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CHARACTERISTIC HOMOMORPHISM FOR (F_1, F_2)-FOLIATED BUNDLES OVER SUBFOLIATED MANIFOLDS

by José Manuel CARBALLÉS

1. Introduction.

Let (F_1, F_2) be a couple of foliations on a differentiable manifold M such that the leaves of F_1 contain those of F_2 ; we shall say such couple (F_1, F_2) a subfoliation on M. While Moussu [9], Feigin [5], Cordero-Gadea [3] and Cordero-Masa [4] have study the (exotic) characteristic homomorphism of a subfoliation (F_1, F_2) using the techniques of Bernstein-Rozenfeld, Bott-Haefliger and Lehmann, our aim in this paper is to present the construction of the characteristic homorphism of (F_1, F_2) using the techniques and language of Kamber-Tondeur for foliated bundles.

Our study is based on the notion of (F_1, F_2)-foliated principal bundle. This is a principal bundle of the form $P = P_1 + P_2 \longrightarrow M$ of structure group $G_1 \times G_2$ endowed with a foliated structure given by a connection of the form $\omega = \omega_1 + \omega_2$ (called adapted connection sum) and where, for each $i = 1, 2$, $P_i \longrightarrow M$ is an F_i-foliated principal bundle of structure group G_i, and ω_i is an adapted connection in P_i. The most meaningful example of (F_1, F_2)-foliated bundle over M is a reduction of the bundle of linear frames of the so called normal bundle of (F_1, F_2) defined by $\nu(F_1, F_2) = (F_1/F_2) \oplus \nu F_1$. This vector bundle $\nu(F_1, F_2)$ has been used in [4] in order to define the characteristic homomorphism of (F_1, F_2) adapting the Bott [2] well-known construction of the characteristic homomorphism of a foliation; our construction of the characteristic homorphism of an (F_1, F_2)-foliated principal bundle generalizes that of Cordero-Masa in the same way as Kamber-Tondeur theory of characteristic classes of foliated bundles generalizes Bott theory. This approach allows, moreover, to initiate the study of the holonomy homomorphism of a "leaf" of a subfoliation, in the line of Goldman's paper [6] for the leaf of a foliation.

The paper is structured as follows. In § 2, we introduce the basic definitions and deduce the filtration preserving properties of the Weil homomorphism $k(\omega)$ of an adapted connection sum in an (F_1, F_2)-foliated bundle. As a particular consequence, the vanishing theorem for the normal bundle of a subfoliation [4], [5] is reobtained. These properties of $k(\omega)$ are used in order to prove the vanishing of $k(\omega)$ on a differential ideal I of the product Weil algebra $W(g_1 \oplus g_2)$ (firstly considered by Feigin [5]) and thus, following Kamber-Tondeur's theory, we introduce the generalized characteristic homomorphism of an (F_1, F_2)-foliated principal bundle P:

$$\Delta_* = \Delta_{(F_1, F_2)}(P) : H(W(g, H)_I) \longrightarrow H_{DR}(M)$$

where $H \subset G$ is a closed Lie subgroup such that P admits an H-reduction. We show that Δ_* does not depend on the connection sum ω and that it satisfies the usual functorial properties (i.e. naturality under pull-backs and ρ-extensions). We also deal with the case where ω_1 and ω_2 both are basic connections.

In § 3, we relate the generalized characteristic homomorphism $\Delta_*(P)$ with the generalized characteristic homomorphism (as defined in [7] of each P_i, $i = 1, 2$. Taking into account that any adapted connection sum in P is F_2-adapted, we deduce some properties of the characteristic homomorphism as F_2-foliated bundle of an (F_1, F_2)-foliated bundle as well as of any F_2-extension of it. This section ends with the construction of the generalized characteristic homomorphism $\Delta_*(P)$ when considering a foliation F as a subfoliation in the three possible forms.

In § 4 we apply the general results of Kamber-Tondeur on the cohomology of g-DG-algebras in order to calculate the cohomology $H(W(g, H)_I)$. In particular, this allows to refind the characteristic homomorphism of (F_1, F_2) as defined in [4]. The algebra of secondary characteristic invariants is constructed and a geometric interpretation of the generalized characteristic homomorphism is also given for the general situation.

Finaly, in § 5, we restrict the (F_1, F_2)-foliated bundle P to the leaves of each foliation $F_i, i = 1, 2$; this leads us, on the one hand to a slightly generalization of Goldman's study, and, on the other, to define the holonomy homomorphism of a "leaf" of a subfoliation and to discuss an example of Reinhart [10].

Through all this paper, the manifolds, maps, etc, will be assumed differentiable of class C^∞. Also, we shall adopt the notation of [7].

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2. Characteristic homomorphism of an (F_1, F_2)-foliated bundle.

Let M be an n-dimensional differentiable manifold, TM its tangent bundle. Through all this paper, we always assume M endowed with a (q_1, q_2)-codimensional subfoliation (F_1, F_2), that is, of a couple of integrable subbundles F_i of TM of dimension $n - q_i$, $i = 1, 2$, and F_2 being a subbundle of F_1. Therefore, for each i, F_i defines a q_i-codimensional foliation on M, $d = q_2 - q_1 \geqslant 0$ and the leaves of F_1 contain those of F_2.

Let $Q_i = TM/F_i$ be the normal bundle of F_i, $i = 1, 2$, and Q_0 the quotient bundle F_1/F_2; then, there is a short exact sequence of vector bundles, canonically associated to (F_1, F_2), $0 \longrightarrow Q_0 \xrightarrow{i} Q_2 \xrightarrow{\pi} Q_1 \longrightarrow 0$ and the vector bundle $\nu(F_1, F_2) = Q_0 \oplus Q_1$ is called the normal bundle of (F_1, F_2).

Let $P_i(M, G_i)$ be an F_i-foliated principal bundle, $i = 1, 2$, and let ω_i be an adapted connection. Let

$$P(M, G_1 \times G_2) = P_1(M, G_1) + P_2(M, G_2)$$

be the principal bundle sum of P_1 and P_2; then $\omega = \omega_1 + \omega_2$ defines two partial connections in P and ω is adapted to both; endowed with these two partial connections, P will be said (F_1, F_2)-foliated and $\omega = \omega_1 + \omega_2$ an adapted connection sum. Let us remark that, in particular, P is F_2-foliated and if both ω_1 and ω_2 are basic, then $\omega = \omega_1 + \omega_2$ is also basic with respect to F_2.

Let $L(Q_i)$ be the frame bundle of Q_i, $i = 0, 1$, and $L(Q_1) + L(Q_0)$ the bundle sum. As it can be easily shown using the results in [4], $L(Q_1) + L(Q_0)$ is (F_1, F_2)-foliated and it will be called the bundle of transverse frames of (F_1, F_2). Other examples can be obtained as follows; let $P_i \longrightarrow M$ be a G_i-principal bundle, $i = 1, 2$, endowed with an F_i-foliated structure, F_i being the orbit foliation defined on M by a left almost free action of a Lie subgroup $K_i \subset G_i$ (see 2.4 in [7]); then, if $K_2 \subset K_1$, $P = P_1 + P_2$ is an (F_1, F_2)-foliated bundle. In particular, if $P \longrightarrow M$ is a G-principal bundle which is F_1-foliated by the orbits of the action of a Lie subgroup $K_1 \subset G$ on M, as above, then for each Lie subgroup $K_2 \subset K_1$, the bundle $P + P$ is (F_1, F_2)-foliated.

Let $P = P_1 + P_2$ be an (F_1, F_2)-foliated bundle over M, $\omega = \omega_1 + \omega_2$ an adapted connection sum. If we denote $G = G_1 \times G_2$, its Lie algebra by $g = g_1 \oplus g_2$ and $k(\omega), k(\omega_1), k(\omega_2)$ the respective Weil homomorphisms, the following commutative diagram allows to write $k(\omega) = k(\omega_1) \otimes k(\omega_2)$:

$$\begin{tikzcd}
W(g) \arrow[r, "L"] \arrow[dd, "k(\omega)"] & W(g_1) \otimes W(g_2) \arrow[d, "k(\omega_1) \otimes k(\omega_2)"] \\
 & \Omega(P_1) \otimes \Omega(P_2) \arrow[d, "\pi"] \\
\Omega(P) & \Omega(P_1 \times P_2) \arrow[l, "\bar{\Delta}^*"']
\end{tikzcd}$$

where L denotes the canonical isomorphism, π is defined by $\pi(\alpha \otimes \beta) = p_1^* \alpha \wedge p_2^* \beta$, $p_i : P_1 \times P_2 \longrightarrow P_i$ the canonical projection, and $\bar{\Delta}^*$ being induced by the canonical homomorphism $\bar{\Delta} : P = P_1 + P_2 \longrightarrow P_1 \times P_2$.

Using $L : W(g) \cong W(g_1) \otimes W(g_2)$, the canonical even decreasing filtration of $W(g)$ by G-DG-ideals can be written as

$$F^{2p} W(g) = \bigoplus_{j \geqslant p} \Lambda^{\cdot} g^* \otimes S^j g^*$$

$$= \bigoplus_{j_1 + j_2 \geqslant p} \Lambda^{\cdot} g^* \otimes S^{j_1} g_1^* \otimes S^{j_2} g_2^* , \quad p \geqslant 0$$

and we can define a new even decreasing filtration of $W(g)$, also by G-DG-ideals, by

$$'F^{2p} W(g) = \bigoplus_{j \geqslant p} \Lambda^{\cdot} g^* \otimes S^j g_1^* \otimes S^{\cdot} g_2^* , p \geqslant 0 .$$

Also, $\Omega^{\cdot}(P)$ has two decreasing filtrations by G-DG-ideals defined by the sheaves $\underline{Q}_i^*$, $i = 1, 2$, of local 1-forms annihilating the foliation F_i on the base space M ; they are given by

$$F^p \Omega(P) = \Gamma(P, \pi^* \Lambda^p \underline{Q}_2^* \cdot \Omega_p),$$

$$'F^p \Omega(P) = \Gamma(P, \pi^* \Lambda^p \underline{Q}_1^* \cdot \Omega_p), \quad p \geqslant 0.$$

Then, the Weil homomorphism $k(\omega)$ of an adapted connection sum $\omega = \omega_1 + \omega_2$ is filtration-preserving, that is

$$k(\omega)(F^{2p} W(g)) \subset F^p \Omega(P), \quad p \geqslant 0,$$

and if ω_1 and ω_2 are basic, then

$$k(\omega)(F^{2p} W(g)) \subset F^{2p} \Omega(P), \quad p \geqslant 0 .$$

Moreover, one easily proves

PROPOSITION 2.1. – *Let* $\omega = \omega_1 + \omega_2$ *be an adapted connection sum in* $P = P_1 + P_2$. *Then* $k(\omega)('F^{2p} W(g)) \subset {}'F^p \Omega(P)$, $p \geqslant 0$. *If* ω_1 *and* ω_2 *are basic, then* $k(\omega)('F^{2p} W(g)) \subset {}'F^{2p} \Omega(P)$, $p \geqslant 0$.

COROLLARY 2.2. – *For an adapted connection sum* $\omega = \omega_1 + \omega_2$,

$$k(\omega)\, F^{2(q_2+1)} W(g) = 0, \quad k(\omega)\, 'F^{2(q_1+1)} W(g) = 0 .$$

If ω_1 *and* ω_2 *are basic,*

$$k(\omega)\, F^{2([q_2/2]+1)}\, W(g) = 0\,, \quad k(\omega)\, 'F^{2([q_1/2]+1)}\, W(g) = 0\,.$$

If we now consider the algebras of G-basic elements, we obtain similar properties for the Chern-Weil homomorphism $h(\omega) : I(G) = I(G_1 \times G_2) \longrightarrow \Omega(M)$ with respect to the following filtrations of $I(G)$ and $\Omega(M)$:

$$F^{2p}\, I(G) = \bigoplus_{j \geqslant p} I^{2j}(G)\,, \quad 'F^{2p}\, I(G) = \bigoplus_{j \geqslant p} I^{2j}(G_1) \otimes I^{\bullet}(G_2), \quad p \geqslant 0$$

$$F^p \Omega(M) = \Gamma(M\,, \Lambda^p\, \underline{Q}_2^* \cdot \Omega_M)\,,$$

$$'F^p\, \Omega(M) = \Gamma(M\,, \Lambda^p\, \underline{Q}_1^* \cdot \Omega_M)\,, \quad p \geqslant 0\,.$$

That is, since $F^{q_2+1}\Omega(M) = 0$ and $'F^{q_1+1}\Omega(M) = 0$, we have

COROLLARY 2.3. – *Let* $\omega = \omega_1 + \omega_2$ *be an adapted connection sum in an* $(F_1\,, F_2)$*-foliated bundle* $P = P_1 + P_2$, *and let* $h(\omega)$ *denote the Chern-Weil homomorphism of* P. *Then*

$$h(\omega)\, F^{2(q_2+1)} I(G) = 0\,, \quad h(\omega)\, 'F^{2(q_1+1)} I(G) = 0\,.$$

If, moreover, ω_1 *and* ω_2 *are basic, then*

$$h(\omega)\, F^{2([q_2/2]+1)} I(G) = 0\,, \quad h(\omega)\, 'F^{2([q_1/2]+1)} I(G) = 0\,.$$

In particular, if P is the bundle of transverse frames of $(F_1\,, F_2)$, then Corollary 2.3 is the Vanishing Theorem for subfoliations stated in [4].

Next, let $I \subset W(g)$ be the G-DG-ideal given by

$$I = F^{2(q_2+1)} W(g) + 'F^{2(q_1+1)} W(g)\,. \tag{2.1}$$

Then, by virtue of Corollary 2.2, $I \subset \mathrm{Ker}(k(\omega))$ and there is an induced G-DG-homomorphism $k(\omega) \colon W(g)_I = W(g)/I \longrightarrow \Omega(P)$.

For any subgroup $H \subset G$, there is the relative ideal I_H of $W(g\,, H) = W(g)_H$, and thus if we construct

$$W(g\,, H)_I = W(g\,, H)/I_H = (W(g)_I)_H\,,$$

we can consider the induced DG-homomorphism

$$k(\omega)_H : W(g\,, H)_I \longrightarrow \Omega(P)_H\,.$$

Now, if we assume H to be closed and P having an H-reduction given by a section $s : M \longrightarrow P/H$ of the induced map $\hat{\pi} : P/H \longrightarrow M$, we can construct a DG-homomorphism as the composition

$$\Delta(\omega) = s^* \circ k(\omega)_H : W(g, H)_I \longrightarrow \Omega(P)_H \cong \Omega(P/H) \longrightarrow \Omega(M).$$

DEFINITION 2.4. – *We shall call generalized characteristic homomorphism of the* (F_1, F_2)*-foliated bundle* P *the homomorphism* $\Delta_* = \Delta_{(F_1, F_2)}(P) : H(W(g, H)_I) \longrightarrow H_{DR}(M)$ *induced by* $\Delta(\omega)$ *in cohomology.*

Remark. – If both ω_1 and ω_2 are basic connections, then $k(\omega)$ vanishes on the ideal

$$I' = F^{2([q_2/2]+1)} W(g) + {'F}^{2([q_1/2]+1)} W(g)$$

and the generalized characteristic homomorphism of P will be $\Delta_* : H(W(g, H)_{I'}) \longrightarrow H_{DR}(M)$ because, under these conditions, $\Delta(\omega)$ factorizes through $p : W(g, H)_I \longrightarrow W(g, H)_{I'}$, the canonical projection induced by the injection $I \subset I'$.

$\Delta_* = \Delta_{(F_1, F_2)}(P)$ is independent of the choice of $\omega = \omega_1 + \omega_2$ in the following sense. Let $\omega^0 = \omega_1^0 + \omega_2^0$, $\omega^1 = \omega_1^1 + \omega_2^1$ be two adapted connections sum in P. Let an H-reduction of P be given by a section $s : M \longrightarrow P/H$, and

$$\Delta_*^i = \Delta(\omega^i)_* : H(W(g, H)_I) \longrightarrow H_{DR}(M)$$

the homomorphism constructed using the connection ω^i, $i = 0, 1$. Then,

PROPOSITION 2.5. – $\Delta_*^0 = \Delta_*^1$.

Proof. – Let $f : M \times [0, 1] \longrightarrow M$ be the canonical projection, and let $f^{-1}(F_k)$, $k = 1, 2$, the foliation inverse image of F_k via f. If $P = P_1 + P_2$ is an (F_1, F_2)-foliated bundle over M then the inverse image $P' = f^*(P) = f^*(P_1) + f^*(P_2)$ of P via f is $f^{-1}(F_1, F_2) = (f^{-1}(F_1), f^{-1}(F_2))$-foliated. Moreover, the connection $\bar{\omega}$ given by

$$\bar{\omega}(X) = t(f^* \omega^1)(X) + (1 - t)(f^* \omega^0)(X), \quad X \in T_{(u, t)}(P')$$

is obviously an adapted connection sum in P'.

On the other hand, if $j_t : M \longrightarrow M \times [0, 1]$ is the canonical injection $j_t(x) = (x, t)$, for each $t \in [0, 1]$, then $j_t^*(P') = P$ for any $t \in [0, 1]$, $\bar{j}_0^* \bar{\omega} = \omega^0$, $\bar{j}_1^* \bar{\omega} = \omega^1$ where $\bar{j}_t : P \longrightarrow P'$

denotes the canonical lift of j_t. Thus, using $\bar{\omega}$ to construct the generalized characteristic homomorphism of $P' : \bar{\Delta}_* = \Delta_*(\bar{\omega})$, we have $\Delta_*^i = (j_i^*)_{DR} \circ \bar{\Delta}_*$, $i = 0, 1$. But, since $(j_0^*)_{DR} = (j_1^*)_{DR}$, then $\Delta_*^0 = \Delta_*^1$.

It is clear from the construction that Δ_* depends a priori upon the H-reduction of P given by s. However, this construction is visibly independent of s if the closed subgroup $H \subset G$ contains a maximal compact subgroup of G.

Δ_* has also the following properties of functoriality.

(A) Δ_* *is functorial under pullbacks.*

This means more precisely the following. Let (F_1', F_2') and (F_1, F_2) be (q_1, q_2)-codimensional subfoliations on M' and M respectively, and let $f : M' \longrightarrow M$ be a differentiable map such that $f_*(F_i') \subset F_i$, $i = 1, 2$. Let $P = P_1 + P_2$ be an (F_1, F_2)-foliated bundle over M, and let

$$P' = f^*P = f^*P_1 + f^*P_2$$

be the inverse image of P via f. Since each f^*P_i is F_i'-foliated ([1], Prop. 1.7), then P' is, in fact, an (F_1', F_2')-foliated bundle over M'. Then, if $H \subset G$ is a closed subgroup and $s : M \longrightarrow P/H$ the section given an H-reduction of P, $s' = f^*s : M' \longrightarrow P'/H$ gives an H-reduction of P' and we can easily prove

PROPOSITION 2.6. – $\Delta_*(P') = f_{DR}^* \circ \Delta_*(P)$.

It is clear that this result is applied in the particular case of f being transversal to the subfoliation (F_1, F_2) on M [4].

(B) Δ_* *is functorial under* ρ*-extensions.*

This means more precisely the following. Let

$$\rho = (\rho_1, \rho_2) : G = G_1 \times G_2 \longrightarrow G' = G_1' \times G_2'$$

a homomorphism of product Lie groups, that is, each $\rho_i : G_i \longrightarrow G_i'$ a Lie group homomorphism, $i = 1, 2$. If P is an (F_1, F_2)-foliated principal bundle over M and ω an adapted connection sum in P, then P', the extension of P by ρ, is (F_1, F_2)-foliated and ω', extension of ω by ρ, is an adapted connection sum in P'.

Let H, H' be closed subgroups of G and G', respectively, such that $\rho(H) \subset H'$; let I' and I be the ideals of $W(g')$ and $W(g)$ given by (2.1). Since $W(d\rho)$ is graduation-preserving, then $W(d\rho)(I') \subset I$ and diagram (4.72) in [7] can be used to state

PROPOSITION 2.7. – $\Delta_*(P') = \Delta_*(P) \circ W(d\rho)^*$.

3. Relation between $\Delta_*(P)$ and $\Delta_*(P_i)$, $i = 1, 2$.

Between the generalized characteristic homomorphism $\Delta_*(P)$ of an (F_1, F_2)-foliated principal bundle $P = P_1 + P_2$ and the generalized characteristic homomorphism $\Delta_*(P_i)$ ([7]) of the F_i-foliated principal bundle P_i, $i = 1, 2$, there exists a canonical relation given as follows.

Let $\rho_i : G = G_1 \times G_2 \longrightarrow G_i$ be the canonical projection, $H_i \subset G_i$ a closed subgroup, $i = 1, 2$, and $H = H_1 \times H_2 \subset G$. Let $s : M \longrightarrow P/H$ be a section defining an H-reduction of P and let $s_i : M \longrightarrow P_i/H_i$ be the induced section defining an induced H_i-reduction of P_i. Then.

PROPOSITION 3.1. – *The diagram*

$$\begin{tikzcd}
H(W(g_i, H_i)_{q_i}) \arrow[rr, "W(d\rho_i)^*"] \arrow[dr, "\Delta_*(P_i)"'] & & H(W(g, H)_I) \arrow[dl, "\Delta_*(P)"] \\
& H_{DR}(M) &
\end{tikzcd}$$

is commutative for each $i = 1, 2$. *In fact, this diagram is also commutative at the cochain level.*

Proof. – Since P_i is isomorphic (as F_i-foliated bundle) to the ρ_i-extension of P, and because $\omega_i = (\rho_i)_* \omega$ is an adapted connection in P_i, $\omega = \omega_1 + \omega_2$ being an adapted connection sum in P, the following diagram commutes for each $i = 1, 2$:

$$\begin{tikzcd}
W(g_i, H_i) \arrow[rr, "W(d\rho_i)"] \arrow[d, "k(\omega_i)_{H_i}"] & & W(g, H) \arrow[d, "k(\omega)_H"] \\
\Omega(P_i/H_i) \arrow[dr, "s_i^*"] & & \Omega(P/H) \arrow[dl, "s^*"'] \\
& \Omega(M) &
\end{tikzcd} \tag{3.1}$$

and we are reduced to show that $W(d\rho_i)\,(F^{2(q_i+1)}W(g_i)) \subset I$ for each $i = 1, 2$.

For $i = 2$, this follows easily because $W(d\rho_i)$ preserves the bigraduation and then

$$W(d\rho_2)\, W^{p,2q}(g_2) \subset W^{p,2q}(g).$$

For $i = 1$, the result follows from the fact that

$$W(d\rho_1)\,(\Lambda^u g_1^* \otimes S^v g_1^*) \subset \Lambda^u g^* \otimes S^v g_1^* \otimes S^0 g_2^*, \quad u, v \geqslant 0$$

since $(d\rho_1)^* : Sg_1^* \longrightarrow Sg^* = Sg_1^* \otimes Sg_2^*$ is given by

$$(d\rho_1)^*(\alpha) = \alpha \otimes 1.$$

Remarks. – 1) Since both $\omega = \omega_1 + \omega_2$ and ω_i are F_2-adapted connections, we can truncate the Weil algebras in diagram (3.1) at the degree q_2 and thus, going into cohomology, obtain a commutative diagram relating the generalized characteristic homomorphisms of P and P_i as F_2-foliated principal bundles.

2) We can use $\omega = \omega_1 + \omega_2$ to construct the generalized characteristic homomorphism of the F_2-foliated bundle P :

$$\Delta_{F_2}(P) : H(W(g, H)_{q_2}) \longrightarrow H_{DR}(M).$$

Then, taking into account that the inclusion $F^{2(q_2+1)}W(g) \subset I$ induces a projection $p : W(g, H)_{q_2} \longrightarrow W(g, H)_I$, we obtain a commutative diagram

$$\begin{tikzcd}
H(W(g, H)_{q_2}) \arrow[rr, "p^*"] \arrow[dr, "\Delta_{F_2}(P)"'] & & H(W(g, H)_I) \arrow[dl, "\Delta_{(F_1, F_2)}(P)"] \\
& H_{DR}(M) &
\end{tikzcd} \tag{3.2}$$

and, therefore, $\mathrm{Im}\ \Delta_{F_2}(P) \subset \mathrm{Im}\ \Delta_{(F_1, F_2)}(P)$.

3) Let $\rho : G = G_1 \times G_2 \longrightarrow G'$ be a homomorphism of Lie groups and consider the structure of F_2-foliated bundle on the ρ-extension $P' = \rho_* P$ induced by the structure of F_2-foliated bundle underlying the (F_1, F_2)-foliated structure of $P = P_1 + P_2$.

Then, for suitable closed subgroups $H \subset G$, $H' \subset G'$, the functoriality under ρ-extensions of the generalized characteristic homomorphism of foliated bundles ([7]) implies that the following diagram is commutative

$$\begin{tikzcd}
H(W(g', H')_{q_2}) \arrow[rr, "W(d\rho)^*"] \arrow[dr, "\Delta_{F_2}(P')"'] & & H(W(g, H)_{q_2}) \arrow[dl, "\Delta_{F_2}(P)"] \\
& H_{DR}(M) &
\end{tikzcd}$$

which combined with (3.2) leads to the following

PROPOSITION 3.2. – *Let* $P' \longrightarrow M$ *be an* F_2*-foliated principal bundle with structure group* G' *and let* $P = P_1 + P_2$ *be an* (F_1, F_2)*-foliated* G*-reduction of* P. *Assume* $i : P \longrightarrow P'$. *be* F_2*-foliated compatibly with the homomorphism*

$$\rho : G = G_1 \times G_2 \longrightarrow G',$$

and let H, H' *be closed subgroups of* G, G' *respectively, verifying the suitable hypothesis. Then, the generalized characteristic homomorphism* $\Delta_{F_2}(P')$ *of* P' *as* F_2*-foliated bundle factorizes through the generalized characteristic homomorphism* $\Delta_{(F_1, F_2)}(P)$

of P *as* (F_1, F_2)*-foliated bundle, that is, the following diagram is commutative:*

$$\begin{tikzcd}
H(W(g', H')_{q_2}) \arrow[rr, "p^* \circ W(d\rho)^*"] \arrow[dr, "\Delta_{F_2}(P')"'] & & H(W(g, H)_I) \arrow[dl, "\Delta_{(F_1, F_2)}(P)"] \\
& H_{DR}(M) &
\end{tikzcd}$$

Example. – Let $P' \doteq L(Q_2) \cong L(\nu(F_1, F_2))$ be the canonically F_2-foliated bundle of transverse frames of F_2, and P the (F_1, F_2)-foliated bundle of transverse frames of (F_1, F_2), which is a (not F_2-foliated) reduction of P' compatible with the canonical homomorphism $\rho : Gl(q_1, \mathbb{R}) \times Gl(d, \mathbb{R}) \longrightarrow Gl(q_2, \mathbb{R})$.

If we consider in P' the ρ-extension of the F_2-foliated structure of P, this is not the canonical F_2-foliated structure of P'; but, as it can be easily shown using the Lemma 5.3 in [4] (see [8]), both are integrably homotopic. Then, for suitable H, H' the proposition 3.2 provides the corresponding commutative diagram. If, moreover, $H = O(q_1) \times O(d)$ and $H' = O(q_2)$, then $\Delta_{F_2}(P')$ is just the characteristic homomorphism of the foliation F_2, whereas $\Delta_{(F_1, F_2)}(P)$ is the characteristic homomorphism of the subfoliation (F_1, F_2) [4], as it will be established later.

Now, let us remark that a q-codimensional foliation F on M can be considered as a subfoliation on M in three different ways; $(C_1) : F_1 = F_2 = F$, $q_1 = q_2 = q$; (C_2): $F_1 = TM$, $F_2 = F$, $q_1 = 0, q_2 = q$; (C_3): $F_1 = F$, $F_2 = 0$, $q_1 = q$, $q_2 = n$. Then, all the previous results particularize to these cases as follows:

Case (C_1). – Here an (F_1, F_2)-foliated bundle $P = P_1 + P_2$ is, in fact, an F-foliated bundle, the ideal I coincides with $F^{2(q+1)} W(g)$, p^* in diagram (3.2) is an isomorphism and $\Delta_{(F, F)}(P) = \Delta_F(P)$.

Case (C_2). – Here, $P = P_1 + P_2$ is the sum of a flat bundle P_1 and an F-foliated bundle P_2; since

$$I = F^{2(q+1)}\, W(g) + {}'F^2 W(g),$$

making calculations we obtain

$$W(g)_I \cong \bigoplus_{j=0}^{q} \Lambda^\bullet g^* \otimes S^0 g_1^* \otimes S^j g_2^*$$

$$W(g)_q \cong W(g)_I \oplus \left(\bigoplus_{j=0}^{q-1} \bigoplus_{i=0}^{q-j} \Lambda^\bullet g^* \otimes S_1^i\, g^* \otimes S_2^j\, g^* \right)$$

and then p^* in diagram (3.2) is surjective. Hence

$$\operatorname{Im}(\Delta_F(P)) = \operatorname{Im}(\Delta_{(TM\,,\,F)}(P)).$$

Case (C_3). – In this case, $P = P_1 + P_2$ is simply an ordinary bundle (that is, 0-foliated) which is not necessarily F-foliated. Thus, if we take $H = G$ in diagram (3.2) and denote $\tau : I(G) \longrightarrow I(G)_n$ the canonical projection, we have a commutative diagram

$$\begin{tikzcd}
I(G) \arrow[r, "\tau"] \arrow[dr, "h^*"'] & I(G)_n \arrow[r, "p^*"] \arrow[d, "\Delta_0(P)"] & I(G)_I \arrow[dl, "\Delta_{(F,0)}(P)"] \\
& H_{DR}(M) &
\end{tikzcd}$$

where h^* denotes the Chern-Weil homomorphism of P. Thus, we can assert the following: if $P = P_1 + P_2$ where P_1 is a foliated bundle, then the Chern-Weil homomorphism of P vanishes on Ker $(p^* \circ \tau)$. Again, since any connection in P_2 is basic with respect to the foliation by points on M, if P_1 admits a basic connection then h^* vanishes on the kernel of the composition

$$I(G) \xrightarrow{\quad \tau' \quad} I(G)_{[n/2]} \xrightarrow{\quad p'^* \quad} I(G)_{I'}\,.$$

4. Difference construction for $\Delta_{(F_1, F_2)}(P)$. Secondary invariants.

The computation of $H(W(g, H)_I)$ can be done from the general results in [7], Chapter 5, from where we shall take the notation.

We assume throughout that G is either connected or $I(G) \cong I(G_0) \equiv I(g)$ for the connected component G_0 of G; the closed subgroup $H \subset G$ is assumed to have finitely many connected components.

Then, let us consider in the G-DG-algebra $W(g)_I$ the canonical connection given by the projection $k : W(g) \longrightarrow W(g)_I$.

If the pair (g, h) is reductive (h = Lie algebra of H), in accordance with Theorem 5.82 in [7] there exists a homomorphism $\zeta(W(g)_I, H) : A(W(g)_I, H) \longrightarrow (W(g)_I)_H = W(g, H)_I$ which induces an isomorphism in cohomology. In this way, the generalized characteristic homomorphism $\Delta_{(F_1, F_2)}(P)$ of P will have the same image as the composition

$$H(A(W(g)_I, H)) \xrightarrow[\cong]{\zeta(W(g)_I, H)_*} H(W(g, H)_I) \xrightarrow{\Delta_{(F_1, F_2)}(P)} H_{DR}(M)$$

induced by the cochain map $\widetilde{\Delta}(\omega) = \Delta(\omega) \circ \zeta(W(g)_I, H)$. In fact, the evaluation of $\widetilde{\Delta}(\omega)$ on the complex

$$A(W(g)_I, H) = \Lambda P_g \otimes (W(g)_I)_g \otimes I(H) = \Lambda P_g \otimes I(G)_I \otimes I(H)$$

is equal to that of Theorem 5.95 in [7] for the case of a foliated bundle.

If we now assume the pair (g, h) to be special Cartan (CS), then, by Theorem 5.107 in [7], there is an isomorphism

$$\overline{\beta} : H(\hat{A}(W(g)_I)) \otimes_{I(g)} I(H) \xrightarrow{\cong} H(A(W(g)_I, H))$$

where $\hat{A}(W(g)_I) = \Lambda\hat{P} \otimes (W(g)_I)_g$. Thus $\Delta_{(F_1, F_2)}(P)$ has the same image as the composition $\Delta_{(F_1, F_2)}(P) \circ \zeta(W(g)_I, H) \circ \overline{\beta}$. Then taking into account that $\hat{A}(W(g)_I) \subset A(W(g)_I, H)$, we consider the composition

$$\hat{\Delta}(\omega) : \hat{A}(W(g)_I) \longrightarrow A(W(g)_I, H) \xrightarrow{\widetilde{\Delta}(\omega)} \Omega(M)$$

and, thus, the characteristic homomorphism $\Delta_{(F_1, F_2)}(P)$ will be realized by $\hat{\Delta}_* \otimes h'_* : H(\hat{A}(W(g)_I)) \otimes_{I(G)} I(H) \longrightarrow H_{DR}(M), h'_*$ being the characteristic homomorphism of the H-reduction P' of P. See 5.112 in [7] for more details.

- In particular, let us assume that $P = P_1 + P_2$ is the bundle of transverse frames of (F_1, F_2), and take

$$H = O(q_1) \times O(d) \subset Gl(q_1, \mathbf{R}) \times Gl(d, \mathbf{R}) = G.$$

Since $gl(q_1, \mathbf{R})$ and $gl(d, \mathbf{R})$ are reductive Lie algebras and $(gl(q_1, \mathbf{R}) \times gl(d, \mathbf{R}), o(q_1) \times o(d))$ is symmetric, this pair will be special Cartan and the previous construction can be used. Then, $\Delta_{(F_1, F_2)}(P)$ can be considered as defined on $H(\hat{A}_I) \otimes_{I(G)} I(H)$, where $\hat{A}_I = \hat{A}(W(g)_I) = \Lambda\hat{P} \otimes I(gl(q_1, \mathbf{R}) \times gl(d, \mathbf{R}))_I$. But, as it happens in the case of the bundle of transverse frames of a foliation [7], $H(\hat{A}_I) \otimes_{I(G)} I(H) \cong H(\hat{A}_I)$, and then

$$\hat{\Delta}_{(F_1, F_2)}(P) = \hat{\Delta}_* : H(\hat{A}_I) \longrightarrow H_{DR}(M).$$

On the other hand, $I(gl(q_1, \mathbf{R})) = \mathbf{R}[c_1, \ldots, c_{q_1}]$, $I(gl(d, \mathbf{R})) = \mathbf{R}[c'_1, \ldots, c'_d]$ and $\Lambda\hat{P} = \Lambda\hat{P}_1 \otimes \Lambda\hat{P}_2$, $\hat{P}_i$ being the Samelson subspace of the pair (g_i, h_i), $i = 1, 2$; since both pairs are special Cartan,

$$\Lambda\hat{P}_1 = \Lambda(y_1, y_2, \ldots, y_{q'_1}), \quad \Lambda\hat{P}_2 = \Lambda(y'_1, y'_3, \ldots, y'_{d'})$$

where $y_i = \sigma c_i$, $y'_i = \sigma' c'_i$ and $q'_1 = 2[(q_1 + 1)/2] - 1$, $d' = 2[(d + 1)/2] - 1$, σ and σ' being the suspension maps. Therefore

$$\hat{A}_I = \Lambda(y_1, y_3, \ldots, y_{q'_1}) \otimes \Lambda(y'_1, y'_3, \ldots, y'_{d'}) \otimes \frac{\mathbf{R}[c_1, \ldots, c_{q_1}] \otimes \mathbf{R}[c'_1, \ldots, c'_d]}{I_g}$$

where

$$I_g = I \cap I(g) = \langle \{\alpha \otimes \beta \in I^{j_1}(g_1) \otimes I^{j_2}(g_2) / j_1 > q_1 \text{ or } j_1 + j_2 > q_2\} \rangle.$$

That is, $\hat{A}_I = WO_I$, the graded differential algebra defined in [4]. Therefore, the generalized characteristic homomorphism of the bundle of transverse frames of the subfoliation (F_1, F_2) coincides with the characteristic homomorphism of (F_1, F_2) as defined in [4]

$$\lambda^*_{(F_1, F_2)} : H(WO_I) \longrightarrow H_{DR}(M).$$

From this point of view, the generalized characteristic homomorphism $\Delta_{(F_1,F_2)}(P)$ of an (F_1, F_2)-foliated bundle $P = P_1 + P_2$ generalizes the characteristic homomorphism of the subfoliation (F_1, F_2) in the same way as Kamber-Tondeur's characteristic homomorphism of a foliated bundle generalizes Bott's characteristic homomorphism of a foliation ([7], [2]).

In order to construct the algebra of secondary characteristic invariants, from now on, we shall consider an (F_1, F_2)-foliated bundle $P = P_1 + P_2$, $H \subset G$ a closed subgroup with finitely many connected components and such that the pair of Lie algebras (g, h) be reductive. Let us denote P' the H-reduction of P used to define the characteristic homomorphism $\Delta_{(F_1,F_2)}(P)$ of P and, to simplify the notation, put $A_I = A(W(g)_I, H)$.

Let $p : A_I \longrightarrow I(G)_I \otimes_{I(G)} I(H)$ the composition of the canonical projection along ΛP_g, $\lambda : A_I \longrightarrow I(G)_I \otimes I(H)$ with the canonical map.

DEFINITION 4.1. – $H(K_I)$, *where* $K_I = \operatorname{Ker} p$, *is called the algebra of secondary characteristic invariants of* P.

PROPOSITION 4.2. – *There is a short exact sequence of algebras*

$$0 \longrightarrow H(K_I) \longrightarrow H(W(g, H)_I) \longrightarrow I(G)_I \otimes_{I(G)} I(H) \longrightarrow 0 . \quad (4.1)$$

Proof. – Consider the short exact sequence of complexes

$$0 \longrightarrow K_I \longrightarrow A_I \longrightarrow I(G)_I \otimes_{I(G)} I(H) \longrightarrow 0 .$$

Then (4.1) appears by writing up the associated long exact sequence of homology whose connecting homomorphism is null, and because $H(A_I) \cong H(W(g, H)_I)$.

□

The non-triviality of $\Delta_{(F_1,F_2)}(P)/_{H(K_I)}$ is a measure for the incompatibility of the (F_1, F_2)-foliated structure of $P = P_1 + P_2$ with its H-reduction P'; that is,

PROPOSITION 4.3. – *Let* $P = P_1 + P_2$ *be an* (F_1, F_2)*-foliated bundle,* $H = H_1 \times H_2 \subset G$ *a closed subgroup and* P' *an* H*-reduction of* P *which is* (F_1, F_2)*-foliated and such that, if*

$\iota : H \longrightarrow G$ is the injection, then the (F_1, F_2)-foliated structure of P is, in fact, the ι-extension of that of P'. Then

$$\Delta_{(F_1, F_2)}(P)/_{H(K_I)} = 0 .$$

Proof. – Applying Proposition 2.7 to the homomorphism $\iota : (H, H) \longrightarrow (G, H)$ we obtain the commutative diagram

$$\begin{tikzcd}
H(W(g, H)_I) \arrow[dd, "W(d\iota)^*"'] \arrow[rrd, "\Delta_*(P)"] & & \\
& & H_{DR}(M) \\
H(W(h, H)_I) \cong I(H)_I \arrow[rru, "\Delta_*(P')"'] & &
\end{tikzcd} \tag{4.2}$$

and hence $\Delta_*(P)/_{\mathrm{Ker}(W(d\iota)^*)} = 0$.

Moreover, there is a commutative diagram

$$\begin{tikzcd}
W(g, H)_I \arrow[r, "W(d\iota)"] & I(H)_I \\
A_I \arrow[u, "\zeta(W(g)_I, H)"] \arrow[r] & I(G)_I \otimes_{I(G)} I(H) \arrow[u, "\psi"']
\end{tikzcd}$$

where ψ is the canonical projection of

$$I(G)_I \otimes_{I(G)} I(H) \cong I(H)/I \cdot I(H)$$

onto $I(H)_I = I(H)/I$. Thus, going into cohomology, we obtain a factorization $H(W(g, H)_I) \longrightarrow I(G)_I \otimes_{I(G)} I(H) \longrightarrow I(H)_I$ of the vertical homomorphism in (4.2). Then, because

$$H(K_I) = \mathrm{Ker}\,\{H(W(g, H)_I) \longrightarrow I(G) \otimes_{I(G)} I(H)\}$$

by virtue of Proposition 4.2, we have $H(K_I) \subset \mathrm{Ker}\,(W(d\iota)^*)$.

□

Moreover, as in the usual case of foliated bundles [7], we have

PROPOSITION 4.4. – *There is a splitting homomorphism*

$$\kappa : I(G)_I \otimes_{I(G)} I(H) \longrightarrow H(W(g, H)_I)$$

of the short exact sequence (4.1) *and the composition* $\Delta_*(P) \circ \kappa$ *is induced by the characteristic homomorphism of* P' :

$$h_*(P') : I(H) \longrightarrow H_{DR}(M).$$

5. Restriction to the leaves.

In this section we shall discuss the restriction of an (F_1, F_2)-foliated bundle $P = P_1 + P_2$ to the leaves of each foliation F_i, $i = 1, 2$. In order to do that, let us previously discuss the restriction to the leaves of an F_2-foliated bundle.

So, let (F_1, F_2) be a (q_1, q_2)-codimensional subfoliation on M, L a leaf of F_1 and $j : L \longrightarrow M$ the canonical immersion. Since $F_2 \subset F_1$, F_2 induces on L a foliation which will be denoted by F_L ; note that codim $(F_L) = d = q_2 - q_1$ while codim $(F_2) = q_2$. Obviously, j maps the leaves of F_L into leaves of F_2 .

Now, let $\pi : P \longrightarrow M$ be a G-principal fibre bundle and denote $P' = j^*P$ the inverse image of P via j. Then $\pi' : P' \longrightarrow L$, the restriction of P to L, is a G-principal fibre bundle and we shall denote $\bar{j} : P' \longrightarrow P$ the canonical injection. The following result is known [1] :

PROPOSITION 5.1. – *If* P *is* F_2*-foliated then* P' *is* F_L*-foliated. Moreover, if* ω *is an adapted connection in* P *then* $\bar{j}^* \omega$ *is an adapted connection in* P'.

Precisely the latter condition allows to consider, using connections ω and $\bar{j}^*\omega$, a commutative diagram

$$\begin{tikzcd} H(W(g, H)_{q_2}) \arrow[r, "\Delta_*(P)"] \arrow[d, "p^*"'] & H_{DR}(M) \arrow[d, "j^*"] \\ H(W(g, H)_d) \arrow[r, "\Delta_*(P')"] & H_{DR}(L) \end{tikzcd} \tag{5.1}$$

where $p : W(g, H)_{q2} \longrightarrow W(g, H)_d$ is the canonical projection $(d \leqq q)$, $H \subset G$ is a subgroup satisfying the usual hypothesis and $\Delta_*(P)$, $\Delta_*(P')$ are the generalized characteristic homomorphisms of P and P′.

For example, $Q_0 = F_1/F_2$ (the normal bundle of F_2 relative to F_1) is an F_2-foliated vector bundle on account of the existence on it of the so-called Bott connection [4], [1]. Moreover, $Q_L = TL/F_L$, the normal bundle of F_L, is canonically isomorphic to $j^* Q_0$ [1] in such way that the Bott connection in Q_0 pulls back via j to the Bott connection in Q_L. Therefore, the frame bundle of Q_0, P, is an F_2-foliated $Gl(d, \mathbf{R})$-principal bundle, and $P' = j^* P$ is precisely the bundle of transverse frames of F_L. Thus, through the corresponding isomorphisms, diagram (5.1) becomes:

$$\begin{tikzcd}
H(W(gl(d, \mathbf{R}), O(d))_{q_2}) \arrow[r, "\Delta_*"] \arrow[d, "p^*"'] & H_{DR}(M) \arrow[d, "j^*"] \\
H(W(gl(d, \mathbf{R}), O(d))_d) \cong H(WO_d) \arrow[r, "\Delta'_*"] & H_{DR}(L)
\end{tikzcd}$$

where Δ'_* is just the usual characteristic homomorphism of foliation F_L on L.

Next, we shall discuss the restriction to a leaf of F_2. Thus, provided that we do not need to use the foliation F_1, we shall assume only one foliation F on M, L a leaf of F and $j : L \longrightarrow M$ the canonical immersion. Now if $\pi : P \longrightarrow M$ is a G-principal bundle and $P' = j^* P$ is the inverse image of P via j, we have

PROPOSITION 5.2. – *Each F-foliated bundle structure on P determines a flat bundle structure on* P′ *in such way that if* ω *is an adapted connection in* P, *then* $\omega' = \bar{j}^* \omega$ *is a flat connection in* P′.

Therefore, if we consider on M the subfoliation (F, F) then the foliation F_L induced on L is trivial, that is, $F_L = TL$, and taking into account that $W(g,H)_0 \cong (\Lambda g^*)_H \cong \Lambda(g/h)^{*H}$, diagram (5.1) becomes

$$\begin{tikzcd}
H(W(g,H)_q) \arrow[r, "\Delta_*"] \arrow[d, "p^*"'] & H_{DR}(M) \arrow[d, "j^*"] \\
H(g,H) \arrow[r, "\Delta'_*"] & H_{DR}(L)
\end{tikzcd} \qquad (5.2)$$

Δ'_* being the generalized characteristic homomorphism of P' as flat bundle [7].

Example. – Let P be the bundle of transverse frames of F. Then, if $\nu F = TM/F$ is the normal bundle of F, $\nu L = \nu F/_L$ is the normal bundle of the leaf L of F and $P' = j^*P$ is just the bundle of frames of νL. Following Goldman [6], any connection in P adapted to its canonical structure of F-foliated bundle will be said a foliation connection, and a connection in P' obtained as inverse image of a foliation connection will be said a leaf connection. In fact, Goldman showed that there is an unique leaf connection which is flat, and one easily checks that Δ'_* in diagram (5.2) is nothing but the so-called holonomy homomorphism of the leaf L [6].

Again, let (F_1, F_2) be a (q_1, q_2)-codimensional subfoliation on M, L_1 a leaf of F_1, $j_1 : L_1 \longrightarrow M$ the canonical immersion, F_{L_1} the foliation on L_1 induced by F_2, $P = P_1 + P_2$ an (F_1, F_2)-foliated bundle on M and $P' = j_1^*P$ its inverse image via j_1. Then, since P is also F_2-foliated we can apply to it all previous results; so, in particular, we can construct a diagram (5.1) for this $P = P_1 + P_2$. On the other hand,

$$P' = j_1^*P_1 + j_1^*P_2 ;$$

then, applying the previous results to each $j_1^*P_i$, $i = 1, 2$, it follows that P' is (TL_1, F_{L_1})-foliated over L_1. Moreover, if ω is an adapted connection sum in P then $\omega' = j_1^*\omega$ is an adapted connection sum in P'. If I and I' are the ideals given by (2.1) for the pairs (q_1, q_2) and $(0, d)$, respectively, then

$I \subset I'$ and, for an appropiate subgroup H, we obtain a commutative diagram

$$\begin{tikzcd}
H(W(g,H)_I) \arrow[r, "\Delta_{(F_1,F_2)}(P)"] \arrow[d, "p'^*"'] & H_{DR}(M) \arrow[d, "j_1^*"] \\
H(W(g,H)_{I'}) \arrow[r, "\Delta_{(TL_1,F_{L_1})}(P')"] & H_{DR}(L_1)
\end{tikzcd} \tag{5.3}$$

where p'^* is induced by the canonical projection. If we now combine (5.3) with (5.1) through (3.2), we obtain

$$\begin{tikzcd}
H(W(g,H)_{q_2}) \arrow[rr, "\Delta_{F_2}(P)"] \arrow[dr, "q^*"] \arrow[dd, "p^*"'] & & H_{DR}(M) \arrow[dd, "j_1^*"] \\
& H(W(g,H)_I) \arrow[ur, "\Delta_{(F_1,F_2)}(P)"'] \arrow[dd, "p'^*"'] & \\
H(W(g,H)_d) \arrow[rr, "\Delta_{F_{L_1}}(P')"] \arrow[dr, "q'^*"'] & & H_{DR}(L_1) \\
& H(W(g,H)_{I'}) \arrow[ur, "\Delta_{(TL_1,F_{L_1})}(P')"'] &
\end{tikzcd} \tag{5.4}$$

Now, if L_2 is a leaf of F_2 and $j_2 : L_2 \longrightarrow M$ is its canonical immersion, then (F_1, F_2) induces on L_2 the trivial subfoliation (TL_2, TL_2). Therefore, the restriction to L_2 of an (F_1, F_2)-foliated bundle $P = P_1 + P_2$ is a flat bundle, and hence we obtain a commutative diagram similar to (5.4):

$$\begin{tikzcd}
H(W(g,H)_{q_2}) \arrow[rr, "\Delta_{F_2}(P)"] \arrow[dr, "q^*"] \arrow[dd, "p^*"'] & & H_{DR}(M) \arrow[dd, "j_2^*"] \\
& H(W(g,H)_I) \arrow[ur, "\Delta_{(F_1,F_2)}(P)"'] \arrow[dl, "p'^*"'] & \\
H(g,H) \arrow[rr, "\Delta_*(P')"] & & H_{DR}(L_2)
\end{tikzcd} \tag{5.5}$$

If we now assume that L_1 contains L_2, $j_0 : L_2 \longrightarrow L_1$ being the canonical immersion with $j_2 = j_1 \circ j_0$, then, using Proposition 3.1 and taking the closed subgroups

$$H_1 \subset G_1 \,,\; H_2 \subset G_2 \,,\; H = H_1 \times H_2$$

and $\rho_i : G = G_1 \times G_2 \longrightarrow G_i$, $i = 1, 2$, the canonical projections, there is a commutative diagram

$$\begin{tikzcd}
H(g_1 H_1) \arrow[r, "\rho_1^*"] \arrow[d, "\Delta_*(j_1^* P_1)"] & H(g, H) \arrow[d, "\Delta_*(P')"] & H(g_2, H_2) \arrow[l, "\rho_2^*"'] \arrow[dl, "\Delta_*(j_2^* P_2)"] \\
H_{DR}(L_1) \arrow[r, "j_0^*"] & H_{DR}(L_2) &
\end{tikzcd} \tag{5.6}$$

All these results, when particularized in certain examples, provide a starting point for a study of the holonomy of the leaves of a subfoliation similar to that of Goldman [6] for the leaves of a foliation.

Example. – With the previous notations, let

$$P = L(Q_1) + L(Q_0)$$

be the bundle of transverse frames of (F_1, F_2) and let (L_1, L_2) be a "leaf" of (F_1, F_2) (that is, L_i leaf of F_i and $L_2 \subset L_1$); then $P' = j_2^* P = L(j_0^*(\nu L_1)) + L(j_2^*(Q_0))$ is a reduction of the bundle of frames of the "normal bundle of (L_1, L_2)" defined as $\nu(L_1, L_2) = j_0^*(\nu L_1) \oplus j_2^*(Q_0)$, νL_1 being the normal bundle of the leaf L_1 [6], that is, $\nu L_1 = Q_1/L_1$. With a terminology analogous to that of Goldman, we call leaf connection any connection in P' obtained by pull-back of any adapted connection sum in P. Then, the following proposition can be easily proved :

PROPOSITION 5.3. – *There exists a unique leaf connection in* P'. *Moreover, this connection is flat.*

Through this result, we can state easily vanishing and obstruction theorems for the leaves of a subfoliation similar to those in [6] for the leaves of a foliation, because the real Pontrjagin ring of $\nu(L_1, L_2)$ is trivial and this fact provides a necessary condition for a pair (N_1, N_2) of connected manifolds, with injective immersions $j_i : N_i \longrightarrow M$, $i = 1, 2$, $j_0 : N_2 \longrightarrow N_1$ and $j_2 = j_1 \circ j_0$, to be a leaf of a subfoliation on M.

Now, if $P'' = L(Q_2)$ and

$$\rho : \mathrm{Gl}(q_1, \mathbf{R}) \times \mathrm{Gl}(d, \mathbf{R}) \longrightarrow \mathrm{Gl}(q_2, \mathbf{R})$$

is the canonical homomorphism, then, using the example that follows Proposition 3.2 and taking a closed subgroup

$$H' \subset \mathrm{Gl}(q_2, \mathbf{R})$$

such that $\rho(H) \subset H'$, we first obtain a commutative diagram combining (5.2) vith (5.5):

$$\begin{tikzcd}
H(W(gl(q_1, \mathbf{R}) \oplus gl(d, \mathbf{R}), H)_I) \arrow[rr, "\Delta_{(F_1, F_2)}(P)"] \arrow[dd, "p'^*"'] & & H_{DR}(M) \arrow[dd, "j_2^*"] \\
& H(W(gl(q_2, \mathbf{R}), H')_{q_2}) \arrow[ul, "q^* \circ W(d\rho)^*"'] \arrow[ur, "\Delta_{F_2}(P'')"'] \arrow[dd, "p^*"] & \\
H(gl(q_1, \mathbf{R}) \oplus gl(d, \mathbf{R}), H) \arrow[rr, "\Delta_*(P')"] & & H_{DR}(L_2) \\
& H(gl(q_2, \mathbf{R}), H') \arrow[ul, "\rho^*"] \arrow[ur, "\Delta_*(j_2^* P'')"'] &
\end{tikzcd} \tag{5.7}$$

and then, taking into account diagram (5.6):

$$\begin{tikzcd}
H(gl(q_2, \mathbf{R}), H') \arrow[d, "\rho^*"] \arrow[dr, "\Delta_*(j_2^* P'')"] & \\
H(gl(q_1, \mathbf{R}) \oplus gl(d, \mathbf{R}), H) \arrow[r, "\Delta_*(P')"] & H_{DR}(L_2) \\
H(gl(q_1, \mathbf{R}), H) \arrow[u, "\rho_1^*"] \arrow[r, "\Delta_*(j_1^* P_1)"] & H_{DR}(L_1) \arrow[u, "j_0^*"']
\end{tikzcd} \tag{5.8}$$

Now, we assume $H = O(q_1) \times O(d)$ and $H' = O(q_2)$. In this case Goldman shows that p^* in diagram (5.7) is the zero homomorphism and concludes that the secondary foliation classes of F_2 vanish in the leaves L_2. Essentially with the same arguments, one can prove that the homomorphism p'^* in diagram (5.7) is also zero and assert that the restriction to L_2 of every secondary subfoliation class of (F_1, F_2) vanishes. Moreover, the homomorphism $\Delta_*(P')$ is similar to the holonomy homomorphism defined by Goldman, and hence it can be called the holonomy homomorphism of the leaf (L_1, L_2) and denoted by $\phi^*_{F,L}$. Then, diagram (5.8) relate the holonomy homomorphism of (L_1, L_2) with that of each L_i, $i = 1, 2$. Through the canonical isomorphisms we obtain the following commutative diagram:

$$\begin{tikzcd}
\Lambda(h_1, h_3, \ldots, h_{\ell_2}) \arrow[d, "\rho^*"] \arrow[dr, "\phi^*_{F_2, L_2}"] & \\
\Lambda(h'_1, h'_3, \ldots, h'_{\ell'}) \otimes \Lambda(h''_1, h''_3, \ldots, h''_{\ell''}) \arrow[r, "\phi^*_{F,L}"] & H_{DR}(L_2) \\
\Lambda(h_1, h_3, \ldots, h_{\ell_1}) \arrow[u, "\rho_1^*"] \arrow[r, "\phi^*_{F_1, L_1}"] & H_{DR}(L_1) \arrow[u, "j_0^*"']
\end{tikzcd}$$

where $\ell_i = 2[(q_i + 1)/2] - 1$, $i = 1, 2$; $\ell' = 2[(d + 1)/2] - 1$; $\ell'' = \ell_1$.

Obviously, the case of a subfoliation with trivialized normal bundle can be also discussed; to do that, it suffices to take H' as the trivial subgroup, and the diagram (5.8) becomes

$$\begin{tikzcd}
\Lambda(h_1, h_2, \ldots, h_{q_2}) \arrow[d, "\rho^*"] \arrow[dr, "\phi^*_{F_2, L_2}"] & \\
\Lambda(h'_1, h'_2, \ldots, h'_{d'}) \otimes \Lambda(h''_1, h''_2, \ldots, h''_{q_1}) \arrow[r, "\phi^*_{F, L}"] & H_{DR}(L_2) \\
\Lambda(h_1, h_2, \ldots, h_{q_1}) \arrow[u, "\rho^*_1"] \arrow[r, "\phi^*_{F_1, L_1}"] & H_{DR}(L_1) \arrow[u, "j^*_0"']
\end{tikzcd}$$

This result may be used in order to obtain topological obstructions to the existence of subfoliations. Reinhart [10] exhibits a first example of these obstructions which can be expressed in our language as follows.

Let (F_1, F_2) be a $(1, 2)$-codimensional subfoliation on a manifold M with trivialized normal bundle; suppose F_1 defined by the global 1-form α_2 and F_2 defined by the global 1-forms α_1, α_2. Hence there exist 1-forms $\tau_{11}, \tau_{21}, \tau_{22}$ on M such that $d\alpha_1 = \alpha_1 \wedge \tau_{11} + \alpha_2 \wedge \tau_{21}$, $d\alpha_2 = \alpha_2 \wedge \tau_{22}$.

If (L_1, L_2) is a leaf of (F_1, F_2), let us consider the 1-forms on L_2 given by

$$\tau^L_{11} = j^*_2(\tau_{11}), \quad \tau^L_{21} = j^*_2(\tau_{21}), \quad \tau^L_{22} = j^*_2(\tau_{22}).$$

In this case, the previous diagram writes, at the cochain level, as

$$\begin{tikzcd}
\Lambda(h_1, h_2) \arrow[d, "\rho^*"] \arrow[dr, "\phi_{F_2, L_2}"] & \\
\Lambda(h'_1) \otimes \Lambda(h''_1) \arrow[r, "\phi_{F, L}"] & \Omega(L_2) \\
\Lambda(h_1) \arrow[u, "\rho^*_1"] \arrow[r, "\phi_{F_1, L_1}"] & \Omega(L_1) \arrow[u, "j^*_0"']
\end{tikzcd}$$

and, from it, we obtain the following holonomy classes :

a) for L_1 as leaf of F_1 :

$$\phi^*_{F_1,L_1}(h_1) = [\tau_{22/L_1}] \in H_{DR}(L_1)$$

b) for L_2 as leaf of F_2 :

$$\phi^*_{F_2,L_2}(h_1) = [\tau^L_{11} + \tau^L_{22}] \in H_{DR}(L_2), \quad \phi^*_{F_2,L_2}(h_2) = 0$$

since $h_2 \in \operatorname{Ker} \rho^*$. In fact, Reinhart shows the vanishing of $\phi^*_{F_2,L_2}(h_2)$ through a direct computation.

c) for (L_1, L_2) as leaf of (F_1, F_2) :

$$\phi^*_{F,L}(h'_1) = [\tau^L_{11}] \in H_{DR}(L_2), \quad \phi^*_{F,L}(h''_1) = [\tau^L_{22}] \in H_{DR}(L_2)$$

$$\phi^*_{F,L}(h'_1 + h''_1) = [\tau^L_{11} + \tau^L_{22}] \in H_{DR}(L_2).$$

Now, by comparing with Reinhart results one can deduce :

1) the vanishing of certain holonomy classes of L_2 follows from the fact that they are obtained from elements of $\operatorname{Ker} \rho^*$.

2) the image of $\Lambda(h'_1)$ by $\phi^*_{F,L}$ gives holonomy classes which cannot be obtained if we consider each leaf separately.

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