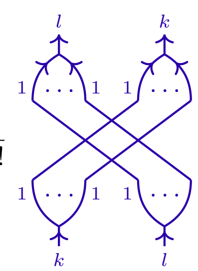
An inductive argument using the recursion formulas yields the following closed formula for the clasps. This can be seen as a super analogue of the classical formula of the Jones–Wenzl projector in terms of permutations, (e.g. see [21, Section 3.2]).

Lemma 4.10. *For $k \geqslant 1$ we have*

(35) $$Cl_k = \frac{1}{k!} \sum_{\sigma \in \Sigma_k}$$  .

4.5. Symmetry on $\mathfrak{q}$-**Web**$_\uparrow$. We now introduce braidings between arbitrary objects of $\mathfrak{q}$-**Web**$_\uparrow$ and show that this provides a symmetric braiding. We first define the braiding between two generating objects.

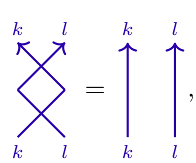
Definition 4.11. *For $k, l \in \mathbb{Z}_{>0}$ define*

$$\beta_{\uparrow_k, \uparrow_l} =$$  $$= \frac{1}{k!\, l!}$$  .

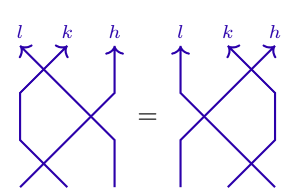
The following lemma shows that $\beta_{\uparrow_k, \uparrow_l}$ is its own inverse. It also shows that if one web can be obtained from another by isotopies and sliding dots, merges, and splits along strands, then they are equal in $\mathfrak{q}$-**Web**$_\uparrow$.

Lemma 4.12. *The following relations hold in $\mathfrak{q}$-**Web**$_\uparrow$ for all $h, k, l \in \mathbb{Z}_{>0}$.*

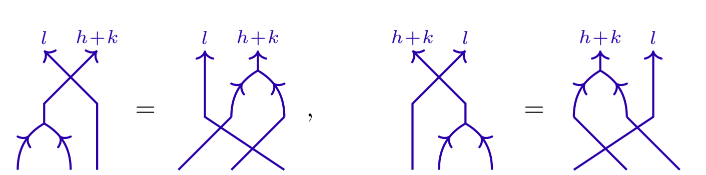
(a)

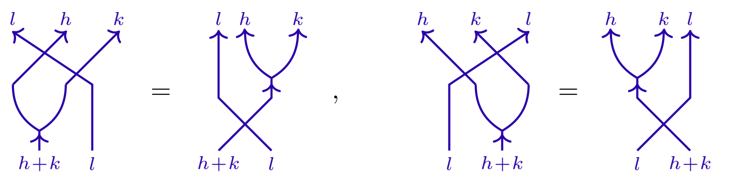


(b)

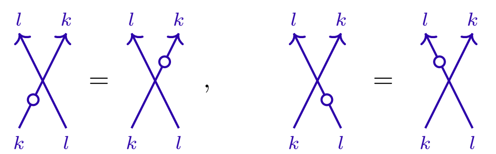


(c)





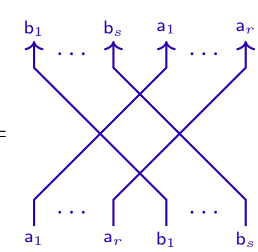
(d)



Proof. The first three sets of relations do not have dots and so hold from the analogous results in type A. If the reader prefers, they can verify that all four sets of relations follow from straightforward calculations where Definition 4.13 is used to rewrite the crossings as crossings of strands labeled by 1, Lemma 4.10 is used to rewrite clasps as crossings labeled by 1, and by applications of the relations given in Section 4.2 and Lemma 4.6. $\square$

We now define the braiding for arbitrary objects in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$.

Definition 4.13. *Let* $\mathsf{a} = (a_1, \dots, a_r)$ *and* $\mathsf{b} = (b_1, \dots, b_s)$ *be two tuples of nonnegative integers and let* $\uparrow_{\mathsf{a}}$ *and* $\uparrow_{\mathsf{b}}$ *be the corresponding objects of* $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$. *Define*

$\beta_{\uparrow_{\mathsf{a}},\uparrow_{\mathsf{b}}} =$  .

With these and Lemma 4.12, the following is immediate.

Theorem 4.14. *The morphisms* $\beta_{\uparrow_{\mathsf{a}},\uparrow_{\mathsf{b}}}$ *define a symmetric braiding on the monoidal supercategory* $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$.

5. Functors Π_m and Ψ, and $\operatorname{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\uparrow_1^k)$

We next relate the combinatorial category $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ to the supercategory $\dot{\mathbf{U}}(\mathfrak{q}(m))$ and to the representations of $\mathfrak{q}(n)$. Namely, as in [29, 37], we will show for all m there is a functor of supercategories $\Pi_m : \dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0} \to \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ and for all n a monoidal functor of supercategories $\Psi_n : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ such that the following diagram commutes:

$$\begin{tikzcd} \dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0} \arrow[r, "\Phi_{m,n}"] \arrow[d, "\Pi_m"'] & \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}} \\ \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \arrow[ur, "\Psi_n"'] & \end{tikzcd} \tag{36}$$

5.1. The Functor Π_m. In what follows, recall if a diagram has an edge labeled by a negative integer then the morphism given by that diagram is understood to be zero and an edge labeled by zero in a web can always be erased (or added). Also we recall the definition of the divided power morphisms can be found in Section 3.7.

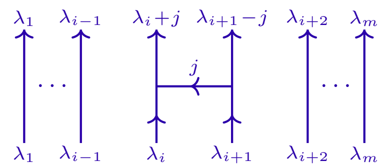
Proposition 5.1. *For every $m \geqslant 1$ there exists a functor of monoidal supercategories*

$$\Pi_m : \dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0} \to \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$$

given on objects $\lambda = \sum_{i=1}^{m} \lambda_i \varepsilon_i \in X(T)_{\geqslant 0}$ by

$$\Pi_m(\lambda) = \uparrow_{\lambda} = \uparrow_{\lambda_1} \uparrow_{\lambda_2} \cdots \uparrow_{\lambda_m},$$

and on the divided powers of the generating morphisms by

$\Pi_m(e_i^{(j)} 1_\lambda) =$ 

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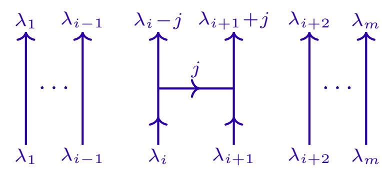
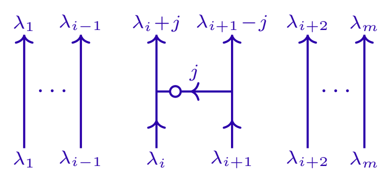
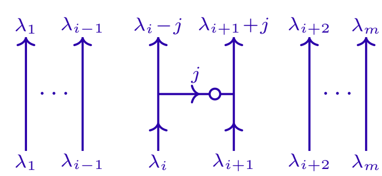
$\Pi_m(f_i^{(j)} 1_\lambda) =$ 

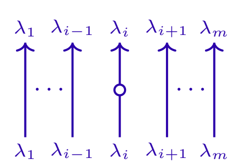
,

$\Pi_m(e_{\bar{i}}^{(j)} 1_\lambda) =$ 

,

$\Pi_m(f_{\bar{i}}^{(j)} 1_\lambda) =$ 

,

$\Pi_m(h_{\bar{i}} 1_\lambda) =$ 

.

Moreover, Π_m is a full functor onto the full subcategory of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ consisting of objects $\left\{\uparrow_{\mathsf{a}} \mid \mathsf{a} \in \mathbb{Z}_{\geqslant 0}^m\right\}$.

Proof. To show Π_m is well-defined it suffices to verify the defining relations of $\dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0}$ hold in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$. This follows by direct calculations using the defining relations of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ along with the identities proven in Section 4.2. For example, the first two equations of (Q4) hold because of (18) and (30). The first two equations of (Q7) hold because of (29) and (31). The other defining relations of $\dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0}$ are similar.

That the image of Π_m lies in the given subcategory is immediate. That the functor is full amounts to the fact that the images of the generating morphisms of $\dot{\mathbf{U}}(\mathfrak{q}(m))_{\geqslant 0}$ also generate the morphisms of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$. As discussed in Section 4.1, every merge and split is a special case of a ladder. Using this observation it it is straightforward to verify one can obtain every possible merge, split, and dot (possibly monoidally multipled by identity strands on the left and/or right) which lies in this subcategory. Since every web which is a morphism in this subcategory is a composition of such diagrams, it follows that Π_m is full. □

REMARK 5.2. Recall the functors $\Theta_{m',m}$ from Remark 3.15. The Π functors are compatible in the sense that $\Pi_{m'} \circ \Theta_{m',m}$ and Π_m are isomorphic functors for all positive integers $m' \geqslant m$.

5.2. THE FUNCTOR $\Psi^{\uparrow}$. Let $w_1, \dots, w_{k+l}$ be homogeneous elements of V_n. Given sets $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_l\}$ such that $I \cup J = \{1, \cdots, k+l\}$, then $w_{i_1} \cdots w_{i_k} w_{j_1} \cdots w_{j_l}$ and $w_1 \cdots w_{k+l}$ are elements of $S^{k+l}(V_n)$ and are equal up to a sign. Define $\varepsilon_{I,J} \in \mathbb{Z}/2\mathbb{Z}$ by the formula

$$w_{i_1} \cdots w_{i_k} w_{j_1} \cdots w_{j_l} = (-1)^{\varepsilon_{I,J}} w_1 \cdots w_{k+l}.$$

PROPOSITION 5.3. *For every $n \geqslant 1$ there exists an essentially surjective functor of monoidal supercategories,*

$$\Psi_n^{\uparrow} : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}},$$

given on objects by

$$\Psi_n^{\uparrow}(\uparrow_\lambda) = S_\lambda = S^\lambda(V_n) = S^{\lambda_1}(V_n) \otimes \cdots \otimes S^{\lambda_t}(V_n),$$

where $\lambda = (\lambda_1, \dots, \lambda_t)$. On morphisms the functor is defined by sending the dot, merge, and split, respectively, to the following maps (where all w_i are homogeneous elements of V_n):

$$S^k(V_n) \to S^k(V_n),$$
$$w_1 \cdots w_k \mapsto \sum_{t=1}^{k} (-1)^{|w_1|+\cdots+|w_{t-1}|} w_1 \cdots c(w_t) \cdots w_k;$$

$$S^k(V_n) \otimes S^l(V_n) \to S^{k+l}(W_n),$$
$$w_1 \cdots w_k \otimes u_1 \cdots u_l \mapsto w_1 \cdots w_k u_1 \cdots u_l;$$

$$S^{k+l}(C_n) \to S^k(V_n) \otimes S^l(W_n),$$
$$w_1 \cdots w_{k+l} \mapsto \sum_{\substack{I=\{i_1<\cdots<i_k\} \\ J=\{j_1<\cdots<j_l\} \\ I\cup J=\{1,\cdots,k+l\}}} (-1)^{\varepsilon_{I,J}} w_{i_1} \cdots w_{i_k} \otimes w_{j_1} \cdots w_{j_l}.$$

Proof. That the functor is essentially surjective is clear. Direct calculations verify these maps are $\mathfrak{q}(n)$-linear and satisfy the defining relations of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$. For example, if $w_1, \dots, w_k \in V_n$ are homogeneous, then we can verify (15) as follows. Under the morphism defined on the left-hand side of (15) $w_1 \cdots w_k$ maps to

$$\sum_{1 \leqslant u < t \leqslant k} (-1)^{|w_u|+\cdots+|w_{t-1}|} w_1 \cdots c(w_u) \cdots c(w_t) \cdots w_k$$
$$+ \sum_{t=1}^{k} w_1 \cdots c^2(w_t) \cdots w_k$$
$$+ \sum_{1 \leqslant t < u \leqslant k} (-1)^{|c(w_t)|+\cdots+|w_{u-1}|} w_1 \cdots c(w_t) \cdots c(w_u) \cdots w_k.$$

Each term in the first sum pairs up with a term in the third sum. Since $|c(w_u)| = |w_u| + \bar{1}$ they have opposite sign and cancel. Since $c^2 : V_n \to V_n$ equals the identity the second sum simplifies to $kw_1 \cdots w_k$, as desired.

Similarly, (17) can be proven as follows. The leftmost diagram in (17) is the map

$$w_1 \otimes w_2 \mapsto w_1 w_2 \mapsto w_1 \otimes w_2 + (-1)^{|w_1||w_2|} w_2 \otimes w_1.$$

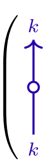
The second diagram in (17) is the map

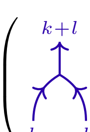
$$\begin{aligned} w_1 \otimes w_2 &\mapsto (-1)^{|w_1|} c(w_1) \otimes c(w_2) \\ &\mapsto (-1)^{|w_1|} c(w_1) c(w_2) \\ &\mapsto (-1)^{|w_1|} c(w_1) \otimes c(w_2) + (-1)^{|w_1|+|c(w_1)||c(w_2)|} c(w_2) \otimes c(w_1) \\ &\mapsto (-1)^{|w_1|+|c(w_1)|} w_1 \otimes w_2 + (-1)^{|w_1|+|c(w_1)||c(w_2)|+|c(w_2)|} w_2 \otimes w_1 \\ &= (-1)^{|w_1|+(|w_1|+\bar{1})} w_1 \otimes w_2 + (-1)^{|w_1|+(|w_1|+\bar{1})(|w_2|+\bar{1})+(|w_2|+\bar{1})} w_2 \otimes w_1 \\ &= -w_1 \otimes w_2 + (-1)^{|w_1||w_2|} w_2 \otimes w_1. \end{aligned}$$

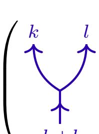
Subtracting the second from the first yields the map $w_1 \otimes w_2 \mapsto 2w_1 \otimes w_2$, as desired. We leave the others to the reader. $\square$

One can verify $\beta_{\uparrow_k, \uparrow_l}$ is sent to the graded flip map $S^k(V_n) \otimes S^l(V_n) \to S^l(V_n) \otimes S^k(V_n)$ given by $v_1 \cdots v_k \otimes w_1 \cdots w_l \mapsto (-1)^{(|v_1|+\cdots+|v_k|)(|w_1|+\cdots+|w_l|)} w_1 \cdots w_l \otimes v_1 \cdots v_k$. Also, we could have defined the functor $\Psi_n^{\uparrow}$ using the functors $\Phi_{m,n}$ using the following remark.

Remark 5.4. Recall the functors $\Phi_{m,n}$ from Proposition 3.14. Then,

$\Psi_n^{\uparrow}$ () $= \Phi_{1,n}^{\uparrow}(h_{\bar{1}} 1_{(k)})$,

Ψ_n () $= \Phi_{2,n}^{\uparrow}(e_1^{(l)} 1_{(k,l)})$,

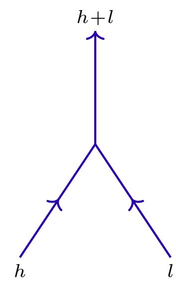
Ψ_n () $= \Phi_{2,n}(f_1^{(l)} 1_{(k+l,0)})$.

In particular, for any $m, n \geqslant 1$ we have $\Psi_n^{\uparrow} \circ \Pi_m = \Phi_{m,n}$. The well-definedness of $\Psi_n^{\uparrow}$ can also be deduced from the well-definedness of the functors $\Phi_{m,n}$ and the compatibility given in Remark 3.15.

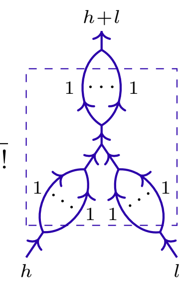
5.3. Description of $\mathrm{End}_{\mathbf{q\text{-}Web}_{\uparrow}}(\uparrow_1^k)$. We next describe the endomorphism algebras $\mathrm{End}_{\mathbf{q\text{-}Web}_{\uparrow}}(\uparrow_1^k)$. In particular, as promised in the discussion after Lemma 4.6, $\mathrm{End}_{\mathbf{q\text{-}Web}_{\uparrow}}(\uparrow_1^k)$ will turn out to be isomorphic to the Sergeev algebra.

Lemma 5.5. *The superalgebra* $\mathrm{End}_{\mathbf{q\text{-}Web}_{\uparrow}}(\uparrow_1^k)$ *is generated by* $c_1, \ldots, c_k, s_1, \ldots, s_{k-1}$.

Proof. It suffices to prove that every web diagram $w \in \mathrm{End}_{\mathbf{q\text{-}Web}_{\uparrow}}(\uparrow^k)$ can be written as a linear combination of webs containing only upward crossings and dotted 1-strands. Given w, we may assume without loss of generality that every dot in w is on a 1-strand by (23). Next, every merge and split in w can have its edges expanded into 1-strands using (14). For example, here is an expanded merge:



$$= \frac{1}{h!\ l!\ (h+l)!}$$



.

By (13), the web enclosed by the dashed rectangle above is $(h+l)!\,Cl_{h+l}$. By Lemma 4.10 each such clasp idempotent can be rewritten as a sum of upward crossings. The result follows. □

Proposition 5.6. *For every $k \in \mathbb{Z}_{>0}$ the map*

$$\xi_k : \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow_1^k)$$

of (32) *is a superalgebra isomorphism. Moreover, the map obtained by composing this homomorphism with $\Psi_n^\uparrow : \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow_1^k) \to \mathrm{End}_{\mathfrak{q}(n)}\left(V^{\otimes k}\right)$ coincides with the map ψ given in Theorem 3.3.*

Proof. As discussed regarding (32), ξ_k gives a well-defined homomorphism. By Lemma 5.5, ξ_k is surjective.

It only remains to prove injectivity. Fix $n > k$ and let $1^k = (1, \dots, 1) \in \mathbb{Z}^k$. Since $\mathcal{S}^{1^k} = V_n^{\otimes k}$, the functor Ψ_n induces a map of superalgebras $\mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow_1^k) \to \mathrm{End}_{\mathfrak{q}(n)}(V_n^{\otimes k})$ which we call by the same name. Taken together with the superalgebra map $\psi : \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}(n)}(V_n^{\otimes k})$ from Theorem 3.3 we have the following diagram of superalgebra maps.

$$\begin{tikzcd} \mathrm{Ser}_k \arrow[r, "\psi"] \arrow[d, "\xi_k"'] & \mathrm{End}_{\mathfrak{q}(n)}(V_n^{\otimes k}) \\ \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow_1^k) \arrow[ur, "\Psi_n^\uparrow"'] & \end{tikzcd} .$$

A direct calculation on generators verifies this diagram commmutes. The injectivity of ξ_k follows from the fact that ψ is an isomorphism by Theorem 3.3 (since $n > k$). □

5.4. The fullness of $\Psi_n^\uparrow$.

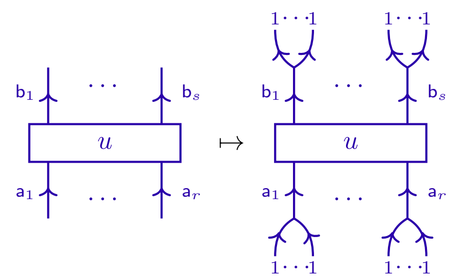
Theorem 5.7. *For every $n \geqslant 1$, the functor $\Psi_n^\uparrow : \mathfrak{q}\text{-}\mathbf{Web}_\uparrow \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ is full.*

Proof. Let $\mathsf{a} = (a_1, \dots, a_r)$ and $\mathsf{b} = (b_1, \dots, b_s)$ be objects of $\mathfrak{q}\text{-}\mathbf{Web}_\uparrow$. We first observe that $\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\mathsf{a}, \mathsf{b}) = 0$ unless $|\mathsf{a}| = |\mathsf{b}|$. Likewise, by weight considerations $\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow(n)}(S^{\mathsf{a}}(V_n), S^{\mathsf{b}}(V_n)) = 0$ unless $|\mathsf{a}| = |\mathsf{b}|$. Thus we can assume $|\mathsf{a}| = |\mathsf{b}|$ in what follows.

There is a map of superspaces

$$\alpha = \alpha_{\mathsf{a},\mathsf{b}} : \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\mathsf{a}, \mathsf{b}) \to \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}\left(\uparrow_1^{|\mathsf{a}|}, \uparrow_1^{|\mathsf{b}|}\right)$$

given by using merges and splits to “explode” all the boundary strands:



This map is an embedding. Up to a scalar the left inverse is given by applying a complementary set of splits and merges to rejoin the strands and applying relation (14). Since the map is given by composing and monoidally multiplying diagrams, via the functor there is corresponding map, $\tilde{\alpha}$, such that the following commutative diagram of superspace maps,

(37)
$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\mathsf{a},\mathsf{b}) \arrow[r, hook, "\alpha"] \arrow[d, "\Psi_n^\uparrow"] & \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}\left(\uparrow_1^{|\mathsf{a}|},\uparrow_1^{|\mathsf{b}|}\right) \arrow[d, "\Psi_n^\uparrow"] \\
\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(S^{\mathsf{a}}(V_n),S^{\mathsf{b}}(V_n)) \arrow[r, hook, "\tilde{\alpha}"] & \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}\left(V_n^{\otimes|\mathsf{a}|},V_n^{\otimes|\mathsf{b}|}\right).
\end{tikzcd}$$

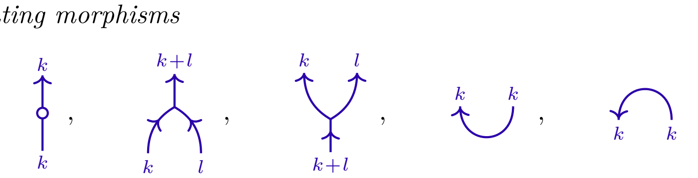
Furthermore, $e_{\mathsf{a}} = Cl_{a_1}\otimes\cdots\otimes Cl_{a_r}, e_{\mathsf{b}} = Cl_{b_1}\otimes\cdots\otimes Cl_{b_s} \in \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow_1^{|\mathsf{a}|})$ are idempotents, the image of α is precisely $e_{\mathsf{b}}\,\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_\uparrow}(\uparrow^{\mathsf{a}},\uparrow^{\mathsf{b}})e_{\mathsf{a}}$, and the image of $\tilde{\alpha}$ is $\Psi_n^\uparrow(e_{\mathsf{b}})\,\mathrm{Hom}_{q(n)}(V^{\mathsf{a}},V^{\mathsf{b}})e_{\mathsf{a}}\Psi_n^\uparrow(e_{\mathsf{a}})$. By Proposition 5.6 and Theorem 3.3 the $\Psi_n^\uparrow$ on the right side of (37) is surjective. This along with a diagram chase implies $\Psi_n^\uparrow$ is surjective on the left side of the diagram. Together with the discussion in the first paragraph it follows $\Psi_n^\uparrow$ is a full functor. □

6. Oriented Webs

In this section we introduce oriented webs. On the representation theory side this corresponds to including the duals of symmetric powers of the natural supermodule for $\mathfrak{q}(n)$. Our approach follows that of oriented Brauer and tangle categories (e.g. as in [39, 38]). We were particularly inspired by [6] where we learned the handy technique of declaring a morphism to be invertible as part of defining a monoidal category.

6.1. Oriented Webs.

Definition 6.1. *The category* $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ *is the monoidal supercategory with generating objects*
$$\{\uparrow_k,\downarrow_k \mid k\in\mathbb{Z}_{\geqslant 0}\}$$
and generating morphisms

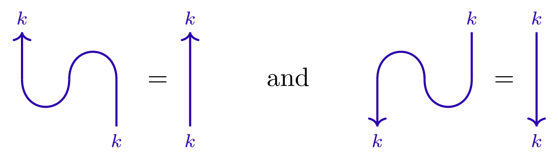


for all $k,l\in\mathbb{Z}_{>0}$*. We call these the* dot*,* merge*,* split*,* cup*, and* cap*, respectively. The parity is given by declaring the dot to be odd and the other generating morphisms to be even.*

The relations imposed on the generators of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ *are the relations* (13)–(18)*,* (20)*, and* (21) *of* $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$*, along with leftward and rightward oriented versions of the relations given in* (38)*, declaring the morphism given in* (39) *to be invertible, and relation* (42)*.*

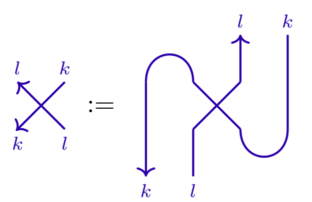
The first relation is the following straightening rules for cups and caps:

(38)

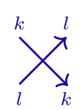


To state the second relation, we first define the *leftward crossing* in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ for all $k, l \geqslant 1$ by

(39)

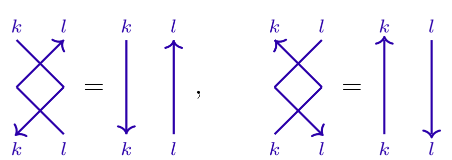


We impose on $\mathfrak{q}(n)$-$\mathbf{Web}_{\uparrow\downarrow}$ the requirement that every leftward crossing is invertible. In other words, for every $k, l \geqslant 1$ we assume the existence of another generating morphism of type $\uparrow_l\downarrow_k \to \downarrow_k\uparrow_l$ which we draw as the rightward crossing



Furthermore, we assume it is a two-sided inverse to leftward crossing under composition:

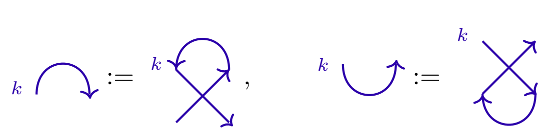
(40)



Note that dots freely move through leftward crossings and, consequently, through rightward crossings.

Using the rightward crossing, we define the rightward cup and cap by

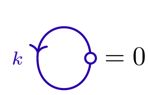
(41)



We impose the relation that these morphisms satisfy the rightward oriented versions of (38).

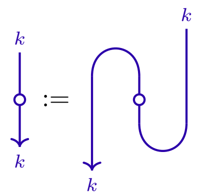
The final relation we impose on $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ is that the following holds for all $k \geqslant 1$:

(42)

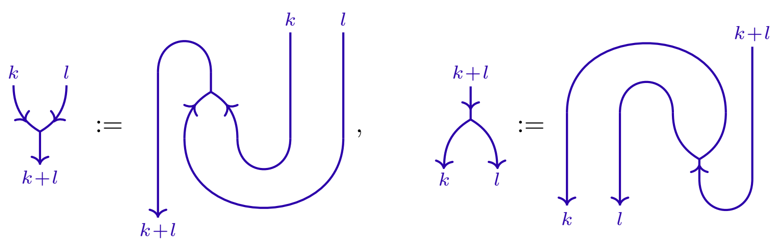


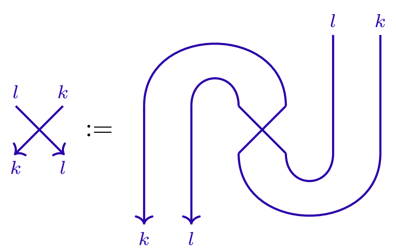
Note that, like $\mathfrak{q}$-**Web**$_{\uparrow}$, the relations for $\mathfrak{q}$-**Web**$_{\uparrow\downarrow}$ are symmetric with respect to reflection across a vertical axis. We will sometimes invoke this symmetry to reduce the number of cases which need to be checked in various calculations.

6.2. Additional Relations. We define the downward dot by

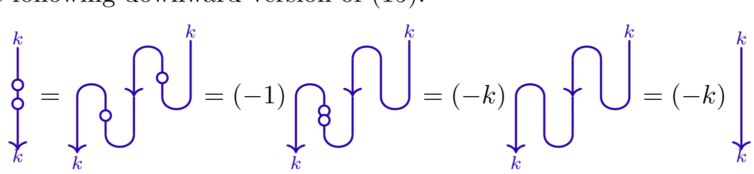
(43) 

We define downward merges, splits, and crossings by:

(44) 

(45) 

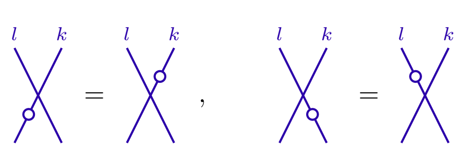
By applying cups and caps to (13)–(18), (20), and (21) one obtains a parallel set of relations on downward oriented diagrams. We freely use these downward relations in what follows. For relations with one or fewer dots, the rotated relations are straightforward. However, when computing relations the reader is advised to keep in mind the effect of the super interchange law when diagrams have multiple dots. For example, applying the definition of the downward dot, the super interchange law, (15), and (38) yields the following downward version of (15):

(46) 

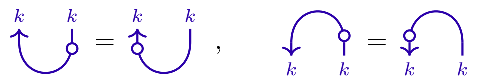
The following additional relations hold in $\mathfrak{q}$-**Web**$_{\uparrow\downarrow}$.

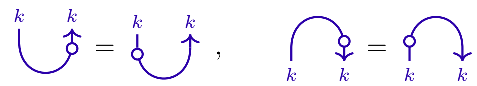
Lemma 6.2. *For any $k, l \geqslant 1$, the following relations hold. When then orientation of a strand is omitted, the given relation holds regardless of the orientation of the strand.*

(a)

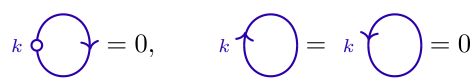
(47) 

(b)

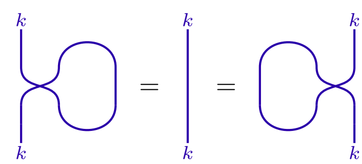
(48) 

(49) 

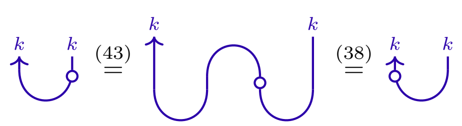
(c)

(50) 

(d)

(51) 

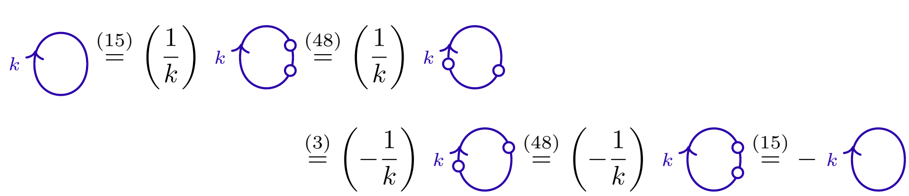
Proof. For the first relation given in (48), we compute:



Proof of the second relation given in (48) is similar.

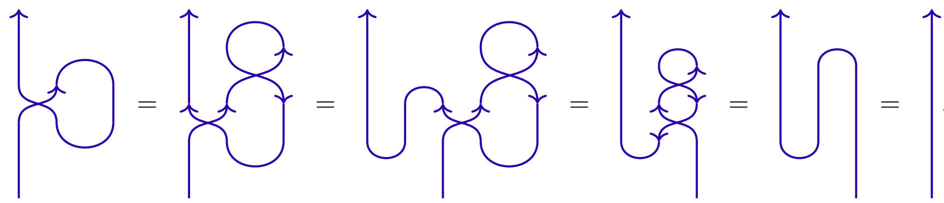
With (48) in hand we next prove the relations given in (47). These relations were already proven in Lemma 4.12 when both strands are upward oriented. This along with the fact dots freely go over leftward cups and caps proves that dots freely move through leftward and downward crossings. Finally, the fact dots move through leftward crossings implies dots also move through their inverses, the rightward crossings. The fact that dots freely move through leftward caps and cups and through rightward crossings shows the relations given in (49).

The first relation of (50) follows by writing the writing the rightward cup using (41), passing the dot through the rightward crossing, using (41) again to rewrite the resulting diagram with a rightward cup, and applying (42). The second follows from



The counterclockwise oriented circle is handled similarly

We prove the first equality in (51) for the upward orientation and then the second equality for the upward orientation will follow from reflecting the first across the vertical axis. As the label will be irrelevant to the calculation, we chose to omit it. The first of the following equalities is the definition of the rightward oriented cap, the second is (38), the third is the definition of the leftward crossing, the fourth is the the fact the rightward crossing is, by definition, the inverse of the leftward crossing, and the last is another application of (38):



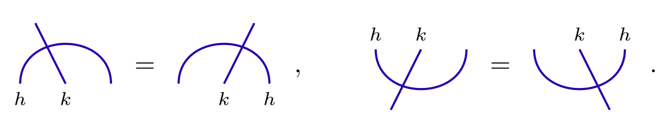
The downward oriented versions of (51) are entirely similar. □

Lemma 6.3. *The relations given in Lemma 4.12 hold for all possible strand orientations and all possible $h, k, l \in \mathbb{Z}_{\geqslant 1}$.*

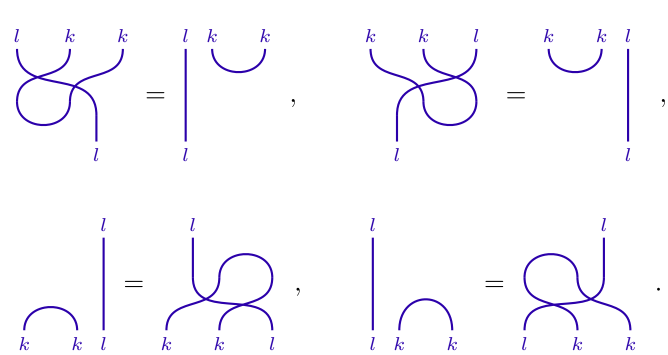
Proof. That appropriately oriented crossings are inverses is immediate from the definition. The braid relation on crossings follows from [39, Lemma 3.3] once we note that our choice of strand orientations is the opposite of Turaev's choice. Namely, the relations verified so far for $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ show that the defining relations of the category of colored ribbon tangles are satisfied by the strands in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. Since the braid relation holds in the tangle category, it holds here as well. The fact that merges, splits and dots pass through strands follows from the fact they freely pass through upward crossings and over leftward oriented cups and caps. □

Lemma 6.4. *The following relations hold in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ for all possible strand orientations and all possible $h, k, l \in \mathbb{Z}_{\geqslant 1}$.*

(a)

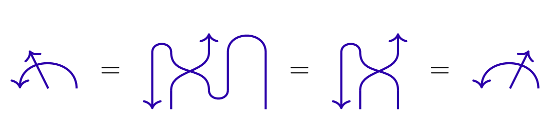


(b)

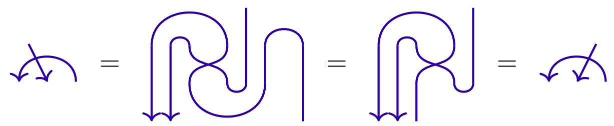


Proof. In the following calculations we leave the edges unlabeled as the labels play no role in the arguments.

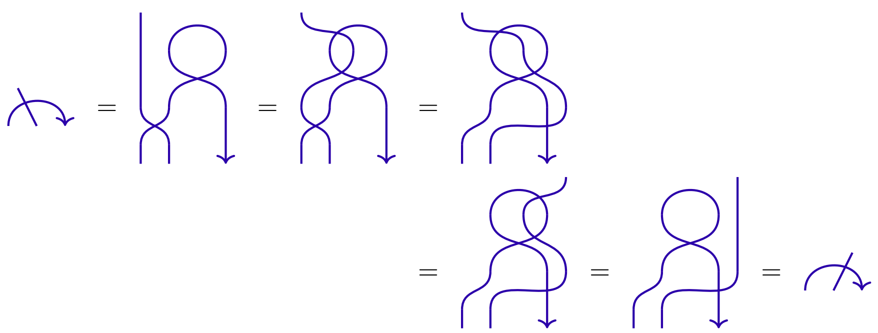
For (a) we only prove the first equality as the second follows by entirely similar arguments. If the web is of type $\downarrow\uparrow\uparrow \to \uparrow$ then using the definition of the leftward crossing and (38), we have:



If the diagram is of type $\downarrow\downarrow\uparrow\to\downarrow$, then using the definition of the downward crossing and (38) we have:



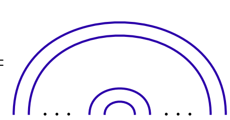
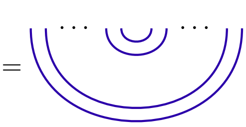
We next consider the left equality in (a) in the cases when the cap has a rightward orientation. As the orientation of the other strand will not matter, we handle both cases at once. By applying the definition of the rightward cap, Lemma 6.3, and the fact the first equality for (a) holds for leftward oriented caps we have:



The identities in (b) follow from those in (a) by applying a suitably oriented crossing to both sides and simplifying. $\square$

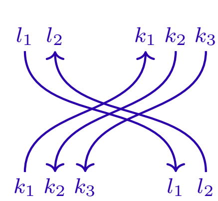
6.3. Duals and braidings in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. We next show that $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ has duals and braidings for arbitrary objects. These constructions are standard and can be found in [39, XII.2.2], for example. We let $|_k$ represent either $\uparrow_k$ or $\downarrow_k$. More generally, for any tuple of nonnegative integers $\mathsf{a} = (a_1, \dots, a_r)$ we write $|_{\mathsf{a}} = |_{a_1}|_{a_2}\cdots|_{a_r}$. For a tuple $\mathsf{a} = (a_1, \dots, a_r)$ we write $\overleftarrow{\mathsf{a}} = (a_r, \dots, a_1)$. Let $\uparrow'_k = \downarrow_k$ and $\downarrow'_k = \uparrow_k$. More generally, let $|'_{\mathsf{a}} = |'_{a_1}\cdots|'_{a_r}$.

For an arbitrary object a in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$, we can use appropriately oriented caps (or cups) to define morphisms $\mathrm{ev}_{\mathsf{a}} : |'_{\overleftarrow{\mathsf{a}}} \otimes |_{\mathsf{a}} \to \mathbb{1}$ and $\mathrm{coev}_{\mathsf{a}} : \mathbb{1} \to |_{\mathsf{a}} \otimes |'_{\overleftarrow{\mathsf{a}}}$ via

$\mathrm{ev}_{\mathsf{a}} =$  and $\mathrm{coev}_{\mathsf{a}} =$  .

Repeated use of the leftward and rightward oriented versions of (38) shows that that for any object $|_{\mathsf{a}}$ the evaluation and coevaluation morphisms satisfy the multi-strand version of (38) as well as its rightward oriented analogue. That is, $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ is a rigid category with $|_{\mathsf{a}}$ having $|'_{\overleftarrow{\mathsf{a}}}$ as its left and right dual.

For arbitrary objects a, b in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$, one can use the appropriately oriented version of Definition 4.13 to define the braiding isomorphism $\beta_{|_{\mathsf{a}},|_b} : |_{\mathsf{a}} \otimes |_{\mathsf{b}} \to |_{\mathsf{b}} \otimes |_{\mathsf{a}}$. For example, the braiding $\beta_{\uparrow_{k_1}\downarrow_{k_2}\downarrow_{k_3},\downarrow_{l_1}\uparrow_{l_2}} : \uparrow_{k_1}\downarrow_{k_2}\downarrow_{k_3} \otimes \downarrow_{l_1}\uparrow_{l_2} \to \downarrow_{l_1}\uparrow_{l_2} \otimes \uparrow_{k_1}\downarrow_{k_2}\downarrow_{k_3}$ is given by



.

Using the evaluation, coevaluation, and braiding morphisms defined above for arbitrary objects, one can generalize the diagrams given in Lemma 6.3 and Lemma 6.4, the leftward and rightward oriented versions of (38), and (51) to arbitrary objects of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. Using repeated applications of these relations in the cases already established allows one to verify the analogous identities also hold for the generalizations. We record this fact in the following lemma.

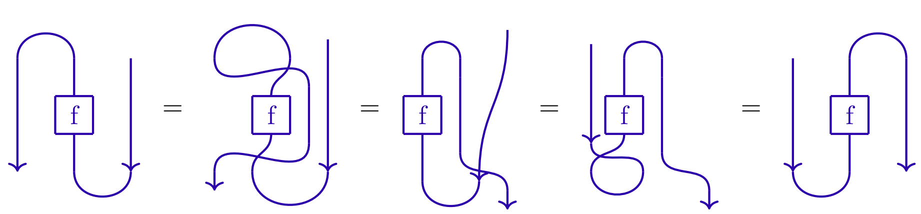
Lemma 6.5. *The analogues of the relations in Lemma 6.3 and Lemma 6.4, the leftward and rightward oriented versions of* (38), *and* (51) *hold for arbitrary objects in* $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$.

Theorem 6.6. *The supercategory* $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ *is a rigid, symmetric, monoidal supercategory. Moreover, any two webs in* $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ *which are related by an isotopy which fixes all dots are equal as morphisms in* $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$.

Proof. The fact that the evaluation, coevaluation, and braiding morphisms introduced in this section make $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ a rigid, symmetric, monoidal supercategory follows from Lemma 6.5 and the other relations verified in this section.

The second assertion follows from [39, Lemma 3.4]. Namely, let $\mathfrak{q}$-$\mathbf{Web}^0_{\uparrow\downarrow}$ denote the subcategory of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ consisting of all objects and all morphisms which can be written as a linear combination of webs which have no dots. Then $\mathfrak{q}$-$\mathbf{Web}^0_{\uparrow\downarrow}$ is a strict monoidal category with duality and a symmetric braiding. Let $\mathcal{RIB}_{\mathfrak{q}\text{-}\mathbf{Web}^0_{\uparrow\downarrow}}$ denote the category of ribbon graphs over $\mathfrak{q}$-$\mathbf{Web}^0_{\uparrow\downarrow}$ as in [39, Section 2.3]. By [39, Lemma 3.4] there is a full functor $\mathcal{RIB}_{\mathfrak{q}\text{-}\mathbf{Web}^0_{\uparrow\downarrow}} \to \mathfrak{q}$-$\mathbf{Web}^0_{\uparrow\downarrow}$ where the generators listed in [39, Lemma 3.1.1] go as expected (keeping in mind that we have the opposite convention on edge orientation) to crossings, evaluations, and coevaluations in $\mathfrak{q}$-$\mathbf{Web}^0_{\uparrow\downarrow}$ and where the twist maps go to identity morphisms. The fact the relations listed in [39, Lemma 3.4] are satisfied follows from the relations established above. Since $\mathcal{RIB}_{\mathfrak{q}\text{-}\mathbf{Web}^0_{\uparrow\downarrow}}$ is a ribbon category and the functor is full, this implies the relations of a ribbon category hold in $\mathfrak{q}$-$\mathbf{Web}^0_{\uparrow\downarrow}$. In particular, since twist maps are identities, strands of arbitrary label and orientation satisfy the Reidemeister moves and, hence, isotopic diagrams in $\mathfrak{q}$-$\mathbf{Web}^0_{\uparrow\downarrow}$ are equal. Therefore two diagrams in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ which are related by an isotopy which fixes all dots are equal as morphisms in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. ☐

It is worth observing that one can combine the previous result and the fact that dots freely move through crossings to show that rotating morphisms using leftward oriented cups and caps agrees with rotating using rightward oriented cups and caps. That is, for any morphism $f : \mathsf{a} \to \mathsf{b}$ in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ we have:



6.4. Isomorphisms between morphism spaces in $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$. We now introduce isomorphisms between various morphism spaces which will be useful in what follows. These isomorphisms are well-known to experts and can be found in the literature (e.g. see [17, Section 5]). We include them here for completeness.

The symmetric group on t letters, Σ_t, acts by place permutation on the set of all objects of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ which are the monoidal product of exactly t generating objects. For example, if $\mathsf{a} = |_{a_1}|_{a_2}\cdots|_{a_t}$ and $\sigma \in \Sigma_t$, then $\sigma \cdot \mathsf{a} = |_{a_{\sigma^{-1}(1)}}|_{a_{\sigma^{-1}(2)}}\cdots|_{a_{\sigma^{-1}(t)}}$.

Moreover, for any object a and $\sigma \in \Sigma_t$ the braiding morphisms in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ give a morphism,

$$d_\sigma = d_{\sigma,\mathsf{a}} \in \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\mathsf{a}, \sigma \cdot \mathsf{a}),$$

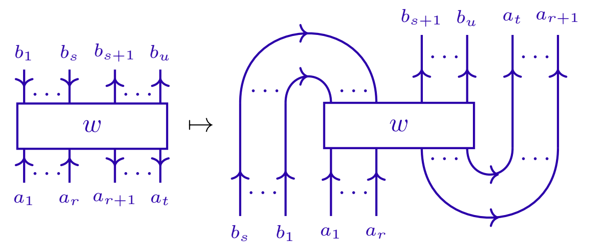
which is an invertible morphism in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ with $d_\sigma^{-1} = d_{\sigma^{-1}}$. More generally, given objects a and b which are the monoidal product of t and u generators, respectively, and any $\sigma \in \Sigma_t$ and $\tau \in \Sigma_u$, then the map $w \mapsto d_\tau \circ w \circ d_{\sigma^{-1}}$ defines a superspace isomorphism

$$\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\mathsf{a}, \mathsf{b}) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\sigma \cdot \mathsf{a}, \tau \cdot \mathsf{b}). \tag{52}$$

In addition, given objects of the form $\mathsf{a} = \uparrow_{a_1} \cdots \uparrow_{a_r} \downarrow_{a_{r+1}} \cdots \downarrow_{a_t}$ and $\mathsf{b} = \downarrow_{b_1} \cdots \downarrow_{b_s} \uparrow_{b_{s+1}} \cdots \uparrow_{b_u}$, then there is a superspace map,

$$\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\mathsf{a}, \mathsf{b}) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}\left(\uparrow_{b_s} \cdots \uparrow_{b_1} \uparrow_{a_1} \cdots \uparrow_{a_r}, \uparrow_{b_{s+1}} \cdots \uparrow_{b_u} \uparrow_{a_t} \cdots \uparrow_{a_{r+1}}\right), \tag{53}$$

given on diagrams by



The inverse is given by a similar map (thanks to relation (38)).

As in the proof of Theorem 5.7, whenever a morphism between Hom-spaces in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ is defined by applying some combination of compositions and monoidal products of morphisms we can apply the functor Ψ and obtain a corresponding morphism in the category of $\mathfrak{q}(n)$-modules. In particular, commuting diagrams go to commuting diagrams and isomorphisms go to isomorphisms.

6.5. Further Functors.

Theorem 6.7. *There is a full functor of symmetric monoidal supercategories*

$$\Omega : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$$

which takes objects and morphisms in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ to objects and morphisms of the same name in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$.

Proof. By construction the defining relations on morphisms of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ hold in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$. Consequently, there is a well-defined functor of monoidal supercategories. Now let a, b be objects of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ and let d be a web diagram which lies in $\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\mathsf{a}, \mathsf{b})$. Using the generating morphisms of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$, we may assume d is assembled from vertical and horizontal concatenations of upward oriented dots, merges, splits, cups and caps, and rightward crossings. Since isotopic webs in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ are equal as morphisms, we may use cups and caps to rotate rightward crossings so they become upward oriented crossings. We may also assume each dot, merge, split, cup, cap, and crossing are at different heights in d.

We call a strand a *segment* if it connects any two elements of the set of merges, splits, crossings, and upper and lower boundary points of d, and if there is nothing between the two other than cups, caps, and dots. In particular, both ends of a segment connect to upward oriented merges, splits, crossings or endpoints (the latter are upward oriented because a and b are objects of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$). Since dots freely move through cups and caps, each segment can be isotopied while leaving its ends fixed to a segment which is continuously upward oriented, possibly with with dots. Doing this to all the segments in d results in a diagram which is upward oriented everywhere. Furthermore, by (42) and Lemma 6.2 there are no bubbles in d.

Taken together, this shows d is equal to a web diagram which lies in the image of $\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\mathsf{a},\mathsf{b})$ in $\mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\mathsf{a},\mathsf{b})$. That is, the functor is full. $\square$

This functor will turn out to be faithful (see Corollary 7.3) and, hence, we will be able to identify $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$ as a full subcategory of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow\downarrow}$ via this functor.

Let $\mathfrak{q}(n)$-$\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$ denote the full monoidal subsupercategory of $\mathfrak{q}(n)$-supermodules generated by the objects

$$\left\{ S^p\left(V_n\right), S^p\left(V_n\right)^* \mid p \geqslant 1 \right\}.$$

We next show the functor $\Psi_n^{\uparrow} : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ may be extended to a functor $\Psi_n^{\uparrow\downarrow} : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$. Set the notation

$$\mathrm{ev}_k : S^k(V_n)^* \otimes S^k(V_n) \to \mathbb{C}, \tag{54}$$

$$\mathrm{coev}_k : \mathbb{C} \to S^k(V_n) \otimes S^k(V_n)^*, \tag{55}$$

for the evaluation and coevaluation maps defined in Section 2.2.

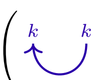
Theorem 6.8. *There is an essentially surjective, full functor of rigid symmetric monoidal supercategories*

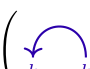
$$\Psi_n^{\uparrow\downarrow} : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$$

given on generating objects by

$$\Psi_n^{\uparrow\downarrow}(\uparrow_k) = S^k(V_n),$$
$$\Psi_n^{\uparrow\downarrow}(\downarrow_k) = S^k(V_n)^*$$

and on generating morphisms by defining it on the dot, merge, and split as in Proposition 5.3 and

$\Psi_n^{\uparrow\downarrow}\Bigg($ 

$\Bigg) = \mathrm{coev}_k,\qquad \Psi_n^{\uparrow\downarrow}\Bigg($ 

$\Bigg) = \mathrm{ev}_k\,.$

Proof. Since $\Psi_n^{\uparrow}$ is already known to preserve (13)–(18), (20), and (21) of $\mathfrak{q}$-$\mathbf{Web}_{\uparrow}$, it suffices to verify the leftward and rightward oriented versions of relation (38), relation (42), and the invertibility of (39). This follows from a direct calculations as follows.

Let W be a finite-dimensional superspace with homogeneous basis $\{w_i \mid i \in I\}$ and let $f_j \in W^*$ be defined by $f_j(w_i) = \delta_{i,j}$ (note that the parity of f_j matches the parity of w_j). For the left relation in (38) the given sequence of maps given by the left hand side of the equality evaluate on a basis element of $W = S^k(V_n)$ by

$$w_t \mapsto \sum_{i\in I} w_i \otimes f_i \otimes w_t \mapsto \sum_{i \in I} w_i f_i(w_t) = w_t.$$

That is, the map is the identity on $W = S^k(V_n)$, as claimed.

For the right relation in (38) the given sequence of maps given by the left hand side of the equality evaluate on a basis element of $W^* = S^k(V_n)^*$ by

$$f_t \mapsto \sum_{i\in I} f_t \otimes w_i \otimes f_i \mapsto \sum_{i\in I} f_t(w_i) f_i = f_t.$$

That is, the map is the identity on $W^* = S^k(V_n)^*$, as claimed.

We next verify that the leftward crossing goes to the graded flip map (see Section 2.1) $S^k(V_n)^* \otimes S^\ell(V_n) \to S^\ell(V_n) \otimes S^k(V_n)^*$. Let $W = S^k(V_n)$ and $W' = S^\ell(V_n)$

with homogeneous bases $\{w_i \mid i \in I\}$ and $\left\{w'_j \mid j \in J\right\}$ and dual bases $\{f_i \mid i \in I\}$ and $\left\{f'_j \mid j \in J\right\}$, respectively. Computing the leftward crossing on a basis element yields

$$\begin{aligned} f_u \otimes w'_t \mapsto \sum_{i\in I} f_u \otimes w'_t \otimes w_i \otimes f_i &\mapsto \sum_{i\in I} (-1)^{|w'_t||w_i|} f_u \otimes w_i \otimes w'_t \otimes f_i \\ &\mapsto \sum_{i\in I} (-1)^{|w'_t||w_i|} f_u\left(w_i\right) w'_t \otimes f_i \\ &= (-1)^{|w'_t||w_u|} w'_t \otimes f_u \\ &= (-1)^{|w'_t||f_u|} w'_t \otimes f_u, \end{aligned}$$

where the last equality follows from the fact the parity of w_u equals the parity of f_u. Thus the leftward crossing goes to the flip map, as claimed. With this in hand, calculations similar to those in the previous paragraph verify the rightward oriented versions of (38).

To verify the relation given in (42), we now assume W admits an odd involution. We further assume we have chosen a basis of even vectors $w_1, \dots, w_s$ for $W_{\bar 0}$ and set $w_{\bar i}$ to be the image in $W_{\bar 1}$ of w_i under the involution. Thus $w_{\bar 1}, \dots, w_{\bar s}$ is a basis for $W_{\bar 1}$. That is, the involution takes w_i to $w_{\bar i}$ and vice versa. Computing the sequence of maps given by the lefthand side of (42) for $W = S^k(V_n)$ yields

$$\begin{aligned} 1 \mapsto \sum_{i=1}^{s} \left(w_i \otimes f_i + w_{\bar i} \otimes f_{\bar i}\right) &\mapsto \sum_{i=1}^{s} \left(f_i \otimes w_i - f_{\bar i} \otimes w_{\bar i}\right) \\ &\mapsto \sum_{i=1}^{s} \left(f_i \otimes w_{\bar i} - f_{\bar i} \otimes w_i\right) \\ &\mapsto \sum_{i=1}^{s} \left(f_i(w_{\bar i}) - f_{\bar i}(w_i)\right) = 0, \end{aligned}$$

where the negative appears due to the grade flip map. Thus the morphism is identically zero as claimed.

Next we show the morphism (39) goes to an invertible map. Let $W = S^k(V_n)$ and $W' = S^\ell(V_n)$ with homogeneous bases $\{w_i \mid i \in I\}$ and $\left\{w'_j \mid j \in J\right\}$ and dual bases $\{f_i \mid i \in I\}$ and $\left\{f'_j \mid j \in J\right\}$, respectively. Computing the morphism on a basis element of $W^* \otimes W' = S^k(V_n) \otimes S^\ell(V_n)^*$ yields

$$\begin{aligned} f_u \otimes w'_t \mapsto \sum_{i\in I} f_u \otimes w'_t \otimes w_i \otimes f_i &\mapsto \sum_{i\in I} (-1)^{|w_i||w'_t|} f_u \otimes w_i \otimes w'_t \otimes f_i \\ &\mapsto \sum_{i\in I} (-1)^{|w_i||w'_t|} f_u\left(w_i\right) w'_t \otimes f_i \\ &\mapsto (-1)^{|w_u||w'_t|} w'_t \otimes f_u = (-1)^{|f_u||w'_t|} w'_t \otimes f_u, \end{aligned}$$

where the last equality follows from the fact the parity of w_u equals the parity of f_u. Thus the diagram on the left hand side of (39) goes to the graded flip map $W^* \otimes W' \to W' \otimes W^*$, which is certainly invertible.

Finally, note that by construction $\Psi_n^{\uparrow\downarrow}$ is a functor of rigid, symmetric, and monoidal supercategories. $\square$

Self-evidently, $\Psi_n^{\uparrow\downarrow} \circ \Omega = \Psi_n^{\uparrow}$.

Remark 6.9. The oriented Brauer–Clifford supercategory $\mathcal{OBC}$ was introduced in [5]. It is given as a monoidal supercategories with generating objects $\uparrow$ and $\downarrow$, three even generating morphisms : $\mathbb{1} \to \uparrow\downarrow$, : $\downarrow\uparrow \to \mathbb{1}$, : $\uparrow\uparrow \to \uparrow\uparrow$, and one odd generating morphism $:\uparrow\to\uparrow$. These morphisms are subject to certain relations which we omit. There is a fully faithful functor of symmetric monoidal supercategories

$$\Upsilon : \mathcal{OBC} \to \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}.$$

On objects one has $\Upsilon(\uparrow) =\uparrow_1$ and $\Upsilon(\downarrow) =\downarrow_1$. The functor Υ sends the generating morphisms of $\mathcal{OBC}$ to the similarly drawn morphisms of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ which have all edges labeled by 1. In this sense $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ is a "thickened" version of $\mathcal{OBC}$.

As explained in [5], for each $n \geqslant 1$ there is a functor of monoidal supercategories

$$\Gamma_n : \mathcal{OBC} \to \mathfrak{q}(n)\text{-modules}.$$

Furthermore, $\Gamma_n = \Psi_n^{\uparrow\downarrow} \circ \Upsilon$.

7. Main Theorems

7.1. Equivalences of Categories. For $n \geqslant 1$, set $k = (n+1)(n+2)/2$ and recall the quasi-idempotent

$$e_{\lambda(n)} \in \mathrm{Ser}_k \cong \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}\left(\uparrow_1^k\right) \cong \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}\left(\uparrow_1^k\right)$$

defined in Section 3.3. The first isomorphism is by Proposition 5.6 and the second isomorphism is the surjective map induced by the functor Ω defined in Theorem 6.7. If we compose this map with the map given by the functor $\Psi_n^{\uparrow\downarrow}$ we obtain a surjective map to $\mathrm{End}_{\mathfrak{q}(n)}\left(V_n^{\otimes k}\right)$. Since this endomorphism space is isomorphic to Ser_k whenever n is sufficiently large (thanks to Proposition 5.6), it follows that the map given by Ω is in fact an isomorphism. We identify these superalgebras via these maps.

Definition 7.1. *Let $\mathcal{I}_\uparrow$ be the tensor ideal of $\mathfrak{q}(n)\text{-}\mathbf{Web}_\uparrow$ generated by $e_{\lambda(n)}$ and let $\mathcal{I}_{\uparrow\downarrow}$ be the tensor ideal of $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}$ generated by $e_{\lambda(n)}$. Set*

$$\mathfrak{q}(n)\text{-}\mathbf{Web}_\uparrow := \mathfrak{q}\text{-}\mathbf{Web}_\uparrow/\mathcal{I}_\uparrow,$$
$$\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow} := \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}/\mathcal{I}_{\uparrow\downarrow}.$$

Informally, $\mathfrak{q}(n)\text{-}\mathbf{Web}_\uparrow$ and $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}$ are given by the same generators and relations as $\mathfrak{q}\text{-}\mathbf{Web}_\uparrow$ and $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ with the extra relation $e_{\lambda(n)} = 0$. Both are monoidal supercategories as discussed in Section 2.5. It is worth emphasizing that both of these categories depend on n (unlike $\mathfrak{q}\text{-}\mathbf{Web}_\uparrow$ and $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$).

We are now prepared to state and prove one of the main theorems of the paper.

Theorem 7.2. *The functors $\Psi_n^\uparrow : \mathfrak{q}\text{-}\mathbf{Web}_\uparrow \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}}$ and $\Psi_n^{\uparrow\downarrow} : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}$ induce functors*

$$\Psi_n^\uparrow : \mathfrak{q}(n)\text{-}\mathbf{Web}_\uparrow \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S}},$$
$$\Psi_n^{\uparrow\downarrow} : \mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow} \to \mathfrak{q}(n)\text{-}\mathbf{Mod}_{\mathcal{S},\mathcal{S}^*}.$$

These functors are equivalences of symmetric monoidal supercategories.

Proof. Since both $\Psi_n^\uparrow$ and $\Psi_n^{\uparrow\downarrow}$ are functors of monoidal categories, and $\Psi_n^\uparrow(e_{\lambda(n)}) = \Psi_n^{\uparrow\downarrow}(e_{\lambda(n)}) = 0$ by Proposition 5.6 and Proposition 3.4, it follows from the discussion in Section 2.5 that $\Psi_n^\uparrow$ and $\Psi_n^{\uparrow\downarrow}$ induce functors of monoidal categories which we call by the same name. By construction they are essentially surjective.

It remains to show they are full and faithful in both cases. By the discussion in Section 6.4 we may assume without loss of generality that both a and b consist of only up arrows. Combining Theorem 6.7 and Theorem 5.7 it follows that $\Psi_n^\uparrow$ and $\Psi_n^{\uparrow\downarrow}$ are full. For faithfulness we consider the case of $\Psi_n^{\uparrow\downarrow}$ since the same argument also applies to $\Psi_n^\uparrow$ and is easier.

Let $\mathcal{I} = \mathcal{I}_{\uparrow\downarrow}$, the tensor ideal of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ which defines $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}$. To show faithfulness amounts to showing that if $f \in \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\mathsf{a},\mathsf{b})$ and $\Psi_n^{\uparrow\downarrow}(f) = 0$, then

$f \in \mathcal{I}(\mathsf{a},\mathsf{b})$. Just as in the proof of Theorem 5.7, we may assume $|\mathsf{a}| = |\mathsf{b}|$ since otherwise $\Psi_n^{\uparrow\downarrow}$ is trivially faithful. We may again assume without loss of generality that both a and b consist of only up arrows. Set $k = |\mathsf{a}| = |\mathsf{b}|$. In what follows we identify Ser_k and $\mathrm{Hom}_{\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}}\left(\uparrow_1^k, \uparrow_1^k\right)$ via the isomorphism given at the top of this section.

As in the proof of Theorem 5.7, we can compose with merges and splits and induce the following commutative diagram of morphisms:

$$\begin{tikzcd} \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\mathsf{a},\mathsf{b}) \arrow[r, hook, "\alpha"] \arrow[d, "\Psi_n^{\uparrow\downarrow}"] & \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}\left(\uparrow_1^k, \uparrow_1^k\right) \arrow[d, "\Psi_n^{\uparrow\downarrow}"] \\ \mathrm{Hom}_{\mathfrak{q}(n)}(S^{\mathsf{a}}(V_n), S^{\mathsf{b}}(V_n)) \arrow[r, hook, "\tilde{\alpha}"] & \mathrm{Hom}_{\mathfrak{q}(n)}\left(V_n^{\otimes k}, V_n^{\otimes k}\right). \end{tikzcd} \tag{56}$$

Consequently, for $f \in \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\mathsf{a},\mathsf{b})$ we have $\Psi_n^{\uparrow\downarrow}(f) = 0$ if and only if $\Psi_n^{\uparrow\downarrow}(\alpha(f)) = 0$. We claim $\alpha(f) \in \mathcal{I}\left(\uparrow_1^k, \uparrow_1^k\right)$. It follows from Proposition 5.6 that $\alpha(f) = \sum_{i=1}^t g_i e_{\gamma_i} h_i$ for $g_i, h_i \in \mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}(\uparrow_1^k)$, where e_{γ_i} are Sergeev quasi-idempotents labeled by strict partitions of k with $\ell(\gamma_i) > n$.

We next claim if γ is a strict partition of k with $\ell(\gamma) > n$, then $e_\gamma \in \mathcal{I}\left(\uparrow_1^k, \uparrow_1^k\right)$. Since $\mathcal{I}$ is closed under composition and linear combinations, it will follow that $\alpha(f) \in \mathcal{I}\left(\uparrow_1^k, \uparrow_1^k\right)$. Let $m > 0$ be fixed and large enough so that $\psi : \mathrm{Ser}_k \to \mathrm{End}_{\mathfrak{q}(m)}(V_m^{\otimes k})$ is an isomorphism. Furthermore, recall this map is compatible with the isomorphism given in Proposition 5.6. We identify the Sergeev superalgebra and these endomorphism superalgebras via these isomorphisms.

In showing the claim we make use of the fact that, through Schur–Weyl–Sergeev duality, idempotents of Ser_r correspond to projection onto direct summands of $V_m^{\otimes r}$ as a $\mathfrak{q}(m)$-module. In particular, for any strict partition of r, μ, the image of the Sergeev quasi-idempotent $e_\mu \in \mathrm{Ser}_r$ is a direct summand of the $L_m(\mu)$-isotypic component of $V_m^{\otimes r}$. By Corollary 3.9, since γ is a strict partition of k with $\ell(\gamma) > n$, $L_m(\gamma)$ is a direct summand of $L_m(\lambda(n)) \otimes V_m^{\otimes(k-|\lambda(n)|)}$ in $V_m^{\otimes k}$. That is, there are elements $a, b \in \mathrm{Ser}_k$ so that $x_\gamma := a\left(e_{\lambda(n)} \otimes \mathrm{Id}_{V_m}^{\otimes(k-|\lambda(n)|)}\right) b$ is a nonzero idempotent of Ser_k which projects onto a summand of $V_m^{\otimes k}$ isomorphic to $L_m(\gamma)$ and, hence, lies in the direct summand of (6) labeled by γ. However, since $e_{\lambda(n)}$ is an element of the tensor ideal $\mathcal{I}$, it follows that $x_\gamma \in \mathcal{I}(\uparrow_1^k, \uparrow_1^k)$. Furthermore, since the direct summand of (6) labeled by γ is a simple ideal of $\mathrm{End}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}}\left(\uparrow_1^k\right) = \mathrm{Ser}_k$, this in turn implies that the entire summand lies in $\mathcal{I}(\uparrow_1^k, \uparrow_1^k)$. In particular, e_γ lies in $\mathcal{I}(\uparrow_1^k, \uparrow_1^k)$ as originally claimed and, hence, $\alpha(f) \in \mathcal{I}(\uparrow_1^k, \uparrow_1^k)$.

Finally, recall that α has a left inverse given by composing by merges and splits. Since $\mathcal{I}$ is a tensor ideal, α^{-1} takes elements of $\mathcal{I}(\uparrow_1^k, \uparrow_1^k)$ to $\mathcal{I}(\mathsf{a},\mathsf{b})$. In particular, $f = \alpha^{-1}(\alpha(f)) \in \mathcal{I}(\mathsf{a},\mathsf{b})$. As explained in the third paragraph, this implies $\Psi_n^{\uparrow\downarrow}$ is faithful. ☐

Recall the functor $\Omega : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ from Theorem 6.7. Since $\Omega(\mathcal{I}_{\uparrow}) \subseteq \mathcal{I}_{\uparrow\downarrow}$, it induces a functor $\Omega_n : \mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}$.

Corollary 7.3. *The functors* $\Omega : \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$ *and* $\Omega_n : \mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow} \to \mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow\downarrow}$ *are faithful.*

Proof. Given a web in $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ of type $\mathsf{a} \to \mathsf{b}$, if one draws a horizontal line through the web at any height which does not intersect a merge, split, or dot, then the sum of the labels of the strands which cross that line is constant. Since the sum of the labels in any web which appears in $e_{\lambda(n)}$ equals $k := (n+1)(n+2)/2$ and this sum can only weakly increase under vertical and horizontal concatenations, it follows that $\mathcal{I}_{\uparrow}(\mathsf{a}, \mathsf{b}) = 0$ whenever $|\mathsf{a}| < k$. That is, for any fixed morphism $d \in \mathrm{Hom}_{\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}}(\mathsf{a}, \mathsf{b})$ one can choose n sufficiently large and ensure the functor $\Psi_n^{\uparrow}$ is faithful on that morphism space. Choosing n in this way means the first commutative square of functors in Section 1.4 induces a commutative square of linear maps on morphism spaces which is injective on the top row. The right side is evidently also injective. Taken together this implies the left side must be injective. Thus the functor Ω is faithful. Using Ω we identify $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$ as a full subcategory of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow\downarrow}$. Since Ω is also full we have $\mathcal{I}_{\uparrow} = \mathfrak{q}\text{-}\mathbf{Web}_{\uparrow} \cap \mathcal{I}_{\uparrow\downarrow}$ and, hence, Ω_n is also faithful. □

Let $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{V,V^*}$ denote the full monoidal subcategory of all $\mathfrak{q}(n)$-supermodules generated by V_n and V_n^*. Recall from Remark 6.9 that the diagrammatic supercategory $\mathcal{OBC}$ can be identified as a full subcategory of $\mathfrak{q}\text{-}\mathbf{Web}_{\uparrow}$. Set $\mathcal{OBC}(n)$ to be the monoidal supercategory given by imposing the relation $e_{\lambda(n)} = 0$ on $\mathcal{OBC}$. Then $\mathcal{OBC}(n)$ can be identified with a monoidal subsupercategory of $\mathfrak{q}(n)\text{-}\mathbf{Web}_{\uparrow}$. This identification along with the previous theorem readily yields the following result.

Corollary 7.4. *The monoidal supercategories $\mathcal{OBC}(n)$ and $\mathfrak{q}(n)\text{-}\mathbf{Mod}_{V,V^*}$ are equivalent.*

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